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On the Fidel-Vakarelov construction for monadic Gödel algebras

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Abstract

The classical Fidel-Vakarelov construction connects Heyting algebras with centered Nelson algebras through a twist representation. We investigate whether this construction is compatible with monadic operators. We introduce monadic centered prelinear Nelson algebras and prove that the Fidel–Vakarelov construction yields a categorical equivalence between monadic Gödel algebras and this new class. We also show, by a finite counterexample, why the additional monadic Gödel equation is necessary, and we establish an order isomorphism between the corresponding congruence lattices.

Keywords: Heyting algebras, Nelson algebras, monadic Gödel algebras, twist structures.

1 Introduction

Monadic expansions provide an algebraic framework for fragments of predicate logic in which quantification is restricted to one variable. The classical starting point is Halmos's theory of monadic Boolean algebras [16]; analogous operators were later studied on Heyting algebras, MV-algebras, basic algebras, De Morgan algebras, and several related ordered structures [1, 3, 7, 8, 10, 11, 15, 21]. These constructions are useful because they allow logical questions involving quantifiers to be studied by algebraic methods, including representations, congruences, subdirect decompositions, and categorical equivalences.

Nelson algebras form the algebraic semantics of constructive logic with strong negation [22, 23]. Their close relationship with Heyting algebras was developed independently by Fidel and Vakarelov and later placed in the broader framework of twist structures [9, 14, 17, 24, 25, 26]. The Fidel–Vakarelov construction is particularly important: from a Heyting algebra it produces a centered Nelson algebra whose elements are compatible pairs. This construction permits results and examples to be transferred between intuitionistic and strongly negated settings.

Recent work has extended Kalman-style categorical methods to further enriched settings, including weak monadic residuated distributive lattices and Heyting algebras equipped with Kalman-Galois connections [4, 27]. These developments provide additional motivation for determining precisely which monadic assumptions allow the Fidel-Vakarelov construction to be lifted.

The purpose of this paper is to determine when this transfer remains valid after monadic operators are added. A naive componentwise definition of the quantifiers does not work for all monadic prelinear Heyting algebras. We identify the additional equation that makes the construction possible and use it to introduce the variety of monadic centered prelinear Nelson algebras. The resulting equivalence is relevant both algebraically and logically: it supplies twist representations for monadic Nelson-style structures and provides a systematic method for transferring congruences and finite examples from monadic Gödel algebras.

Our main contributions are the following. First, we define the induced monadic operators on the Fidel–Vakarelov construction and prove that they produce a monadic N_c-algebra. Second, we prove that the functors K and C establish

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a categorical equivalence between monadic Gödel algebras and monadic N_c-algebras. Third, we show that the corresponding congruence lattices are order-isomorphic and derive consequences for simplicity and subdirect irreducibility. Finally, we present explicit finite examples showing both the necessity of the additional equation and the behavior of the congruence correspondence.

The paper is organized as follows. Section 2 collects the algebraic background and fixes notation. Section 3 introduces monadic Nelson algebras. Section 4 proves the categorical equivalence, and Section 5 studies congruences and gives a finite example.

2 Preliminaries

We collect here the definitions and results used later. Standard background on distributive lattices and Heyting algebras can be found in [2]; the Kleene–Nelson constructions used below are treated in [6, 9, 14, 26].

2.1 Kleene algebras and the Kalman construction

A Kleene algebra is a De Morgan algebra $\langle T, \vee, \wedge, \sim, 0, 1\rangle$ satisfying $x \wedge \sim x \leq y \vee \sim y$. It is centered if it has an element c such that $c = \sim c$; this element is unique. We write BDL for the category of bounded distributive lattices and KA_c for the category of centered Kleene algebras, with the corresponding homomorphisms.

For $A \in \mathsf{BDL}$, define

$$K(A) = \{(a, b) \in A \times A : a \wedge b = 0\},$$

with operations

$$\begin{aligned}
(a, b) \vee (d, e) &= (a \vee d, b \wedge e),\\
(a, b) \wedge (d, e) &= (a \wedge d, b \vee e),\\
\sim(a, b) &= (b, a),\\
0 &= (0, 1),\\
1 &= (1, 0),\\
c &= (0, 0).
\end{aligned}$$

Then $K(A)$ is a centered Kleene algebra. If $f : H \to A$ is a morphism in BDL, then $K(f)(a, b) = (f(a), f(b))$ defines a morphism in KA_c, and hence $K : \mathsf{BDL} \to \mathsf{KA}_c$ is a functor.

Conversely, for $T \in \mathsf{KA}_c$ set

$$C(T) = \{x \in T : x \geq c\}.$$

With bottom element c, this is a bounded distributive lattice. A centered Kleene homomorphism $g : T \to U$ restricts to a bounded lattice homomorphism $C(g) : C(T) \to C(U)$, so $C : \mathsf{KA}_c \to \mathsf{BDL}$ is a functor. The maps

$$\alpha_A(a) = (a, 0), \qquad \beta_T(x) = (x \vee c, \sim x \vee c),$$

are natural; α_A is an isomorphism and β_T is an injective centered Kleene homomorphism [6, 9].

A centered Kleene algebra satisfies condition (CK) when, for all $x, y \geq c$,

$$x \wedge y = c \quad \Longrightarrow \quad (\exists z)\big(z \vee c = x \ \&\ \sim z \vee c = y\big).$$

This condition is equivalent to the surjectivity of β_T. Thus the functors K and C restrict to an equivalence between BDL and the full subcategory $\mathsf{KA}_c^{\mathrm{CK}}$ of centered Kleene algebras satisfying (CK). This algebraic formulation is closely related to Cignoli's interpolation property [9, Theorem 2.4]; see also [6].

2.2 Heyting and Nelson algebras

A Heyting algebra is an algebra $A = \langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and

$$x \wedge y \leq z \quad \text{if and only if} \quad x \leq y \Rightarrow z.$$

Equivalently, it may be axiomatized by the standard identities used in [2]. We write HA for the category of Heyting algebras.

A Nelson algebra is an algebra $T = \langle T, \vee, \wedge, \to, \sim, 0, 1\rangle$ satisfying

(N1) $\langle T, \vee, \wedge, \sim, 0, 1\rangle$ is a Kleene algebra;

(N2) $x \to x = 1$;

(N3) $x \to (y \to z) = (x \wedge y) \to z$;

(N4) $x \wedge (x \to y) = x \wedge (\sim x \vee y)$.

See [18, 20]. It is centered when its Kleene reduct is centered, and it is prelinear when $(x \to y) \vee (y \to x) = 1$ [19]. In every Nelson algebra,

$$x \wedge z \leq \sim x \vee y \quad \iff \quad z \leq x \to y. \tag{1}$$

We denote by NA_c the category of centered Nelson algebras.

If $A \in \mathsf{HA}$, then $K(A)$ becomes a centered Nelson algebra by defining

$$(a, b) \to (d, e) = (a \Rightarrow d, a \wedge e). \tag{2}$$

Here $\Rightarrow$ is the Heyting implication on A, whereas $\to$ is the induced Nelson implication on $K(A)$. The functors K and C establish a categorical equivalence between HA and NA_c [9, Theorem 3.14].

2.3 Monadic Gödel algebras

A monadic Heyting algebra is a structure $(A, \forall, \exists)$, where A is a Heyting algebra, and the unary operations satisfy [3, 21]

(m1) $\forall x \leq x$;

(m2) $\forall(x \wedge y) = \forall x \wedge \forall y$;

(m3) $\forall 1 = 1$;

(m4) $\forall \exists x = \exists x$;

(m5) $\forall(x \Rightarrow y) \leq \exists x \Rightarrow \exists y$;

(m6) $x \leq \exists x$;

(m7) $\exists(x \vee y) = \exists x \vee \exists y$;

(m8) $\exists 0 = 0$;

(m9) $\exists \forall x = \forall x$.

A Gödel algebra is a prelinear Heyting algebra. A monadic Gödel algebra is a monadic prelinear Heyting algebra satisfying

$$\forall(\exists x \vee y) = \exists x \vee \forall y. \tag{3}$$

This presentation and the properties below follow from the theory of monadic BL-algebras developed in [5].

Lemma 2.1. *Let $(A, \forall, \exists)$ be a monadic Gödel algebra. Then:*

1. $\exists(A) = \forall(A)$*;*
2. $\exists(A)$ *is a subalgebra of* A*;*
3. *for every* $a \in A$*,* $\exists a = \min\{b \in \exists(A) : a \leq b\}$ *and* $\forall a = \max\{b \in \exists(A) : b \leq a\}$*;*
4. *the congruence lattices of* $(A, \forall, \exists)$ *and* $\exists(A)$ *are isomorphic.*

Lemma 2.2. *Let $(A, \forall, \exists)$ be a monadic Gödel algebra. For all $a, b \in A$ and every u in the common range $\exists(A) = \forall(A)$, the following hold:*

1. $\exists 1 = 1$ *and* $\forall 0 = 0$*;*
2. $\forall u = u = \exists u$*;*

3. $\forall a \leq a \leq \exists a$;
4. *if* $a \leq b$, *then* $\forall a \leq \forall b$ *and* $\exists a \leq \exists b$;
5. $\forall(a \vee u) = \forall a \vee u$;
6. $\exists(a \wedge u) = \exists a \wedge u$;
7. $\forall(a \Rightarrow u) = \exists a \Rightarrow u$;
8. $\exists(a \Rightarrow u) \leq \forall a \Rightarrow u$;
9. $\forall(u \Rightarrow a) = u \Rightarrow \forall a$;
10. $\exists(u \Rightarrow a) \leq u \Rightarrow \exists a$;
11. $\exists a \wedge \forall b \leq \exists(a \wedge b)$;
12. $\forall(a \Rightarrow b) \leq \forall a \Rightarrow \forall b$;
13. $\forall \neg a = \neg \exists a$;
14. $\exists \neg a \leq \neg \forall a$.

3 Monadic Nelson algebras

In this section, we will introduce the concept of monadic centered Nelson algebras and present some fundamental properties.

Definition 3.1. *An algebra* $\mathbf{U} = (T, \exists)$ *is a* monadic Nelson algebra *if* $T = \langle T, \vee, \wedge, \rightarrow, \sim, 0, 1\rangle$ *is a Nelson algebra and the following conditions hold:*

(n1) $\exists 0 = 0$,

(n2) $x \leq \exists x$,

(n3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

(n4) $\exists(x \vee y) = \exists x \vee \exists y$,

(n5) $\forall \exists x = \exists x$,

(n6) $\forall(x \rightarrow y) \leq \exists x \rightarrow \exists y$,

(n7) $\forall(x \rightarrow y) \leq \forall x \rightarrow \forall y$, *where* $\forall a := \sim\exists(\sim a)$.

Furthermore, if T *is a prelinear Nelson algebra with center* c, *and* $c \in \exists(T)$, *we refer to* $\mathbf{U}$ *as a* monadic centered prelinear Nelson algebra *(or* monadic N_c-algebra *for brevity).*

Remark 3.2. *Let* $\mathbf{U} = (T, \exists)$ *be a monadic Nelson algebra. Then from (n1) to (n5), we have that the reduct* $\langle T, \vee, \wedge, \sim, \exists, 0, 1\rangle$ *is a monadic De Morgan algebra (see* [8]*).*

The proofs of the following algebraic properties are straightforward.

Lemma 3.3. *Let* $\mathbf{U} = (T, \exists)$ *be a monadic Nelson algebra. Then the following properties hold:*

1. $\forall 1 = 1$,
2. $\forall x \leq x$,
3. $\forall(x \vee \forall y) = \forall x \vee \forall y$,
4. $\forall(x \wedge y) = \forall x \wedge \forall y$,
5. $\exists \forall x = \forall x$,
6. $\forall \exists x = \exists x$.

4 Fidel-Vakarelov construction

In this section, we lift the classical Fidel–Vakarelov construction recalled in Section 2 to the monadic setting. Let $(A, \forall, \exists)$ be a monadic Gödel algebra. We use $\Rightarrow$ for the Heyting implication of A and $\rightarrow$ for the Nelson implication on $K(A)$; thus,

$$(a, b) \rightarrow (d, e) = (a \Rightarrow d, a \wedge e). \tag{4}$$

The underlying centered Nelson algebra is denoted A_K. We define unary operations on $K(A)$ by

$$\exists_K(a, b) = (\exists a, \forall b), \qquad \forall_K(a, b) = (\forall a, \exists b). \tag{5}$$

Lemma 4.1. *Let $\mathbf{G} = (A, \forall, \exists)$ be a monadic Gödel algebra and let $(a, b) \in K(A)$. Then the following hold:*

(a) $\exists_K(a, b) \in K(A)$,

(b) $\forall_K(a, b) = \sim\exists_K(\sim(a, b))$,

(c) $\forall_K(a, b) \in K(A)$.

Proof. We will only prove (a). Let $(a, b) \in K(A)$. Therefore, $a \wedge b = 0$. Then by using (m8) and item 11 of Lemma 2.2, we have $\exists a \wedge \forall b \leq \exists(a \wedge b) = \exists 0 = 0$. Consequently, $\exists_K(a, b) \in K(A)$. ☐

Lemma 4.2. *Let $\mathbf{G} = (A, \forall, \exists)$ be a monadic Gödel algebra. Then,*

$$K(\mathbf{G}) = (A_K, \exists_K),$$

is a monadic N_c-algebra.

Proof. By the classical Fidel–Vakarelov construction, A_K is a centered Nelson algebra. Recall that $\Rightarrow$ denotes implication in A, whereas $\rightarrow$ denotes the induced Nelson implication on $K(A)$. We first verify prelinearity. Let $(a, b), (x, y) \in K(A)$. Then,

$$\begin{aligned}
[(a, b) \rightarrow (x, y)] \vee [(x, y) \rightarrow (a, b)] &= (a \Rightarrow x, a \wedge y) \vee (x \Rightarrow a, x \wedge b) \\
&= ((a \Rightarrow x) \vee (x \Rightarrow a), a \wedge y \wedge x \wedge b) \\
&= (1, 0).
\end{aligned}$$

Lemma 4.1 shows that $\exists_K$ and $\forall_K$ are well-defined. We now verify axioms (n1)–(n7) of Definition 3.1.

(n1) From (m3) and (m8), we have that $\exists_K(0, 1) = (\exists 0, \forall 1) = (0, 1)$.

(n2) From (m1) and (m6), we obtain

$$(a, b) \wedge (\exists a, \forall b) = (a \wedge \exists a, b \vee \forall b) = (a, b).$$

Therefore, $(a, b) \leq \exists_K(a, b)$.

(n3) From items 5 and 6 of Lemma 2.2, we have

$$\begin{aligned}
\exists_K[(a, b) \wedge \exists_K(x, y)] &= \exists_K[(a, b) \wedge (\exists x, \forall y)] \\
&= \exists_K(a \wedge \exists x, b \vee \forall y) \\
&= (\exists(a \wedge \exists x), \forall(b \vee \forall y)) \\
&= (\exists a \wedge \exists x, \forall b \vee \forall y) \\
&= (\exists a, \forall b) \wedge (\exists x, \forall y) \\
&= \exists_K(a, b) \wedge \exists_K(x, y).
\end{aligned}$$

(n4) From (m2) and (m7), we have

$$\begin{aligned}
\exists_K[(a, b) \vee (x, y)] &= \exists_K(a \vee x, b \wedge y) \\
&= (\exists(a \vee x), \forall(b \wedge y)) \\
&= (\exists a \vee \exists x, \forall b \wedge \forall y) \\
&= (\exists a, \forall b) \vee (\exists x, \forall y) \\
&= \exists_K(a, b) \vee \exists_K(x, y).
\end{aligned}$$

(n5) From (m4) and (m9), we deduce that

$$\forall_K \exists_K (a, b) = \forall_K (\exists a, \forall b) = (\forall \exists a, \exists \forall b) = (\exists a, \forall b) = \exists_K (a, b).$$

This is precisely axiom (n5).

(n6) From (m5), we deduce that $\forall(a \Rightarrow x) \leq \exists a \Rightarrow \exists x$, and from item 11 in Lemma 2.2, we conclude $\exists a \wedge \forall y \leq \exists(a \wedge y)$. Therefore,

$$(\forall(a \Rightarrow x), \exists(a \wedge y)) \leq (\exists a \Rightarrow \exists x, \exists a \wedge \forall y).$$

Thus, we can establish

$$\forall_K((a, b) \to (x, y)) \leq \exists_K(a, b) \to \exists_K(x, y).$$

(n7) From Lemma 2.2, we have that $\forall(a \Rightarrow x) \leq \forall a \Rightarrow \forall x$, and $\forall a \wedge \exists y \leq \exists(a \wedge y)$. This implies that

$$(\forall(a \Rightarrow x), \exists(a \wedge y)) \leq (\forall a \Rightarrow \forall x, \forall a \wedge \exists y).$$

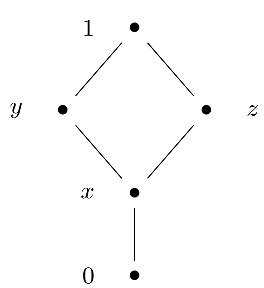
Therefore, by the order on $K(A)$, we can conclude that

$$\forall_K((a, b) \to (x, y)) \leq \forall_K(a, b) \to \forall_K(x, y).$$

□

The following remark illustrates that equation (3) cannot be omitted in Lemma 4.2. This observation, in our view, justifies the study of the Fidel–Vakarelov construction in the case of monadic Gödel algebras.

Remark 4.3. *Note that monadic prelinear Heyting algebras may not satisfy equation* (3)*. A counterexample is given by the monadic Heyting algebra* $(A, \exists, \forall)$ *depicted in the Hasse diagram below, with the monadic operators defined as in the table.*



a	0	x	y	z	1
$\exists a$	0	z	1	z	1
$\forall a$	0	0	0	z	1

Figure 1: A monadic prelinear Heyting algebra that does not satisfy equation (3).

Indeed, note that $\forall(y \vee \exists z) = \forall(y \vee z) = \forall 1 = 1$*, whereas* $\forall y \vee \exists z = 0 \vee z = z$*. Upon applying the previously described Fidel–Vakarelov construction, we arrive at the ensuing centered Nelson algebra:*

$$K(A) = \{(0, 0), (0, 1), (1, 0), (0, x), (x, 0), (0, y), (y, 0), (0, z), (z, 0)\}.$$

This is elucidated in the subsequent Hasse diagram.

Let us observe that if we consider the operator $\exists_K$ *defined as in equation* (5)*, axiom (n3) of Definition 3.1 is not satisfied. Indeed,*

$$\exists_K\big((x, 0) \wedge \exists_K(0, x)\big) = (0, z) \neq (0, 0) = \exists_K(x, 0) \wedge \exists_K(0, x).$$

Hence, $(K(A), \exists_K)$ *is not a monadic De Morgan algebra, and consequently, it is not a monadic* N_c*-algebra.*

We write mG for the category whose objects are monadic Gödel algebras and mN_c for the category whose objects are monadic N_c-algebras. In both cases, the morphisms are the corresponding algebra homomorphisms. Moreover, if $\mathbf{G} = (A, \forall, \exists)$ and $\mathbf{M} = (B, \forall, \exists)$ are monadic Gödel algebras and $f : \mathbf{G} \to \mathbf{M}$ is a morphism in mG, then it is straightforward to verify that the map $K(f) : K(A) \to K(B)$ given by $K(f)(x, y) = (f(x), f(y))$ is a morphism in mN_c from $K(\mathbf{G}) = (A_K, \exists_K)$ to $K(\mathbf{M}) = (B_K, \exists_K)$. It is clear that these assignments establish a functor K from mG to mN_c.

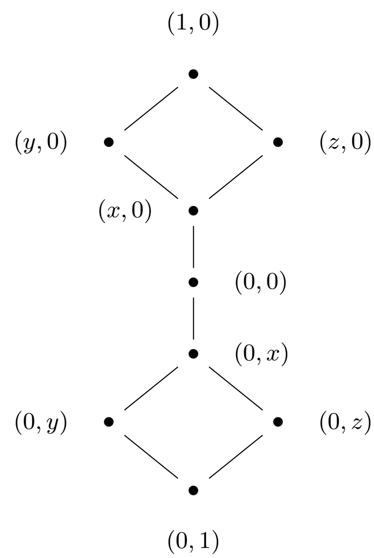


Figure 2: The underlying order of the centered Nelson algebra $K(A)$.

Lemma 4.4. *Let $T = \langle T, \wedge, \vee, \to, \sim, c, 0, 1\rangle$ be a centered prelinear Nelson algebra and define*

$$C(T) := \{x \in T : x \geq c\}.$$

Then, the structure

$$T_C = \langle C(T), \wedge, \vee, \to, c, 1\rangle,$$

is a Gödel algebra. Moreover, if $f : T \to S$ is a homomorphism of centered Nelson algebras, then it follows that $C(f) : C(T) \to C(S)$, defined by $C(f)(x) = f(x)$, is a homomorphism of Gödel algebras.

Proof. The set $C(T)$ is a bounded distributive lattice with bottom element c. Let $x, y \in C(T)$. Since $x \wedge c = c \leq y \leq {\sim}x \vee y$, equation (1) gives $c \leq x \to y$, and hence $x \to y \in C(T)$. For $x, y, z \in C(T)$, if $x \wedge y \leq z$, then $y \wedge x \leq {\sim}y \vee z$, so (1) yields $x \leq y \to z$. Conversely, if $x \leq y \to z$, then

$$x \wedge y \leq (y \to z) \wedge y = y \wedge ({\sim}y \vee z) \leq z,$$

because ${\sim}y \leq {\sim}c = c \leq z$. Thus $\to$ is the relative pseudocomplement on $C(T)$. Prelinearity is inherited from T, so T_C is a Gödel algebra. Finally, the restriction $C(f)$ preserves all the displayed operations, and therefore it is a Gödel-algebra homomorphism. □

Lemma 4.5. *Let $\mathbf{U} = (T, \exists)$ be a monadic N_c-algebra. Then, $C(\mathbf{U}) = (T_C, \forall, \exists)$ is a monadic Gödel algebra. Moreover, if $f : (T, \exists) \to (S, \exists)$ is a morphism in mN_c, then $C(f)$ is a morphism in mG.*

Proof. By Lemma 4.4, the reduct T_C is a Gödel algebra. It remains to verify that $C(T)$ is closed under the monadic operators. Since $c \in \exists(T)$, then $\exists c = c = \forall c$. If $x \geq c$, monotonicity yields $\exists x \geq \exists c = c$, so $\exists x \in C(T)$. Moreover, $x \geq c$ implies ${\sim}x \leq c$, and therefore $\exists({\sim}x) \leq \exists c = c$, whence $c \leq {\sim}\exists({\sim}x) = \forall x$. Thus $\forall x \in C(T)$. All monadic Gödel identities are inherited by restriction. If f is a monadic N_c-homomorphism, then its restriction preserves $\forall$ and $\exists$, so $C(f)$ is a morphism in mG. □

Let $f : (T, \exists) \to (S, \exists)$ be a homomorphism of monadic N_c-algebras. By Lemmas 4.4 and 4.5, the assignments $T \mapsto C(T)$, $f \mapsto C(f)$ determine a functor C from mN_c to mG.

Lemma 4.6. *Let $\mathbf{G} = (A, \forall, \exists)$ be a monadic Gödel algebra. Then the map $\alpha : \mathbf{G} \to C(K(\mathbf{G}))$ given by $\alpha(x) = (x, 0)$ is an isomorphism in mG.*

Proof. We will only prove that α commutes with the unary operators $\exists$ and $\forall$. Let $a \in A$. Then

- $\alpha(\forall a) = (\forall a, 0) = (\forall a, \exists 0) = \forall_K((a, 0)) = \forall_K(\alpha(a))$.

- $\alpha(\exists a) = (\exists a, 0) = (\exists a, \forall 0) = \exists_K((a, 0)) = \exists_K(\alpha(a))$.

□

Cignoli proved that every Nelson algebra satisfies the interpolation property [9, Theorem 3.14]. By the equivalence between that property and condition (CK) recalled in Section 2, the map $\beta : T \to K(C(T))$, $\beta(x) = (x \vee c, {\sim}x \vee c)$, is an isomorphism of the underlying Nelson algebras. It remains only to verify compatibility with the monadic operator.

Next, we establish that the previously mentioned result can be extended to the monadic context.

Lemma 4.7. *Let $\mathbf{U} = (T, \exists)$ be a monadic N_c-algebra. Then the map β is an isomorphism in mN_c.*

Proof. It remains to prove that β commutes with $\exists$. Let $a \in T$. Since $c \in \exists(T)$, we have $\exists c = c = \forall c$. Using (n4), item 3 of Lemma 3.3, and the identity ${\sim}\exists a = \forall({\sim}a)$, we obtain

$$\begin{aligned} \beta(\exists a) &= (\exists a \vee c, {\sim}\exists a \vee c) \\ &= (\exists(a \vee c), \forall({\sim}a \vee c)) \\ &= \exists_K(a \vee c, {\sim}a \vee c) \\ &= \exists_K(\beta(a)). \end{aligned}$$

Thus β is an isomorphism in mN_c. □

Straightforward computations based on previous results of this section prove the following result.

Theorem 4.8. *The functors K and C establish a categorical equivalence between mG and mN_c with natural isomorphisms α and β.*

Corollary 4.9. *The constructions of Lemmas 4.2 and 4.5 are converse to one another up to natural isomorphism. More precisely, every monadic Gödel algebra $\mathbf{G}$ is isomorphic to $C(K(\mathbf{G}))$, and every monadic N_c-algebra $\mathbf{U}$ is isomorphic to $K(C(\mathbf{U}))$.*

Proof. The first assertion is Lemma 4.6, and the second is Lemma 4.7. Naturality follows from Theorem 4.8. □

5 Congruences

Write $\mathrm{Con}(A)$ for the lattice of congruences of an algebra A. Let L be a bounded distributive lattice. If $\theta \in \mathrm{Con}(L)$, we can define a congruence γ_θ of $K(L)$ by

$$(a, b)\ \gamma_\theta\ (x, y) \quad \text{if and only if} \quad (a, x) \in \theta \text{ and } (b, y) \in \theta.$$

Conversely, if $\gamma \in \mathrm{Con}(K(L))$, we can also define a congruence θ^γ of L as

$$(a, b) \in \theta^\gamma \quad \text{if and only if} \quad (a, 0)\ \gamma\ (b, 0).$$

In Lemma 5.3 of [6], it was proved that the assignments $\theta \mapsto \gamma_\theta$ and $\gamma \mapsto \theta^\gamma$ establish an order isomorphism between $\mathrm{Con}(L)$ and $\mathrm{Con}(K(L))$.

The following results show that the latter assignment can also be extended to monadic Gödel algebras and monadic N_c-algebras.

Lemma 5.1. *Let $\mathbf{G} = (A, \forall, \exists)$ be a monadic Gödel algebra and let $\theta \in \mathrm{Con}(\mathbf{G})$ and $\gamma \in \mathrm{Con}(K(\mathbf{G}))$. Then, $\gamma_\theta \in \mathrm{Con}(K(\mathbf{G}))$ and $\theta^\gamma \in \mathrm{Con}(\mathbf{G})$.*

Proof. Let $(a, b)\ \gamma_\theta\ (x, y)$ and $(v, w)\ \gamma_\theta\ (t, u)$, meaning that $(a, x) \in \theta$, $(b, y) \in \theta$, $(v, t) \in \theta$, and $(w, u) \in \theta$. Consequently, we have $(a \Rightarrow v, x \Rightarrow t) \in \theta$ and $(a \wedge w, x \wedge u) \in \theta$. By the definition of γ_θ, it follows that

$$(a \Rightarrow v, a \wedge w)\ \gamma_\theta\ (x \Rightarrow t, x \wedge u).$$

Hence,

$$(a, b) \to (v, w)\ \gamma_\theta\ (x, y) \to (t, u).$$

On the other hand, if $(a, b)\ \gamma_\theta\ (x, y)$, which implies that $(a, x) \in \theta$ and $(b, y) \in \theta$, then, because θ is compatible with $\exists$ and $\forall$, we can conclude that $(\exists a, \exists x) \in \theta$ and $(\forall b, \forall y) \in \theta$. This leads to

$$(\exists a, \forall b)\ \gamma_\theta\ (\exists x, \forall y).$$

So, $\exists_K(a, b)\ \gamma_\theta\ \exists_K(x, y)$. Therefore, $\gamma_\theta \in \mathrm{Con}(K(\mathbf{G}))$.

Finally, we prove that θ^γ belongs to $\mathrm{Con}(\mathbf{G})$. Take $(a, b) \in \theta^\gamma$ and $(x, y) \in \theta^\gamma$. This implies $(a, 0)\ \gamma\ (b, 0)$ and $(x, 0)\ \gamma\ (y, 0)$. Consequently, we have

$$(a, 0) \to (x, 0)\ \gamma\ (b, 0) \to (y, 0).$$

However, $(a, 0) \to (x, 0) = (a \Rightarrow x, 0)$ and $(b, 0) \to (y, 0) = (b \Rightarrow y, 0)$. Thus, we can conclude that $(a \Rightarrow x, b \Rightarrow y) \in \theta^\gamma$.

We next show that θ^γ is compatible with both $\exists$ and $\forall$. Assume $(a, b) \in \theta^\gamma$. This means $(a, 0)\ \gamma\ (b, 0)$. Given that γ is compatible with $\exists_K$, we can deduce $(\exists a, 0)\ \gamma\ (\exists b, 0)$. Hence, we have $(\exists a, \exists b) \in \theta^\gamma$. Additionally, since γ is compatible with $\sim$ and $\exists_K$, we know that $\forall_K(a, 0)\ \gamma\ \forall_K(b, 0)$, which leads to $(\forall a, 0)\ \gamma\ (\forall b, 0)$. In other words, $(\forall a, \forall b) \in \theta^\gamma$. □

Theorem 5.2. *Let $\mathbf{G} = (A, \forall, \exists)$ be a monadic Gödel algebra. Then, the mapping*

$$f : \mathrm{Con}(\mathbf{G}) \longrightarrow \mathrm{Con}(K(\mathbf{G})), \qquad f(\theta) = \gamma_\theta,$$

establishes an order isomorphism.

In [5], Castaño et al. obtained a characterization of simple and subdirectly irreducible monadic Gödel algebras. The preceding theorem provides a complementary perspective in terms of Kalman's construction.

Corollary 5.3. *Let $\mathbf{G}$ be a monadic Gödel algebra. Then:*

1. *$\mathbf{G}$ is simple if and only if $K(\mathbf{G})$ is simple.*
2. *$\mathbf{G}$ is subdirectly irreducible if and only if $K(\mathbf{G})$ is subdirectly irreducible.*

The following example explicitly illustrates Theorem 5.2.

Example 5.4. *Let $A_4 = \{0, a, b, 1\}$ be the four-element Gödel chain whose underlying order is $0 < a < b < 1$. The Gödel implication on A_4 is defined by*

$$x \Rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y. \end{cases}$$

Define the monadic operators by

x	0	a	b	1
$\forall x$	0	0	b	1
$\exists x$	0	b	b	1

Then $\mathbf{G}_4 = \langle A_4, \wedge, \vee, \Rightarrow, \forall, \exists, 0, 1\rangle$ is a monadic Gödel algebra. The algebra $K(\mathbf{G}_4)$ has universe

$$K(A_4) = \{(0, 1), (0, b), (0, a), (0, 0), (a, 0), (b, 0), (1, 0)\},$$

and its underlying order is

$$(0, 1) < (0, b) < (0, a) < (0, 0) < (a, 0) < (b, 0) < (1, 0).$$

The induced monadic operators are given by

z	$(0,1)$	$(0,b)$	$(0,a)$	$(0,0)$	$(a,0)$	$(b,0)$	$(1,0)$
$\forall_K z$	$(0,1)$	$(0,b)$	$(0,b)$	$(0,0)$	$(0,0)$	$(b,0)$	$(1,0)$
$\exists_K z$	$(0,1)$	$(0,b)$	$(0,0)$	$(0,0)$	$(b,0)$	$(b,0)$	$(1,0)$

The Gödel reduct A_4 has exactly four congruences:

$$\begin{aligned} \Delta_{A_4} &= \big\{\{0\}, \{a\}, \{b\}, \{1\}\big\}, \\ \theta &= \big\{\{0\}, \{a\}, \{b, 1\}\big\}, \\ \rho &= \big\{\{0\}, \{a, b, 1\}\big\}, \\ \nabla_{A_4} &= \big\{\{0, a, b, 1\}\big\}. \end{aligned}$$

However, ρ is not a congruence of the monadic Gödel algebra $\mathbf{G}_4$. Therefore,

$$\operatorname{Con}(\mathbf{G}_4) = \{\Delta_{A_4}, \theta, \nabla_{A_4}\}.$$

The corresponding congruences of $K(\mathbf{G}_4)$ are

$$\begin{aligned}
\Delta_{K(A_4)} &= \big\{\{(0,1)\},\{(0,b)\},\{(0,a)\},\{(0,0)\},\{(a,0)\},\{(b,0)\},\{(1,0)\}\big\},\\
\gamma_\theta &= \big\{\{(0,1),(0,b)\},\{(0,a)\},\{(0,0)\},\{(a,0)\},\{(b,0),(1,0)\}\big\},\\
\nabla_{K(A_4)} &= \big\{K(A_4)\big\}.
\end{aligned}$$

Consequently,

$$\operatorname{Con}(K(\mathbf{G}_4)) = \{\Delta_{K(A_4)}, \gamma_\theta, \nabla_{K(A_4)}\}.$$

Thus, the correspondence

$$\Delta_{A_4} \longmapsto \Delta_{K(A_4)}, \qquad \theta \longmapsto \gamma_\theta, \qquad \nabla_{A_4} \longmapsto \nabla_{K(A_4)},$$

is an order isomorphism.

Conclusion and open problems

We have shown that the usual componentwise definition of the monadic operators on the Fidel–Vakarelov twist construction does not work for arbitrary monadic prelinear Heyting algebras: equation (3) is essential for axiom (n3). Under this hypothesis, the construction yields a categorical equivalence between monadic Gödel algebras and monadic centered prelinear Nelson algebras. We have also described the induced correspondence between congruence lattices and derived consequences for simplicity and subdirect irreducibility.

The results suggest several concrete directions for further work. First, it would be useful to characterize, by equations or quasi-equations, the largest class of monadic Heyting algebras for which the componentwise twist quantifiers are well-defined and satisfy the monadic Nelson axioms. Second, the Sendlewski representation of Nelson algebras by pairs (A, F), where A is a Heyting algebra and F is a Boolean filter, should be extended to the monadic setting. This requires an appropriate compatibility condition between the filter and the quantifiers. Related categorical constructions for Nelson algebras with consistency operators appear in [12, 13]. Finally, a logical application of the present equivalence would be to transfer algebraic semantics, completeness results, and finite counterexamples between the corresponding monadic Gödel and Nelson-style logics.

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Data availability

This article does not use any particular data, or human participant. Indeed, the results obtained have been established from the articles cited in the references.

Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Human participants and/or animals

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