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Relaxed upper stability of Ekeland's variational principle for fuzzy-valued functions

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Abstract

We study the relaxed upper stability of Ekeland's variational principle for fuzzy functions under fuzzy order relations induced by the lower and upper set orders of α-level sets. Under a $\mathbf{D}$-convergence assumption for fuzzy-valued functions, we prove that every Painlevé–Kuratowski outer limit point of exact fuzzy Ekeland variational points for perturbed problems belongs to the relaxed variational set of the limit problem. We also clarify the relation between exact and relaxed variational sets by explicit examples, show that exact upper stability fails in general and prove that it can be recovered under an additional separation condition on the limit problem. Several open questions concerning, lower and exact upper stability are discussed.

Keywords: Ekeland's variational principle, fuzzy-valued function, relaxed upper stability, fuzzy order relation, Painlevé–Kuratowski outer limit.

1 Introduction

Ekeland's variational principle, introduced by Ekeland [12], is one of the fundamental tools in nonlinear analysis, optimization, control theory and equilibrium theory; see, e.g., [3, 10, 14]. In its classical scalar form, it asserts that if (Z, d) is a complete metric space and $f : Z \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, lower bounded and not identically $+\infty$, then for every $\varepsilon > 0$, every $\lambda > 0$, and every ε-minimizer $z_0 \in Z$ satisfying

$$f(z_0) \leq \inf_{z \in Z} f(z) + \varepsilon,$$

there exists $\bar{z} \in Z$ such that

$$f(\bar{z}) \leq f(z_0), \quad d(z_0, \bar{z}) \leq \lambda,$$

and

$$f(z) > f(\bar{z}) - \frac{\varepsilon}{\lambda} d(z, \bar{z}), \quad \forall z \in Z \setminus \{\bar{z}\}.$$

This principle provides a variational characterization of approximate minimizers and has become a cornerstone in the study of existence, approximation and stability questions.

Following the classical work of Ekeland, many extensions of Ekeland's variational principle have been developed for vector-valued functions, set-valued mappings and bifunctions; see, for instance, [1] for a vector-valued version, [6, 7, 15], for equilibrium-type and bifunction settings, and [16, 35] for set-valued formulations.

On the other hand, since the introduction of fuzzy sets by Zadeh [31] and fuzzy mappings by Chang and Zadeh [8], fuzzy analysis has developed into an active area with important applications in optimization, game theory and differential equations; see, e.g., [4, 5, 13, 21, 22, 25, 26, 28, 30, 32, 34, 38]. Fuzzy-valued models are particularly well

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suited for describing uncertainty, vagueness and imprecision in mathematical frameworks. In particular, for recent developments in fuzzy games and fuzzy equilibrium problems, including existence, stability, convergence and well-posedness results, we refer the reader to [20, 24, 29, 37] and the references therein. In the fuzzy setting, Zhang and Huang [33] established interval-valued versions of Ekeland's variational principle with several applications. Although interval values may be regarded as special fuzzy numbers, such results do not extend directly to general fuzzy sets. This limitation was recently overcome by Zhang and Zeng [36], who established fuzzy versions of Ekeland's variational principle by introducing order relations on fuzzy sets through the lower and upper set orders of their α-level sets. In that framework, for every $\varepsilon > 0$ and every $u_0 \in X$, if F is a fuzzy-valued mapping on a complete metric space (X, d) and satisfies suitable $\preceq$-lower boundedness and $\preceq$-lower semicontinuity assumptions, then there exists a point $\widetilde{u} \in X$ such that

$$F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}} \preceq F(u_0),$$

and

$$F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}), \quad \forall u \in X \setminus \{\widetilde{u}\}.$$

Such a point is called a *fuzzy Ekeland variational point.*

Because of the fundamental role of Ekeland's variational principle in optimization and variational analysis, its stability under perturbations has also attracted considerable attention. In the scalar setting, Attouch and Riahi [2] established stability results for Ekeland's variational principle and cone extremal solutions. In vector-valued and set-valued settings, further stability results were obtained by Chen and Huang [9] and Huang [18, 19]. These studies show that, beyond existence, the behavior of Ekeland variational points under perturbations of the underlying data is a natural and important issue, especially in approximation methods and variational models. Motivated by these developments, together with the fuzzy versions of Ekeland's variational principle recently established by Zhang and Zeng [36], the aim of the present work is to investigate corresponding stability properties for fuzzy Ekeland variational points.

For perturbation analysis, however, the mere existence of fuzzy Ekeland variational points is not sufficient. Given a sequence of fuzzy functions $\{F_m\}$ converging to a limit mapping F, together with perturbations of the base point and the penalty parameter, a natural question is whether fuzzy Ekeland variational points are stable under such perturbations. More precisely, if $\widetilde{u}_m$ is a fuzzy Ekeland variational point associated with $(u_{0,m}, F_m, \varepsilon_m)$ and $\widetilde{u}_m \to \widetilde{u}$, what variational property is inherited by the limit point $\widetilde{u}$?

The present paper addresses this question directly in the fuzzy setting, without passing to vector-valued or set-valued epigraphic reformulations in product spaces. Working with the metric $\mathbf{D}$ on E^n, the induced convergence of fuzzy-valued mappings and closedness properties of the associated fuzzy order, we establish a relaxed upper stability result formulated in terms of outer limits in the sense of Painlevé–Kuratowski.

The main contributions of the paper are as follows. First, we introduce a $\mathbf{D}$-convergence framework for fuzzy functions and establish several auxiliary stability properties of the associated fuzzy order. Second, we prove a relaxed upper stability theorem showing that every outer limit point of exact fuzzy Ekeland variational points for perturbed problems is a relaxed fuzzy Ekeland variational point of the limit problem. Third, we clarify the gap between exact and relaxed variational sets by explicit examples, including a sharp example showing that exact upper stability fails in general and another showing that the relaxed variational set need not coincide with the whole ambient space. Fourth, we show that exact upper stability can nevertheless be recovered under an additional separation condition on the limit problem. Finally, we formulate open questions on lower stability and on weaker conditions guaranteeing exact upper stability.

The rest of the paper is organized as follows. Section 2 recalls basic notions on fuzzy sets, fuzzy order relations, set limits and convergence of fuzzy-valued mappings. Section 3 introduces exact and relaxed fuzzy Ekeland variational points, establishes the main relaxed upper stability theorem, presents examples illustrating the relation between exact and relaxed variational sets, derives an exact upper stability consequence under Assumption ($\mathbf{A}$) and discusses several open problems. Finally, Section 4 concludes the paper.

2 Preliminaries

In this section, we collect the basic notions and auxiliary facts that will be used throughout the paper. We begin with the fuzzy setting and the metric structure on E^n, then recall the order relations needed in the sequel and finally introduce the regularity and convergence concepts associated with fuzzy-valued mappings.

Throughout the paper, (X, d) denotes a complete metric space,

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \ldots, n\},$$

and $\operatorname{int}\mathbb{R}^n_+$ denotes the interior of $\mathbb{R}^n_+$. We denote by $\mathcal{K}_{cc}(\mathbb{R}^n)$ the family of all nonempty convex compact subsets of $\mathbb{R}^n$.

We now introduce the fuzzy setting, the metric structure on E^n, and the basic order relations that will be used throughout the paper.

A mapping $z : \mathbb{R}^n \to [0,1]$ is called a fuzzy set on $\mathbb{R}^n$. Let E^n be the set of all fuzzy sets $z : \mathbb{R}^n \to [0,1]$ satisfying:

(a) z is normal, i.e., its core is nonempty; equivalently, there exists $x_0 \in \mathbb{R}^n$ such that $z(x_0) = 1$.

(b) z is fuzzy convex, i.e., for all $x, y \in \mathbb{R}^n$ and all $\lambda \in [0,1]$, $z(\lambda x + (1-\lambda)y) \geq \min\{z(x), z(y)\}$.

(c) z is upper semicontinuous.

(d) $[z]^0 = \operatorname{cl}\{x \in \mathbb{R}^n : z(x) > 0\}$ is compact, where cl stands for the closure in $\mathbb{R}^n$.

Clearly, if $B \in \mathcal{K}_{cc}(\mathbb{R}^n)$, then $\chi_B \in E^n$, where $\chi_B(z) = 1$ if $z \in B$ and $\chi_B(z) = 0$ otherwise.

For $z \in E^n$ and $\alpha \in [0,1]$, define the α-level set by $[z]^\alpha = \{x \in \mathbb{R}^n : z(x) \geq \alpha\}$. Then $[z]^\alpha \in \mathcal{K}_{cc}(\mathbb{R}^n)$ for all $\alpha \in [0,1]$.

A fuzzy-valued function is a mapping $F : X \to E^n$. For each $z \in X$, the α-level set of $F(z)$ is denoted by $[F(z)]^\alpha$, where

$$[F(z)]^\alpha = \{y \in \mathbb{R}^n : F(z)(y) \geq \alpha\}, \forall \alpha \in (0,1] \text{ and } [F(z)]^0 = \operatorname{cl}\{y \in \mathbb{R}^n : F(z)(y) > 0\}.$$

It follows that $[F(z)]^\alpha \in \mathcal{K}_{cc}(\mathbb{R}^n)$. For $z_1, z_2 \in E^n$, define the metric $\mathbf{D}$ on E^n as follows

$$\mathbf{D}(z_1, z_2) = \sup_{\alpha \in [0,1]} d_H([z_1]^\alpha, [z_2]^\alpha), \tag{1}$$

where $d_H\big([z_1]^\alpha, [z_2]^\alpha\big)$ denotes the Hausdorff distance between the sets $[z_1]^\alpha$ and $[z_2]^\alpha$, defined by

$$d_H\big([z_1]^\alpha, [z_2]^\alpha\big) = \max\left\{\sup_{x \in [z_1]^\alpha} d\big(x, [z_2]^\alpha\big),\ \sup_{y \in [z_2]^\alpha} d\big(y, [z_1]^\alpha\big)\right\}, \tag{2}$$

where $d\big(x, [z_2]^\alpha\big) = \inf_{a \in [z_2]^\alpha} \|x - a\|_{\mathbb{R}^n}$. It is known that $(E^n, \mathbf{D})$ is a complete metric space (see [11]).

Remark 2.1. *It is easy to deduce from* (1) *that, whenever* $\mathbf{D}(z_m, z) \to 0$*, one has* $d_H([z_m]^\alpha, [z]^\alpha) \to 0$ *for every* $\alpha \in [0,1]$*. Consequently, if* $y \in [z]^\alpha$*, then there exists a sequence* $y_m \in [z_m]^\alpha$ *such that* $y_m \to y$ *as* $m \to +\infty$*.*

We next recall the lower and upper set orders and the induced fuzzy order relations, which will play a central role in the stability analysis developed later.

Definition 2.2. [16, 23] *Let* $A, C \in \mathcal{K}_{cc}(\mathbb{R}^n)$*.*

(a) $A \leq^{\boldsymbol{l}} C$ *if and only if* $C \subset A + \mathbb{R}^n_+$*;*

(b) $A <^{\boldsymbol{l}} C$ *if and only if* $C \subset A + \operatorname{int}\mathbb{R}^n_+$*;*

(c) $A \leq^{\boldsymbol{u}} C$ *if and only if* $A \subset C - \mathbb{R}^n_+$*;*

(d) $A <^{\boldsymbol{u}} C$ *if and only if* $A \subset C - \operatorname{int}\mathbb{R}^n_+$*.*

Using these relations, we define the fuzzy order on E^n.

Definition 2.3. [36] *Let* $z_1, z_2 \in E^n$*.*

(a) $z_1 \preceq z_2$ *if and only if,* $\forall \alpha \in [0,1]$, $[z_1]^\alpha \leq^{\boldsymbol{l}} [z_2]^\alpha$ *and* $[z_1]^\alpha \leq^{\boldsymbol{u}} [z_2]^\alpha$*.*

(b) $z_1 \prec z_2$ *if and only if,* $\forall \alpha \in [0,1]$, $[z_1]^\alpha <^{\boldsymbol{l}} [z_2]^\alpha$ *and* $[z_1]^\alpha <^{\boldsymbol{u}} [z_2]^\alpha$*.*

(c) $z_1 \npreceq z_2$ *if and only if,* $\exists \alpha \in [0,1]$ *such that* $[z_1]^\alpha \nleq^{\boldsymbol{l}} [z_2]^\alpha$ *or* $[z_1]^\alpha \nleq^{\boldsymbol{u}} [z_2]^\alpha$*.*

(d) $z_1 \nprec z_2$ *if and only if,* $\exists \alpha \in [0,1]$ *such that* $[z_1]^\alpha \not<^{\boldsymbol{l}} [z_2]^\alpha$ *or* $[z_1]^\alpha \not<^{\boldsymbol{u}} [z_2]^\alpha$*.*

Remark 2.4. [36, Remark 1(I)] *In the one-dimensional case* $n = 1$*, if* $[w]^\alpha = [\underline{w}_\alpha, \overline{w}_\alpha]$ *and* $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ *for all* $\alpha \in [0,1]$*, then*

$$w \preceq u \iff \underline{w}_\alpha \leq \underline{u}_\alpha,\ \overline{w}_\alpha \leq \overline{u}_\alpha \quad \forall \alpha \in [0,1],$$
$$w \prec u \iff \underline{w}_\alpha < \underline{u}_\alpha,\ \overline{w}_\alpha < \overline{u}_\alpha \quad \forall \alpha \in [0,1].$$

Remark 2.5. *Let $z_1, z_2 \in E^n$. If $z_1 \prec z_2$, then $z_1 \preceq z_2$.*

Indeed, for every $\alpha \in [0,1]$, the relations $[z_1]^\alpha <^{\boldsymbol{l}} [z_2]^\alpha$ and $[z_1]^\alpha <^{\boldsymbol{u}} [z_2]^\alpha$ imply $[z_1]^\alpha \leq^{\boldsymbol{l}} [z_2]^\alpha$ and $[z_1]^\alpha \leq^{\boldsymbol{u}} [z_2]^\alpha$, since $\operatorname{int}\mathbb{R}^n_+ \subset \mathbb{R}^n_+$. Hence $z_1 \preceq z_2$.

Fix $e \in \operatorname{int}\mathbb{R}^n_+$. Define the fuzzy singleton $\chi_{\{e\}}$ at e by

$$[\chi_{\{e\}}]^\alpha = \{e\}, \quad \forall \alpha \in [0,1].$$

We now specify the arithmetic operations on E^n used throughout the paper, particularly the fuzzy translations occurring in the fuzzy Ekeland variational principle.

Definition 2.6. [11] *For $z_1, z_2 \in E^n$ and $\lambda \geq 0$, their Minkowski sum $z_1 \oplus z_2$ and the nonnegative scalar multiple $\lambda \odot z_1$ are defined levelwise by*

$$[z_1 \oplus z_2]^\alpha = [z_1]^\alpha + [z_2]^\alpha = \{x + y : x \in [z_1]^\alpha,\ y \in [z_2]^\alpha\},$$

and

$$[\lambda \odot z_1]^\alpha = \lambda [z_1]^\alpha, \quad \forall \alpha \in [0,1].$$

These operations are well defined in E^n.

Remark 2.7. *For the fuzzy singleton $\chi_{\{e\}}$ at e and every $t \geq 0$, one has*

$$[t \odot \chi_{\{e\}}]^\alpha = \{te\}, \quad \forall \alpha \in [0,1].$$

Consequently, for every $z \in E^n$,

$$[z \oplus (t \odot \chi_{\{e\}})]^\alpha = [z]^\alpha + te, \quad \forall \alpha \in [0,1].$$

In particular, every expression $F(u) + \varepsilon d(u_0, u)\chi_{\{e\}}$ occurring below is a well-defined element of E^n, whose α-level set is

$$[F(u)]^\alpha + \varepsilon d(u_0, u)e.$$

Throughout the paper, we write $+$ in place of $\oplus$ and omit the symbol $\odot$. Thus,

$$F(u) + \varepsilon d(u_0, u)\chi_{\{e\}} = F(u) \oplus \big((\varepsilon d(u_0, u)) \odot \chi_{\{e\}}\big).$$

The following lemma expresses the closedness of the non-strict fuzzy order $\preceq$ with respect to the metric $\mathbf{D}$.

Lemma 2.8. *Let $w_m, u_m, w, u \in E^n$ for all $m \in \mathbb{N}$. If*

$$\mathbf{D}(w_m, w) \to 0, \quad \mathbf{D}(u_m, u) \to 0, \quad w_m \preceq u_m,\ \forall m \in \mathbb{N},$$

then $w \preceq u$.

Proof. Fix $\alpha \in [0,1]$. Since $w_m \preceq u_m$ for all $m \in \mathbb{N}$, we have $[w_m]^\alpha \leq^{\boldsymbol{l}} [u_m]^\alpha$ and $[w_m]^\alpha \leq^{\boldsymbol{u}} [u_m]^\alpha$ for all m. We first prove that $[w]^\alpha \leq^{\boldsymbol{l}} [u]^\alpha$. Let $y \in [u]^\alpha$. By Remark 2.1, there exists $y_m \in [u_m]^\alpha$ such that $y_m \to y$. Since $[w_m]^\alpha \leq^{\boldsymbol{l}} [u_m]^\alpha$, for each m there exist $x_m \in [w_m]^\alpha$ and $c_m \in \mathbb{R}^n_+$ such that $y_m = x_m + c_m$.

Since $d_H([w_m]^\alpha, [w]^\alpha) \to 0$ and $[w]^\alpha$ is compact, the sequence $\{x_m\}$ is bounded. Passing to a subsequence if necessary, we may assume that $x_m \to x$ for some $x \in \mathbb{R}^n$. Moreover, since $x_m \in [w_m]^\alpha$, we have $\operatorname{dist}(x_m, [w]^\alpha) \leq d_H([w_m]^\alpha, [w]^\alpha) \to 0$. As $[w]^\alpha$ is closed, it follows that $x \in [w]^\alpha$. On the other hand, $c_m = y_m - x_m \to y - x$, and since each $c_m \in \mathbb{R}^n_+$ and $\mathbb{R}^n_+$ is closed, we get $y - x \in \mathbb{R}^n_+$. Therefore $y = x + (y - x) \in [w]^\alpha + \mathbb{R}^n_+$, proving that $[w]^\alpha \leq^{\boldsymbol{l}} [u]^\alpha$.

By the same argument, from $[w_m]^\alpha \leq^{\boldsymbol{u}} [u_m]^\alpha$ for all m, we obtain $[w]^\alpha \leq^{\boldsymbol{u}} [u]^\alpha$. Since $\alpha \in [0,1]$ is arbitrary, we conclude that $w \preceq u$. □

The following lemma shows that a positive shift in the direction of $\chi_{\{e\}}$ strengthens the weak order relation $\preceq$ into the strict order relation $\prec$.

Lemma 2.9. *Let $w, v \in E^n$ and let $\delta > 0$. If $w + \delta\chi_{\{e\}} \preceq v$, then $w \prec v$.*

Proof. Fix $\alpha \in [0,1]$. Since $w + \delta\chi_{\{e\}} \preceq v$, we have $[w]^\alpha + \delta e \leq^{\boldsymbol{l}} [v]^\alpha$ and $[w]^\alpha + \delta e \leq^{\boldsymbol{u}} [v]^\alpha$. Hence $[v]^\alpha \subset [w]^\alpha + \delta e + \mathbb{R}^n_+$. Since $\delta e \in \operatorname{int}\mathbb{R}^n_+$, we have $\delta e + \mathbb{R}^n_+ \subset \operatorname{int}\mathbb{R}^n_+$, and therefore $[v]^\alpha \subset [w]^\alpha + \operatorname{int}\mathbb{R}^n_+$. Thus $[w]^\alpha <^{\boldsymbol{l}} [v]^\alpha$. Similarly, from $[w]^\alpha + \delta e \leq^{\boldsymbol{u}} [v]^\alpha$, we obtain $[w]^\alpha + \delta e \subset [v]^\alpha - \mathbb{R}^n_+$, hence $[w]^\alpha \subset [v]^\alpha - (\delta e + \mathbb{R}^n_+) \subset [v]^\alpha - \operatorname{int}\mathbb{R}^n_+$. Thus $[w]^\alpha <^{\boldsymbol{u}} [v]^\alpha$. Since $\alpha \in [0,1]$ is arbitrary, we conclude that $w \prec v$. □

We now collect the regularity notions needed for the fuzzy Ekeland variational principle, together with the auxiliary variational constructions used later.

Definition 2.10. [34] *A fuzzy-valued function $F : X \to E^n$ is called $\mathbf{D}$-continuous at $z_0 \in X$ if, for any sequence $\{z_m\} \subset X$ with $z_m \to z_0$, one has $\mathbf{D}(F(z_m), F(z_0)) \to 0$ as $m \to +\infty$. We say that F is $\mathbf{D}$-continuous on X if F is $\mathbf{D}$-continuous at every $z_0 \in X$.*

Definition 2.11. [34] *A fuzzy-valued function $F : X \to E^n$ is called $\preceq$-lower semicontinuous (briefly, $\preceq$-l.s.c.) if for every $v \in E^n$, the set $\{u \in X : F(u) \preceq v\}$ is closed in X.*

Lemma 2.12. [36, Remark 2(II)] *If $F : X \to E^n$ is $\mathbf{D}$-continuous on X, then F is $\preceq$-l.s.c.*

Definition 2.13. [36] *A fuzzy-valued function $F : X \to E^n$ is called $\preceq$-lower bounded if there exists $w \in E^n$ such that $w \preceq F(u), \quad \forall u \in X$.*

We next recall the fuzzy Ekeland variational principle and the associated point-to-set constructions that will be used in the main stability argument.

Theorem 2.14. [36, Theorem 1] *Let $F : X \to E^n$ be $\preceq$-lower bounded and $\preceq$-l.s.c. Then for every $\varepsilon > 0$ and $u_0 \in X$, there exists $\widetilde{u} \in X$ such that*

(i) $F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}} \preceq F(u_0)$;

(ii) $F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}), \quad \forall u \in X \setminus \{\widetilde{u}\}$.

Definition 2.15. *For $\varepsilon > 0$, $u_0 \in X$ and a fuzzy-valued function $F : X \to E^n$, define*

$$\mathbf{\Pi}_\varepsilon(u_0; F) = \left\{ \widetilde{u} \in X : \begin{array}{l} F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}} \preceq F(u_0), \\ F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}), \ \forall u \in X \setminus \{\widetilde{u}\} \end{array} \right\}.$$

Any element of $\mathbf{\Pi}_\varepsilon(u_0; F)$ is called a fuzzy Ekeland variational point of F associated with (u_0, ε).

Remark 2.16. *Under the assumptions of Theorem 2.14, the set $\mathbf{\Pi}_\varepsilon(u_0; F)$ is nonempty for every $\varepsilon > 0$ and $u_0 \in X$.*

We conclude this preliminary section with the convergence notions for fuzzy functions that will be used in the stability analysis of fuzzy Ekeland variational points. Motivated by [21, Definition 3.2], we introduce the following notion of convergence adapted to our setting.

Definition 2.17. *Let $F, F_m : X \to E^n$ $(m \in \mathbb{N})$. We say that F_m $\mathbf{D}$-converges to F and write $F_m \xrightarrow{\mathbf{D}\text{-c.}} F$ if, for every $u \in X$ and every sequence $\{u_m\} \subset X$ with $u_m \to u$, one has $\mathbf{D}(F_m(u_m), F(u)) \to 0$ as $m \to +\infty$.*

In the spirit of [17, Definition 3.2] and [21, Definition 3.3], we adopt the following localized notion of uniform convergence for fuzzy-valued mappings.

Definition 2.18. *Let $F, F_m : X \to E^n$ $(m \in \mathbb{N})$. We say that F_m converges uniformly to F on compact subsets of X if, for every compact set $K \subset X$, one has*

$$\sup_{u \in K} \mathbf{D}(F_m(u), F(u)) \to 0 \text{ as } m \to +\infty.$$

Remark 2.19. *The above notion is a localized version of the uniform convergence considered in* [17, 21]*. In particular, when X is compact, it reduces to the usual uniform convergence on X.*

The following theorem provides an equivalent characterization of $\mathbf{D}$-convergence for fuzzy-valued mappings.

Theorem 2.20. *Let $F, F_m : X \to E^n$ $(m \in \mathbb{N})$. Then the following are equivalent:*

(i) $F_m \xrightarrow{\mathbf{D}\text{-c.}} F$;

(ii) $F_m \to F$ *uniformly on compact subsets of X and F is $\mathbf{D}$-continuous on X.*

Proof. [(i)⇒(ii)] We first prove that F is $\mathbf{D}$-continuous on X. Let $u_k \to u$ in X. Since (i) implies pointwise convergence, for each $k \in \mathbb{N}$ one can choose m_k such that $\mathbf{D}(F_{m_k}(u_k), F(u_k)) < 1/k$. Passing to a subsequence if necessary, we may assume that $\{m_k\}$ is strictly increasing. Define

$$y_m = \begin{cases} u, & m < m_1, \\ u_k, & m_k \leq m < m_{k+1}. \end{cases}$$

Then $y_m \to u$. By (i), $\mathbf{D}(F_m(y_m), F(u)) \to 0$, hence $\mathbf{D}(F_{m_k}(u_k), F(u)) \to 0$. Therefore,

$$\mathbf{D}(F(u_k), F(u)) \leq \mathbf{D}(F(u_k), F_{m_k}(u_k)) + \mathbf{D}(F_{m_k}(u_k), F(u)) \to 0.$$

This implies that F is $\mathbf{D}$-continuous on X.

It remains to prove that $F_m \to F$ uniformly on compact subsets of X. Assume to the contrary that this fails. Then there exist a compact set $K \subset X$, $\varepsilon_0 > 0$, a subsequence $\{m_k\}$ and points $u_{m_k} \in K$, such that $\mathbf{D}(F_{m_k}(u_{m_k}), F(u_{m_k})) \geq \varepsilon_0$, for all $k \in \mathbb{N}$. By compactness, passing to a further subsequence if necessary, we may assume that $u_{m_k} \to u_0 \in K$. Then (i) yields $\mathbf{D}(F_{m_k}(u_{m_k}), F(u_0)) \to 0$, while the $\mathbf{D}$-continuity of F gives $\mathbf{D}(F(u_{m_k}), F(u_0)) \to 0$. Hence, we obtain

$$\mathbf{D}(F_{m_k}(u_{m_k}), F(u_{m_k})) \leq \mathbf{D}(F_{m_k}(u_{m_k}), F(u_0)) + \mathbf{D}(F(u_0), F(u_{m_k})) \to 0,$$

a contradiction. Therefore, $F_m \to F$ uniformly on compact subsets of X.

[(ii)⇒(i)] Let $u_m \to u$ in X and set $K = \{u\} \cup \{u_m : m \in \mathbb{N}\}$. Then K is compact. By uniform convergence on compact subsets, $\sup_{v \in K} \mathbf{D}(F_m(v), F(v)) \to 0$, and hence $\mathbf{D}(F_m(u_m), F(u_m)) \to 0$. Since F is $\mathbf{D}$-continuous and $u_m \to u$, we also have $\mathbf{D}(F(u_m), F(u)) \to 0$. Therefore,

$$\mathbf{D}(F_m(u_m), F(u)) \leq \mathbf{D}(F_m(u_m), F(u_m)) + \mathbf{D}(F(u_m), F(u)) \to 0,$$

which shows that $F_m \xrightarrow{\mathbf{D}\text{-}c.} F$. □

3 Main results

In this section, we establish the relaxed upper stability of fuzzy Ekeland variational points. We first introduce the relaxed solution concept and its basic relation to exact fuzzy Ekeland points, then prove a perturbation lemma under a positive gap and finally derive the main upper stability theorem together with sharpness examples and several natural open problems.

We first introduce a relaxed solution set that is stable under arbitrarily small right perturbations of the penalty parameter.

Definition 3.1. *Let $F : X \to E^n$, let $\varepsilon > 0$ and let $u_0 \in X$. Define*

$$\mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F) = \left\{ \widetilde{u} \in X : \begin{array}{c} F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}} \preceq F(u_0), \\ F(u) + (\varepsilon + \eta) d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}), \\ \forall u \in X \setminus \{\widetilde{u}\}, \ \forall \eta > 0 \end{array} \right\}.$$

The set $\mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F)$ is called the set of relaxed fuzzy Ekeland variational points of F associated with (u_0, ε).

Proposition 3.2. *For every $\varepsilon > 0$ and $u_0 \in X$,*

$$\mathbf{\Pi}_\varepsilon(u_0; F) \subset \mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F).$$

Proof. Let $\widetilde{u} \in \mathbf{\Pi}_\varepsilon(u_0; F)$. Then $F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}} \preceq F(u_0)$ and

$$F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}), \quad \forall u \in X \setminus \{\widetilde{u}\}. \tag{3}$$

It remains to prove that

$$F(u) + (\varepsilon + \eta) d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}), \quad \forall u \in X \setminus \{\widetilde{u}\}, \ \forall \eta > 0.$$

Assume to the contrary that there exist $\eta_0 > 0$ and $u^* \in X \setminus \{\widetilde{u}\}$ such that

$$F(u^*) + (\varepsilon + \eta_0) d(\widetilde{u}, u^*)\chi_{\{e\}} \preceq F(\widetilde{u}). \tag{4}$$

Set

$$w = F(u^*) + \varepsilon d(\widetilde{u}, u^*)\chi_{\{e\}}, \quad v = F(\widetilde{u}), \quad \delta = \eta_0\, d(\widetilde{u}, u^*).$$

Since $u^* \neq \widetilde{u}$, one has $\delta > 0$ and the inequality (4) becomes $w + \delta\chi_{\{e\}} \preceq v$. Since $w + \delta\chi_{\{e\}} \preceq v$ with $\delta > 0$, Lemma 2.9 yields $w \prec v$, and hence Remark 2.5 gives $w \preceq v$. Therefore,

$$F(u^*) + \varepsilon d(\widetilde{u}, u^*)\chi_{\{e\}} \preceq F(\widetilde{u}),$$

which contradicts (3). Thus $\widetilde{u} \in \mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F)$. □

Remark 3.3. *The parameter $\eta > 0$ introduces an additional margin in the second variational inequality, so that $\mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F)$ may be viewed as a relaxed version of $\mathbf{\Pi}_{\varepsilon}(u_0; F)$, stable under right perturbations of the penalty parameter. In particular, Proposition 3.2 shows that $\mathbf{\Pi}_{\varepsilon}(u_0; F) \subset \mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F)$. In general, this inclusion may be strict, as will be illustrated by the sharpness example below.*

Before turning to the main upper stability theorem, we record the following perturbation lemma, which shows that the fuzzy order is preserved under convergence whenever a positive gap is available.

Lemma 3.4. *Let $u_m, u, v_m, v \in E^n$. If $\mathbf{D}(u_m, u) \to 0$, $\mathbf{D}(v_m, v) \to 0$ and there exists $\delta > 0$ such that $u + \delta\chi_{\{e\}} \preceq v$, then there exists $m_0 \in \mathbb{N}$ such that $u_m \preceq v_m$ for all $m \geq m_0$.*

Proof. Fix $\alpha \in [0, 1]$. Since $u + \delta\chi_{\{e\}} \preceq v$, we have $[v]^\alpha \subset [u]^\alpha + \delta e + \mathbb{R}^n_+$ and $[u]^\alpha + \delta e \subset [v]^\alpha - \mathbb{R}^n_+$. Choose $r > 0$ such that $\delta e + r\mathbb{B} \subset \operatorname{int} \mathbb{R}^n_+$, where $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$. Since $\mathbf{D}(u_m, u) \to 0$ and $\mathbf{D}(v_m, v) \to 0$, there exists $m_0 \in \mathbb{N}$ such that $\mathbf{D}(u_m, u) < \frac{r}{2}$ and $\mathbf{D}(v_m, v) < \frac{r}{2}$ for all $m \geq m_0$. Hence, by (1),

$$d_H([u_m]^\alpha, [u]^\alpha) < \frac{r}{2}, \quad d_H([v_m]^\alpha, [v]^\alpha) < \frac{r}{2} \quad \forall \alpha \in [0, 1],\ \forall m \geq m_0.$$

Therefore,

$$[u_m]^\alpha \subset [u]^\alpha + \frac{r}{2}\mathbb{B}, \quad [u]^\alpha \subset [u_m]^\alpha + \frac{r}{2}\mathbb{B}, \quad [v_m]^\alpha \subset [v]^\alpha + \frac{r}{2}\mathbb{B}, \quad [v]^\alpha \subset [v_m]^\alpha + \frac{r}{2}\mathbb{B}.$$

For every $m \geq m_0$, we obtain

$$[v_m]^\alpha \subset [v]^\alpha + \frac{r}{2}\mathbb{B} \subset [u]^\alpha + \delta e + \mathbb{R}^n_+ + \frac{r}{2}\mathbb{B} \subset [u_m]^\alpha + \delta e + \mathbb{R}^n_+ + r\mathbb{B} \subset [u_m]^\alpha + \mathbb{R}^n_+,$$

because $\delta e + r\mathbb{B} \subset \mathbb{R}^n_+$. Thus $[u_m]^\alpha \leq^{\boldsymbol{l}} [v_m]^\alpha$.

Similarly,

$$[u_m]^\alpha + \delta e \subset [u]^\alpha + \delta e + \frac{r}{2}\mathbb{B} \subset [v]^\alpha - \mathbb{R}^n_+ + \frac{r}{2}\mathbb{B} \subset [v_m]^\alpha - \mathbb{R}^n_+ + r\mathbb{B},$$

and hence

$$[u_m]^\alpha \subset [v_m]^\alpha - \mathbb{R}^n_+ + (r\mathbb{B} - \delta e).$$

Since $\mathbb{B}$ is symmetric, we have $\delta e - r\mathbb{B} = \delta e + r\mathbb{B} \subset \mathbb{R}^n_+$, so

$$r\mathbb{B} - \delta e = -(\delta e - r\mathbb{B}) \subset -\mathbb{R}^n_+.$$

Therefore $[u_m]^\alpha \subset [v_m]^\alpha - \mathbb{R}^n_+$, that is, $[u_m]^\alpha \leq^{\boldsymbol{u}} [v_m]^\alpha$. Since $\alpha \in [0, 1]$ is arbitrary and m_0 is independent of α, we conclude that $u_m \preceq v_m$ for all $m \geq m_0$. □

Following the sequential characterization of set limits in variational analysis, we recall the notions of inner and outer limits of a sequence of sets.

Definition 3.5. [27] *Let $\{K_m\}_{m\in\mathbb{N}}$ be a sequence of subsets of X.*

(a) *The outer limit of $\{K_m\}$ is defined by*

$$\limsup_{m\to\infty} K_m = \{u \in X :\ \exists\, m_k \to \infty,\ \exists\, u_{m_k} \in K_{m_k} \text{ such that } u_{m_k} \to u\}.$$

(b) *The inner limit of $\{K_m\}$ is defined by*

$$\liminf_{m\to\infty} K_m = \{u \in X :\ \exists\, u_m \in K_m \text{ for all sufficiently large } m \text{ such that } u_m \to u\}.$$

We are now in a position to establish the main stability result for sequences of fuzzy Ekeland variational points under **D**-continuous convergence of the underlying fuzzy-valued mappings.

Theorem 3.6. *Let (X, d) be a complete metric space. Let $F_m, F : X \to E^n$, $\varepsilon_m, \varepsilon > 0$, and let $u_{0,m}, u_0 \in X$. Assume that:*

(i) *each F_m is $\preceq$-lower bounded and $\preceq$-l.s.c.;*

(ii) *F is $\preceq$-lower bounded and $\preceq$-l.s.c.;*

(iii) *$F_m \xrightarrow{\mathbf{D}\text{-}c.} F$;*

(iv) *$\varepsilon_m \to \varepsilon$ and $u_{0,m} \to u_0$.*

Then

$$\limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m) \subset \mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F).$$

Proof. Let $\widetilde{u} \in \limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m)$. Then there exist a subsequence $\{m_k\}$ and points $\widetilde{u}_{m_k} \in \mathbf{\Pi}_{\varepsilon_{m_k}}(u_{0,m_k}; F_{m_k})$ such that $\widetilde{u}_{m_k} \to \widetilde{u}$. For simplicity, we relabel this subsequence by $\{\widetilde{u}_m\}$. Hence

$$\widetilde{u}_m \in \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m) \quad \text{and} \quad \widetilde{u}_m \to \widetilde{u}.$$

By the definition of $\mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m)$, for each $m \in \mathbb{N}$ we have

$$F_m(\widetilde{u}_m) + \varepsilon_m d(u_{0,m}, \widetilde{u}_m)\chi_{\{e\}} \preceq F_m(u_{0,m}),$$

and

$$F_m(u) + \varepsilon_m d(\widetilde{u}_m, u)\chi_{\{e\}} \npreceq F_m(\widetilde{u}_m), \quad \forall u \in X \setminus \{\widetilde{u}_m\}. \tag{5}$$

Set

$$a_m = F_m(\widetilde{u}_m) + \varepsilon_m d(u_{0,m}, \widetilde{u}_m)\chi_{\{e\}}, \quad b_m = F_m(u_{0,m}),$$

and

$$a = F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}}, \quad b = F(u_0).$$

Since $\widetilde{u}_m \to \widetilde{u}$ and $u_{0,m} \to u_0$, assumption (iii) yields

$$\mathbf{D}(F_m(\widetilde{u}_m), F(\widetilde{u})) \to 0 \quad \text{and} \quad \mathbf{D}(F_m(u_{0,m}), F(u_0)) \to 0.$$

Moreover, since $\varepsilon_m \to \varepsilon$ and $d(u_{0,m}, \widetilde{u}_m) \to d(u_0, \widetilde{u})$, for every $\alpha \in [0, 1]$ we have

$$[a_m]^\alpha = [F_m(\widetilde{u}_m)]^\alpha + \varepsilon_m d(u_{0,m}, \widetilde{u}_m)e, \quad [a]^\alpha = [F(\widetilde{u})]^\alpha + \varepsilon d(u_0, \widetilde{u})e.$$

Hence,

$$\begin{aligned} d_H([a_m]^\alpha, [a]^\alpha) &\le d_H\big([F_m(\widetilde{u}_m)]^\alpha + \varepsilon_m d(u_{0,m}, \widetilde{u}_m)e,\ [F(\widetilde{u})]^\alpha + \varepsilon_m d(u_{0,m}, \widetilde{u}_m)e\big) \\ &\quad + d_H\big([F(\widetilde{u})]^\alpha + \varepsilon_m d(u_{0,m}, \widetilde{u}_m)e,\ [F(\widetilde{u})]^\alpha + \varepsilon d(u_0, \widetilde{u})e\big) \\ &= d_H([F_m(\widetilde{u}_m)]^\alpha, [F(\widetilde{u})]^\alpha) + \big|\varepsilon_m d(u_{0,m}, \widetilde{u}_m) - \varepsilon d(u_0, \widetilde{u})\big| \, \|e\|. \end{aligned}$$

Taking the supremum over $\alpha \in [0, 1]$, we obtain $\mathbf{D}(a_m, a) \to 0$. Also, $\mathbf{D}(b_m, b) \to 0$. Since $a_m \preceq b_m$ for all m, Lemma 2.8 implies that $a \preceq b$, i.e.,

$$F(\widetilde{u}) + \varepsilon d(u_0, \widetilde{u})\chi_{\{e\}} \preceq F(u_0). \tag{6}$$

It remains to prove the relaxed variational condition. Fix $\eta > 0$ and suppose, by contradiction, that there exists $u^* \in X \setminus \{\widetilde{u}\}$ such that

$$F(u^*) + (\varepsilon + \eta)d(\widetilde{u}, u^*)\chi_{\{e\}} \preceq F(\widetilde{u}).$$

Define

$$p = F(u^*) + \varepsilon d(\widetilde{u}, u^*)\chi_{\{e\}}, \quad q = F(\widetilde{u}), \quad \delta = \eta\, d(\widetilde{u}, u^*).$$

Since $u^* \neq \widetilde{u}$, we have $\delta > 0$ and hence

$$p + \delta\chi_{\{e\}} \preceq q. \tag{7}$$

Now set
$$p_m = F_m(u^*) + \varepsilon_m d(\widetilde{u}_m, u^*)\chi_{\{e\}}, \quad q_m = F_m(\widetilde{u}_m).$$
By assumption (iii), applied to the constant sequence $u_m \equiv u^*$, we have $\mathbf{D}(F_m(u^*), F(u^*)) \to 0$, and since $\widetilde{u}_m \to \widetilde{u}$, we also get $\mathbf{D}(F_m(\widetilde{u}_m), F(\widetilde{u})) \to 0$. Moreover, by the same argument used above for (a_m, a), from $\mathbf{D}(F_m(u^*), F(u^*)) \to 0$, $\varepsilon_m \to \varepsilon$, and $d(\widetilde{u}_m, u^*) \to d(\widetilde{u}, u^*)$, we infer that $\mathbf{D}(p_m, p) \to 0$. Also, since $\widetilde{u}_m \to \widetilde{u}$, assumption (iii) gives $\mathbf{D}(q_m, q) \to 0$.

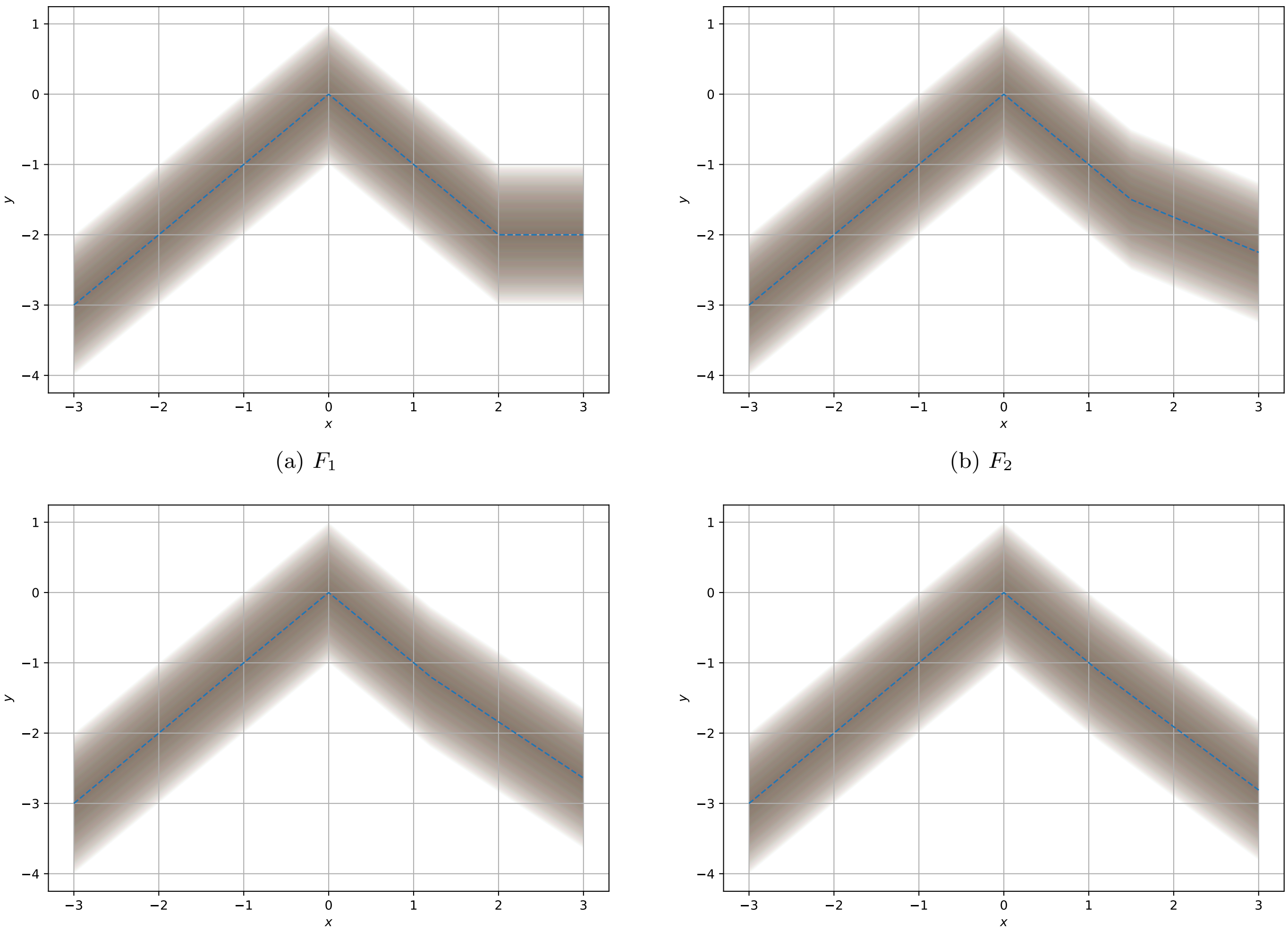
From (7) and the convergences $\mathbf{D}(p_m, p) \to 0$ and $\mathbf{D}(q_m, q) \to 0$, Lemma 3.4 yields that there exists $m_1 \in \mathbb{N}$ such that
$$p_m \preceq q_m, \quad \forall m \geq m_1.$$
That is,
$$F_m(u^*) + \varepsilon_m d(\widetilde{u}_m, u^*)\chi_{\{e\}} \preceq F_m(\widetilde{u}_m), \quad \forall m \geq m_1.$$
On the other hand, since $\widetilde{u}_m \to \widetilde{u}$ and $u^* \neq \widetilde{u}$, we have $u^* \neq \widetilde{u}_m$ for all sufficiently large m. Hence, for all sufficiently large m, the above inequality contradicts (5) with $u = u^*$. Thus, for every $\eta > 0$, we obtain
$$F(u) + (\varepsilon + \eta) d(\widetilde{u}, u)\chi_{\{e\}} \npreceq F(\widetilde{u}) \quad \forall u \in X \setminus \{\widetilde{u}\}. \tag{8}$$
Combining (6) and (8), we conclude that $\widetilde{u} \in \mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F)$. □

Remark 3.7. *The second condition in the definition of $\mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F)$ may equivalently be written with every $\varepsilon' > \varepsilon$ by setting $\varepsilon' = \varepsilon + \eta$. The present formulation emphasizes that the limiting variational property is preserved under every arbitrarily small right perturbation of the penalty parameter.*

The following example shows that the relaxed conclusion in Theorem 3.6 is, in general, sharp and cannot be improved to exact upper stability without additional assumptions.

Example 3.8. *Fix $\varepsilon_0 > 0$. Let $X = [-3, 3]$ be equipped with the usual metric, and set $u_{0,m} = u_0 = 0$ and $\varepsilon_m = \varepsilon = \varepsilon_0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, define*
$$c_m(x) = -\varepsilon_0|x| + \frac{\left(x - 1 - \frac{1}{m}\right)^+}{m}, \quad c(x) = -\varepsilon_0|x|,$$
where $(t)^+ := \max\{t, 0\}$. Next, define fuzzy-valued maps $F_m, F : X \to E^1$ by
$$[F_m(x)]^\alpha = [\, c_m(x) - (1-\alpha),\ c_m(x) + (1-\alpha)\,],$$
$$[F(x)]^\alpha = [\, c(x) - (1-\alpha),\ c(x) + (1-\alpha)\,],$$
for all $x \in X$ and $\alpha \in [0, 1]$. Since c_m and c are continuous on X, the mappings F_m and F are $\mathbf{D}$-continuous on X; hence, by Lemma 2.12, they are $\preceq$-lower semicontinuous on X. Moreover, since $c_m(x) \geq -\varepsilon_0|x| \geq -3\varepsilon_0$ and $c(x) \geq -\varepsilon_0|x| \geq -3\varepsilon_0$ for all $x \in X$, both F_m and F are $\preceq$-lower bounded on X; for instance, one may choose $\underline{z} \in E^1$ so that
$$[\underline{z}]^\alpha = [-3\varepsilon_0 - (1-\alpha),\ -3\varepsilon_0 + (1-\alpha)],$$
for every $\alpha \in [0, 1]$. Moreover, for every $x \in [-3, 3]$,
$$|c_m(x) - c(x)| = \frac{\left(x - 1 - \frac{1}{m}\right)^+}{m} \leq \frac{2}{m} \to 0,$$
so $\sup_{x \in [-3,3]} |c_m(x) - c(x)| \to 0$ as $m \to \infty$. Consequently, $F_m \to F$ uniformly on X, and therefore $F_m \xrightarrow{\mathbf{D}\text{-c.}} F$.

By Remark 2.4, in the one-dimensional case the fuzzy order is characterized by the lower and upper endpoints of the α-level intervals. In particular, for intervals of the form
$$I_\alpha(s) = [\, s - (1-\alpha),\ s + (1-\alpha)\,],$$
one has
$$I_\alpha(s) \preceq I_\alpha(t) \iff s \leq t,$$
because the two intervals have the same radius $1 - \alpha$. Hence the defining conditions of $\mathbf{\Pi}_{\varepsilon_0}(0; F_m)$ and $\mathbf{\Pi}_{\varepsilon_0}(0; F)$ reduce to scalar inequalities for the center functions c_m and c.



(a) F_1 (b) F_2

(c) F_5 (d) F_{10}

Figure 1: Geometric illustrations of the fuzzy functions F_m for $\varepsilon_0 = 1$ and $m = 1, 2, 5, 10$. The shaded band represents the values $y \in [F_m(x)]^\alpha$ for $\alpha \in [0, 1]$ and the dashed curve is the center function c_m.

Computation of $\mathbf{\Pi}_{\varepsilon_0}(0; F_m)$**.** *Since the first defining condition is* $F_m(u) + \varepsilon_0|u|\chi_{\{e\}} \preceq F_m(0)$ *and the order relation reduces to the comparison of centers, it is equivalent to* $c_m(u) + \varepsilon_0|u| \le c_m(0)$*. Now* $c_m(u) + \varepsilon_0|u| = \frac{\left(u-1-\frac{1}{m}\right)^+}{m}$ *and* $c_m(0) = 0$*, so this condition is equivalent to* $\frac{\left(u-1-\frac{1}{m}\right)^+}{m} \le 0$*, i.e.,* $u \le 1 + \frac{1}{m}$*.*

For the second one, if $u = 1 + \frac{1}{m}$*, then for every* $z \ne 1 + \frac{1}{m}$*,*

$$c_m(z) + \varepsilon_0 \left| z - \left(1 + \frac{1}{m}\right) \right| > c_m\left(1 + \frac{1}{m}\right) = -\varepsilon_0\left(1 + \frac{1}{m}\right).$$

If $u \in \left[0, 1 + \frac{1}{m}\right)$*, taking* $z = 1 + \frac{1}{m}$ *yields*

$$c_m\left(1 + \frac{1}{m}\right) + \varepsilon_0 \left| u - \left(1 + \frac{1}{m}\right) \right| = -\varepsilon_0\left(1 + \frac{1}{m}\right) + \varepsilon_0\left(1 + \frac{1}{m} - u\right) = -\varepsilon_0 u = c_m(u).$$

If $u \in (-3, 0)$*, taking any* $z < u$*, one gets*

$$c_m(z) + \varepsilon_0|u - z| = \varepsilon_0 z + \varepsilon_0(u - z) = \varepsilon_0 u = c_m(u).$$

Finally, if $u = -3$*, then*

$$c_m(z) + \varepsilon_0|-3 - z| > -3\varepsilon_0 = c_m(-3) \quad \forall z \in X \setminus \{-3\}.$$

Therefore, we obtain $\mathbf{\Pi}_{\varepsilon_0}(0; F_m) = \left\{-3, 1 + \frac{1}{m}\right\}, \quad \forall m \in \mathbb{N}$.

Computation of $\mathbf{\Pi}_{\varepsilon_0}(0; F)$***.*** *Arguing as above, the defining conditions reduce to scalar inequalities for the center function* c*. For the limit mapping* F*, the first defining condition holds for every* $u \in X$*, since* $c(u) + \varepsilon_0|u| = 0 = c(0)$*. Thus it remains to check the second condition*

$$c(z) + \varepsilon_0|u - z| > c(u), \quad \forall z \in X \setminus \{u\}.$$

If $u \in (0, 3)$*, choosing any* $z > u$ *gives equality; if* $u \in (-3, 0)$*, choosing any* $z < u$ *gives equality; and if* $u = 0$*, choosing any* $z > 0$ *again gives equality. Hence the second condition fails for every* $u \in (-3, 3)$*. On the other hand, for* $u = \pm 3$*, the strict inequality holds for all* $z \neq u$*. Therefore,* $\mathbf{\Pi}_{\varepsilon_0}(0; F) = \{-3, 3\}$*.*

Computation of $\mathbf{\Pi}^{\mathcal{R}}_{\varepsilon_0}(0; F)$***.*** *As above, the first defining condition holds for every* $u \in X$*, since* $c(u)+\varepsilon_0|u| = 0 = c(0)$*. It remains to verify the relaxed second condition. Let* $u \in X$*,* $\eta > 0$ *and* $z \neq u$*. Then*

$$c(z) + (\varepsilon_0 + \eta)|u - z| - c(u) = \varepsilon_0\big(|u - z| + |u| - |z|\big) + \eta|u - z|.$$

By the triangle inequality, the first term is nonnegative, while the second is strictly positive. Hence, we have

$$c(z) + (\varepsilon_0 + \eta)|u - z| > c(u), \quad \forall z \in X \setminus \{u\},$$

and so, we obtain $\mathbf{\Pi}^{\mathcal{R}}_{\varepsilon_0}(0; F) = X = [-3, 3]$*.*

Since $1 + \frac{1}{m} \to 1$ *as* $m \to +\infty$*, we obtain*

$$\limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_0}(0; F_m) = \limsup_{m\to\infty} \left\{-3, 1 + \frac{1}{m}\right\} = \{-3, 1\}.$$

Therefore

$$\{-3, 1\} = \limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_0}(0; F_m) \subset \mathbf{\Pi}^{\mathcal{R}}_{\varepsilon_0}(0; F) = [-3, 3],$$

while

$$1 \notin \mathbf{\Pi}_{\varepsilon_0}(0; F) = \{-3, 3\}.$$

In particular, the target set in Theorem 3.6 cannot, in general, be replaced by $\mathbf{\Pi}_{\varepsilon}(u_0; F)$*.*

Remark 3.9. *The enlargement from* $\mathbf{\Pi}_{\varepsilon}(u_0; F)$ *to* $\mathbf{\Pi}^{\mathcal{R}}_{\varepsilon}(u_0; F)$ *need not be extreme. For instance, let* $X = [-3, 3]$*,* $u_0 = 0$*,* $\varepsilon = 1$*, and define* $F : X \to E^1$ *by*

$$[F(x)]^\alpha = [\, c(x) - (1 - \alpha),\ c(x) + (1 - \alpha)\,], \quad \forall x \in X,\ \forall \alpha \in [0, 1],$$

where $c(x) = -|x| + (x - 1)^+$*. By a similar computation as in Example 3.8, one obtains*

$$\mathbf{\Pi}_1(0; F) = \{-3, 1\}, \quad \mathbf{\Pi}^{\mathcal{R}}_1(0; F) = [-3, 1].$$

Hence, $\mathbf{\Pi}_1(0; F) \subsetneq \mathbf{\Pi}^{\mathcal{R}}_1(0; F) \subsetneq X$*. This shows that the relaxed variational set may be strictly larger than the exact one while still remaining a proper subset of the ambient space.*

The failure of exact upper stability in the preceding examples suggests the need for an additional separation condition on the limit problem. Under such a condition, exact upper stability can be recovered, as we show next.

Assumption (A). Let $F : X \to E^n$, $\varepsilon > 0$ and $u_0 \in X$. We assume that, for every $\widetilde{u} \in X$ and every $u \in X \setminus \{\widetilde{u}\}$, $F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \preceq F(\widetilde{u})$ implies that there exists $\delta > 0$ such that

$$F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} + \delta\chi_{\{e\}} \preceq F(\widetilde{u}).$$

Example 3.10. *Let* $X = [0, 1]$*,* $u_0 = 0$*,* $\varepsilon = 1$ *and define* $F : X \to E^1$ *by*

$$[F(x)]^\alpha = [\, c(x) - (1 - \alpha),\ c(x) + (1 - \alpha)\,], \quad \forall x \in X,\ \forall \alpha \in [0, 1],$$

where $c(x) = -2x$*. Then Assumption* $(\mathbf{A})$ *holds.*

Indeed, the order relation is determined by the center function c*, so*

$$F(u) + d(\widetilde{u}, u)\chi_{\{e\}} \preceq F(\widetilde{u}) \iff -2u + |u - \widetilde{u}| \leq -2\widetilde{u}.$$

If $u < \widetilde{u}$*, this becomes* $\widetilde{u} \leq u$*, a contradiction. Hence necessarily,* $u > \widetilde{u}$*. In that case,*

$$-2u + |u - \widetilde{u}| = -2u + (u - \widetilde{u}) = -u - \widetilde{u} < -2\widetilde{u}.$$

Thus, there is a positive gap and choosing, for instance, $\delta = (u - \widetilde{u})/2 > 0$*, we obtain*

$$F(u) + d(\widetilde{u}, u)\chi_{\{e\}} + \delta\chi_{\{e\}} \preceq F(\widetilde{u}).$$

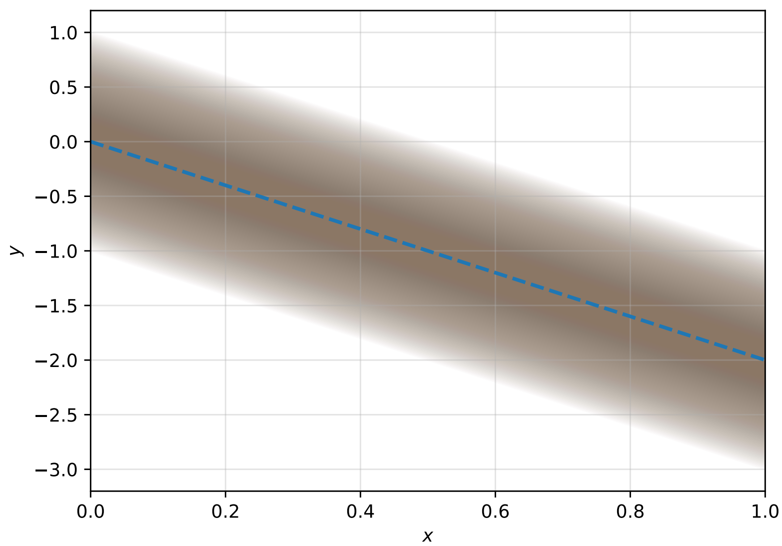


Figure 2: Geometric illustration of the fuzzy-valued mapping F on $X = [0, 1]$ with center function $c(x) = -2x$. The shaded band represents the values $y \in [F(x)]^\alpha$ for $\alpha \in [0, 1]$ and the dashed curve is the center function c.

The next corollary shows that, under Assumption (**A**), the relaxed upper stability theorem reduces to an exact upper stability statement.

Corollary 3.11. *Assume the hypotheses of Theorem 3.6. In addition, suppose that* (u_0, F, ε) *satisfies Assumption* (**A**)*. Then*

$$\limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m) \subset \mathbf{\Pi}_\varepsilon(u_0; F).$$

Proof. By Theorem 3.6, it is enough to show that $\mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F) \subset \mathbf{\Pi}_\varepsilon(u_0; F)$. Let $\widetilde{u} \in \mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F)$ and suppose, by contradiction, that $\widetilde{u} \notin \mathbf{\Pi}_\varepsilon(u_0; F)$. Then there exists $u \in X \setminus \{\widetilde{u}\}$ such that

$$F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \preceq F(\widetilde{u}).$$

By Assumption (**A**), there exists $\delta > 0$ such that

$$F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} + \delta\chi_{\{e\}} \preceq F(\widetilde{u}).$$

Since $u \neq \widetilde{u}$, setting $\eta = \delta/d(\widetilde{u}, u) > 0$, we obtain

$$F(u) + (\varepsilon + \eta)d(\widetilde{u}, u)\chi_{\{e\}} \preceq F(\widetilde{u}),$$

which contradicts the definition of $\mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F)$. Hence $\mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F) \subset \mathbf{\Pi}_\varepsilon(u_0; F)$ and the conclusion follows. $\square$

Remark 3.12. *Assumption* (**A**) *excludes boundary-contact phenomena at the level* ε*: whenever*

$$F(u) + \varepsilon d(\widetilde{u}, u)\chi_{\{e\}} \preceq F(\widetilde{u}),$$

the comparison holds with a strictly positive margin. In this sense, it acts as a separation or slack condition for the fuzzy order. Under Assumption (**A**)*, the relaxed and exact variational sets coincide, and Theorem 3.6 yields exact upper stability.*

Remark 3.13. *It is useful to compare our results with earlier stability analyses for Ekeland variational points. In the scalar setting, Attouch and Riahi* [2] *studied stability properties of* ε*-variational solutions and cone extremal solutions. The vector-valued case was subsequently considered by Chen and Huang* [9]*, while further results for vector-valued and set-valued mappings were obtained by Huang* [18, 19]*. The present framework differs from these settings because the fuzzy order is imposed levelwise through the lower and upper set orders of the* α*-level sets. The appearance of the relaxed variational set* $\mathbf{\Pi}_\varepsilon^{\mathcal{R}}(u_0; F)$ *is caused by a boundary phenomenon: the noncomparison relation* $\npreceq$ *need not be preserved* *under* $\mathbf{D}$*-convergence. The additional parameter* $\eta > 0$ *in the relaxed condition produces the strictly positive margin required by Lemma 3.4, thereby allowing the relevant order comparison to be transferred to the perturbed problems.*

Our Painlevé–Kuratowski outer-limit formulation is close in spirit to the convergence analyses of fuzzy optimization and fuzzy set-optimization problems in [21, 34]*. Here, however, the analysis is carried out directly for fuzzy Ekeland variational points, without passing to an epigraphical reformulation in a product space. To the best of our knowledge, Theorem 3.6 and Corollary 3.11 provide the first direct upper-stability results for Ekeland variational points in this fuzzy-valued setting.*

Remark 3.14 (Open problems)**.** *Theorem 3.6 shows that relaxed upper stability holds under natural assumptions on* F_m *and* F*:*

$$\limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m) \subset \mathbf{\Pi}_{\varepsilon}^{\mathcal{R}}(u_0; F).$$

Two related stability questions then arise naturally. The first is the lower stability (inner semicontinuity) problem:

$$\mathbf{\Pi}_{\varepsilon}(u_0; F) \subset \liminf_{m\to\infty} \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m).$$

The second is the exact upper stability problem:

$$\limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_m}(u_{0,m}; F_m) \subset \mathbf{\Pi}_{\varepsilon}(u_0; F).$$

Example 3.8 shows that both inclusions fail in general. Indeed,

$$3 \in \mathbf{\Pi}_{\varepsilon_0}(0; F) \quad \textit{but} \quad 3 \notin \liminf_{m\to\infty} \mathbf{\Pi}_{\varepsilon_0}(0; F_m) = \{-3, 1\},$$

so lower stability fails. Similarly,

$$1 \in \limsup_{m\to\infty} \mathbf{\Pi}_{\varepsilon_0}(0; F_m) \quad \textit{but} \quad 1 \notin \mathbf{\Pi}_{\varepsilon_0}(0; F) = \{-3, 3\},$$

so exact upper stability also fails in general.

Corollary 3.11 shows that exact upper stability can nevertheless be recovered under Assumption ($\mathbf{A}$)*. It remains an interesting problem to identify weaker or more transparent conditions ensuring exact upper stability, as well as to determine conditions under which lower stability holds.*

4 Conclusion

In this paper, we established a relaxed upper stability result for fuzzy Ekeland variational points in complete metric spaces. More precisely, under $\mathbf{D}$-convergence of fuzzy-valued mappings, together with the convergence of the perturbation parameters and base points, every Painlevé–Kuratowski outer limit point of exact fuzzy Ekeland variational points for perturbed problems belongs to the relaxed variational set of the limit problem. This shows that the relaxed notion provides a natural and robust stability framework in the fuzzy setting.

We also clarified the relation between exact and relaxed fuzzy Ekeland variational sets through explicit examples. In particular, exact upper stability fails in general, while the relaxed variational set may be strictly larger than the exact one without coinciding with the whole ambient space. On the other hand, we showed that exact upper stability can be recovered under Assumption ($\mathbf{A}$), which acts as a separation condition on the limit problem.

These results suggest several directions for further research. In particular, it would be of interest to identify weaker or more transparent conditions ensuring exact upper stability and to investigate lower stability for fuzzy Ekeland variational points.

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