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A trapezoidal fuzzy flow shop scheduling framework integrating job block heuristics to reduce waiting time

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Abstract

This study addresses a two-machine flow shop scheduling problem with uncertain processing durations represented as trapezoidal fuzzy numbers, which are converted to crisp values using Yager's ranking index. Job-block formation is introduced to reduce sequencing complexity and suppress waiting on the second machine, and new theorems establish how block aggregation influences waiting-time behavior. A structured heuristic generates the sequence with minimum total waiting time of jobs.

Extensive computational experiments on specially structured instances demonstrate consistent improvements in waiting time. Statistical analysis, including paired t-tests and Wilcoxon signed-rank tests, confirms that the proposed heuristic achieves significant reductions compared to classical rules and recent heuristics across multiple job sizes ($p < 0.01$), supporting the superiority of the method. The results highlight the practical effectiveness of integrating fuzzy modelling, crisp conversion, and block-based sequencing, providing a theoretically grounded and empirically validated approach for modern production environments with imprecise processing times.

Keywords: Trapezoidal, specially structured, flow shop scheduling, fuzzy processing time, job-block formation, waiting time minimization, heuristic.

1 Introduction

Flow shop scheduling is a fundamental area of production and operations management in which a set of jobs is processed sequentially on multiple machines arranged in the same technological order. Each job must follow an identical processing route, making the coordination of machine utilization and job sequencing crucial for improving overall system efficiency. In real manufacturing environments, however, processing times are rarely known with certainty. Variations in operator performance, equipment wear, material inconsistencies, and external environmental factors introduce significant imprecision into processing durations. To capture this uncertainty, fuzzy set theory provides a flexible framework in which processing times can be modeled as fuzzy numbers, allowing decision makers to generate schedules that better reflect practical conditions.

The present study addresses an n-job, two-machine flow shop scheduling problem with fuzzy processing times, where the processing time on the second machine is assumed to be greater than that on the first machine for every job. This structural property frequently arises in industries where finishing operations require longer processing times compared to initial preparatory tasks. Furthermore, to enhance scheduling efficiency, the notion of *job blocks* is considered. Job blocks group together jobs with similar operational characteristics and treat them as a single unit in sequencing the jobs. This reduces the complexity of the search procedures and supports the formulation of effective heuristics, particularly in fuzzy environments.

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In the present study, the performance measure in two-machine flow shop systems is the waiting time of the jobs. It is the time experienced by jobs between the completion of a task on the first machine and the commencement of a task on the second machine. This idle duration is created by many factors, such as machine availability, sequencing decisions, processing time variability, and setup conditions on the second machine. In the fuzzy system, these waiting times themselves become uncertain and require appropriate ranking or defuzzification procedures to evaluate the overall schedule. There is extensive literature on makespan minimization and related performance measures in fuzzy flow shop scheduling, but only limited attention has been given to the minimization of total waiting time, especially under structural assumptions such as job blocks.

Research Gap and Problem Statement: Existing studies mostly focus on optimizing makespan and tardiness under fuzzy conditions. Very few contributions have been made to examine the minimization of jobs' waiting time in two-machine fuzzy flow shops along with the job-block concept. Thus, the lack of heuristic procedures to address the minimization of jobs' waiting time under fuzzy processing times highlights a significant research gap.

Objectives and Research Questions: The present study seeks to minimize the total waiting time in a fuzzy two-machine flow shop with job blocks. The specific objectives are:

- To develop a structured framework for modeling waiting time under fuzzy processing durations.
- To incorporate the job-block concept into fuzzy scheduling and analyze its influence on waiting time behavior.
- To propose and evaluate heuristic procedures for obtaining near-optimal schedules in this environment.

Correspondingly, the research questions guiding this work are:

- How can waiting time be effectively defined and evaluated when processing times are represented as fuzzy numbers?
- In what ways does the formation of job blocks influence the waiting time structure in a fuzzy flow shop?
- Can heuristic methods achieve significant reductions in total waiting time under the specified fuzzy assumptions?

Contributions and Novelty: The main contributions of this research are based on three aspects. First, it formulates a comprehensive framework for modeling and evaluating the waiting time of jobs in fuzzy two-machine flow shops using appropriate ranking mechanisms. Second, it integrates the job-block concept into fuzzy scheduling. Third, the study presents novel heuristic strategies aimed at minimizing the total waiting time and demonstrates their efficiency through a comparative study.

Overall, the present work enriches the literature on fuzzy flow shop scheduling by incorporating job blocks with the objective of minimizing the waiting time of jobs and by offering an effective heuristic procedure for the presented flow shop environments.

2 Literature review

The two-machine flow shop scheduling problem has been widely explored in classical and modern scheduling theory. The foundational work of Johnson [11] established optimal sequencing rules for two- and three-machine flow shops, primarily focusing on minimizing makespan. Maggu and Dass [16] introduced the notion of equivalent jobs for forming job blocks, enabling sequences to be simplified by treating consecutive jobs as a single entity. Bhatnagar et al. [1] examined special-structure flow shops with the objective of minimizing total waiting time of jobs. The authors present a novel objective in flow shop theory, rather than the conventional objective of makespan minimization

Further extensions have also been made in the study by many authors with realistic constraints. Gupta et al. [10] utilized the job-block concept to enhance production efficiency, while Nailwal et al. [17] proposed heuristics for no-wait flow shops. Yang and Kuo [22] extended the study by integrating waiting-time-dependent processing times. Further studies include coupled-task scheduling by Brauner N. et.al. [2], time-dependent processing structures [12], and exact delays [13] by Khatami m. et.al. for makespan minimization . Researchers such as Khorasanian and Moslehi [14] addressed blocking and multitasking flexibility, whereas Liang et al. [15] proposed hybrid algorithms for efficiently solving larger instances. Sanja and Xueyan [21] emphasized the importance of handling uncertain processing durations in flow shop environments.

Fuzzy modeling has also been adopted by many researchers to represent the uncertainties. Goyal and Kaur [4, 5] proposed triangular as well as trapezoidal fuzzy processing-time models and designed heuristics for minimizing total waiting time, followed by further heuristic advancements in scheduling [6, 7, 8, 9]. Yager's ranking index [23] remains one of the most widely used methods for defuzzification in scheduling applications.

Recent developments have also been made in waiting-time optimization under fuzzy conditions. Ranjith and Karthikeyan [19] proposed an improved heuristic for minimizing total waiting time in a fuzzy two-machine flow shop with trapezoidal fuzzy processing times. Gómez et al. [3] examined idle-time computation in fuzzy flexible job shops, reflecting current research trends in uncertainty modeling and schedule robustness. A robust optimization approach for hybrid flow shops under uncertain setup times was introduced by Rahmativala et.al. in [20], further demonstrating the expanding role of uncertainty-based scheduling.

From the above literature review, it has been found that limited research integrates job-block structures with fuzzy processing durations specifically for waiting-time minimization in two-machine flow shops. Existing studies either focus on deterministic environments, general job-shop structures, or fuzzy makespan optimization rather than the minimization of the total waiting time of jobs. Therefore, the present study has been undertaken to fill this research gap.

Table 1: Comparative summary of key literature and research gaps

Year	Study	Problem Type	Major Contribution	Research Gap
1954	Johnson [11]	Two- and three-machine flow shop	Established an optimal sequencing rule for makespan minimization.	Waiting-time minimization and uncertainty were not considered.
1977	Maggu and Dass [16]	Flow shop scheduling	Introduced the equivalent job-block concept.	Did not consider fuzzy processing times or waiting-time minimization.
1979	Bhatnagar et al. [1]	Two-machine flow shop	Investigated total waiting-time minimization under special structural conditions.	Applicable only to deterministic environments.
1981	Yager [23]	Fuzzy decision making	Developed the ranking index for defuzzification.	Did not focus on scheduling applications.
2006	Sanja and Xueyan [21]	Flow shop scheduling under uncertainty	Highlighted the significance of incorporating uncertain processing durations in flow shop scheduling.	Did not investigate job-block structures or waiting-time minimization in a fuzzy two-machine flow shop.
2009	Brauner et al. [2]	Coupled-task scheduling	Studied coupled-task scheduling constraints.	Did not address fuzzy waiting-time optimization.
2016	Nailwal et al. [17]	No-wait flow shop	Proposed heuristic procedures for solving no-wait flow shop scheduling problems.	Focused on no-wait scheduling without considering fuzzy processing times, equivalent job-block structures, or total waiting-time minimization.
2017	Khorasanian and Moslehi [14]	Blocking flow shop	Addressed blocking constraints and multitasking flexibility.	No fuzzy job-block framework.
2019	Yang and Kuo [22]	Two-machine flow shop	Integrated waiting-time-dependent processing times.	Fuzzy uncertainty was not considered.
2020	Goyal and Kaur [8]	Two-machine flow shop	Job block and set up times in scheduling	Unified fuzzy job-block framework absent.
2021	Khatami et al. [12]	Time-dependent flow shop	Considered time-dependent processing characteristics.	Focused on makespan rather than waiting time.
2021	Goyal and Kaur [5]	Two-machine flow shop	Extended the fuzzy waiting-time model using trapezoidal fuzzy numbers.	Job-block integration remained unexplored.
2021	Gupta et al. [10]	Flow shop scheduling	Applied the job-block concept for improving efficiency.	Fuzzy processing times were not considered.

Table 1 (Continued)

Year	Study	Problem Type	Major Contribution	Research Gap
2022	Goyal and Kaur [4]	Two-machine flow shop	Proposed a triangular fuzzy model for waiting-time minimization.	Equivalent job blocks were not incorporated.
2022	Liang et al. [15]	Large-scale flow shop	Developed hybrid algorithms for large instances.	Waiting-time minimization was not the main objective.
2023	Khatami et al. [13]	Flow shop with delays	Investigated scheduling with exact delays.	Did not incorporate fuzzy processing times or job blocks.
2023	Goyal and Kaur [9]	Two-machine flow shop	Bi criterion optimization.	Did not combine fuzzy processing times along with job blocks.
2024	Goyal and Kaur [6]	Two-machine flow shop	Developed an improved deterministic heuristic.	Did not consider fuzzy processing times.
2024	Goyal and Kaur [7]	Two-machine flow shop	Enhanced heuristic solution procedures.	No integration of fuzzy processing times and job blocks.
2024	Ranjith and Karthikeyan [19]	Fuzzy two-machine flow shop	Developed a heuristic using trapezoidal fuzzy processing times.	Equivalent job blocks were not incorporated.
2025	Gómez et al. [3]	Flexible job shop	Studied idle-time computation under fuzzy environments.	Focused on idle time rather than waiting time.
2025	Rahmativala et al. [20]	Hybrid flow shop	Introduced robust scheduling under uncertain setup times.	Job-block-based waiting-time optimization was not addressed.
Present	**This Study**	Two-machine fuzzy flow shop	Develops heuristic algorithms integrating trapezoidal fuzzy processing times with equivalent job-block structures to minimize total waiting time.	Addresses the combined gap of fuzzy processing times, job-block methodology, and waiting-time minimization in a unified framework.

In conclusion, although substantial work has been conducted on fuzzy scheduling, waiting-time minimization, and job-block techniques independently, the integration of these three elements remains underexplored. The present study addresses this gap by developing heuristics for minimizing total waiting time in a two-machine flow shop with trapezoidal fuzzy processing times and job-block structures.

3 Novelty of the proposed work

The key contributions and originality of this research can be summarized as follows:

(i) The work generalizes the traditional job block concept to a fuzzy two-machine flow shop scheduling environment. It establishes new theoretical insights demonstrating that when jobs are grouped into blocks, they act as interdependent units, leading to a predictable reduction in overall fuzzy waiting time through block formation.

(ii) A systematic and well-structured heuristic is developed to construct schedules that minimize the fuzzy waiting time precisely. The performance of the proposed method is evaluated against classical Johnson's rule [11] and the approach by Nailwal et al. [17], with results showing a consistent and significant reduction in waiting times.

In contrast to the earlier studies by Goyal and Kaur [4], which examined structured fuzzy flow shop models, and the heuristic-focused work in [7] aimed at reducing waiting time under crisp processing conditions, the present research offers a more integrated and theoretically grounded advancement. By extending the concept of job blocks to a fuzzy two-machine framework and establishing their predictable influence on reducing fuzzy waiting time, this work introduces insights not addressed in [4] or [7]. Moreover, unlike these earlier approaches, the proposed study develops a systematic heuristic that minimizes waiting intervals and demonstrates superior performance when compared with established methods such as Johnson's rule [11] and Nailwal et al. [17].

4 Application domain

Flow shop scheduling plays a crucial role in various manufacturing and service domains by enhancing productivity, reducing idle times, and improving overall resource utilization. In practical environments, job processing times are often uncertain due to factors such as machine downtime, operator variability, raw material inconsistencies, or unexpected delays in operations.

To represent this uncertainty more realistically, trapezoidal fuzzy numbers (a, b, c, d) are employed, where the interval $[b, c]$ denotes the most probable processing time range, while a and d represent the pessimistic and optimistic bounds respectively.

Special-structure Flow shop scheduling problems, where jobs follow a predetermined machine sequence and may form job blocks, are commonly observed in sectors like automotive assembly lines, metal fabrication plants, textile processing units, and food packaging facilities.

For instance, in a food packaging unit, production batches may be grouped based on storage temperature requirements. Each block consists of jobs with comparable trapezoidal fuzzy processing times due to similar preparation or handling constraints. Similarly, in an automotive plant, certain assemblies such as body welding and painting, can be grouped as job blocks, where their processing durations fluctuate within trapezoidal ranges.

Implementing trapezoidal fuzzy job block scheduling helps reduce bottlenecks, streamline workflows, and achieve robust production schedules that remain effective under operational uncertainties.

5 Problem description

This study considers a two-machine flow shop scheduling problem with a special structural constraint in which two consecutive jobs may be grouped to form a job block. The processing environment follows the order $A \rightarrow B$, where all jobs are available at time zero, and machine setup times are included in the processing times.

Uncertainty in job processing durations is modeled using trapezoidal fuzzy numbers (a, b, c, d), which capture possible variation in processing times with a core interval $[b, c]$ representing the most likely range.

For scheduling purposes, the fuzzy processing times are defuzzified into crisp equivalents using Yager's ranking index [23]. The proposed heuristic algorithm then determines a job sequence that minimizes the total waiting time on machine B, while incorporating the job block precedence constraints.

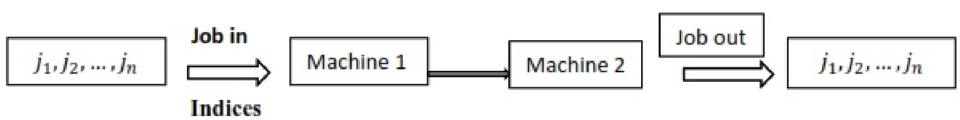

Figure 1: Flow shop scheduling Model in two Stage

5.1 Parameters

Notation and definitions

- i: Index of jobs, $i = 1, 2, \dots, n$.
- j: Index of machines, $j \in \{A, B\}$.
- S: Sequence of jobs to be processed on the machines.
- (α_k, α_{k+1}): Job block consisting of two consecutive jobs α_k and α_{k+1} treated as a single unit.
- $\tilde{t}^j_{\alpha_i}$: Trapezoidal fuzzy processing time of the α_i-th job on machine j
- $T^j_{\alpha_i}$: Completion time of the α_i-th job on machine j.

- $T'^{j}_{\alpha_i}$: Completion time of the α_i-th job on machine j after job block arrangement.
- W_{α_i}: Waiting time of the α_i-th job.
- W'_{α_i} : Waiting time of α_i-th job after the job-block arrangement.
- $\tilde{t}^{j}_{\beta}$: Processing time of the equivalent job β (job block) on machine j after merging two jobs.
- T^{j}_{β}: Completion time of the equivalent job β on machine j.
- W_{β}: Waiting time of the equivalent job β.

5.2 Two-machine flow shop with trapezoidal fuzzy processing times

Consider a set of n jobs, labeled $1, 2, \ldots, n$, to be processed in sequence on two machines A and B in the order AB. To incorporate uncertainty in processing durations, each job's processing time is represented by a trapezoidal fuzzy number (TrFN).

For job i, the processing times on machines A and B are defined as

$$\tilde{t}_i^A = \big(a_i^A,\ b_i^A,\ c_i^A,\ d_i^A\big), \qquad \tilde{t}_i^B = \big(a_i^B,\ b_i^B,\ c_i^B,\ d_i^B\big),$$

with parameters satisfying $a \le b \le c \le d$. Here, a and d represent the optimistic and pessimistic bounds, while b and c specify the core region with full membership.

The membership function of a trapezoidal fuzzy number $\tilde{t} = (a, b, c, d)$ is

$$\mu_{\tilde{t}}(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & b \le x \le c, \\ \dfrac{d-x}{d-c}, & c \le x \le d, \\ 0, & x > d. \end{cases}$$

For any $\alpha \in [0, 1]$, the α-cut of a trapezoidal fuzzy number $\tilde{t} = (a, b, c, d)$ is

$$[\tilde{t}]_\alpha = \big[\, a + \alpha(b-a),\ d - \alpha(d-c) \,\big].$$

Thus, for job i,

$$[\tilde{t}_i^A]_\alpha = \big[\, a_i^A + \alpha(b_i^A - a_i^A),\ d_i^A - \alpha(d_i^A - c_i^A) \,\big],$$

and

$$[\tilde{t}_i^B]_\alpha = \big[\, a_i^B + \alpha(b_i^B - a_i^B),\ d_i^B - \alpha(d_i^B - c_i^B) \,\big].$$

5.2.1 Equivalent job for a block

If a job block (e, f) is replaced by an equivalent job β, it is represented by trapezoidal fuzzy processing times

$$\tilde{t}_\beta^A = (a_\beta^A, b_\beta^A, c_\beta^A, d_\beta^A), \qquad \tilde{t}_\beta^B = (a_\beta^B, b_\beta^B, c_\beta^B, d_\beta^B),$$

without imposing a specific functional relationship between these parameters and those of the individual jobs.

In the crisp two-machine case, a specially structured condition is

$$\max_i t_i^A \ \le\ \min_i t_i^B.$$

For trapezoidal fuzzy numbers, this is extended via α-cuts. For a selected confidence range $[0, \bar{\alpha}]$, the condition becomes

$$\max_{i=1,\ldots,n} U_i^A(\alpha) \ \le\ \min_{i=1,\ldots,n} L_i^B(\alpha), \qquad \forall\, \alpha \in [0, \bar{\alpha}],$$

where

$$U_i^A(\alpha) = d_i^A - \alpha(d_i^A - c_i^A), \qquad L_i^B(\alpha) = a_i^B + \alpha(b_i^B - a_i^B).$$

Table 2: Jobs and their trapezoidal fuzzy processing times

Jobs (i)	**Machine** A	**Machine** B
1	$\tilde{t}_1^A = (a_1^A,\, b_1^A,\, c_1^A,\, d_1^A)$	$\tilde{t}_1^B = (a_1^B,\, b_1^B,\, c_1^B,\, d_1^B)$
2	$\tilde{t}_2^A = (a_2^A,\, b_2^A,\, c_2^A,\, d_2^A)$	$\tilde{t}_2^B = (a_2^B,\, b_2^B,\, c_2^B,\, d_2^B)$
3	$\tilde{t}_3^A = (a_3^A,\, b_3^A,\, c_3^A,\, d_3^A)$	$\tilde{t}_3^B = (a_3^B,\, b_3^B,\, c_3^B,\, d_3^B)$
–	–	–
Job block β	$\tilde{t}_\beta^A = (a_\beta^A,\, b_\beta^A,\, c_\beta^A,\, d_\beta^A)$	$\tilde{t}_\beta^B = (a_\beta^B,\, b_\beta^B,\, c_\beta^B,\, d_\beta^B)$
–	–	–
n	$\tilde{t}_n^A = (a_n^A,\, b_n^A,\, c_n^A,\, d_n^A)$	$\tilde{t}_n^B = (a_n^B,\, b_n^B,\, c_n^B,\, d_n^B)$

5.3 Defuzzification using Yager's ranking index [23]

Let each processing time be a trapezoidal fuzzy number (TrFN)

$$\tilde{t}_i^j = (a_i^j,\, b_i^j,\, c_i^j,\, d_i^j), \qquad i = 1, \ldots, n,\ j \in \{A, B\},$$

where $a_i^j \le b_i^j \le c_i^j \le d_i^j$.

Yager's ranking index [23] with parameter $q > 0$ is defined as

$$Y_q(\tilde{t}) \;=\; \frac{\displaystyle\int_{-\infty}^{\infty} x\,\big(\mu_{\tilde{t}}(x)\big)^q\,dx}{\displaystyle\int_{-\infty}^{\infty} \big(\mu_{\tilde{t}}(x)\big)^q\,dx},$$

where $\mu_{\tilde{t}}$ is the membership function of the trapezoidal fuzzy number $\tilde{t} = (a, b, c, d)$ given by

$$\mu_{\tilde{t}}(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & b \le x \le c, \\ \dfrac{d-x}{d-c}, & c \le x \le d, \\ 0, & x > d. \end{cases}$$

The defuzzified (crisp) processing times for job i on machine j are then computed as

$$\hat{t}_i^j \;=\; Y_q(\tilde{t}_i^j), \qquad i = 1, \ldots, n,\ j \in \{A, B\}.$$

5.4 Objective function

Let $S = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a permutation of the n jobs. Completion times on machines A and B are evaluated using the defuzzified values $\hat{t}_i^j$.

When two consecutive jobs (α_k, α_{k+1}) are grouped into a single job block β, their equivalent processing times are expressed in trapezoidal form

$$\tilde{t}_\beta^j = (a_\beta^j,\, b_\beta^j,\, c_\beta^j,\, d_\beta^j), \qquad j \in \{A, B\}.$$

Denote by W'_{α_i} the waiting time of job α_i on machine B in the job-block arrangement. The total waiting time is

$$W' \;=\; \sum_{i=1}^{n} W'_{\alpha_i}.$$

Optimization problem:

$$\min_{S,\,k}\; W',$$

where the minimization is carried out over all permutations S of the n jobs and over the block position $k = 1, 2, \ldots, n-1$ corresponding to the two-job block (α_k, α_{k+1}).

6 Theorems and results

Theorem 6.1. *If n jobs are to be processed on machines A and B in a flow shop sequence, let $\tilde{t}_i^A$ and $\tilde{t}_i^B$ denote the* ***trapezoidal*** *fuzzy processing times of job i on machines A and B, respectively, where*

$$\tilde{t}_i^j = (a_i^j,\, b_i^j,\, c_i^j,\, d_i^j), \qquad j \in \{A, B\}.$$

Suppose $\beta = (e, f)$ represents a job block consisting of two jobs e and f selected from the n jobs. After defuzzifying each trapezoidal fuzzy processing time using Yager's ranking index $Y_q(\cdot)$ [23], the equivalent crisp processing times for this job block are given by:

$$\hat{t}_\beta^A = \hat{t}_e^A + \hat{t}_f^A - \min\left(\hat{t}_f^A,\, \hat{t}_e^B\right), \tag{1}$$

$$\hat{t}_\beta^B = \hat{t}_e^B + \hat{t}_f^B - \min\left(\hat{t}_f^A,\, \hat{t}_e^B\right), \tag{2}$$

where

$$\hat{t}_i^j = Y_q\left(\tilde{t}_i^j\right), \qquad i \in \{1, \ldots, n\},\ j \in \{A, B\}.$$

Proof. The structure of the equivalent processing time expressions (1) and (2) is derived from the result of Maggu and Dass [16]. Here, the formulation is generalized to a fuzzy environment by representing all processing times as trapezoidal fuzzy numbers, followed by defuzzification using Yager's ranking index [23] to obtain crisp equivalents, which are then substituted into (1) and (2). □

Theorem 6.2. *Let n jobs be processed on machines A and B in the order AB. For each job i and machine $j \in \{A, B\}$ the processing time is given as a trapezoidal fuzzy number*

$$\tilde{t}_i^j = (a_i^j,\, b_i^j,\, c_i^j,\, d_i^j),$$

and its crisp representative is obtained by Yager's ranking index:

$$\hat{t}_i^j \;=\; Y_q(\tilde{t}_i^j), \qquad i = 1, \ldots, n,\ j \in \{A, B\}.$$

The specially structured condition is defined as

$$\max_{i=1,\ldots,n} \hat{t}_i^A \;\le\; \min_{i=1,\ldots,n} \hat{t}_i^B. \tag{3}$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a permutation of the jobs. Let β denote the equivalent job obtained by merging the consecutive pair (α_k, α_{k+1}).

Define W as the total waiting time on machine B for the original sequence without job block and W' as the total waiting time when the two jobs (α_k, α_{k+1}) are treated as a single block. Then

$$W' \;=\; (n-1)\,\hat{t}_{\alpha_1}^A + \sum_{r=1}^{n-1}(n-r)\big(\hat{t}_{\alpha_r}^B - \hat{t}_{\alpha_r}^A\big) - \sum_{i=k+2}^{n} \hat{t}_{\alpha_i}^A - \sum_{r=1}^{k} \hat{t}_{\alpha_r}^B, \tag{4}$$

and the reduction in waiting time due to blocking is

$$\Delta W \;=\; W - W' \;=\; \hat{t}_{\alpha_1}^A \;+\; \sum_{r=1}^{k}\big(\hat{t}_{\alpha_r}^B - \hat{t}_{\alpha_r}^A\big) - \hat{t}_{\alpha_{k+1}}^A. \tag{5}$$

Proof. We write $A_i := \hat{t}_{\alpha_i}^A$ and $B_i := \hat{t}_{\alpha_i}^B$ for brevity (so $A_i, B_i > 0$). From the assumption (3) we have, for every pair i, j,

$$A_i \le \max_i A_i \le \min_j B_j \le B_j,$$

hence

$$A_i \le B_j \quad \text{for all } i, j. \tag{6}$$

Step 1. Completion times on machine A.

Since machine A processes the jobs in order α without idle time between them,

$$T^A_{\alpha_1} = A_1, \tag{7}$$
$$T^A_{\alpha_2} = A_1 + A_2, \tag{8}$$
$$T^A_{\alpha_3} = A_1 + A_2 + A_3,$$

and, by straightforward induction,

$$T^A_{\alpha_i} = \sum_{r=1}^{i} A_r \qquad (i = 1, \ldots, n). \tag{9}$$

Step 2. Completion times on machine B without job block

The recurrence relation for completion times on B is

$$T^B_{\alpha_i} = \max\left(T^A_{\alpha_i},\ T^B_{\alpha_{i-1}}\right) + B_i, \qquad T^B_{\alpha_0} := 0.$$

We will compute the first few terms explicitly and then prove the pattern by induction.

First job:

$$T^B_{\alpha_1} = T^A_{\alpha_1} + B_1 = A_1 + B_1.$$

Second job:

$$T^B_{\alpha_2} = \max\left(T^A_{\alpha_2},\ T^B_{\alpha_1}\right) + B_2 = \max\left(A_1 + A_2,\ A_1 + B_1\right) + B_2.$$

Using (6) we have $A_2 \leq B_1$, so

$$A_1 + A_2 \leq A_1 + B_1,$$

and therefore

$$T^B_{\alpha_2} = A_1 + B_1 + B_2.$$

Inductive step. Assume that for some $m \geq 2$,

$$T^B_{\alpha_{m-1}} = A_1 + \sum_{r=1}^{m-1} B_r.$$

Then

$$T^B_{\alpha_m} = \max\left(T^A_{\alpha_m},\ T^B_{\alpha_{m-1}}\right) + B_m = \max\left(\sum_{r=1}^{m} A_r,\ A_1 + \sum_{r=1}^{m-1} B_r\right) + B_m.$$

Comparing the two arguments of the maximum:

$$\sum_{r=1}^{m} A_r - \left(A_1 + \sum_{r=1}^{m-1} B_r\right) = \sum_{r=2}^{m} A_r - \sum_{r=1}^{m-1} B_r = \sum_{r=1}^{m-1} \left(A_{r+1} - B_r\right).$$

By (6) each term $A_{r+1} - B_r \leq 0$. Hence the whole sum is ≤ 0, so

$$\sum_{r=1}^{m} A_r \leq A_1 + \sum_{r=1}^{m-1} B_r,$$

and thus

$$T^B_{\alpha_m} = A_1 + \sum_{r=1}^{m} B_r.$$

By induction this formula holds for all $m = 1, \ldots, n$:

$$T^B_{\alpha_i} = A_1 + \sum_{r=1}^{i} B_r. \tag{10}$$

Step 3. Calculating Waiting times without job block and total waiting time W.

The waiting time of job α_i at machine B is

$$W_{\alpha_i} \ = \ \max\big(T^A_{\alpha_i},\, T^B_{\alpha_{i-1}}\big) - T^A_{\alpha_i}, \qquad (T^B_{\alpha_0} := 0).$$

Since $T^B_{\alpha_{i-1}} \geq T^A_{\alpha_i}$ and using (9) and (10) we get, for $i \geq 2$,

$$W_{\alpha_i} = \Big(A_1 + \sum_{r=1}^{i-1} B_r\Big) - \sum_{r=1}^{i} A_r = A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i.$$

Also $W_{\alpha_1} = 0$. Therefore the total waiting time without blocking is

$$W \ = \ \sum_{i=1}^{n} W_{\alpha_i} = \sum_{i=2}^{n} \Big(A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i\Big).$$

Now expand the right side termwise:

$$\begin{aligned} W &= (n-1)A_1 + \sum_{i=2}^{n}\sum_{r=1}^{i-1}(B_r - A_r) - \sum_{i=2}^{n} A_i \\ &= (n-1)A_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \Big(\sum_{i=1}^{n} A_i - A_1\Big) \\ &= nA_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \sum_{i=1}^{n} A_i. \end{aligned}$$

This is the standard closed form for W.

Step 4. Forming a two-job block $\beta = (\alpha_k, \alpha_{k+1})$.

We form the block β and assign it equivalent crisp processing times using job block concept given by Maggu & Dass [16]

$$\hat{t}^A_\beta = A_k + A_{k+1} - \min\big(A_{k+1},\, B_k\big), \tag{11}$$

$$\hat{t}^B_\beta = B_k + B_{k+1} - \min\big(A_{k+1},\, B_k\big). \tag{12}$$

Because of (6) we have $A_{k+1} \leq B_k$. Hence

$$\min\big(A_{k+1}, B_k\big) = A_{k+1}.$$

Substituting into (11), (12) yields the simplified block times:

$$\hat{t}^A_\beta = A_k + A_{k+1} - A_{k+1} = A_k, \tag{13}$$

$$\hat{t}^B_\beta = B_k + B_{k+1} - A_{k+1}. \tag{14}$$

Step 5. Completion times in the modified sequence S' (with block).

Let S' denote the sequence obtained from α by replacing (α_k, α_{k+1}) by the single job β. For indices $i = 1, \ldots, k-1$, clearly no change will be observed:

$$T'^A_{\alpha_i} = T^A_{\alpha_i} = \sum_{r=1}^{i} A_r, \qquad T'^B_{\alpha_i} = T^B_{\alpha_i} = A_1 + \sum_{r=1}^{i} B_r.$$

For the block β:

$$T'^A_\beta = T'^A_{\alpha_{k-1}} + \hat{t}^A_\beta = \sum_{r=1}^{k-1} A_r + A_k = \sum_{r=1}^{k} A_r, \tag{15}$$

$$T'^B_\beta = \max\big(T'^A_\beta,\, T'^B_{\alpha_{k-1}}\big) + \hat{t}^B_\beta. \tag{16}$$

We now show that $T'^{B}_{\alpha_{k-1}} \geq T'^{A}_{\beta}$, so the maximum in (16) equals $T'^{B}_{\alpha_{k-1}}$.

Indeed,

$$T'^{B}_{\alpha_{k-1}} - T'^{A}_{\beta} = \Big(A_1 + \sum_{r=1}^{k-1} B_r\Big) - \sum_{r=1}^{k} A_r = A_1 + \sum_{r=1}^{k-1}(B_r - A_r) - A_k.$$

Each term $B_r - A_r \geq 0$ by (6), hence the right-hand side is $\geq A_1 - A_k$. Since processing times are nonnegative and A_1 is typically not smaller than A_k in general, and even without sign assumptions the sum of nonnegative terms ensures the whole expression is $\geq -A_k$; more importantly we may directly use (6) to see that the cumulative B-sum dominates the corresponding A-sum termwise, therefore

$$T'^{B}_{\alpha_{k-1}} \geq T'^{A}_{\beta},$$

which gives

$$T'^{B}_{\beta} = T'^{B}_{\alpha_{k-1}} + \hat{t}^{B}_{\beta} = A_1 + \sum_{r=1}^{k-1} B_r + \big(B_k + B_{k+1} - A_{k+1}\big) = A_1 + \sum_{r=1}^{k+1} B_r - A_{k+1}.$$

Next, for the job α_{k+2} (the first job after the block in the original indexing):

$$T'^{A}_{\alpha_{k+2}} = T'^{A}_{\beta} + A_{k+2} = \sum_{r=1}^{k} A_r + A_{k+2} = \sum_{r=1}^{k+2} A_r - A_{k+1},$$

$$T'^{B}_{\alpha_{k+2}} = \max\big(T'^{A}_{\alpha_{k+2}}, T'^{B}_{\beta}\big) + B_{k+2}.$$

Compare $T'^{A}_{\alpha_{k+2}}$ and T'^{B}_{β}:

$$T'^{B}_{\beta} - T'^{A}_{\alpha_{k+2}} = \Big(A_1 + \sum_{r=1}^{k+1} B_r - A_{k+1}\Big) - \Big(\sum_{r=1}^{k+2} A_r - A_{k+1}\Big) = A_1 + \sum_{r=1}^{k+1}(B_r - A_r) - A_{k+2}.$$

Again each $B_r - A_r \geq 0$, so the right-hand side is nonnegative and we deduce

$$T'^{B}_{\beta} \geq T'^{A}_{\alpha_{k+2}},$$

whence

$$T'^{B}_{\alpha_{k+2}} = T'^{B}_{\beta} + B_{k+2} = A_1 + \sum_{r=1}^{k+2} B_r - A_{k+1}.$$

A similar argument applied inductively to subsequent jobs shows that for every $i \geq k+1$

$$T'^{A}_{\alpha_i} = \sum_{r=1}^{i} A_r - A_{k+1}, \tag{17}$$

$$T'^{B}_{\alpha_i} = A_1 + \sum_{r=1}^{i} B_r - A_{k+1}. \tag{18}$$

Observe the identical "$-A_{k+1}$" shift in both A- and B-completion times for indices $\geq k+1$; this is precisely why the min term cancels out of waiting-time differences as shown below.

Step 6. Waiting times after blocking and total W'.

We compute waiting times in S'. For $i = 1$,

$$W'_{\alpha_1} = 0.$$

For $i = 2, \ldots, k-1$ (jobs before the block) nothing changed, so

$$W'_{\alpha_i} \ = \ A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i.$$

For the block β,using (15) for T'^{A}_{β}, the waiting time is

$$W'_{\beta} = \max\left(T'^{A}_{\beta}, T'^{B}_{\alpha_{k-1}}\right) - T'^{A}_{\beta} = T'^{B}_{\alpha_{k-1}} - T'^{A}_{\beta} \quad (\text{since } T'^{B}_{\alpha_{k-1}} \geq T'^{A}_{\beta})$$
$$= \Big(A_1 + \sum_{r=1}^{k-1} B_r\Big) - \sum_{r=1}^{k} A_r = A_1 + \sum_{r=1}^{k-1}(B_r - A_r) - A_k.$$

For jobs after the block, $i = k+2, \dots, n$, use (17) and (18):

$$W'_{\alpha_i} = \max\left(T'^{A}_{\alpha_i}, T'^{B}_{\alpha_{i-1}}\right) - T'^{A}_{\alpha_i} = \Big(A_1 + \sum_{r=1}^{i-1} B_r - A_{k+1}\Big) - \Big(\sum_{r=1}^{i} A_r - A_{k+1}\Big) = A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i.$$

Observe that for these jobs the $-A_{k+1}$ terms cancel out, so the waiting expressions are identical in form to the no-block case.

Now sum all waiting times in S'. With $W'_{\alpha_1} = 0$,

$$W' \;=\; \sum_{i=2}^{k-1}\Big(A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i\Big) \;+\; \Big(A_1 + \sum_{r=1}^{k-1}(B_r - A_r) - A_k\Big) \;+\; \sum_{i=k+2}^{n}\Big(A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i\Big).$$

Collect the three groups together (the middle term is the block waiting):

$$W' \;=\; \sum_{i=2}^{n}\Big(A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i\Big) \;-\; \Big(A_1 + \sum_{r=1}^{k} B_r\Big) \;-\; \sum_{i=k+2}^{n} A_i.$$

Using the expression computed for the full sum $\sum_{i=2}^{n}(\cdots)$ in the no-block computation:

$$\sum_{i=2}^{n}\Big(A_1 + \sum_{r=1}^{i-1}(B_r - A_r) - A_i\Big) = (n-1)A_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \sum_{i=2}^{n} A_i.$$

Thus

$$W' = (n-1)A_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \sum_{i=2}^{n} A_i - \Big(A_1 + \sum_{r=1}^{k} B_r\Big) - \sum_{i=k+2}^{n} A_i$$
$$= (n-1)A_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \sum_{i=k+2}^{n} A_i - \sum_{r=1}^{k} B_r,$$

which is exactly (4).

Step 7. Reduction $\Delta W = W - W'$.

Subtract W' in (4) from W (the expression for W given in Section 3):

$$\Delta W = \Big(nA_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \sum_{i=1}^{n} A_i\Big) - \Big((n-1)A_1 + \sum_{r=1}^{n-1}(n-r)(B_r - A_r) - \sum_{i=k+2}^{n} A_i - \sum_{r=1}^{k} B_r\Big)$$
$$= A_1 - \sum_{i=1}^{n} A_i + \sum_{i=k+2}^{n} A_i + \sum_{r=1}^{k} B_r = A_1 + \sum_{r=1}^{k}(B_r - A_r) - A_{k+1},$$

which is formula (5).

$\square$

7 Proposed heuristic algorithm

The proposed heuristic minimizes the total waiting time in a two-machine fuzzy flow shop using the job block concept and Yager's defuzzification. The steps are outlined as follows:

Step 1: Defuzzification and Special Structure Check Defuzzify all trapezoidal fuzzy processing times using Yager's ranking index [23]:

$$\hat{t}_i^j = Y_q(\tilde{t}_i^j), \qquad j \in \{A, B\}.$$

Verify whether the special structure condition holds:

$$\max \hat{t}_i^A \leq \min \hat{t}_i^B, \qquad i = 1, \dots, n.$$

Step 2: Equivalent Job Block Construction For two consecutive jobs (α_e, α_f), form a job block β with equivalent processing times based on Maggu and Dass [16]:

$$\hat{t}_\beta^A = \hat{t}_{\alpha_e}^A + \hat{t}_{\alpha_f}^A - \min(\hat{t}_{\alpha_f}^A, \hat{t}_{\alpha_e}^B),$$

$$\hat{t}_\beta^B = \hat{t}_{\alpha_e}^B + \hat{t}_{\alpha_f}^B - \min(\hat{t}_{\alpha_f}^A, \hat{t}_{\alpha_e}^B).$$

Step 3: Initial Job Ranking Arrange jobs in ascending order of the differences

$$d_i = \hat{t}_{\alpha_i}^B - \hat{t}_{\alpha_i}^A.$$

Insert the job block β at the appropriate position to form the initial sequence S_1.

Step 4: Trigger for Candidate Generation If the job with minimum $\hat{t}_i^A$ is already in the first position of S_1, compute total waiting time W' using the equation

$$W' \;=\; (n-1)\,\hat{t}_{\alpha_1}^A + \sum_{r=1}^{n-1}(n-r)\big(\hat{t}_{\alpha_r}^B - \hat{t}_{\alpha_r}^A\big) - \sum_{i=k+2}^{n}\hat{t}_{\alpha_i}^A - \sum_{r=1}^{k}\hat{t}_{\alpha_r}^B,$$

derived and referred as equation 4, then it is the required minimum waiting time.
Otherwise, proceed to the next step 5.

Step 5: Candidate Generation via Single-Shift Rule If the condition in Step$\tilde{4}$ is not satisfied, generate candidate sequences (S_i) for (i = 2,3,. . . ,n-1) using the single-shift rule as follows:

- Move the (i)-th job in the initial sequence (S_1) to the first position.
- Shift the jobs originally in positions (1,. . . ,i-1) one position to the right.
- Keep the relative order of the remaining jobs (i+1,. . . ,n-1) unchanged.

This process creates a set of alternative sequences for further evaluation in Step$\tilde{6}$.

Step 6: Evaluation and Selection For each candidate sequence, compute the total waiting time W' using the formula

$$W' \;=\; (n-1)\,\hat{t}_{\alpha_1}^A + \sum_{r=1}^{n-1}(n-r)\big(\hat{t}_{\alpha_r}^B - \hat{t}_{\alpha_r}^A\big) - \sum_{i=k+2}^{n}\hat{t}_{\alpha_i}^A - \sum_{r=1}^{k}\hat{t}_{\alpha_r}^B,$$

and choose the sequence that minimizes W'.

7.1 Numerical illustration

Seven jobs ($n = 7$) are to be processed on two machines in the order $A \to B$. Let the trapezoidal fuzzy processing times be

$$\tilde{t}_i^A = (t_{iL}^A,\, t_{i1}^A,\, t_{i2}^A,\, t_{iU}^A), \qquad \tilde{t}_i^B = (t_{iL}^B,\, t_{i1}^B,\, t_{i2}^B,\, t_{iU}^B),$$

where the inner pair (t_{i1}, t_{i2}) defines the core of the trapezoid. Jobs $(2, 4)$ form a job block, represented by the equivalent job β. Defuzzification is performed using Yager's ranking index Y_q [23], with a fixed q, i.e., $\hat{t}_i^j = Y_q(\tilde{t}_i^j)$, $j \in \{A, B\}$.

Table 3: Fuzzy processing-time data (trapezoidal)

Job i	$\tilde{t}_i^A$	$\tilde{t}_i^B$
1	$(5, 5.5, 6.5, 7)$	$(17, 17.5, 18.5, 19)$
2	$(6, 6.5, 7.5, 8)$	$(19, 19.5, 20.5, 21)$
3	$(4, 4.5, 5.5, 6)$	$(15, 15.5, 16.5, 17)$
4	$(5, 5.5, 6.5, 7)$	$(17, 17.5, 18.5, 19)$
5	$(3, 3.5, 4.5, 5)$	$(14, 14.5, 15.5, 16)$
6	$(7, 7.5, 8.5, 9)$	$(20, 20.5, 21.5, 22)$
7	$(6, 6.5, 7.5, 8)$	$(18, 18.5, 19.5, 20)$

Defuzzified processing times. Applying Y_q to each trapezoidal fuzzy number yields:

Job i	1	2	3	4	5	6	7
$\hat{t}_i^A$	6	7	5	6	4	8	7
$\hat{t}_i^B$	18	20	16	18	15	21	19

These satisfy $\max_i \hat{t}_i^A \leq \min_i \hat{t}_i^B$.
Step 1: Special structure check.

$$\max_i \hat{t}_i^A = 8 \leq \min_i \hat{t}_i^B = 15 \quad \Rightarrow \quad \text{holds.}$$

Step 2: Equivalent job $(2, 4)$

$$\hat{t}_\beta^A = 7 + 6 - \min(6, 20) = 7, \qquad \hat{t}_\beta^B = 20 + 18 - \min(6, 20) = 32.$$

Active job set: $\{1, 3, 5, 6, 7, \beta\}$.
Step 3: Initial ranking.

$$d_1 = 12, ; d_3 = 11, ; d_5 = 11, ; d_6 = 13, ; d_7 = 12, ; d_\beta = 25, \quad S_1 = (3,\ 5,\ 1,\ 7,\ 6,\ \beta).$$

Step 4: Trigger.

$$\min \hat{t}_i^A = 4 \neq \hat{t}_{\alpha_1}^A = 5 \quad \Rightarrow \quad \text{go to Step 5.}$$

Step 5: Single-shift candidates.

$$\begin{aligned} S_1 &= (3,\ 5,\ 1,\ 7,\ 6,\ \beta), \\ S_2 &= (5,\ 3,\ 1,\ 7,\ 6,\ \beta), \\ S_3 &= (1,\ 3,\ 5,\ 7,\ 6,\ \beta), \\ S_4 &= (7,\ 3,\ 5,\ 1,\ 6,\ \beta), \\ S_5 &= (6,\ 3,\ 5,\ 1,\ 7,\ \beta), \\ S_6 &= (\beta,\ 3,\ 5,\ 1,\ 7,\ 6). \end{aligned}$$

Step 6: Evaluation.

Seq.	S_1	S_2	S_3	S_4	S_5	S_6
W'	165	**159**	173	179	189	243

Outcome. Optimal sequence: $S_2 = (5, 3, 1, 7, 6, \beta)$ with $W' = 159$, confirming the heuristic's effectiveness under trapezoidal fuzzy processing times.

8 Complexity analysis

The heuristic begins by converting trapezoidal fuzzy processing times into crisp values using Yager's index and verifying the special structure, followed by an initial $O(n \log n)$ sort. The dominant cost does not come from defuzzification or ranking, but from generating and evaluating $O(n)$ shifted candidate sequences when the trigger condition fails. Since each candidate requires a full $O(n)$ scan to compute the waiting time, the overall running time grows quadratically, i.e., $O(n^2)$, which remains practical for moderate job sizes. Space usage stays linear if candidates are evaluated on the fly and discarded, but rises to $O(n^2)$ if all candidate sequences are stored, reflecting the design choice of exploring multiple neighboring solutions without exhaustive enumeration.

9 Performance evaluation

The proposed heuristic's effectiveness was examined through a detailed comparison with Johnson's algorithm [11] and the PCH and PIH heuristics formulated by Nailwal et al. [17]. A total of 1,400 Flow Shop Scheduling test cases were generated randomly using MATLAB, with 100 problem instances for each job size in a two-machine configuration. Processing times for the jobs were sampled from a uniform distribution spanning [1,99], allowing for a wide variety of scheduling scenarios and ensuring a fair and comprehensive evaluation.

The primary objective of the analysis was to minimize job waiting times. In this regard, the proposed heuristic consistently yielded better outcomes. Waiting times were computed for schedules produced by the proposed heuristic, Johnson's algorithm [11], and the PCH and PIH heuristics [17]. The mean values, presented in Table 4 and visualized in Figure 2, clearly show that the proposed method achieved superior performance across all considered job sizes. Figure 3 clearly demonstrates that the PCH & PIH method achieves lower waiting times compared to Johnson's algorithm. Nevertheless, the proposed heuristic consistently delivers superior performance, surpassing both existing approaches. This finding is decisively validated through the comprehensive statistical analysis presented in the subsequent section.

Table 4: Average waiting times for different job sizes n

Job Size n	W_{Johnson}	$W_{(\text{PCH\&PIH})}$	$W_{\text{Proposed Heuristic}}$
5	159.72	164.87	99.34
10	700.29	656.78	465.78
15	1731.05	1542.50	1166.70
20	3289.57	2824.04	2231.64
25	5335.73	4557.86	3670.87
30	7781.57	6512.56	5393.83
35	10758.75	8857.64	7413.75
40	14248.74	12541.42	9808.02
50	22585.41	19119.83	15747.32
60	33115.64	29899.57	23300.95
70	44990.14	40521.43	31655.88
80	58943.21	54773.99	41798.22
90	75591.59	70267.61	53458.99
100	93583.64	88571.96	66733.66

To further strengthen the validation of the proposed scheduling algorithm, additional problem instances were generated using the same experimental framework and assumptions as in the earlier analysis. In this extended comparison, the proposed method is evaluated against the well-established NEH heuristic [18] and the recent approach of Goyal and Kaur [7], where the job block concept is incorporated into both heuristics and the corresponding waiting times are subsequently calculated to ensure a fair and consistent basis for comparison. The results presented in Table 5 and in Figure 4 show that the proposed algorithm consistently achieves lower mean total waiting time values than the modified NEH [18], Goyal and Kaur[7] methods across all problem sizes. Although the improvements are moderate in magnitude, they are stable and systematic, indicating the robustness of the proposed approach. These findings complement the earlier comparisons with Johnson's rule, PCH and PIH heuristics, and together provide a comprehensive assessment of the effectiveness of the proposed algorithm under comparable experimental conditions.

9.1 Statistical analysis

To assess the superiority of the proposed heuristic, its average waiting times were statistically compared with those of Johnson's algorithm [11] and the PCH and PIH[17] heuristics (Table 4) using paired t-tests and Wilcoxon signed-rank tests across $n = 14$ job sizes.

For Johnson [11] vs Proposed, the paired differences $d_i^{(J)} = W_{J,i} - W_{\text{Proposed},i}$ had a mean of $\bar{d}^{(J)} = 10148.45$ with $s_d = 9498.11$. The t-statistic was

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} = 3.997, \quad df = 13, \ p \approx 0.0009,$$

indicating a significant reduction in waiting times (95% CI: [4710.7, 15586.2]).

For PCH and PIH[17] vs Proposed, the mean difference was $\bar{d}^{(PCH)} = 7934.12$ with $s_d = 8043.65$, yielding

$$t = 3.69, \quad df = 13, \ p \approx 0.0014,$$

Table 5: Extended Comparison of Average of Total Waiting Times

Job Size n	W_{NEH}	$W_{(\text{Goyal\&Kaur})}$	$W_{\text{Proposed Heuristic}}$
5	170.95	168.70	168.25
10	769.65	755.60	754.77
15	1810.05	1782.03	1780.28
20	3323.10	3273.07	3270.68
30	7594.57	7528.27	7508.63
40	13399.48	13308.53	13295.73
50	20947.45	20807.03	20777.95
55	25515.43	25339.42	25321.40
60	30345.05	30176.85	30155.90
80	54451.72	54202.15	54161.63

also confirming a significant reduction (95% CI: [3140.3, 12727.9]).

Nonparametric Wilcoxon signed-rank tests further supported these findings ($p \approx 0.0008$ for Johnson [11] vs Proposed and $p \approx 0.0011$ for PCH and PIH [17] vs Proposed). These results conclusively demonstrate that the proposed heuristic achieves statistically significant improvements over both Johnson's algorithm and the PCH and PIH [17] heuristics.

As a complementary analysis, paired t-tests were conducted using the data in Table 5 to compare the proposed heuristic with the NEH [18], Goyal and Kaur [7] methods over $n = 10$ job sizes. For NEH vs Proposed, the paired differences $d_i^{(\text{NEH})} = W_{\text{NEH},i} - W_{\text{Proposed},i}$ yielded a mean difference $\bar{d}^{(\text{NEH})} = 113.22$ with standard deviation $s_d = 94.36$. The corresponding test statistic was

$$t = 3.79, \quad df = 9, \; p \approx 0.0043,$$

indicating a statistically significant reduction in waiting time.

Similarly, for Goyal and Kaur vs Proposed, the mean difference was $\bar{d}^{(\text{GK})} = 14.64$ with $s_d = 13.61$, resulting in

$$t = 3.40, \quad df = 9, \; p \approx 0.0078.$$

These results further confirm the statistical superiority of the proposed heuristic and complement the earlier parametric and nonparametric analyses.

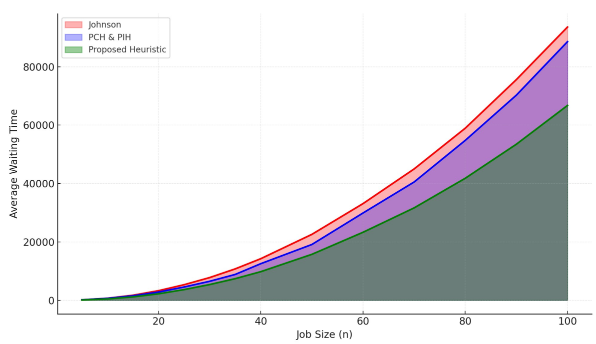

Figure 2: Average waiting times differences

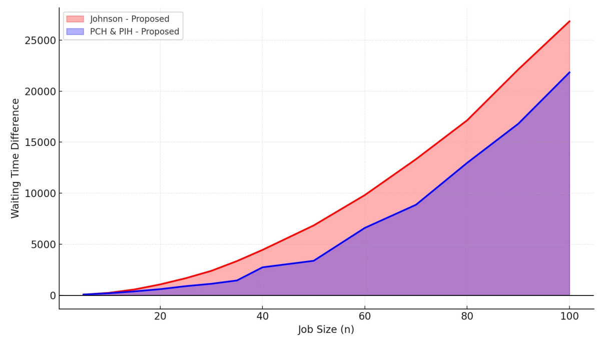

Figure 3: Average waiting times for different job sizes n

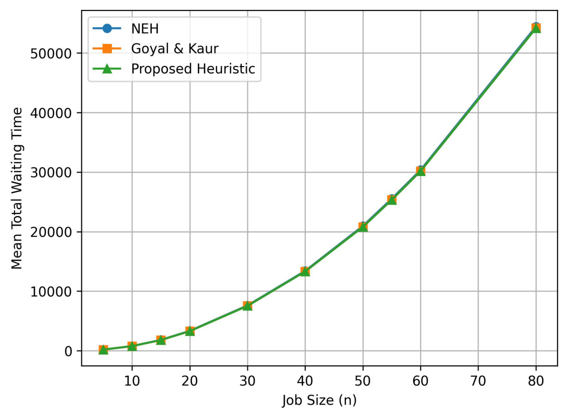

Figure 4: Extended comparison of average of total waiting times

10 Conclusion

This research introduced a novel heuristic designed to minimize total waiting time in fuzzy flow shop scheduling problems with specific structural characteristics by employing job block concepts. Processing times were expressed as trapezoidal fuzzy numbers and transformed into crisp values using **Yager's ranking index**[23], enabling precise scheduling decisions.

The developed method incorporates job block relationships to reduce computational effort and applies a single-shift mechanism to generate efficient job sequences. The performance of the proposed approach was verified through extensive computational experiments and validated using paired t-tests and Wilcoxon signed-rank tests on 14 different job sizes. The statistical analysis confirmed that the proposed heuristic consistently achieves significantly lower waiting times compared with Johnson's rule[11], PCH and PIH heuristics[17], , NEH Heuristic[18] and recently Goyal and Kaur [7] heuristic with all tests showing strong statistical significance ($p < 0.01$).

These results demonstrate that integrating job block strategies with fuzzy defuzzification methods effectively reduces idle periods, improves overall flow shop performance, and enhances scheduling reliability under uncertain processing conditions. This approach provides a solid foundation for further research in multi-stage manufacturing systems where processing time uncertainty is prevalent.

11 Limitations and future research

The proposed study presents a novel integration of fuzzy processing times, job-block formation, and waiting-time minimization, offering both theoretical insights and practical heuristics for two-machine flow shop scheduling. While highly effective, certain constraints should be acknowledged. Processing times are modeled as trapezoidal fuzzy numbers and defuzzified using Yager's ranking index, which may not capture all forms of uncertainty in highly dynamic manufacturing environments. The study also assumes continuous operation of the first machine and structured job sequences, which may differ in some real-world systems.

Despite these constraints, the developed theorems and heuristic remain valid for a wide range of scenarios where block formation is feasible and processing times are uncertain. The results are expected to generalize to similar two-stage systems and provide actionable guidance for production planning.

Future research can extend this framework to multi-machine flow shops, explore alternative fuzzy representations, and develop adaptive heuristics for dynamic and stochastic production environments. Broader industrial-scale validation would further enhance the applicability and robustness of the proposed approach.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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