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Hesitant fuzzy sets and hesitant fuzzy relations with membership values in power set quantales

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Abstract

A hesitant fuzzy set extends the notion of a fuzzy set by allowing membership values to be represented as sets of values drawn from a predefined domain, typically the real unit interval. In this paper, we introduce a novel framework in which membership values are instead drawn from the power set quantale of a semigroup. This shift is motivated by the fact that such a power set quantale admits residuation, thereby offering a rich algebraic setting for the development of the theory of hesitant fuzzy sets and hesitant fuzzy relations. We provide several characterizations of hesitant fuzzy relations, with particular emphasis on hesitant fuzzy equivalences, their classes and partitions, as well as hesitant fuzzy quasi-orders.

Keywords: Hesitant fuzzy sets, power set quantale, hesitant fuzzy equivalences, hesitant fuzzy partitions, hesitant fuzzy quasi-orders.

1 Introduction

Fuzzy set theory was introduced by Lotfi A. Zadeh in 1965 [38] as a mathematical framework for dealing with imprecise, vague, or uncertain information. Unlike classical (crisp) set theory, where an element either belongs or does not belong to a set, fuzzy set theory allows for degrees of membership. Fuzzy sets are used in any field where knowledge is imprecise or qualitative, such as control systems, decision making, image processing and pattern recognition, artificial intelligence, expert systems, and medical diagnosis.

Over time, various generalizations of fuzzy sets have been proposed to better capture different forms of uncertainty. Gehrke et al. [16] provided a solid theoretical foundation for interval-valued fuzzy sets, showing how they can be naturally studied within the framework of lattice theory. In 1976, Zadeh [40] proposed type-2 fuzzy sets, where the membership function is represented as a fuzzy set rather than a single crisp value, thus enabling a more nuanced representation of uncertainty. Other important developments include rough sets by Pawlak [27] and intuitionistic fuzzy sets by Atanassov [2].

The theory of fuzzy sets has provided a foundation for the development of various relational structures, including fuzzy equivalence relations and fuzzy quasi-orders, which are essential in modeling similarity, clustering, and hierarchical structures under uncertainty. Zadeh originally introduced the notions of fuzzy equivalence relations and quasi-orders in [39]. Bělohlávek further elaborated these ideas and he established a formal algebraic and relational framework [6]. Later, Bělohlávek and Vychodil (2005) introduced fuzzy equational logic, providing a method for reasoning about fuzzy equality and relations [7]. Ćirić et al. conducted a comprehensive analysis of fuzzy equivalence relations and the corresponding equivalence classes and partitions [11], while Ignjatović et al. explored the properties of fuzzy quasi-orders [20]. More

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recently, Ghodousian et al. investigated structural and optimization aspects of fuzzy relation equations, including the characterization of their feasible solution sets and conditions for feasibility [17].

In many real-world applications, especially those involving multiple experts or incomplete information, it is more appropriate to assign several possible membership values to an element rather than a single one. While classical fuzzy set theory effectively models gradual transitions between truth values, it is limited by assigning a single membership degree to each element, which may fail to capture the hesitation arising from multiple plausible evaluations. To address this limitation, Torra and Narukawa defined hesitant fuzzy sets in 2009, [35], allowing the membership grade of an element to be expressed as a finite set of values, typically from the unit interval $[0, 1]$. This formulation explicitly captures hesitation and provides a natural extension of Zadeh's original fuzzy set theory.

This concept was further developed in subsequent works. Xu and Xia [37] introduced hesitant fuzzy elements as the basic building blocks of hesitant fuzzy sets, while Dehmiry et al. [15] extended the framework to hesitant $\mathcal{L}$-fuzzy sets by replacing the unit interval with a lattice structure. In addition, Dehmiry et al. generalized hesitant fuzzy relations to hesitant $\mathcal{L}$-fuzzy relations in [14], further increasing the expressive power of the framework. More recently, Talafha et al. [32] proposed a new structure of hesitant fuzzy relations based on a hesitant fuzzy Cartesian product and a lattice-based framework, with an application to academic performance assessment. Other recent studies have explored hesitant fuzzy structures from different perspectives. For instance, Sup-hesitant fuzzy ideals and p-ideals have been studied in the setting of BCK/BCI-algebras by Takallo et al. [24], while Tilahun [33] investigated hesitant fuzzy subalgebras, ideals, and congruences on autometrized algebras, as well as characteristic equations and level subsets of hesitant fuzzy sets and relations. In addition, hesitant fuzzy soft sets have been investigated through new numerical characteristics such as scored-energy, with applications to clustering and decision-making [1].

Despite these developments, most existing approaches to hesitant fuzzy sets are primarily based on numerical structures and aggregation operators (cf. [34, 35]). As a result, they do not provide a general algebraic framework supporting residuation and the systematic study of relational structures, such as quasi-orders and equivalence relations. Bustince et al. showed that hesitant fuzzy sets equipped with Torra's operations do not satisfy the law of absorption [10]. Jara et al. addressed this problem and, in their work [21], proposed a definition of a symmetric order on hesitant fuzzy sets, which resulted in a lattice structure. Nevertheless, the corresponding symmetric meet and join operations do not coincide with those defined by Torra. Although residuated structures have been considered in some works (e.g., [28]), they are typically developed under additional assumptions or restricted to specific subclasses of hesitant fuzzy sets. Recently, Merino et al. [23] proposed an order-oriented framework for hesitant fuzzy elements, demonstrating that several classical orders fail to induce lattice structures, whereas the symmetric order yields scores that satisfy key normative criteria for scoring functions.

In addition, previous studies on hesitant fuzzy equations have primarily focused on solving specific systems using numerical or parametric methods. For example, Babakordi and Allahviranloo [3] introduced methods for solving hesitant fuzzy systems of the form $AX = B$, defining norms of hesitant fuzzy vectors and introducing the concepts of hesitant fuzzy zero, "almost equal", "less than" and "equal" of two hesitant fuzzy numbers. The solution of the system is obtained via a minimization problem. Furthermore, Babakordi and Taghi-Nezhad [4] studied partial and half hesitant fuzzy equations and classified solution types. Further developments include fully hesitant parametric fuzzy equations [31].

While the previously mentioned contributions offer practical computational techniques, they do not provide a fully algebraic framework for reasoning over relational structures. In contrast, our approach provides a unified algebraic setting in which systems of relational inequalities and equations can be analyzed via residuation. In this work, we establish the foundational framework, leaving detailed applications and computational studies for future research.

A major open problem in this area is the construction of an algebraic structure for hesitant fuzzy sets that is both residuated and supports efficient solving of equations using residuals. Such a structure would generalize classical fuzzy logic while preserving the expressive power of hesitant evaluations.

This paper proposes a novel framework for hesitant fuzzy sets and relations based on power set quantales induced by semigroups. The proposed approach provides an algebraic framework for the study of hesitant fuzzy relational structures, with particular emphasis on residuation and relational properties. It enables a systematic analysis of hesitant fuzzy relations within a power set quantale setting, without requiring further specialization of hesitant fuzzy sets to obtain a residuated algebraic structure.

Our primary contribution lies in proposing a novel framework based on power set quantales, providing a residuated algebraic structure over which hesitant fuzzy sets and relations can be naturally defined and solved straightforwardly. We begin with a semigroup $\langle S, * \rangle$, where $*$ denotes a binary associative operation on S. From this semigroup, we construct its power set $\mathcal{P}(S)$, and define a binary operation on subsets $A, B \subseteq S$ by

$$A \otimes B = \{a * b \mid a \in A \text{ and } b \in B\}.$$

This construction yields an algebraic structure called a *quantale*. We refer to this structure as the *power set quantale of a semigroup S*, and we develop the theory of hesitant fuzzy sets and hesitant fuzzy relations over such quantales.

The approach developed in this paper focuses on the algebraic treatment of hesitant fuzzy sets and relations within the framework of power set quantales induced by semigroups. Although inspired by fuzzy set theory, this approach is not merely a replacement of the underlying space of membership values by a power set structure. In the present framework, hesitant fuzzy sets assign to each element a value from the power set quantale of a semigroup S. This leads to a distinct algebraic setting in which the power set quantale operations are induced by the set-theoretic structure and the operation of the underlying semigroup. The resulting quantale-based framework provides a natural setting for residuation and the study of hesitant fuzzy relational structures, while also laying the foundation for future investigations of relation inequalities and equations. The residuation property provides a natural tool for characterizing and computing greatest solutions of relation inequalities and equations.

Within this setting, we introduce and study hesitant fuzzy equivalence relations and hesitant fuzzy quasi-orders and investigate their connections with hesitant fuzzy partitions and the corresponding classes of equivalence. These results extend the theory of hesitant fuzzy relations to a broader algebraic setting and open new directions for research on uncertainty modeling in non-standard logical frameworks.

While quantale-based approaches and hesitant fuzzy sets have been investigated independently throughout the literature, their combination in the form considered in this paper has not been previously addressed. The contributions of the present work include:

- Introducing a novel algebraic framework for hesitant fuzzy sets based on power set quantales constructed from semigroups;
- Extending classical and fuzzy relational properties to this framework and investigating their structural characteristics;
- Developing a residuated algebraic structure that provides a foundation for the analysis of hesitant fuzzy relational systems and future studies of hesitant fuzzy relation inequalities and equations.

The results of the paper provide an algebraic framework for modeling relations in situations where disagreement among experts, measurement variability, or insufficient information may lead to many degrees of similarity. In such settings, hesitant fuzzy relations allow similarity evaluations to be represented as sets of possible values. The characterization of hesitant fuzzy equivalence derived in this paper offers a mathematical foundation for creating hesitant fuzzy partitions, which may be treated as clusters of related items. This paradigm is effective in making similarity evaluations in uncertain fields including medical diagnosis, document classification, and image recognition. Another important applications include multi-expert decision making. Many decision-making processes involve specialists evaluating alternatives based on several factors. Their assessments often yield many evaluation values rather than a single score. Hesitant fuzzy relations can model evaluations without premature aggregation. Using hesitant fuzzy equivalence relations, experts can detect indistinguishable alternatives, whereas hesitant fuzzy quasi-orders can reflect preference or dominance relations. Such structures are crucial in multi-criteria decision analysis, where options must be ranked or classified despite evaluation uncertainty.

The remainder of this work is organized into the following sections. Section 2 introduces the algebraic structures used throughout the paper, such as lattices, semigroups, monoids, quantales, and investigates their properties. Furthermore, it explores the residuation properties in quantales, as well as the power set quantale construction from a given semigroup. Section 3 provides a generalization of hesitant fuzzy sets using values in the power set quantale. Specifically, it provides the definition of hesitant fuzzy sets and their key properties, such as residuation. Section 4 gives the definition and basic properties of hesitant fuzzy relations. In Section 5, we provide a characterization of hesitant fuzzy equivalences. We define the indistinguishability and extensionality with respect to an equivalence, as in the case of fuzzy equivalences, and provide the construction of equivalences from hesitant fuzzy sets. Finally, we provide their connection with hesitant fuzzy partitions and hesitant fuzzy semi-partitions. In Section 6, hesitant fuzzy quasi-orders are studied, and the natural hesitant fuzzy equivalences induced by them are defined. Related constructions such as aftersets and foresets are also discussed. In the final section, we present several concluding remarks.

2 Preliminaries

Within this section, we present basic notions that we use throughout the paper. The terminology and fundamental concepts are based on references [5, 8, 9, 18]. For further information about the application of quantales, we refer to [29].

A *lattice* is a partially ordered set $\langle S, \leq \rangle$ in which every finite subset of S has a supremum and infimum, or as a tuple $\langle S, \wedge, \vee \rangle$ in which binary operations $\wedge$ and $\vee$ on S satisfy the laws of commutativity, associativity, and absorption. These two definitions are mutually interchangeable. In addition, a lattice is *complete* if supremum and infimum exist for an arbitrary subset of S.

A *semigroup* is an ordered pair $\langle S, * \rangle$, consisting of a nonempty set S equipped with an associative binary operation $*$ (*multiplication* or the *product*). Furthermore, a *monoid* is a tuple $\langle S, *, 1_* \rangle$ in which $\langle S, * \rangle$ is a semigroup and 1_* is the *identity element* on S.

A (partially) *ordered semigroup* (resp. *ordered monoid*) is a tuple $\langle S, *, \leqslant \rangle$ (resp. a tuple $\langle S, *, 1_* \leqslant, \rangle$) such that $\langle S, * \rangle$, is a semigroup (not necessarily commutative), $\langle S, \leqslant \rangle$ is a (partially) ordered set, and the (partial) order $\leqslant$ is compatible relative to the multiplication $*$, i.e. for all $a, b, c \in S$, it holds

$$a \leqslant b \quad \Rightarrow \quad c * a \leqslant c * b \quad \text{and} \quad a * c \leqslant b * c.$$

A *quantale* is an algebra $\mathcal{Q} = \langle Q, \vee, \wedge, \otimes, 0, 1 \rangle$, in which the following conditions hold:

(Q1) $\langle Q, \vee, \wedge, 0, 1 \rangle$ forms a complete lattice with least element 0 and greatest element 1;

(Q2) $\langle Q, \otimes \rangle$ is a semigroup, which does not have to be commutative;

(Q3) for arbitrary $a \in Q$ and $\{b_i\}_{i \in I} \subseteq Q$ the following infinite distributive laws hold

$$a \otimes \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \otimes b_i), \quad \left(\bigvee_{i \in I} b_i \right) \otimes a = \bigvee_{i \in I} (b_i \otimes a). \tag{1}$$

The operation $\otimes$ is called multiplication. If $\otimes$ is a commutative operation on Q, then $\mathcal{Q}$ is called a *commutative quantale*. If the semigroup $\langle Q, \otimes \rangle$ has the identity element (i.e., if there exists $1_\otimes$ such that $\langle Q, \otimes, 1_\otimes \rangle$ is a monoid), then $\mathcal{Q}$ is called a *unital quantale*, and if this identity coincides with the greatest element 1, then $\mathcal{Q}$ is called an *integral quantale*. The least element 0 is the zero of the semigroup $\langle Q, \otimes \rangle$, i.e. $a \otimes 0 = 0 \otimes a = 0$, for every $a \in Q$. A commutative integral quantale in which $1_\otimes$ is also the greatest element (i.e., $1 = 1_\otimes$), is known as a *complete residuated lattice*.

In what follows, let $\mathcal{Q} = \langle Q, \wedge, \vee, \otimes, 0, 1 \rangle$ denotes a quantale. Due to the infinite distributivity, quantale $\mathcal{Q}$ admits two binary operations $\backslash$ and $/$ on Q, defined as:

$$a \backslash b = \bigvee \{ c \in Q \mid a \otimes c \leqslant b \}, \tag{2}$$

$$b / a = \bigvee \{ c \in Q \mid c \otimes a \leqslant b \}, \tag{3}$$

for any $a, b \in Q$. We call $a \backslash b$ the *right residual of* b *by* a, and b/a the *left residual of* b *by* a. The following *residuation property* holds:

$$a \otimes b = c \quad \Leftrightarrow \quad a \leqslant c/b \quad \Leftrightarrow \quad b \leqslant a \backslash c, \tag{4}$$

for each $a, b, c \in Q$. The following is valid for each $a, b, c \in Q$:

$$b \leqslant c \quad \Rightarrow \quad a \otimes b \leqslant a \otimes c, \quad b \otimes a \leqslant c \otimes a, \quad a \backslash b \leqslant a \backslash c, \quad b/a \leqslant c/a.$$

If $\otimes$ is a commutative operation on Q, then it holds $a \backslash b = b/a$, for all $a, b \in Q$. In that case, the operations $\backslash$ and $/$ are denoted by $\to$. Then, the residuation property (4) becomes

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad a \leqslant b \to c, \tag{5}$$

for all $a, b, c \in Q$. Also, in that case, we define a new operation on Q, called the *biresiduum*, as:

$$a \leftrightarrow b \equiv (a \to b) \wedge (b \to a),$$

for all $a, b \in Q$.

Let $1_\otimes$ be the identity element for $\otimes$ in a unital commutative quantale $\mathcal{Q} = \langle Q, \vee, \wedge, \otimes, 0, 1 \rangle$. Then, for any $a, b, c \in Q$, the following properties are satisfied (cf. [30]):

$$1_\otimes \to a = a, \tag{6}$$

$$(a \to b) \otimes (b \to c) \leq (a \to c), \tag{7}$$

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \leq (a \leftrightarrow c). \tag{8}$$

These properties follow directly from the residuation property.

In the sequel, we provide an example of a quantale that is presented in the papers [19, 22]. Kruml and Paseka in [22] referred to this example as the most general example that leads to any quantale. Example 2.1 is particularly significant, as it serves as a starting point for the key results of the present study.

Example 2.1. *Let $\langle S, *\rangle$ be a semigroup, and let $\mathcal{P}(S)$ denote the power set of S, ordered by inclusion. Then, $\langle \mathcal{P}(S), \subseteq\rangle$ is a complete lattice. Define the operation of multiplication $\otimes$ on $\mathcal{P}(S)$ as*

$$A \otimes B = \{a * b \mid a \in A, b \in B\}, \tag{9}$$

for all $A, B \in \mathcal{P}(S)$. From the definition of $\otimes$, it follows that multiplication is isotone in both arguments, that is, for all $A, B, C \in \mathcal{P}(S)$,

$$A \subseteq B \quad \Rightarrow \quad A \otimes C \subseteq B \otimes C \quad \text{and} \quad C \otimes A \subseteq C \otimes B. \tag{10}$$

Moreover, $\otimes$ distributes over arbitrary unions. In particular, for any family $\{A_i\}_{i\in I} \subseteq \mathcal{P}(S)$ and any $B \in \mathcal{P}(S)$, we have:

$$\left(\bigcup_{i\in I} A_i\right) \otimes B = \bigcup_{i\in I}(A_i \otimes B), \quad B \otimes \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \otimes A_i). \tag{11}$$

Then $\mathcal{P}(\mathcal{S}) = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ forms a quantale, known as the power set quantale of S*. It is clear that $\mathcal{P}(\mathcal{S})$ is a commutative quantale (resp. unital quantale) if $\langle S, *\rangle$ is a commutative semigroup (resp. $\langle S, *, 1_*\rangle$ is a commutative monoid). In the latter case, the identity for the multiplication $\otimes$ is given by $1_\otimes = \{1_*\}$. The right residual $B\backslash A$ of A by B and the left residual A/B of A by B are respectively defined as:*

$$B\backslash A = \{c \in S \mid B \otimes \{c\} \subseteq A\}, \tag{12}$$
$$A/B = \{c \in S \mid \{c\} \otimes B \subseteq A\}, \tag{13}$$

for every $A, B \in \mathcal{P}(S)$ (see [19]*). We call $\backslash$ and $/$ the left and right residuum operations on $\mathcal{P}(S)$. The* residuation property *is satisfied for every $A, B, C \in \mathcal{P}(S)$:*

$$A \otimes B \subseteq C \quad \Leftrightarrow \quad A \subseteq C/B \quad \Leftrightarrow \quad B \subseteq A\backslash C. \tag{14}$$

Remark 2.2. *The following alternative definitions for the residuals hold:*

$$\bigcup\{C \in \mathcal{P}(S) \mid B \otimes C \subseteq A\} = \{c \in S \mid B \otimes \{c\} \subseteq A\},$$
$$\bigcup\{C \in \mathcal{P}(S) \mid C \otimes B \subseteq A\} = \{c \in S \mid \{c\} \otimes B \subseteq A\},$$

for all $A, B \in \mathcal{P}(S)$. By definition of $\otimes$ in (9)*, for $B, C \in \mathcal{P}(S)$, we have:*

$$B \otimes C = \{\, b * c \mid b \in B, c \in C\,\} = \bigcup_{c\in C}(B \otimes \{c\}).$$

Hence, for any $C \subseteq S$,

$$B \otimes C \subseteq A \quad \Leftrightarrow \quad B \otimes \{c\} \subseteq A, \quad \text{for all } c \in C.$$

Thus, the largest such set C consists exactly of all elements $c \in S$ satisfying $B \otimes \{c\} \subseteq A$, which proves the claim. The second equality follows analogously. However, we stick to the original definitions (12) and (13) of the residuals in power set quantales for the sake of simplicity.

If the multiplication operation $*$ in the semigroup S is commutative, then the multiplication operation $\otimes$ in $\mathcal{P}(S)$ is also commutative. In that case, the left and right residuum operations coincide, and therefore we denote both by the same symbol $\to$, that is, $A \to B = A\backslash B = B/A$, for all $A, B \in \mathcal{P}(S)$. We refer to $\to$ as the *residuum operation* on $\mathcal{P}(S)$. The *biresiduum operation* on $\mathcal{P}(S)$ is defined by $A \leftrightarrow B \equiv (A \to B) \cap (B \to A)$, for all $A, B \in \mathcal{P}(S)$.

Example 2.3. *We provide some examples of power set quantales.*

1. *Let $S = [0,1]$ be the real unit interval equipped with a t-norm $*$. Then, the structure $\langle S, *, 1\rangle$ is a commutative monoid. According to Example 2.1, it follows that the structure $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ forms a commutative unital power set quantale, where $\otimes$ is defined by (9), and where the identity element is given by $1_\otimes = \{1\}$. The residuals are calculated by (12) and (13), and they are equal as $\otimes$ is a commutative operation on $\mathcal{P}(S)$. In other words, $B\backslash A = A/B$, for every $A, B \in \mathcal{P}(S)$.*
   We further adopt the following conventions:

   - *If $*$ is the* Gödel t-norm, *specified as $a * b = \min\{a, b\}$, for every $a, b \in S$, $\mathcal{P}(S)$ is called the* Gödel power set quantale*;*
   - *If $*$ is the* Łukasiewicz t-norm, *specified as $a * b = \max\{0, a + b - 1\}$, for every $a, b \in S$, $\mathcal{P}(S)$ is called the* Łukasiewicz power set quantale*;*
   - *If $*$ is the* product t-norm, *specified as $a * b = a \cdot b$ (the usual multiplication), for every $a, b \in S$, $\mathcal{P}(S)$ is called the* product power set quantale.

   For example, let $A = \{0.1, 0.3, 0.7\}$ and $B = \{0.3, 0.5, 0.6\}$ be two subsets of S, and let $\mathcal{P}(S)$ be the Gödel power set quantale. Then, $A \otimes B = \{0.1, 0.3, 0.5, 0.6\}$ and $A \to B = B\backslash A = A/B = \{0.1, 0.3\}$. Furthermore, if $\mathcal{P}(S)$ is the product power set quantale, then

   $$A \otimes B = \{0.03, 0.05, 0.06, 0.09, 0.15, 0.18, 0.21, 0.35, 0.42\},$$

   and $B\backslash A = A/B = \emptyset$.

2. *Consider the set $\mathbb{N}$ of natural numbers equipped with the minimum operation $\wedge$. Then, the ordered pair $\langle \mathbb{N}, \wedge\rangle$ forms a commutative semigroup. According to Example 2.1, the structure $\langle \mathcal{P}(\mathbb{N}), \cup, \cap, \otimes, \emptyset, \mathbb{N}\rangle$ forms a commutative power set quantale, where $\otimes$ is defined as (9), which takes the form $A \otimes B = \{a \wedge b \mid a \in A \text{ and } b \in B\}$, for every $A, B \in \mathcal{P}(\mathbb{N})$. For example, if $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8\}$, then $A \otimes B = \{1, 2, 3, 4, 5, 6\}$ and $A \to B = B\backslash A = A/B = \{1, 2, 3, 4, 5, 6\}$.*

3. *Let $S = \{(a_1, a_2, a_3, a_4) \mid a_1, a_2, a_3, a_4 \in \mathbb{Z}\}$ be the set of all four-dimensional vectors with integer coordinates. Define a binary operation $\oplus$ on S by componentwise addition:*

   $$(a_1, a_2, a_3, a_4) \oplus (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4).$$

   Since componentwise addition of vectors is associative, $\langle S, \oplus\rangle$ is a semigroup. Moreover, the neutral element for $\oplus$ is the vector $(0, 0, 0, 0)$, hence, $\langle S, \oplus\rangle$ forms a commutative monoid. Consequently, the structure $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ forms a unital power set quantale, where $\otimes$ is defined by

   $$A \otimes B = \{a \oplus b \mid a \in A, b \in B\},$$

   for all $A, B \subseteq S$. As an illustration, let $A, B \subseteq S$ be defined by

   $$A = \{(1, 2, 1, -1), (1, 0, 2, 1)\}, \quad B = \{(1, 0, 0, 1), (1, -2, 1, 3)\}.$$

   By applying the quantale multiplication and the corresponding residuum operation, we obtain

   $$\begin{aligned} A \otimes B &= \{(2, 2, 1, 0), (2, 0, 2, 2), (2, -2, 3, 4)\}, \\ A \to B &= \{0, -2, -1, 2\}, \\ B \to A &= \{0, 2, 1, -2\}. \end{aligned}$$

   The vectors in S may be interpreted as evaluations with respect to four criteria. Since different experts may assign different evaluations to the same object, a subset of S can be used to represent the collection of all evaluations assigned to that object. In this way, the power set quantale provides an algebraic structure for representing and manipulating hesitant information arising from multiple expert opinions.

4. *Now we present an example of a non-commutative power set quantale. Let*

   $$S = \{123, 132, 213, 231, 312, 321\},$$

   be the set of all permutations of $M = \{1, 2, 3\}$, linearly ordered by the lexicographic order $\leqslant$. Then, the structure $\langle S, \leq, \mathbf{0}, \mathbf{1}\rangle$ constitutes a complete lattice structure with least and greatest elements denoted by $\mathbf{0} = 123$ and $\mathbf{1} = 321$, respectively.

Consider the composition $\circ$ of permutations as the usual composition of mappings. Consequently, $\langle S, \circ\rangle$ is a noncommutative semigroup (for example, $132 \circ 312 = 213$ and $312 \circ 132 = 321$). According to Example 2.1, $\langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ is a power set quantale, where $\otimes$ is defined by (9), that is, $A \otimes B = \{p_1 \circ p_2 \mid p_1 \in A$ and $p_2 \in B\}$, for every $A, B \in \mathcal{P}(S)$. Consider the particular example where $A = \{123, 132, 213\}$ and $B = \{213, 312\}$. According to Example 2.1 we get the residuals $B\backslash A = \{213, 231\}$ and $A/B = \emptyset$.

In the next remark, we point out the connection between the residuum operation in $\mathcal{P}(S)$ and the residuum operation in S, in the case when S is the real unit interval and the multiplication operation on S is a left-continuous or continuous t-norm.

Remark 2.4. Let $\mathcal{P}(S)$ be the power set quantale of a semigroup S. Then, for arbitrary $a, b \in S$, we get

$$\begin{aligned}\{a\}\backslash\{b\} &= \{x \in S \mid \{a\} \otimes \{x\} \subseteq \{b\}\} = \{x \in S \mid a * x = b\},\\ \{b\}/\{a\} &= \{x \in S \mid \{x\} \otimes \{a\} \subseteq \{b\}\} = \{x \in S \mid x * a = b\}.\end{aligned}$$

In other words, we have that $\{a\}\backslash\{b\}$ is the set of all solutions to the equation $a * x = b$ in S, whereas $\{b\}/\{a\}$ is the set of all solutions to the equation $x * a = b$ in S.

In what follows, let $S = [0, 1]$, let $* = \otimes$ be an arbitrary left-continuous t-norm on $[0, 1]$, and let $\to$ be the residuum operation (residual implication) on $[0, 1]$ corresponding to $\otimes$. Since the left-continuous t-norm $\otimes$ is commutative on $S = [0, 1]$, it follows that the multiplication operation in $\mathcal{P}([0, 1])$ is also commutative, and hence the left and right residuum operations on $\mathcal{P}([0, 1])$ coincide. However, in this case we will not denote this operation by $\to$, as noted above, but we will retain the notation $\backslash$ (or $/$), to clearly distinguish it from the residuum operation on $[0, 1]$, for which we already use the symbol $\to$.

As known, for any $a, b \in S$, $a \to b$ is the greatest solution to the inequality $a \otimes x \leqslant b$, and according to [12, Lemma 2], if the set $\{a\}\backslash\{b\} = \{b\}/\{a\}$ is non-empty, i.e., if $a \otimes x = b$ is a solvable equation in $[0, 1]$, then $a \to b$ is also greatest solution to this equation, that is, $a \to b$ is the greatest element in $\{a\}\backslash\{b\}$.

If the set is $\{a\}\backslash\{b\}$ is empty, meaning that the equation $a \otimes x = b$ is not solvable, then the relationship between the $\{a\}\backslash\{b\}$ and $a \to b$ is not so clear. However, if $\otimes$ is a continuous t-norm, this relationship can be made considerably clearer. Recall that a continuous t-norm $\otimes$ satisfies the *divisibility condition*

$$a \otimes (a \to b) = a \wedge b, \tag{15}$$

for every $a, b \in [0, 1]$, whose equivalent form is

$$b \leqslant a \quad\Rightarrow\quad (\exists x \in [0, 1])\ a \otimes x = b, \tag{16}$$

for every $a, b \in [0, 1]$ (cf. [6, Theorem 2.36 and Lemma 2.90] or [7, Lemma 1.39]). Now, according to (16) and the fact that the interval $[0, 1]$ is linearly ordered, for arbitrary $a, b \in [0, 1]$ we have that

$$b \leqslant a \quad\Leftrightarrow\quad \{a\}\backslash\{b\} \neq \emptyset \qquad \text{and} \qquad a < b \quad\Leftrightarrow\quad \{a\}\backslash\{b\} = \emptyset,$$

and therefore

$$a \to b = \begin{cases} \top(\{a\}\backslash\{b\}) & \text{if } \{a\}\backslash\{b\} \neq \emptyset,\\ 1 & \text{if } \{a\}\backslash\{b\} = \emptyset,\end{cases}$$

where $\top X$ represents the greatest element of a set X.

It is interesting to observe that for $a \neq b$, the equation $a \otimes x = b$ has no solution in $[0, 1]$ if and only if all elements of $[0, 1]$ are solutions to $a \otimes x \leqslant b$.

3 Hesitant fuzzy sets over the power set quantale

Various definitions of hesitant fuzzy sets exist in the literature. Assume that X is a universal set. Torra [34] introduced a *hesitant fuzzy set* on X as a function h assigning to each $x \in X$ a subset of $[0, 1]$, with X being a finite set. Each such subset of $[0, 1]$, referred to as a hesitant fuzzy element, represents the membership values of x. In particular, when $h(x)$ forms a nonempty closed interval, the hesitant fuzzy set reduces to an intuitionistic fuzzy set.

In the same work [34], Torra proposed an alternative formulation of hesitant fuzzy sets based on sets of membership functions. Given a collection of n membership functions $M = \{\mu_1, \dots, \mu_n\}$, the corresponding hesitant fuzzy set $h_M : X \to \mathcal{P}([0, 1])$ is defined as:

$$h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}.$$

Table 1: Comparison between the proposed approach and the approach in [34]

Feature	**Quantale-Based Approach**	**Approach by Torra**
Values $h(x)$	Subsets of a monoid S, i.e., elements of the power set $\mathcal{P}(S)$	Multisets, tuples, or fuzzy numerical values
Join and meet operations	Classical set-theoretic union and intersection	Based on t-norms and t-conorms
Underlying structure	Quantale $\mathcal{P}(S)$, equipped with union and intersection	Not a lattice; operations do not satisfy the absorption laws
Main focus	Axiomatic, algebraic approach based on quantales	Semantically driven approach rooted in fuzzy logic context

Pei and Yi [28] provided another definition, describing a hesitant fuzzy set h on X as a collection of ordered pairs $h = \{\langle x, h(x)\rangle \mid x \in X\}$, where $h(x)$ is a collection of distinct values in $[0,1]$, corresponding to the membership degrees of x in h.

In the present section, we consider a notion of a hesitant fuzzy set with values in the power set quantale of an arbitrary monoid. Let $\mathcal{S} = \langle S, *, 1_*\rangle$ be a monoid, and let $Q = \mathcal{P}(S)$ be the power set of S. Recall from Example 2.1 that the structure $\mathcal{P}(\mathcal{S}) = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$, where the multiplication $\otimes$ is defined as (9), forms a unital quantale with the identity element $1_\otimes = \{1_*\}$. This structure is referred to as the *power set quantale of* $\mathcal{S}$.

Definition 3.1. *Let X be a nonempty set (a universal set). Furthermore, let $\mathcal{S} = \langle S, *, 1_*\rangle$ be a monoid, let $Q = \mathcal{P}(S)$ be the power set of S, and let $\mathcal{Q} = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S\rangle$ be the power set quantale of $\mathcal{S}$. Then, a* hesitant fuzzy set on X over $\mathcal{Q}$*, or simply a* hesitant fuzzy set*, is any function $h : X \to Q$.*

Throughout the rest of this work, X denotes a universal set and $\mathcal{Q}$ denotes the unital power set quantale of a monoid $\mathcal{S}$. Some particularly important hesitant fuzzy sets are:

1. Empty hesitant fuzzy set: $h_\emptyset(x) = \emptyset$, for each $x \in X$,
2. Full hesitant fuzzy set: $h_S(x) = S$, for each $x \in X$.

For a hesitant fuzzy set $h : X \to Q$ and $x \in X$, we say that $h(x)$ is the hesitant fuzzy element of h. In fact, $h(x)$ represents the *membership value* of x in h. The set of all hesitant fuzzy sets on X taking membership values in $\mathcal{Q}$ is denoted by Q^X.

As stated in Definition 3.1, in this paper, hesitant fuzzy sets are modeled as functions of the form $h : X \to Q = \mathcal{P}(S)$ where the values are given as subsets of a monoid S. The union and intersection operations between two hesitant fuzzy sets h_1 and h_2 are defined pointwise as follows:

$$(h_1 \cup h_2)(x) = h_1(x) \cup h_2(x), \quad (h_1 \cap h_2)(x) = h_1(x) \cap h_2(x),$$

for all $x \in X$. These operations correspond to the standard quantale structure $\mathcal{Q}$, with union and intersection serving as the join and meet operations, respectively, which differs from the algebraic frameworks proposed by Torra [34] and Pei and Yi [28]. This formulation is significant for two reasons. First, in some sense, it generalizes the well-known definitions of a hesitant fuzzy set proposed by Torra. Second, the algebraic richness of quantales offers a robust foundation for structuring hesitant fuzzy sets, which in turn facilitates the development of the results established in this work. In Table 3 we present a brief comparative overview between our approach and that in [34]. This comparison highlights the fundamental structural and operational distinctions between the two frameworks.

The *crisp part* of a hesitant fuzzy set h on X, denoted by $c(h)$, is a classical (crisp) subset of X described by

$$c(h) = \{x \in X \mid 1_\otimes \subseteq h(x)\}. \tag{17}$$

The height of h, represented by $||h||$, is given by

$$||h|| = \bigcup_{x \in X} h(x). \tag{18}$$

The crisp equivalence relation on X, given by $\ker h = \{(x,y) \in X \times X \mid h(x) = h(y)\}$, is the *kernel* of a hesitant fuzzy set h on X.

Let $h_1, h_2 \in Q^X$. We say that h_1 and h_2 are equal ($h_1 = h_2$) when $h_1(x) = h_2(x)$ for all $x \in X$. Likewise, we say that h_1 is included in h_2 ($h_1 \subseteq h_2$) when $h_1(x) \subseteq h_2(x)$ for all $x \in X$. Endowed with this partial order, the structure $\mathcal{Q}(X) = \langle Q^X, \cup, \cap, h_\emptyset, h_S \rangle$ is a complete lattice where $h_\emptyset$ and h_S are the least and greatest elements, respectively. The meet (intersection) $\bigcap_{i\in I} h_i$ and join (union) $\bigcup_{i\in I} h_i$ of an arbitrary family $\{h_i\}_{i\in I}$ of hesitant fuzzy sets on X are functions from X into Q defined by:

$$\left(\bigcap_{i\in I} h_i\right)(x) = \bigcap_{i\in I} h_i(x), \quad \left(\bigcup_{i\in I} h_i\right)(x) = \bigcup_{i\in I} h_i(x),$$

for every $x \in X$. As usual, we identify the structure $\mathcal{Q}(X)$ with the carrier set Q^X. In addition, the *multiplication* of hesitant fuzzy sets is defined as the binary operation $\otimes$ on $\mathcal{Q}(X)$ by:

$$(h_1 \otimes h_2)(x) = h_1(x) \otimes h_2(x) = \{a * b \mid a \in h_1(x) \text{ and } b \in h_2(x)\},$$

for every $x \in X$. Since $\langle S, * \rangle$ is a semigroup by assumption (see Example 2.1), the induced operation $\otimes$ on Q^X is associative. More precisely, for all $h_1, h_2, h_3 \in Q^X$ and $x \in X$, associativity of $*$ implies:

$$\begin{aligned}
(h_1 \otimes (h_2 \otimes h_3))(x) &= h_1(x) \otimes (h_2(x) \otimes h_3(x)) = h_1(x) \otimes \{\beta * \gamma \mid \beta \in h_2(x), \gamma \in h_3(x)\} \\
&= \{\alpha * (\beta * \gamma) \mid \alpha \in h_1(x), \beta \in h_2(x), \gamma \in h_3(x)\} \\
&= \{(\alpha * \beta) * \gamma \mid \alpha \in h_1(x), \beta \in h_2(x), \gamma \in h_3(x)\} \\
&= h_1(x) \otimes ((h_2(x) \otimes h_3(x)) \\
&= ((h_1 \otimes h_2) \otimes h_3)(x).
\end{aligned}$$

Moreover, since $\otimes$ is isotone under inclusion in each argument, that is, for all $h_1, h_2, h_3 \in Q^X$

$$h_1 \subseteq h_2 \quad \Rightarrow \quad h_1 \otimes h_3 \subseteq h_2 \otimes h_3 \quad \text{and} \quad h_3 \otimes h_1 \subseteq h_3 \otimes h_2,$$

it follows that $\langle Q^X, \otimes \rangle$ forms a partially ordered semigroup.

In addition, since the infinite distributive laws (1) are satisfied in $\mathcal{Q}(X)$, for any family of hesitant fuzzy sets $\{h_i\}_{i\in I} \subseteq \mathcal{Q}(X)$ and any hesitant fuzzy set $g \in \mathcal{Q}(X)$, the following holds for all $x \in X$:

$$\left(\bigcup_{i\in I} h_i(x)\right) \otimes g(x) = \bigcup_{i\in I} \left(h_i(x) \otimes g(x)\right),$$
$$g(x) \otimes \left(\bigcup_{i\in I} h_i(x)\right) = \bigcup_{i\in I} \left(g(x) \otimes h_i(x)\right).$$

Therefore, the structure $\mathcal{Q}(X) = \langle Q^X, \cup, \cap, \otimes, h_\emptyset, h_S \rangle$ forms a quantale. In such a quantale, the residuals are defined as follows. For arbitrary $h_1, h_2 \in \mathcal{Q}(X)$, the *right residual* $h_2 \backslash h_1$ of h_1 by h_2 is defined by:

$$(h_2 \backslash h_1)(x) = \{c \in S \mid h_2(x) \otimes c \subseteq h_1(x)\},$$

and the *left residual* h_1 / h_2 of h_1 by h_2 is defined by:

$$(h_1 / h_2)(x) = \{c \in S \mid c \otimes h_2(x) \subseteq h_1(x)\},$$

for all $x \in X$. For arbitrary $h_1, h_2, h_3 \in \mathcal{Q}(X)$ the following residuation property holds:

$$h_1 \otimes h_2 \subseteq h_3 \quad \Leftrightarrow \quad h_1 \subseteq h_3 / h_2 \quad \Leftrightarrow \quad h_2 \subseteq h_1 \backslash h_3.$$

If $\mathcal{Q}(X)$ is a commutative power set quantale, the right and left residuals coincide and we write:

$$(h_1 \to h_2)(x) = (h_2 \backslash h_1)(x) = (h_1 / h_2)(x),$$

for all $x \in X$. The operation $\to$ is referred to as the *residuum*. Therefore, the residuation property is defined as follows

$$h_1 \otimes h_2 \subseteq h_3 \quad \Leftrightarrow \quad h_1 \subseteq h_2 \to h_3, \tag{19}$$

where $h_1, h_2, h_3 \in \mathcal{Q}(X)$ are arbitrary hesitant fuzzy sets on X. The commutativity of $\otimes$ allows the introduction of the concept of *biresiduum* in a standard way, i.e.,

$$(h_1 \leftrightarrow h_2)(x) \equiv (h_1 \to h_2)(x) \cap (h_2 \to h_1)(x), \tag{20}$$

for all $h_1, h_2 \in \mathcal{Q}(X)$ and all $x \in X$. Recall that the term biresiduum was introduced in Section 2. Here, we extend it to the commutative quantale of hesitant fuzzy sets.

Remark 3.2. *Note that this setting provides all the structural properties needed for the remainder of the paper. In particular, since S is a monoid, the quantale $\mathcal{Q}$ is unital. The existence of the unit element in $\mathcal{Q}$ ensures the existence of the corresponding neutral element $1_\otimes$ in $\mathcal{Q}(X)$, which is used in the definition of a normalized hesitant fuzzy set, ensuring their well-defined normalization property. Moreover, since $\mathcal{Q}$ is a complete residuated quantale, the induced structure $\mathcal{Q}(X)$ is also residuated, so that left and right residuals exist. These properties together ensure consistency of the induced hesitant fuzzy relational framework in the sense that all induced operations are well-defined and compatible, and support all subsequent constructions in the paper.*

Throughout the paper, hesitant fuzzy sets are represented as row vectors. In the following example, we apply the residuation property to compute the greatest solution of hesitant fuzzy inequalities of the forms $h \otimes u \subseteq g$ and $w \otimes h \subseteq g$, where u and w are unknown hesitant fuzzy sets on X.

Example 3.3. *Let $X = \{x_1, x_2, x_3\}$, let $\mathcal{Q}$ be the product power set quantale such that $S = [0,1]$ and $a * b = a \cdot b$ for all $a, b \in S$. Consider the hesitant fuzzy sets h and g on X, defined as follows:*

$$h = \begin{bmatrix} \{0.1, 0.5\} & \{0\} & \{0.5, 1\} \end{bmatrix}, \quad g = \begin{bmatrix} \{0.5\} & \{0, 0.5\} & \{0.5, 1\} \end{bmatrix}.$$

We now consider the following hesitant fuzzy inequalities on $\mathcal{Q}$:

$$h \otimes u \subseteq g, \qquad w \otimes h \subseteq g, \tag{21}$$

where u and w are unknown hesitant fuzzy sets on X. Observe that $\otimes$ is commutative here. Consequently, the left and right residuals coincide and, by the residuation property (19)*, we obtain:*

$$\begin{aligned}
u(x_1) = w(x_1) = h(x_1) \to g(x_1) &= \{0.1, 0.5\} \to \{0.5\} = \{c \in S \mid c \otimes \{0.1, 0.5\} \subseteq \{0.5\}\} = \emptyset, \\
u(x_2) = w(x_2) = h(x_2) \to g(x_2) &= \{0\} \to \{0, 0.5\} = \{c \in S \mid c \otimes \{0\} \subseteq \{0, 0.5\}\} = [0,1], \\
u(x_3) = w(x_3) = h(x_3) \to g(x_3) &= \{0.5, 1\} \to \{0.5, 1\} = \{c \in S \mid c \otimes \{0.5, 1\} \subseteq \{0.5, 1\}\} = \{1\}.
\end{aligned}$$

Hence, the greatest solution to the inequalities (21) *is:*

$$u = w = \begin{bmatrix} \emptyset & [0,1] & \{1\} \end{bmatrix}.$$

4 Hesitant fuzzy relations over the power set quantale

Recall that X denotes a universal set. For the remainder of the paper, $\mathcal{S}$ is presumed to be a commutative monoid, so the power set quantale $\mathcal{Q}$ of $\mathcal{S}$ will be a commutative quantale and will have a unique residuum operation. A hesitant fuzzy relation R on X is any function from $X \times X$ into Q. For hesitant fuzzy relations, the inclusion, equality, ordering, joins, and meets are introduced in the same way as for hesitant fuzzy sets. Given two hesitant fuzzy relations R_1 and R_2 on X, their *composition* is a hesitant fuzzy relation $R_1 \circ R_2$ on X defined by

$$(R_1 \circ R_2)(x, y) = \bigcup_{a \in X} (R_1(x, a) \otimes R_2(a, y)), \tag{22}$$

for any $x, y \in X$.

Given a hesitant fuzzy relation R on X, its inverse is the relation R^{-1} determined by $R^{-1}(y, x) = R(x, y)$, for any $x, y \in X$. By the definition of the inverse relation, $(R^{-1})^{-1} = R$. The *identity hesitant fuzzy relation* Δ_X on X is given by:

$$\Delta_X(x, y) = \begin{cases} 1_\otimes & \text{if } x = y, \\ \emptyset & \text{if } x \neq y. \end{cases}$$

The *zero* and *full hesitant fuzzy relations* on X are given by

$$\mathbf{0}(x,y)=\emptyset \quad \text{and} \quad \mathbf{I}(x,y)=S,$$

respectively, for any $x,y\in X$.

We denote by $Q^{X\times X}$ the collection of all hesitant fuzzy relations on X. For any $h\in Q^X$ and $R\in Q^{X\times X}$, the compositions $h\circ R$ and $R\circ h$ are hesitant fuzzy sets on X defined by

$$(h\circ R)(x)=\bigcup_{y\in X} h(y)\otimes R(y,x), \quad (R\circ h)(x)=\bigcup_{y\in X} R(x,y)\otimes h(y),$$

for all $x\in X$. The composition $h_1\circ h_2$ of two hesitant fuzzy sets $h_1,h_2\in Q^X$ is defined by

$$(h_1\circ h_2)(x)=\bigcup_{x\in X} h_1(x)\otimes h_2(x), \tag{23}$$

for all $x\in X$. The value $h_1\circ h_2$ represents the *degree of overlap* between h_1 and h_2. For a hesitant fuzzy relation R on X we introduce the *n-th power* R^n of R inductively as follows: $R^0=\Delta_X$ and $R^{n+1}=R^n\circ R$, for each $n\in\mathbb{N}_0$, where $\mathbb{N}$, represents the set of positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Proposition 4.1. *Let $R_1,R_2,R_3,T\in Q^{X\times X}$ be hesitant fuzzy relations on X, and let $\{R_i\}_{i\in I}\subseteq Q^{X\times X}$ be a family of hesitant fuzzy relations on X. The properties stated below hold:*

$$R_1\circ(R_2\circ R_3)=(R_1\circ R_2)\circ R_3, \tag{24}$$

$$R_1\subseteq R_2 \quad\Rightarrow\quad R_1\circ R_3\subseteq R_2\circ R_3, \quad R_3\circ R_1\subseteq R_3\circ R_2, \tag{25}$$

$$T\circ\left(\bigcup_{i\in I} R_i\right)=\bigcup_{i\in I}(T\circ R_i)\,, \quad \left(\bigcup_{i\in I} R_i\right)\circ T=\bigcup_{i\in I}(R_i\circ T). \tag{26}$$

Proof. We first prove property (24). By the definition of composition of relations (22) and the associativity of $\otimes$, for arbitrary $x,y\in X$, we obtain:

$$\begin{aligned} R_1\circ(R_2\circ R_3)(x,y) &= \bigcup_{a\in X}\big(R_1(x,a)\otimes(R_2\circ R_3)(a,y)\big)=\bigcup_{a\in X}\left(R_1(x,a)\otimes\bigcup_{b\in X}\big(R_2(a,b)\otimes R_3(b,y)\big)\right)\\ &=\bigcup_{a\in X}\bigcup_{b\in X}\big((R_1(x,a)\otimes R_2(a,b))\otimes R_3(b,y)\big)\\ &=\bigcup_{b\in X}\left(\bigcup_{a\in X}\big(R_1(x,a)\otimes R_2(a,b)\big)\otimes R_3(b,y)\right)\\ &=((R_1\circ R_2)\circ R_3)(x,y). \end{aligned}$$

To prove property (25), assume that $R_1(x,y)\subseteq R_2(x,y)$ for all $x,y\in X$. Then, by the isotonicity of $\otimes$, for all $x,y\in X$ we obtain:

$$(R_1\circ R_3)(x,y)=\bigcup_{a\in X}\big((R_1(x,a)\otimes R_3(a,y)\big)\subseteq\bigcup_{a\in X}\big((R_2(x,a)\otimes R_3(a,y)\big)=(R_2\circ R_3)(x,y).$$

The remaining inclusion can be proved analogously.

Property (26) follows from the distributivity of $\otimes$ over arbitrary joins. For arbitrary $x,y\in X$, we have:

$$T\circ\left(\bigcup_{i\in I} R_i\right)(x,y)=\bigcup_{a\in X}\left(T(x,a)\otimes\left(\bigcup_{i\in I} R_i\right)(a,y)\right)=\bigcup_{i\in I}\bigcup_{a\in X}\big(T(x,a)\otimes R_i(a,y)\big)=\bigcup_{i\in I}(T\circ R_i)(x,y).$$

The remaining equality can be proved analogously. □

The next proposition characterizes the structure formed by all hesitant fuzzy relations on X.

Proposition 4.2. *The algebra $\mathcal{R}(X)=\langle Q^{X\times X},\cup,\cap,\circ,\mathbf{0},\mathbf{I}\rangle$ forms a unital quantale in which the least element is the zero hesitant fuzzy relation **0**, the greatest element is the full hesitant fuzzy relation **I**, and the identity element is the identity hesitant fuzzy relation Δ_X.*

Proof. One can readily verify that the structure $\langle Q^{X\times X}, \cup, \cap, \mathbf{0}, \mathbf{I}\rangle$, is a complete lattice in which $\mathbf{0}$ is the least element and $\mathbf{I}$ is the greatest element. In addition, this lattice is distributive. Since (24) holds, the structure $\langle Q^{X\times X}, \circ\rangle$, forms a semigroup, and moreover, $\langle Q^{X\times X}, \circ, \Delta_X\rangle$ is a monoid. Finally, the infinite distributivity law (26) also holds, which completes the proof of the proposition. □

In what follows, we identify the structure $\mathcal{R}(X)$ with its carrier set $Q^{X\times X}$.

If X is a finite set of cardinality n, then every $R \in \mathcal{R}(X)$ can be interpreted as $n\times n$ hesitant fuzzy matrix over $\mathcal{Q}$, i.e. the matrix in which the elements are subsets of S. For two hesitant fuzzy relations $R_1, R_2 \in \mathcal{R}(X)$ and two hesitant fuzzy sets $h_1, h_2 \in \mathcal{Q}(X)$, $R_1 \circ R_2$ is the matrix product, whereas $h_1 \circ R_1$ can be treated as an $1 \times n$ by $n\times n$ matrix product and $R_1 \circ h_1$ as the product of an $n \times n$ matrix with the column vector h_1^t (the transposed vector h_1), i.e. as vector-matrix products. Also, we can interpret $h_1 \circ h_2$ as the scalar product of h_1 and h_2.

Given a hesitant fuzzy relation R on X, we say that it is:

(R) *Reflexive*, if $1_\otimes \subseteq R(x, x)$, for each $x \in X$, i.e., if $\Delta_X \subseteq R$;

(S) *Symmetric*, if $R(x, y) = R(y, x)$, for each $x, y \in X$, i.e., if $R = R^{-1}$;

(T) *Transitive*, if

$$R(x, y) \otimes R(y, z) \subseteq R(x, z), \tag{27}$$

for each $x, y, z \in X$, i.e., if $R \circ R \subseteq R$.

Moreover, if R is reflexive, then $R \subseteq R \circ R$, and if R is a reflexive and transitive hesitant fuzzy relation then $R \circ R = R$. A reflexive and transitive hesitant fuzzy relation on X is called a hesitant *fuzzy quasi-order on* X. A hesitant fuzzy relation on X, is called a hesitant fuzzy tolerance relation if it is reflexive and symmetric. It is called a hesitant fuzzy equivalence relation (or simply a hesitant fuzzy equivalence) if it is reflexive, symmetric, and transitive.

A hesitant fuzzy equivalence E on X is called a *hesitant fuzzy equality relation on* X, or just a *hesitant fuzzy equality on* X, if for all $x, y \in X$, $1_\otimes \subseteq E(x, y)$ implies $x = y$. For E being a hesitant equality on X, a hesitant fuzzy relation R is referred to as E-*antisymmetric* if for all $x, y \in X$, the condition $R(x, y) \otimes R(y, x) \subseteq E(x, y)$ is satisfied. We call a reflexive, E-antisymmetric and transitive hesitant fuzzy relation a hesitant fuzzy E-*order*.

Consider a hesitant fuzzy relation R on X. The smallest transitive hesitant fuzzy relation R^t satisfying $R \subseteq R^t$ and $R^t = \bigcup_{n\in\mathbb{N}} R^n$ is called *transitive closure* of R. The *transitive closure* R^t of an arbitrary hesitant fuzzy relation R can be expressed as

$$R^t = \bigcup_{n\in\mathbb{N}} R^n. \tag{28}$$

Further, for each $x \in X$, the R-*afterset* of x is the hesitant fuzzy set $xR \in \mathcal{Q}(X)$ given by $xR(y) = R(x, y)$ and the R-*foreset* of x is the hesitant fuzzy set $Rx \in \mathcal{Q}(X)$ given by $Rx(y) = R(y, x)$, for any $y \in X$. The collections of all R-aftersets and R-foresets are represented by X/R and $X\backslash R$, respectively.

If E is a hesitant fuzzy equivalence relation, we denote E_x instead of $xE = Ex$ and we refer to E_x as the *hesitant fuzzy equivalence class* (*equivalence class* for short) of E induced by x. Hence, $E_x(a) = E(x, a)$, for all $a \in X$. The collection of all equivalence classes of E is referred to as the *factor set of* X with respect to E, and is denoted by X/E, i.e.,

$$X/E = \{E_x \mid x \in X\}. \tag{29}$$

We have already noted that if X is a finite set containing n elements, then a hesitant fuzzy relation R on X can be represented as an $n\times n$ hesitant fuzzy matrix over $\mathcal{Q}$. In that case, R-aftersets correspond to row vectors and R-foresets correspond to column vectors of R.

In the following example, a hesitant fuzzy relation on X over the Gödel power set quantale $\mathcal{Q}$ is represented by a matrix, and its transitive closure is computed.

Example 4.3. *Let $X = \{x, y\}$ be a universal set, and let $\mathcal{Q}$ be the Gödel power set quantale defined in Example 2.3, where $S = [0, 1]$ and $a * b = \min(a, b)$ for all $a, b \in S$. Consider a hesitant fuzzy relation R on X that is reflexive and symmetric, represented by the following matrix:*

$$R = \begin{bmatrix} \{0.1, 0.5, 1\} & \{0.1, 0.2\} \\ \{0.1, 0.2\} & \{0.5, 1\} \end{bmatrix}.$$

We compute the transitive closure of R. Applying the algorithm described above, we first obtain R^2:

$$R^2 = \begin{bmatrix} \{0.1, 0.2, 0.5, 1\} & \{0.1, 0.2\} \\ \{0.1, 0.2\} & \{0.1, 0.2, 0.5, 1\} \end{bmatrix}.$$

A direct verification shows that $R^2 \circ R^2 = R^2$, i.e., R^2 is transitive. Therefore, R^2 is the transitive closure of R. Note that the obtained relation is a hesitant fuzzy equivalence on X.

5 Hesitant fuzzy equivalences, hesitant fuzzy equivalence classes and hesitant fuzzy partitions over the power set quantale

The current section provides the characterization of hesitant fuzzy equivalences and their relationship to hesitant fuzzy partitions and hesitant fuzzy semi-partitions.

Let $\mathcal{E}(X)$ be a set of all hesitant fuzzy equivalences on X, i.e., $\mathcal{E}(X)$ is a subset of $\mathcal{R}(X)$. The set $\mathcal{E}(X)$ admits a partial order on $\mathcal{E}(X)$ induced by the $\subseteq$ ordering on $\mathcal{R}(X)$. Moreover, the following characterization holds.

Theorem 5.1. *The structure $\langle \mathcal{E}(X), \cup, \cap \rangle$ forms a complete lattice, in which the meet $\cap$ coincides with the usual intersection of hesitant fuzzy relations, while the join $\cup$ of two hesitant fuzzy equivalence relations E_1 and E_2 is defined as follows:*

$$E_1 \cup E_2 = \bigcap \{H \in \mathcal{E}(X) \mid E_1 \subseteq H \text{ and } E_2 \subseteq H\}.$$

Valverde [36] established that the meet of a family of fuzzy equivalence relations is itself a fuzzy equivalence relation (Proposition 3.1). Motivated by this result, we formulate the following statement as a direct consequence of the preceding theorem.

Corollary 5.2. *Given a family $\{E_i\}_{i \in I}$ of hesitant fuzzy equivalences on X, the relation $E(x, y) = \bigcap_{j \in J} E_j(x, y)$, is itself a hesitant fuzzy equivalence on X.*

Before stating the next results, we introduce several specific types of hesitant fuzzy sets that will be used in the sequel.

Definition 5.3. *A hesitant fuzzy set h on X is said to be normalized if there exists $x \in X$ such that $1_\otimes \subseteq h(x)$.*

Definition 5.4. *A hesitant fuzzy set h on X is said to be extensional with respect to (WRT) a given hesitant fuzzy equivalence E if*

$$h(x) \otimes E(x, y) \subseteq h(y), \tag{30}$$

and it is said to be indistinguishable WRT to E if

$$h(x) \otimes h(y) \subseteq E(x, y), \tag{31}$$

for every $x, y \in X$. Moreover, if h is normalized, extensional, and indistinguishable WRT to E, h is called a hesitant fuzzy point WRT to E.

The following lemma provides two basic properties of a normalized hesitant fuzzy set with respect to the operations $\otimes$ and $\rightarrow$, used in later proofs.

Lemma 5.5. *Let $h, g \in \mathcal{Q}(X)$. If $1_\otimes \subseteq h(a)$, for some $a \in X$ (i.e., if h is normalized), then the following properties hold:*

$$g(x) \subseteq h(a) \otimes g(x), \tag{32}$$
$$h(a) \rightarrow g(x) \subseteq g(x), \tag{33}$$

for every $x \in X$.

Proof. Since $1_\otimes \subseteq h(a)$ and $1_\otimes \otimes g(x) = g(x)$ for each $x \in X$, the monotonicity of $\otimes$ yields $1_\otimes \otimes g(x) \subseteq h(a) \otimes g(x)$, which proves (32). Furthermore, (33) follows from the property (6) and the antitonicity of $\rightarrow$ in the first argument. $\square$

Lemma 5.6. *A normalized hesitant fuzzy set h on X is indistinguishable WRT to $E \in \mathcal{E}(X)$ if and only if it is included in some equivalence class of E.*

Proof. Assume that h is a normalized hesitant fuzzy set on X, i.e. $1_\otimes \subseteq h(a)$, for some $a \in X$. According to (32), $h(x) \subseteq h(a) \otimes h(x)$ for every $x \in X$. Next, suppose that h is indistinguishable WRT to E, that is (31) holds. Then,

$$h(x) \subseteq h(a) \otimes h(x) \subseteq E(a,x) = E_a(x),$$

for all $x \in X$; thus $h \subseteq E_a$. For the converse direction, assume that $h \subseteq E_a$ for some $a \in X$. It follows that, $h(x) \subseteq E(a,x)$, and $h(y) \subseteq E(a,y)$, for all $x, y \in X$. By the monotonocity of $\otimes$ and as E is symmetric, we get:

$$h(x) \otimes h(y) \subseteq E(a,x) \otimes E(a,y) \subseteq E(x,y),$$

for every $x, y \in X$, hence h is indistinguishable WRT to E. □

The following lemma establishes a connection between hesitant fuzzy points and equivalence classes induced by a hesitant fuzzy equivalence relation.

Lemma 5.7. *A hesitant fuzzy set h on X is a hesitant fuzzy point WRT to $E \in \mathcal{E}(X)$ if and only if h is an equivalence class of E.*

Proof. Assume that h is a hesitant fuzzy point WRT to a given $E \in \mathcal{E}(X)$. Since h is normalized, then according to Lemma 5.6, we have $h \subseteq E_a$ for $a \in X$ such that $1_\otimes \subseteq h(a)$. Moreover, from (32) and the fact that h is extensive WRT to E, it follows:

$$E_a(x) \subseteq h(a) \otimes E_a(x) = h(a) \otimes E(a,x) \subseteq h(x),$$

for every $x \in X$, meaning that $E_a \subseteq h$. Hence, we showed that $E_a = h$. To prove the converse implication, assume that $h = E_a$ for some $a \in X$. Since E_a is an equivalence class, $1_\otimes \subseteq E(a,a) = h(a)$, meaning that h is normalized. Moreover, as E is transitive, it holds:

$$h(x) \otimes E(x,y) = E(a,x) \otimes E(x,y) \subseteq E(a,y) = h(y),$$

for every $x, y \in X$, so h is extensive WRT to E. Finally, by using the fact that E is symmetric and transitive, the following holds:

$$h(x) \otimes h(y) = E(a,x) \otimes E(a,y) \subseteq E(x,y),$$

for all $x, y \in X$. Thus, h is indistinguishable WRT to E. □

The properties of hesitant fuzzy equivalences stated below are standard in related settings; however, for completeness, we provide their proofs.

Lemma 5.8. *The following assertions hold for an arbitrary $E \in \mathcal{E}(X)$:*

(a) $||E_x \otimes E_y|| = E(x,y)$,

(b) $1_\otimes \subseteq E(x,y) \Leftrightarrow E_x = E_y$,

for every $x, y \in X$.

Proof. (a) Let $E \in \mathcal{E}(X)$ be arbitrary. Then, for every $x, y \in X$,

$$||E_x \otimes E_y|| = \bigcup_{a \in X} (E_x(a) \otimes E_y(a)) = \bigcup_{a \in X} (E(x,a) \otimes E(a,y)) = E \circ E(x,y) = E(x,y),$$

where the last equality holds due to reflexivity and transitivity of E.

(b) Let $E_x = E_y$. Since E is an equivalence relation, it follows $E(y,y) = E_y(y) = E_x(y) = E(x,y)$. Therefore, as $1_\otimes \subseteq E(y,y)$, we also have $1_\otimes \subseteq E(x,y)$. Conversely, suppose that $1_\otimes \subseteq E(x,y)$. For every $a \in X$, due to the transitivity of E and (32):

$$E_y(a) = E(y,a) \subseteq E(x,y) \otimes E(y,a) \subseteq E(x,a) = E_x(a).$$

Analogously, we prove that $E_x(a) \subseteq E_y(a)$. Hence, $E_x = E_y$. □

To construct hesitant fuzzy equivalence relation from a given hesitant fuzzy set h on X, we define the relation E_h by:

$$E_h(x,y) = h(x) \leftrightarrow h(y) \quad \text{for every } x,y \in X. \tag{34}$$

In what follows, we show that a E_h is a hesitant fuzzy equivalence on X. First, for every $x \in X$, $1_\otimes \otimes h(x) = h(x)$, and by the residuation property (5), it follows that $1_\otimes \subseteq h(x) \to h(x)$. Hence,

$$1_\otimes \in h(x) \leftrightarrow h(x) = E_h(x,x),$$

for every $x \in X$. Consequently, E_h is reflexive. Symmetry follows from the commutativity of the biresiduum, i.e., for every $x,y \in X$, we have:

$$E_h(x,y) = h(x) \leftrightarrow h(y) = h(y) \leftrightarrow h(x) = E_h(y,x).$$

Finally, the transitivity of E_h follows directly from property (8). Therefore, E_h is a hesitant fuzzy equivalence on X.

Example 5.9. *This example illustrates how formula* (34) *can be used to construct the corresponding hesitant fuzzy equivalences* E_h^G, E_h^L *and* E_h^P *in a Gödel, Łukasiewicz, and product power set quantales, respectively, which are defined in Example* (2.3)*.*

Let $X = \{x,y,z\}$ *be the underlying universe in each of the three cases. First, consider a hesitant fuzzy set* h *on* X *over the Gödel power set quantale given by* $h(x) = h(y) = \{0.1, 0.6\}$ *and* $h(z) = \{0.3, 1\}$*. The corresponding equivalence is:*

$$E_h^G = \begin{bmatrix} \{0.1\} \cup [0.6,1] & \{0.1\} \cup [0.6,1] & \emptyset \\ \{0.1\} \cup [0.6,1] & \{0.1\} \cup [0.6,1] & \emptyset \\ \emptyset & \emptyset & \{0.3,1\} \end{bmatrix}.$$

Next, we consider a hesitant fuzzy set h *on* X *over the Łukasiewicz power set quantale, defined by* $h(x) = \{0, 0.9\}$, $h(y) = \{0.6, 1\}$ *and* $h(z) = \{0, 0.4, 0.7\}$*. In this case:*

$$E_h^L = \begin{bmatrix} \{0, 0.1, 1\} & \emptyset & \{0\} \\ \emptyset & \{1\} & \emptyset \\ \{0\} & \emptyset & [0, 0.3] \cup \{1\} \end{bmatrix}.$$

Finally, consider a hesitant fuzzy set h *on* X *over the product power set quantale, defined by* $h(x) = \{0, 0.7\}$, $h(y) = \{0.2, 0.7, 1\}$ *and* $h(z) = \{0.2, 0.6\}$*. The resulting equivalence is:*

$$E_h^P = \begin{bmatrix} \{0,1\} & \emptyset & \emptyset \\ \emptyset & \{1\} & \emptyset \\ \emptyset & \emptyset & \{1\} \end{bmatrix}.$$

Example 5.10. *Let* $\mathcal{Q} = \langle \mathcal{P}(\mathbb{N}_0), \cup, \cap, \otimes, \emptyset, \mathbb{N}_0 \rangle$ *be the power set quantale of the semigroup* $\langle \mathbb{N}_0, \cdot \rangle$*, in which*

$$A \otimes B = \{a \cdot b \mid a \in A, b \in B\},$$

for every $A, B \subseteq \mathbb{N}_0$*. Let* $X = \{x,y\}$ *and take* h *to be the hesitant fuzzy set on* X *over* $\mathcal{Q}$*, given by*

$$h = \begin{bmatrix} \{0,4,8,12\} \\ \{0,8,16,24\} \end{bmatrix}.$$

Applying formula (34)*, we obtain the relation*

$$E_h = \begin{bmatrix} \{0,1\} & \{0\} \\ \{0\} & \{0,1\} \end{bmatrix}.$$

It is straightforward to verify that E_h *is reflexive and symmetric. Moreover,* $E_h \circ E_h = E_h$*, and therefore* E_h *is transitive. Consequently,* E_h *is a hesitant fuzzy equivalence on* X *over* $\mathcal{Q}$*.*

The previous example illustrates the construction of hesitant fuzzy equivalences generated by a single hesitant fuzzy set. We now extend this idea to families of hesitant fuzzy sets. Consider a family of hesitant fuzzy sets $\mathcal{H}$ on X. According to Corollary 5.2, the hesitant fuzzy relation on X given by:

$$E_{\mathcal{H}}(x,y) = \bigcap_{h\in\mathcal{H}} E_h(x,y), \tag{35}$$

for all $x,y \in X$, is a hesitant fuzzy equivalence on X.

Remark 5.11. *Recall that the collection of fuzzy sets defined on the same universe forms a residuated lattice where the greatest element coincides with the unit element, denoted by 1. In contrast, in the hesitant fuzzy setting the unit element does not, in general, coincide with the greatest element.*

In the classical fuzzy setting, a fuzzy equivalence can be obtained from a given fuzzy set in several ways. In addition to the construction via biresiddum considered above, the following two fuzzy relations are also known to be fuzzy equivalence relations:

$$E'_f(x,y) = \begin{cases} f(x)\otimes f(y) & \text{if } x\neq y,\\ 1 & \text{if } x = y, \end{cases} \qquad E''_f(x,y) = \begin{cases} f(x)\otimes f(y) & \text{if } f(x)\neq f(y),\\ 1 & \text{if } f(x) = f(y), \end{cases} \tag{36}$$

for every $x,y\in X$ (see [11]).

We can extend the definitions in (36) to the hesitant fuzzy setting and define the relations $E'_h, E''_h \in Q^{X\times X}$ as follows

$$E'_h(x,y) = \begin{cases} h(x)\otimes h(y) & \text{if } x\neq y,\\ T & \text{if } x = y, \end{cases} \qquad E''_h(x,y) = \begin{cases} h(x)\otimes h(y) & \text{if } h(x)\neq h(y),\\ T & \text{if } h(x) = h(y), \end{cases} \tag{37}$$

for every $x,y\in X$, where $T\subseteq S$ is an arbitrary subset containing the multiplicative identity 1_* of S (that is, $1_\otimes \subseteq T$). However, E'_h and E''_h need not be hesitant fuzzy equivalences on X for any such T. The following example illustrates this observation.

Example 5.12. *Let $X = \{x,y,z\}$, let $T\subseteq S = [0,1]$ be an arbitrary subset such that $1\in T$, and let h be a hesitant fuzzy set on X over the Gödel power set quantale $\mathcal{Q}$ given by*

$$h = \begin{bmatrix}\{0.3\} & \{0.4,0.5\} & \{0.1\}\end{bmatrix}. \tag{38}$$

Using (37), we obtain that E'_h is given by

$$E'_h = \begin{bmatrix} T & \{0.3\} & \{0.1\}\\ \{0.3\} & T & \{0.1\}\\ \{0.1\} & \{0.1\} & T \end{bmatrix}.$$

Since $a\wedge b\in\{a,b\}\subseteq T$ for all $a,b\in T$, we have that $T\otimes T = T\wedge T = \{a\wedge b \mid a,b\in T\}\subseteq T$. On the other hand, since $1\in T$, we have that $a = a\wedge 1\in T\wedge T$, for each $a\in T$, so we get that $T\subseteq T\wedge T = T\otimes T$. Therefore, $T\otimes T = T$.

Now, if $0.3\notin T$ or $0.1\notin T$, then

$$E'_h\circ E'_h(x,x) = T\otimes T\cup\{0.3\}\otimes\{0.3\}\cup\{0.1\}\otimes\{0.1\}\nsubseteq T = T\cup\{0.3\}\cup\{0.1\}\nsubseteq T = E'_h(x,x).$$

Otherwise, if $0.3\in T$ and $0.1\in T$, then

$$E'_h\circ E'_h(x,y) = T\otimes\{0.3\}\cup\{0.3\}\otimes T\cup\{0.1\}\nsubseteq\{0.3\} = E'_h(x,y).$$

Therefore, in both cases we obtain that E'_h is not transitive, and consequently, it is not a hesitant fuzzy equivalence relation on X.

On the other hand, since, $h(x)\neq h(y)$, $h(y)\neq h(z)$, and $h(z)\neq h(x)$, it follows that $E'_h = E''_h$, and hence E''_h is not a hesitant fuzzy equivalence relation on X either.

We now turn to the notion of partitions induced by equivalence relations. A family of crisp subsets $\mathcal{X} = \{X_i\}_{i\in I}$ of a set X is said to be a *semi-partition* of X if $X_i\neq\emptyset$ and $X_i\cap X_j = \emptyset$, for all $i,j\in I$. The family $\mathcal{X} = \{X_i\}_{i\in I}$ satisfying $\bigcup_{i\in I} X_i = X$ is referred to as a *cover* of X. A semi-partition of X which is also a cover of X is termed a partition of X. In the crisp case, the collection of all equivalence classes of an equivalence relation on X is a partition of X and every element of X belongs to one and only one equivalence class X_i.

A precise definition of partitions induced by fuzzy equivalence relations can be found in the work of Murali [25]. Ćirić et al., in [11] defined fuzzy partitions and fuzzy semi-partitions following the approaches of De Baets [13] and Ovchinnikov [26]. According to them, a fuzzy partition of X is a collection of all distinct equivalence classes generated by a fuzzy equivalence relation on X. A *fuzzy semi-partition* is a collection of equivalence classes (not necessarily all) of some fuzzy equivalence on X.

In this section, we define analogous concepts in the context of hesitant fuzzy equivalences. A family $\mathcal{X}$ of normalized hesitant fuzzy sets on X forms a *hesitant fuzzy partition* of X if it coincides with the collection of all mutually distinct equivalence classes induced by a hesitant fuzzy equivalence on X. Any subset of a hesitant fuzzy partition of X is referred to as a hesitant fuzzy semi-partition of X.

Lemma 5.13. *A hesitant fuzzy set on X is normalized iff it contains an equivalence class of a hesitant fuzzy equivalence on X.*

Proof. ($\Rightarrow$) Let h be a normalized hesitant fuzzy set on X. Hence, there exists $a \in X$ such that $1_\otimes \in h(a)$. Let define the hesitant fuzzy equivalence E on X via biresiduum (34),

$$E(x, y) = h(x) \leftrightarrow h(y), \quad \text{for all } x, y \in X.$$

Consider the equivalence class E_a. By definition of biresiduum and property (33) of Lemma 5.5, we obtain:

$$E_a(x) = E(a, x) = h(a) \leftrightarrow h(x) \subseteq h(a) \to h(x) \subseteq h(x),$$

for all $x \in X$. Consequently, $E_a \subseteq h$.

($\Leftarrow$) Consider a hesitant fuzzy set h on X that contains an equivalence class E_a of some hesitant fuzzy equivalence E on X. Observe that $1_\otimes \subseteq E_a(a)$ for every $a \in X$. Since $E_a \subseteq h$, h is normalized. □

Example 5.14. *Observe that equivalence classes of an arbitrary hesitant fuzzy equivalence are normalized hesitant fuzzy sets. For illustration, consider the hesitant fuzzy equivalence $E = E_h^G$ given in Example 5.9. Denote by E_x the equivalence class induced by $x \in X$, i.e., $E_x(y) = E(x, y)$ for every $y \in X$. In particular, since $E_x = E_y$, and by reflexivity of E, we obtain:*

$$1_\otimes \subseteq E_x(x) = E_y(x).$$

The same reasoning applies to the class E_y. Moreover, E_z is also a normalized hesitant fuzzy set since $1_\otimes \subseteq E_z(z)$.

To illustrate the converse, consider the normalized hesitant fuzzy set h on $X = \{x, y, z\}$ over the Gödel power set quantale, given by:

$$h = \begin{bmatrix} \{0.3, 0.5, 1\} & \{0.2, 0.3\} & \{0.3, 0.6\} \end{bmatrix}.$$

Using formula (34), we obtain the following hesitant fuzzy equivalence E on X:

$$E = \begin{bmatrix} \{1\} & \emptyset & \{0.3\} \\ \emptyset & \{0.2\} \cup [0.3, 1] & \emptyset \\ \{0.3\} & \emptyset & \{0.3\} \cup [0.6, 1] \end{bmatrix}.$$

Thus, h contains the equivalence class E_x.

Remark 5.15. *Observe that, in the classical fuzzy setting, a normalized fuzzy subset coincides exactly with equivalence class of a fuzzy equivalence on the same universe (see Theorem 3.1 in [11]). However, in the hesitant fuzzy setting, a normalized hesitant fuzzy set only contains an equivalence class of a hesitant fuzzy equivalence relation. This an observation is illustrated in Example 5.14.*

One of the structural reasons for this difference is the lack of integrality of the underlying structure. In the classical setting of complete residuated lattices, the structure is integral, meaning that the multiplicative identity coincides with the greatest element, $1 = \top$. Therefore, the requirement that a normalized fuzzy set contains the multiplicative identity is equivalent to requiring that its membership value reaches the greatest element.

In contrast, in the power set quantale $\mathcal{Q} = \langle \mathcal{P}(S), \cup, \cap, \otimes, \emptyset, S \rangle$, the multiplicative identity $1_\otimes$ and the greatest element $\top = S$ are generally different. Hence, normalization only requires that the multiplicative identity $1_\otimes$ is contained in *a hesitant fuzzy element, while the corresponding hesitant fuzzy element may contain additional elements of S and therefore need not coincide with the greatest element $\top$.*

The lack of integrality of the underlying quantale has another important consequence: the relations defined in (37) need not be transitive. Indeed, for classical fuzzy relations defined by (36), transitivity follows from the fact that the product of two elements a and b of a complete residuated lattice is bounded above by their meet, that is,

$$a \otimes b \leqslant a \wedge b,$$

which is a direct consequence of the fact that the multiplicative identity is the greatest element. However, in a non-integral quantale, this inequality does not hold in general.

Example 5.16. *This example shows that, in contrast to the classical fuzzy setting, a hesitant fuzzy set does not necessarily coincide with the equivalence classes of the induced hesitant fuzzy equivalence relation. Consider a semigroup $\langle \mathbb{Z}_4, +_4\rangle$, where $\mathbb{Z}_4 = \{0,1,2,3\}$ is and $+_4$ denotes addition modulo 4. Let $\mathcal{Q} = \langle \mathbb{Z}_4, \cup, \cap, \otimes, \emptyset, \mathbb{Z}_4\rangle$ be the power set quantale induced by this semigroup. Further, let h be the hesitant fuzzy set on $X = \{x, y, z\}$ over $\mathcal{Q}$ given by*

$$h = \begin{bmatrix} \{0,2\} & \{1,3\} & \{0,1\} \end{bmatrix}. \tag{39}$$

The corresponding hesitant fuzzy equivalence relation on X induced by h using formula (34) is given by

$$E_h = \begin{bmatrix} \{0,2\} & \{1,3\} & \emptyset \\ \{1,3\} & \{0,2\} & \emptyset \\ \emptyset & \emptyset & \{0\} \end{bmatrix}.$$

It is immediately clear that h contains the equivalence class E_x. However, h does not coincide with the equivalence classes of E_x.

We now state a lemma characterizing all hesitant fuzzy equivalence relations WRT to which a hesitant fuzzy set is extensional.

Lemma 5.17. *The collection of all $E \in \mathcal{E}(X)$ WRT to which a hesitant fuzzy set h is extensional forms the principal ideal generated by E_h, defined in (34).*

Proof. Let $E \in \mathcal{E}(X)$ and $h \in \mathcal{Q}(X)$. To establish the claim, it is enough to show that h is extensional w.r.t. E iff $E \subseteq E_h$. Recall that $E_h(x,y) = h(x) \leftrightarrow h(y)$ for every $x, y \in X$ (see (34)).

We first prove that, if h is extensional WRT to E, then $E \subseteq E_h$. By extensionality, for all $x, y \in X$:

$$h(x) \otimes E(x,y) \subseteq h(y), \quad h(y) \otimes E(y,x) \subseteq h(x). \tag{40}$$

Since E is symmetric, the second inclusion is equivalent to $h(y) \otimes E(x,y) \subseteq h(x)$. Applying the residuation property to (40) we obtain:

$$E(x,y) \subseteq h(x) \to h(y), \quad E(x,y) \subseteq h(y) \to h(x).$$

Hence,

$$E(x,y) \subseteq (h(x) \to h(y)) \cap (h(y) \to h(x)) = h(x) \leftrightarrow h(y).$$

Therefore, $E(x,y) \subseteq E_h(x,y)$ for all $x, y \in X$, i.e., $E \subseteq E_h$.

Conversely, assume that $E \subseteq E_h$, i.e., $E(x,y) \subseteq E_h(x,y)$ for every $x, y \in X$. By definition of E_h, we have:

$$E(x,y) \subseteq h(x) \leftrightarrow h(y) \subseteq h(x) \to h(y),$$

and hence,

$$E(x,y) \subseteq h(x) \to h(y). \tag{41}$$

By applying residuation property to (41), we obtain:

$$h(x) \otimes E(x,y) \subseteq h(y),$$

which shows that h is the extensional with respect to E. □

The following theorem gives several equivalent conditions for a family of normalized hesitant fuzzy sets to form a hesitant fuzzy semi-partition of X.

Theorem 5.18. *Given a family $\mathcal{H}$ of normalized hesitant fuzzy sets on X, the following statements are equivalent:*

(i) *$\mathcal{H}$ forms a hesitant fuzzy semi-partition of X;*

(ii) *Given any $h, g \in \mathcal{H}$:*
$$\bigcup_{x\in X} h(x) \otimes g(x) \subseteq \bigcap_{y\in X} h(y) \leftrightarrow g(y); \tag{42}$$

(iii) *Given any $x, y \in X$:*
$$\bigcup_{h\in\mathcal{H}} h(x) \otimes h(y) \subseteq \bigcap_{g\in\mathcal{H}} g(x) \leftrightarrow g(y). \tag{43}$$

Proof. (i) ⇒ (ii). Take $\mathcal{H}$ to be a hesitant fuzzy semi-partition of X. Hence, $\mathcal{H}$ consists of equivalence classes determined by a hesitant fuzzy equivalence E on X. Then, for arbitrary $h, g \in \mathcal{H}$, there exist $a, b \in X$ satisfying $h = E_a$ and $g = E_b$. According to Lemma 5.8,
$$E(a, b) = ||E_a \otimes E_b|| = ||h \otimes g|| = \bigcup_{x\in X} h(x) \otimes g(x).$$

Next, choose an arbitrary $y \in X$ and set $E_y = f$. According to Lemma 5.7, f is a hesitant fuzzy point WRT to E, and specifically, f is extensional WRT to E. Then, by Lemma 5.17, $E \subseteq E_f$. However, we further have:
$$E(a, b) \subseteq E_f(a, b) = f(a) \leftrightarrow f(b) = E_y(a) \leftrightarrow E_y(b) = E_a(y) \leftrightarrow E_b(y) = h(y) \leftrightarrow g(y),$$

which yields:
$$E(a, b) \subseteq \bigcap_{y\in X} h(y) \leftrightarrow g(y).$$

Thus, (ii) holds.

(ii) ⇒ (iii). Assume that (42) holds. Notice that this property can be written in the equivalent form:
$$h(x) \otimes g(x) \subseteq h(y) \leftrightarrow g(y), \tag{44}$$

for every $h, g \in \mathcal{H}$, $x, y \in X$. According to definition of biresiduum, for arbitrary $h, g \in \mathcal{H}$ and $x, y \in X$, we have:
$$h(x) \otimes g(x) \subseteq h(y) \to g(y), \quad h(y) \otimes g(y) \subseteq h(x) \to g(x).$$

Residuation property (5), together with the commutativity of $\otimes$ implies that
$$\begin{aligned} h(x) \otimes g(x) \subseteq h(y) \to g(y) \quad &\Leftrightarrow \quad h(x) \otimes g(x) \otimes h(y) \subseteq g(y) \\ &\Leftrightarrow \quad h(x) \otimes h(y) \otimes g(x) \subseteq g(y) \\ &\Leftrightarrow \quad h(x) \otimes h(y) \subseteq g(x) \to g(y). \end{aligned}$$

Analogously, it can be shown:
$$h(y) \otimes g(y) \subseteq h(x) \to g(x) \quad \Leftrightarrow \quad h(x) \otimes h(y) \subseteq g(y) \to g(x).$$

Hence,
$$h(x) \otimes h(y) \subseteq g(x) \leftrightarrow g(y), \tag{45}$$

for every $h, g \in \mathcal{H}$ and $x, y \in X$. Hence, (iii) follows.

(iii) ⇒ (i). Suppose that condition (iii) is satisfied. It can be equivalently written as (45), for all $h, g \in \mathcal{H}$, $x, y \in X$. Now, choose an arbitrary $h \in \mathcal{H}$. For any $x, y \in X$, it holds that:
$$h(x) \otimes h(y) \subseteq \bigcap_{g\in\mathcal{H}} g(x) \leftrightarrow g(y). \tag{46}$$

Recall that, according to Corollary 5.2, the hesitant fuzzy relation $E_{\mathcal{H}}$ on X given by (35), is a hesitant fuzzy equivalence on X. By (46), for any $x, y \in X$, we obtain:

$$h(x) \otimes h(y) \subseteq E_{\mathcal{H}}(x, y),$$

which means that h is indistinguishable with respect to $E_{\mathcal{H}}$. By the definition of $E_{\mathcal{H}}$, we have $E_{\mathcal{H}} \subseteq E_h$. According to Lemma 5.17, this fact is equivalent to the one that h is extensional WRT to $E_{\mathcal{H}}$. Finally, h is normalized by the assumption; thus h is a hesitant fuzzy point WRT to $E_{\mathcal{H}}$. Thus, by Lemma 5.7, h corresponds to an equivalence class of $E_{\mathcal{H}}$. As h was chosen arbitrarily, it means that each $h \in \mathcal{H}$ is an equivalence class of $E_{\mathcal{H}}$; thus, $\mathcal{H}$ is a hesitant fuzzy semi-partition of X; therefore, (i) follows. □

Theorem 5.18 not only provides algebraic conditions under which a family of normalized hesitant fuzzy sets on X constitutes a hesitant fuzzy semi-partition of X, but also offers computational insights. Rather than providing full computational details, we only outline the main aspects of the computational analysis. To do so, assume that the set X, the monoid of values S and the family of normalized hesitant fuzzy sets $\mathcal{H}$ on X are all finite. Denote $|X| = n_X$, $|S| = n_S$, and $|\mathcal{H}| = n_{\mathcal{H}}$. We begin by analyzing the computation costs for performing the operations $\otimes$ and $\leftrightarrow$ on hesitant fuzzy sets on X.

Let c_* denote the computational cost, for the monoid $\mathcal{S}$. Then, we have $c_{\otimes} = O(n_S^2 \cdot c_*)$ as, according to (9), we perform $*$ on each pair of elements of S in the worst case, and subsequently add the computed element to the set, which does not exceed this complexity bound. Moreover, since we work within the commutative power set quantale, the operation $\leftrightarrow$ between two hesitant fuzzy sets on X can be computed from either of the residuals (12) or (13). In both cases, the computation reduces to performing $\otimes$ between a hesitant fuzzy set and a one-element hesitant fuzzy set $\{c\}$, for each $c \in S$. Consequently, we also obtain $c_{\leftrightarrow} = O(n_S^2 \cdot c_*)$.

According to Theorem 5.18, $\mathcal{H}$ constitutes a hesitant fuzzy semi-partition on X if and only if (42) holds. Let $h, g \in \mathcal{H}$. For the left-hand side of $\subseteq$ in (42), we compute $h(x) \otimes g(x)$ for each element $x \in X$, each at cost of $c_{\otimes} = O(n_S^2 \cdot c_*)$. Hence, the complexity for computing all hesitant fuzzy sets on the left-hand side of $\subseteq$ is $O(n_X \cdot n_S^2 \cdot c_*)$. The subsequent union $\bigcup_{x \in X} h(x) \otimes g(x)$ can be computed linearly in the size of each hesitant fuzzy set $h(x) \otimes g(x)$, giving a complexity of $O(n_X \cdot n_S^2)$. Therefore, the total complexity of the left-hand side of (42) remains $O(n_X \cdot n_S^2 \cdot c_*)$. A similar argument applies to the right-hand side.

Consequently, the total complexity to compute (42) for given $h, g \in \mathcal{H}$ is $O(n_X \cdot n_S^2 \cdot c_*)$. We compute (42) for each pair $h, g \in \mathcal{H}$, making the total complexity to compute (42) equal to $O(n_{\mathcal{H}}^2 \cdot n_X \cdot n_S^2 \cdot c_*)$. Therefore, the total complexity to determine whether $\mathcal{H}$ is a hesitant fuzzy semi-partition on X is

$$O(n_{\mathcal{H}}^2 \cdot n_X \cdot n_S^2 \cdot c_*), \tag{47}$$

which is polynomial in sizes of the input parameters.

On the other hand, Theorem 5.18 also implies that $\mathcal{H}$ forms a hesitant fuzzy semi-partition on X if and only if (43) holds. By the analogous reasoning, the total complexity to compute (43) is

$$O(n_{\mathcal{H}} \cdot n_X^2 \cdot n_S^2 \cdot c_*), \tag{48}$$

which is also the total complexity to determine whether $\mathcal{H}$ is a hesitant fuzzy semi-partition on X. Note that this complexity is different from the one obtained using (42), but remains polynomial in the sizes of the input parameters.

Corollary 5.19. *Given a hesitant fuzzy semi-partition $\mathcal{H}$ of X, for every $h, g \in \mathcal{H}$, it holds that:*

$$\bigcup_{x \in X} h(x) \otimes g(x) = \bigcap_{y \in X} h(y) \leftrightarrow g(y).$$

Proof. Choose arbitrary $h, g \in \mathcal{H}$. It follows that h coincides with an equivalence class of a hesitant fuzzy equivalence, thus h is normalized, meaning that there exists an element $a \in X$ for which $1_{\otimes} \subseteq h(a)$. Thus, by applying (33) and (32) successively, we get:

$$\bigcap_{y \in X} h(y) \leftrightarrow g(y) \subseteq h(a) \leftrightarrow g(a) \subseteq h(a) \to g(a) \subseteq g(a) \subseteq h(a) \otimes g(a) \subseteq \bigcup_{x \in X} h(x) \otimes g(x).$$

Combining the previous inclusion with Theorem 5.18, we obtain that the statement holds. □

Example 5.20. *Let $X = \{x, y, z\}$. We work over the Gödel power set quantale, defined in Example 2.3. Consider a hesitant fuzzy set h on X, given by:*

$$h = \begin{bmatrix} \{0.1, 0.6\} & \{0.3, 1\} & \{0.1, 0.7\} \end{bmatrix}.$$

Using formula (34)*, we construct the corresponding hesitant fuzzy equivalence E_h on X:*

$$E_h = \begin{bmatrix} \{0.1\} \cup [0.6, 1] & \emptyset & \{0.1\} \\ \emptyset & \{0.3, 1\} & \emptyset \\ \{0.1\} & \emptyset & \{0.1\} \cup [0.7, 1] \end{bmatrix}.$$

Now, consider the hesitant fuzzy semi-partition $\mathcal{X} = \{h, g\}$ of X, where:

$$h = \begin{bmatrix} \{0.1\} \cup [0.6, 1] & \emptyset & \{0.1\} \end{bmatrix}, \qquad g = \begin{bmatrix} \{0.1\} & \emptyset & \{0.1\} \cup [0.7, 1] \end{bmatrix}.$$

It can be verified that part (ii) of Theorem 5.18 holds. Furthermore, the condition in Corollary 5.19 also holds, namely,

$$\bigcup_{x \in X} h(x) \otimes g(x) = \bigcap_{x \in X} h(x) \leftrightarrow g(x) = \{0.1\}.$$

However, note that $\mathcal{X}$ does not satisfy condition (H3) of Theorem 5.23 stated below. More precisely, for $y \in X$, we have $h(y) = g(y) = \emptyset$, and

$$1_\otimes \nsubseteq h(y), \qquad 1_\otimes \nsubseteq g(y).$$

This shows that $\mathcal{X}$ does not form a hesitant fuzzy partition.

Example 5.21. *Let $\mathcal{Q} = \langle \mathcal{P}(\mathbb{Z}), \cup, \cap, \otimes, \emptyset, \mathbb{Z} \rangle$ be the power set quantale of the semigroup $\langle \mathbb{Z}, + \rangle$, where*

$$A \otimes B = \{a + b \mid a \in A, b \in B\},$$

for every $A, B \subseteq \mathbb{Z}$. Let $X = \{x, y, z, w\}$ and let h be the hesitant fuzzy set on X over $\mathcal{Q}$, given by

$$h = \begin{bmatrix} \{0, 2\} & \{1, 3\} & \{0, 2\} & \{1, 3\} \end{bmatrix}.$$

Let us observe that $h(y) = h(w) = \{1, 3\} = \{0, 2\} \otimes \{1\} = h(x) \otimes \{1\} = h(z) \otimes \{1\}$. Thus, we have $\{0, 2\} \to \{0, 2\} = \{0\}$, $\{1, 3\} \to \{1, 3\} = \{0\}$, $\{0, 2\} \to \{1, 3\} = \{1\}$ and $\{1, 3\} \to \{0, 2\} = \{-1\}$. By definition, $A \leftrightarrow B = (A \to B) \cap (B \to A)$, for every $A, B \in \mathcal{P}(\mathbb{Z})$, so we get:

$$E_h = \begin{bmatrix} \{0\} & \emptyset & \{0\} & \emptyset \\ \emptyset & \{0\} & \emptyset & \{0\} \\ \{0\} & \emptyset & \{0\} & \emptyset \\ \emptyset & \{0\} & \emptyset & \{0\} \end{bmatrix}.$$

We further observe that E_h has two different hesitant fuzzy equivalence classes:

$$(E_h)_x = (E_h)_z = \begin{bmatrix} \{0\} & \emptyset & \{0\} & \emptyset \end{bmatrix}, \quad \text{and} \quad (E_h)_y = (E_h)_w = \begin{bmatrix} \emptyset & \{0\} & \emptyset & \{0\} \end{bmatrix}.$$

In other words, the residuum detects the translation by 1. The biresiduum then detects whether the translation works in both directions simultaneously, which happens only when the two sets are equal.
It can be verified that the parts (ii) and (iii) of Theorem 5.18 are satisfied. Furthermore, the condition in Corollary 5.19 also holds. For example:

$$\bigcup_{t \in X} E_x(t) \otimes E_x(t) = \bigcap_{t \in X} E_x(t) \leftrightarrow E_x(t) = \{0\}.$$

Corollary 5.22. *Let $\mathcal{X}$ be a hesitant fuzzy semi-partition of X. For each $h, g \in \mathcal{X}$, the following holds:*

(H1) *If $h \neq g$, then $1_\otimes \nsubseteq ||h \otimes g||$;*

(H2) $g(a) = ||h \otimes g||$, *for all* $a \in c(h)$.

Proof. (H1) Let $h, g \in \mathcal{X}$ such that $h \neq g$. According to Corollary 5.19 and definition (18), we obtain:

$$||h \otimes g|| = \bigcap_{y \in X} h(y) \leftrightarrow g(y).$$

Assume that $1_\otimes \subseteq ||h \otimes g||$. Consequently, we get $1_\otimes \subseteq h(x) \leftrightarrow g(x)$, for every $x \in X$. However, according to the residuation property, we obtain:

$$1_\otimes \subseteq h(x) \leftrightarrow g(x) \subseteq h(x) \to g(x) \quad \text{iff} \quad h(x) \subseteq g(x),$$

and dually:

$$1_\otimes \subseteq h(x) \leftrightarrow g(x) \subseteq g(x) \to h(x) \quad \text{iff} \quad g(x) \subseteq h(x),$$

for each $x \in X$. In other words, we obtained $h = g$, which is a contradiction with the initial assumption. Thus, the statement holds.

(H2) Follows directly from the proof of Corollary 5.19. □

We proceed to examine fuzzy partitions, as characterized in the subsequent theorem.

Theorem 5.23. *Let $\mathcal{H}$ be a family of normalized hesitant fuzzy sets on X. The following statements are equivalent:*

(i) *$\mathcal{H}$ is a hesitant fuzzy partition of X,*

(ii) *$\mathcal{H}$ is a hesitant fuzzy semi-partition of X and*

(H3) *for all $x \in X$ there is some $h \in \mathcal{H}$, with $1_\otimes \subseteq h(x)$,*

(iii) *$\mathcal{H}$ satisfies:*

(H4) *Every $x \in X$ uniquely determines $h_x \in \mathcal{H}$ satisfying $1_\otimes \subseteq h_x(x)$,*

(H5) *$h_x(y) = ||h_x \otimes h_y||$, for every $x, y \in X$.*

Proof. (i) ⇒ (ii). A hesitant fuzzy partition $\mathcal{H}$ induces a hesitant fuzzy equivalence E on X such that each $f \in \mathcal{H}$ represents an equivalence class of E. Therefore, $E_x \in \mathcal{H}$ and $1_\otimes \subseteq E_x(x)$, for each $x \in X$. Hence, the condition (ii) follows.

(ii) ⇒ (iii). Assume that (ii) holds. According to (H3), for every $x \in X$ there is some $h \in \mathcal{H}$ with $1_\otimes \subseteq h(x)$. Assume that there exists $g \in \mathcal{H}$ satisfying $g \neq f$ and $1_\otimes \subseteq g(x)$. Then, from the monotonicity of $\otimes$, we get

$$1_\otimes \subseteq h(x) \otimes g(x) \subseteq \bigcup_{u \in X} h(y) \otimes g(y) = ||h \otimes g||,$$

which contradicts (H1). Thus, (H4) is valid. On the other hand, (H5) is reformulation of (H2).

(iii) ⇒ (i). Assume that (iii) holds. Following the notations from (H4) and (H5), let us define a hesitant fuzzy relation $E \in \mathcal{R}(X)$ on X by $E(x, y) = ||h_x \otimes h_y||$, for every $x, y \in X$. From (H4) and (H5), we get $1_\otimes \subseteq h_x(x) = ||h_x \otimes h_x|| = E(x, x)$, for every $x \in X$, thus E is reflexive. Moreover, E is symmetric. In the end, for every $x, y, z \in X$, we have:

$$\begin{aligned} E(x, y) \otimes E(y, z) &= ||h_x \otimes h_y|| \otimes ||h_y \otimes h_z|| = h_x(y) \otimes h_z(y) \\ &\subseteq \bigcup_{t \in X} h_x(t) \otimes h_z(t) = ||h_x \otimes h_z|| = E(x, z), \end{aligned}$$

meaning that E is transitive. In other words, E is a hesitant fuzzy equivalence on X. According to (H5), $E(x, y) = h_x(y)$ for every $x, y \in X$. Hence, $\mathcal{H}$ contains all equivalence classes of E, i.e., it is a hesitant fuzzy partition on X. □

The conditions $(H3)$ and $(H4)$ admit the alternative formulations $(H3')$ and $(H4')$, respectively, given below:

(H3') $\{c(h)\}_{h \in \mathcal{C}}$ forms a cover of X,

(H4') $\{c(h)\}_{h \in \mathcal{C}}$ forms a partition of X.

Theorem 5.23 also provides a basis for analyzing the computational complexity of determining whether a family of normalized hesitant fuzzy sets on X forms a hesitant fuzzy partition of X. Indeed, assume again that the sets X, S and $\mathcal{H}$ are all finite, and denote $|X| = n_X$, $|S| = n_S$, and $|\mathcal{H}| = n_{\mathcal{H}}$.

If the condition (ii) is used, we must first determine whether $\mathcal{H}$ is a hesitant fuzzy semi-partition of X, which can be verified in (47) or (48) complexity. Afterwards, we need to check whether (H3) is satisfied, which is done in $O(n_X \cdot n_{\mathcal{H}} \cdot n_S)$ complexity. However, this bound does not exceed the one given either by (47) or (48). Hence, the overall complexity to determine the condition (ii) remains the same as the complexity of determining whether $\mathcal{H}$ is a hesitant fuzzy semi-partition of X.

On the other hand, if condition (iii) is considered, both conditions (H4) and (H5) must be verified. Regarding condition (H4), we can see that it can be performed in $O(n_X \cdot n_{\mathcal{H}} \cdot n_S)$ complexity. For the condition (H5), for each pair $x, y \in X$, we need to calculate $(h_x \otimes h_y)(z)$, which requires $O(n_X \cdot n_S^2 \cdot c_*)$ operations, and then compute the union over all $z \in X$, which can be done in $O(n_X \cdot n_S^2)$ complexity. We do this for each pair $x, y \in X$, leading to the conclusion that (H5) is performed in $O(n_X^3 \cdot n_S^2 \cdot c_*)$ complexity. Thus, the overall complexity for condition (iii) is $O(n_X \cdot n_{\mathcal{H}} \cdot n_S + n_X^3 \cdot n_S^2 \cdot c_*)$.

A final note on the complexity issue is that, in the case when $n_{\mathcal{H}} = n_X = n$ (which belongs to case (iii) where for every $x \in X$ is uniquely determined $h_x \in \mathcal{H}$ satisfying $1_\otimes \subseteq h_x(x)$), then all final complexities for determining whether a family of normalized hesitant fuzzy sets on X forms a hesitant fuzzy (semi-)partition of X are equal to $O(n^3 \cdot n_S^2 \cdot c_*)$. We emphasize that the above complexity considerations are applicable only when the involved sets are finite. Theorem 5.18 and Theorem 5.23, however, are stated in a general mathematical form and remain valid for arbitrary (possibly infinite) families.

6 Hesitant fuzzy quasi-orders over the power set quantale

This section examines hesitant fuzzy quasi-orders on a finite set X and characterizes the natural hesitant fuzzy equivalences induced by them. Furthermore, we provide necessary and sufficient conditions under which, for every hesitant fuzzy quasi-order F on X and the associated natural hesitant fuzzy equivalence E, there exists a bijection between the collection X/F of all aftersets and the family X/E of all equivalence classes, as well as between the collection X/F of all aftersets and all foresets $X\backslash F$ of F.

Lemma 6.1. *Let F be a hesitant fuzzy quasi-order on X. Then, for all $x, y \in X$ the following conditions are equivalent:*

1. $1_\otimes \subseteq F(x, y)$,
2. $xF \supseteq yF$,
3. $Fx \subseteq Fy$.

Proof. 1. $\Leftrightarrow$ 2. Assume that $1_\otimes \subseteq F(x, y)$. In that case, according to (27) and (32), for every $a \in X$ we obtain:

$$xF(a) = F(x, a) \supseteq F(x, y) \otimes F(y, a) \supseteq F(y, a) = yF(a).$$

Therefore, we conclude $xF \supseteq yF$.

Conversely, assume that $xF \supseteq yF$. Then

$$F(x, y) = xF(y) \supseteq yF(y) = F(y, y).$$

Reflexivity of F implies $1_\otimes \subseteq F(y, y)$, i.e., $1_\otimes \subseteq F(x, y)$.

Similarly, we show that 1. $\Leftrightarrow$ 3. $\square$

Assume that F is a hesitant fuzzy quasi-order on a finite set X with n elements, represented by an $n \times n$ hesitant fuzzy matrix. As noted in Section 5, in this representation, F-aftersets correspond to the row vectors, whereas F-foresets correspond to the column vectors of this matrix.

Before proceeding, recall that every hesitant fuzzy equivalence relation E on X induces a partition of X into equivalence classes E_x, for $x \in X$, forming the factor set X/E defined in (29). This follows directly from the definition of a hesitant fuzzy partition, introduced in Section 5.

We now turn our attention to constructing hesitant fuzzy equivalence relations on X from a hesitant fuzzy quasi-order F on X, as described in Theorems 6.2 and 6.5. Each of these equivalence relations induces a partition of X into equivalence classes, and hence a corresponding factor set X/E. On these factor sets, we define the induced relations $\tilde{E}$ and $\tilde{F}$ on X/E. This construction allows us to work on a reduced domain while preserving the relevant properties.

Theorem 6.2. *For a given hesitant fuzzy quasi-order F on X, the following assertions hold:*

(a) *The mapping*

$$E(x,y) = F(x,y) \cap F(y,x), \quad \text{for all } x,y \in X, \tag{49}$$

defines a hesitant fuzzy equivalence on X.

(b) *For any $x,x',y,y' \in X$ such that $E_x = E_{x'}$ and $E_y = E_{y'}$ it holds $F(x,y) = F(x',y')$, $E(x,y) = E(x',y')$.*

(c) *The hesitant fuzzy relation $\tilde{E}(E_x,E_y) = E(x,y)$, for all $x,y \in X$ is a hesitant fuzzy equality on X/E.*

(d) *The hesitant fuzzy relation $\tilde{F}(E_x,E_y) = F(x,y)$, for every $x,y \in X$ is a hesitant fuzzy quasi-order on X/E.*

Proof. (a) Let $E(x,y) = F(x,y) \cap F(y,x)$ for arbitrary $x,y \in X$. Reflexivity of E follows directly from reflexivity of F, since $1_\otimes \subseteq F(x,x)$ for every $x \in X$. Symmetry holds by construction:

$$E(y,x) = F(y,x) \cap F(x,y) = E(x,y).$$

Transitivity is inherited from transitivity of F. Therefore, E constitutes a hesitant fuzzy equivalence on X.

(b) From $E_x = E_{x'}$, we obtain $1_\otimes \subseteq E(x,x')$, that is,

$$\begin{aligned} 1_\otimes \subseteq F(x,x') \cap F(x',x) \quad &\Rightarrow \quad 1_\otimes \subseteq F(x,x'), 1_\otimes \subseteq F(x',x) \\ &\Rightarrow \quad xF \supseteq x'F \text{ and } x'F \supseteq xF \\ &\Rightarrow \quad xF = x'F. \end{aligned}$$

Similarly, since $E_y = E_{y'}$, then $Fy = Fy'$. Hence, the following holds

$$F(x,y) = xF(y) = x'F(y) = F(x',y) = Fy(x') = Fy'(x') = F(x',y'). \tag{50}$$

Analogously, $F(y,x) = F(y',x')$ and finally $E(x,y) = E(x',y')$.

(c) Let $\tilde{E}(E_x,E_y) = E(x,y)$ for every $x,y \in X$. According to item (b), $\tilde{E}$ is a well-defined hesitant fuzzy relation on X/E. Since E is a hesitant fuzzy equivalence, $\tilde{E}$ inherits reflexivity, symmetry, and transitivity, and hence is itself a hesitant fuzzy equivalence on X/E. To prove that it is also a hesitant fuzzy equality on X/E, assume that $1_\otimes \subseteq \tilde{E}(E_x,E_y) = E(x,y)$, for some $x,y \in X$. Hence, for each $z \in X$, we obtain:

$$E_x(z) = E(x,z) \subseteq E(y,x) \otimes E(x,z) \subseteq E(y,z) = E_y(z).$$

In addition,

$$E_y(z) = E(y,z) \subseteq E(x,y) \otimes E(y,z) \subseteq E(x,z) = E_x(z).$$

Therefore, $E_x = E_y$, thus we have proved that $\tilde{E}$ is a hesitant fuzzy equivalence on X/E.

(d) Again, according to item (b), $\tilde{F}$ is well-defined on X/E, and it is a hesitant fuzzy quasi-order on X/E. □

The hesitant fuzzy equivalence E on X, defined as in Theorem 6.2 given a hesitant fuzzy quasi-order F on a set X, is represented by E_F. We reffer to it as the *natural hesitant fuzzy equivalence of* F.

According to the following result, the i-th and j-th row vectors of the matrix representation of E_F are identical if and only if the i-th and j-th column vectors are equal, and vice versa. Additionally, the same result shows that there are bijective mappings between the collection of all aftersets X/F of a given hesitant fuzzy quasi-order F and the collection of all equivalence classes X/E_F of E_F, as well as between the collection of all aftersets X/F and all foresets $X\backslash F$ of F.

Theorem 6.3. *Let $E = E_F$ denote the natural hesitant fuzzy equivalence associated with F, where F is a hesitant fuzzy quasi-order on X.*

(a) *The following conditions are equivalent for any $x,y \in X$:*

(i) $1_\otimes \subseteq E(x,y)$,

(ii) $E_x = E_y$,

(iii) $xF = yF$,

(iv) $Fx = Fy$.

(b) *Functions* $xF \mapsto E_x$ *of* X/F *to* X/E*, and* $xF \mapsto Fx$ *of* X/F *to* $X\backslash F$*, are bijective functions.*

Proof. (a) For any $x, y \in X$, we have the following implications:

(i) $\Rightarrow$ (ii) This implication was proved in the proof of Theorem 6.2, part (c).

(ii) $\Rightarrow$ (i) Let $E_x = E_y$. Then $E(x,y) = E_x(y) = E_y(y) = E(y,y)$. Since $1_\otimes \subseteq E(y,y)$, we also have $1_\otimes \subseteq E(x,y)$.

(i) $\Rightarrow$ (iii) Let $1_\otimes \subseteq E(x,y)$, i.e., $1_\otimes \subseteq F(x,y) \cap F(y,x)$. It follows that $1_\otimes \subseteq F(x,y)$ and $1_\otimes \subseteq F(y,x)$. Then for every $z \in X$ the following holds

$$yF(z) = F(y,z) \subseteq F(x,y) \otimes F(y,z) \subseteq F(x,z) = xF(z).$$

Hence, $yF(z) \subseteq xF(z)$, also, for every $z \in X$ we have

$$xF(z) = F(x,z) \subseteq F(y,x) \otimes F(x,z) \subseteq F(y,z) = yF(z).$$

So, $xF(z) \subseteq yF(z)$. Finally, we get $xF(z) = yF(z)$, for all $z \in X$.

(iii) $\Rightarrow$ (i) Assume $xF = yF$. Then

$$F(x,y) = xF(y) = yF(y) = F(y,y) \supseteq 1_\otimes.$$

Similarly,

$$F(y,x) = yF(x) = yF(y) = F(y,y) \supseteq 1_\otimes.$$

Hence, $1_\otimes \subseteq F(x,y) \cap F(y,x)$ and $1_\otimes \subseteq E(x,y)$.

The equivalence (i) $\Leftrightarrow$ (iv) can be shown in the same way as (i) $\Leftrightarrow$ (iii).

It follows from part (a) that the functions defined in (b) are both injective and surjective. $\square$

The next theorem presents three important properties of a hesitant fuzzy quasi-order F and the natural hesitant fuzzy equivalence E_F induced by it.

Theorem 6.4. *For a given set* X*, let* F *be a hesitant fuzzy quasi-order on* X*. Then:*

(a) E_F *is the greatest hesitant fuzzy equivalence on* X *contained in* F*.*

(b) *For any* $x, y \in X$

$$F(x,y) = \bigcap_{a\in X} aF(x) \to aF(y) = \bigcap_{a\in X} Fa(y) \to Fa(x), \tag{51}$$

(c) *For any* $x, y \in X$

$$E_F(x,y) = \bigcap_{a\in X} aF(x) \leftrightarrow aF(y) = \bigcap_{a\in X} Fa(x) \leftrightarrow Fa(y). \tag{52}$$

Proof. (a) Assume E is an arbitrary hesitant fuzzy equivalence on X included in F. Thus:

$$E = E \cap E^{-1} \subseteq F \cap F^{-1} = E_F.$$

(b) Since F is a transitive relation, due to the residuation property (5), the following sequence of equivalences holds for all $x, y, a \in X$

$$\begin{aligned} F(x,y) \otimes F(y,a) \subseteq F(x,a) \quad &\Leftrightarrow \quad F(x,y) \subseteq F(y,a) \to F(x,a) \\ &\Leftrightarrow \quad F(x,y) \subseteq \bigcap_{a\in X} F(y,a) \to F(x,a) \\ &\Leftrightarrow \quad F(x,y) \subseteq \bigcap_{a\in X} Fa(y) \to Fa(x). \end{aligned}$$

Let us consider the opposite direction,

$$\bigcap_{a\in X} Fa(y) \to Fa(x) = \bigcap_{a\in X} F(y,a) \to F(x,a) \subseteq F(y,y) \to F(x,y).$$

Since F is a reflexive hesitant fuzzy relation, then $1_\otimes \subseteq F(y,y)$. Taking into account that the residuum is an antitone function by the first argument, we obtain $F(y,y) \to F(x,y) \subseteq F(x,y)$, hence

$$\bigcap_{a\in X} Fa(y) \to Fa(x) \subseteq F(x,y).$$

Finally,

$$F(x,y) = \bigcap_{a\in X} Fa(y) \to Fa(x).$$

The equality $F(x,y) = \bigcap_{a\in X} aF(x) \to aF(y)$ can be proved analogously.

(c) This is a direct consequence of (51) and (49). □

The following theorem introduces an additional hesitant fuzzy equivalence induced by a given hesitant fuzzy quasi-order and describes the relationship between them.

Theorem 6.5. *Let F be a hesitant fuzzy quasi-order on a set X. The following conditions hold.*

(a) *The hesitant fuzzy relation E on X given by*

$$E(x,y) = F(x,y) \otimes F(y,x), \qquad \textit{for every } x,y \in X,$$

is a hesitant fuzzy equivalence on X.

(b) *For all $x,x',y,y' \in X$ such that $E_x = E_{x'}$ and $E_y = E_{y'}$ the next equalities hold*

$$F(x,y) = F(x',y'), \quad E(x,y) = E(x',y').$$

(c) *The hesitant fuzzy relation $\tilde{E}$ on X/E defined by*

$$\tilde{E}(E_x,E_y) = E(x,y), \qquad \textit{for every } x,y \in X,$$

is a hesitant fuzzy equality on X/E.

(d) *The hesitant fuzzy relation $\tilde{F}$ on X/E,*

$$\tilde{F}(E_x,E_y) = F(x,y), \qquad \textit{for every } x,y \in X,$$

is a hesitant fuzzy E-order on X/E.

Proof. (a) Since F is a hesitant fuzzy quasi-order, F is reflexive and transitive. Hence, by the associativity and commutativity of $\otimes$, for every $x,y,z \in X$ the following is true:

$$1_\otimes \subseteq F(x,x) \quad \Rightarrow \quad 1_\otimes \subseteq F(x,x)\otimes F(x,x) \quad \Rightarrow \quad 1_\otimes \subseteq E(x,x),$$

$$E(x,y) = F(x,y)\otimes F(y,x) = F(y,x)\otimes F(x,y) = E(y,x),$$

$$\begin{aligned} E(x,y)\otimes E(y,z) &= (F(x,y)\otimes F(y,x))\otimes(F(y,z)\otimes F(z,y)) \\ &= (F(x,y)\otimes F(y,z))\otimes(F(z,y)\otimes F(y,x)) \\ &\subseteq F(x,z)\otimes F(z,x) \\ &= E(x,z). \end{aligned}$$

Therefore, E is a hesitant fuzzy equivalence on X.

(b) Assume $E_x = E_{x'}$ for arbitrary $x,x' \in X$, i.e., assume $E(x,y) = E(x',y)$ for all $y \in X$. In particular, by taking $y = x$, we obtain $1_\otimes \subseteq E(x,x) = E(x',x) = E(x,x')$. Therefore,

$$1_\otimes \subseteq E(x,x') = F(x,x')\otimes F(x',x).$$

As a direct consequence, $1_\otimes \subseteq F(x, x')$ and $1_\otimes \subseteq F(x', x)$. Then, by Lemma 6.1, $xF \supseteq x'F$ and $x'F \supseteq xF$, and hence $xF = x'F$. Similarly, if $E_y = E_{y'}$, then $Fy = Fy'$. Therefore,

$$F(x, y) = xF(y) = x'F(y) = F(x', y) = Fy(x') = Fy'(x') = F(x', y').$$

The equality $F(y, x) = F(y', x')$ can be proved analogously. Finally,

$$E(x, y) = F(x, y) \otimes F(y, x) = F(x', y') \otimes F(y', x') = E(x', y').$$

(c) This part is a direct consequence of the definition of the relation $\tilde{E}$ and the statement in part (b).
(d) Based on (b), $\tilde{F}$ is well-defined on X/E. So, $\tilde{F}$ is a hesitant fuzzy quasi-order. Consequently, for every $x, y \in X$,

$$\tilde{F}(E_x, E_y) \otimes \tilde{F}(E_y, E_x) = F(x, y) \otimes F(y, x) = E(x, y) = \tilde{E}(E_x, E_y).$$

Therefore, $\tilde{F}$ is $\tilde{E}$-antisymmetric. □

Example 6.6. *Let $\mathcal{Q}$ be the Łukasiewicz power set quantale over the semigroup $\langle [0, 1], * \rangle$, as introduced in Example 2.3. Recall that $*$ is given by $a * b = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$. Hence, by the definition of $\otimes$ (see (9)), we have:*

$$h_1(x) \otimes h_2(x) = \{\max\{0, a + b - 1\} \mid a \in h_1(x) \text{ and } b \in h_2(x)\},$$

for all $h_1, h_2 \in Q^X$ and all $x \in X$. Consider the hesitant fuzzy relation R on $X = \{x, y\}$, given by:

$$R = \begin{bmatrix} \{0, 1\} & \{0, 0.4\} \\ \{0, 0.7\} & \{0, 0.1, 1\} \end{bmatrix}.$$

A direct verification shows that:

$$R \circ R(x, x) = \{0, 0.1, 1\} \nsubseteq R(x, x),$$

and thus R is not a hesitant fuzzy quasi-order. Applying the transitive closure algorithm (28), we obtain a hesitant fuzzy quasi-order F, given by $F = R^2$:

$$F = \begin{bmatrix} \{0, 0.1, 1\} & \{0, 0.4\} \\ \{0, 0.7\} & \{0, 0.1, 1\} \end{bmatrix}.$$

It can be verified that $F \circ F = F$, which implies that F is reflexive and transitive. Consequently, F is a hesitant fuzzy quasi-order.

Next, we construct the hesitant fuzzy equivalences on X induced by F. By applying Theorem 6.2, we obtain the natural hesitant fuzzy equivalence E_F:

$$E_F = \begin{bmatrix} \{0, 0.1, 1\} & \{0\} \\ \{0\} & \{0, 0.1, 1\} \end{bmatrix}.$$

On the other hand, using Theorem 6.5, we obtain the hesitant fuzzy equivalence E:

$$E = \begin{bmatrix} \{0, 0.1, 1\} & \{0, 0.1\} \\ \{0, 0.1\} & \{0, 0.1, 1\} \end{bmatrix}.$$

This example shows that, in general, the hesitant fuzzy equivalences derived from Theorems 6.2 and 6.5 do not coincide.

7 Illustrative applications

In this section, we illustrate the applicability of the theoretical results developed in the previous sections through concrete examples over the power set quantale. The aim is to demonstrate how the developed theory leads to interpretable structures in illustrative examples.

Consider the power set quantale induced by the additive semigroup of non-negative integers. Let $S = \langle \mathbb{N}_0, + \rangle$ be the additive semigroup of non-negative integers, and let $\mathcal{Q} = \langle \mathcal{P}(\mathbb{N}_0), \cup, \cap, \otimes, \emptyset, \mathbb{N}_0 \rangle$ be its power set quantale, as constructed in Example 2.1, where $A \otimes B = \{a + b \mid a \in A, b \in B\}$ for all $A, B \in \mathcal{P}(\mathbb{N}_0)$, and the identity element is $1_\otimes = \{0\}$. Since S is commutative, the left and right residuals coincide and we write

$$A \to B = \{c \in \mathbb{N}_0 \mid A \otimes \{c\} \subseteq B\} = \{c \in \mathbb{N}_0 \mid a + c \in B, \text{ for all } a \in A\}.$$

This setting is well suited to modeling accumulation of uncertain, discretely estimated quantities. For example, when managing a portfolio of projects, the several independent auditors may each report a distinct point estimate of a project's budget overrun, rather than a continuous range of possible values; every additional procurement step then accumulates these overruns rather than replacing them. The resulting hesitant fuzzy element is precisely the finite set of these distinct reported values.

For simplicity of notation, we identify the quantale $\mathcal{Q}$ with its underlying set $\mathcal{P}(\mathbb{N}_0)$ and, in the following examples, use the notation $\mathcal{P}(\mathbb{N}_0)$ for both. We now illustrate how the results of Sections 5 and 6 lead to equivalence-based clustering and preference grouping in the hesitant fuzzy setting over the power set quantale $\mathcal{P}(\mathbb{N}_0)$, within a shared budget-management scenario.

Example 7.1 (Application to clustering over the power set quantale $\mathcal{P}(\mathbb{N}_0)$)**.** *Let $X = \{p_1, p_2, p_3, p_4\}$ represent four ongoing projects. Two auditors independently report point estimates of each project's budget overrun (in thousands of euros), yielding a hesitant fuzzy set h on X over $\mathcal{P}(\mathbb{N}_0)$:*

$$h(p_1) = h(p_2) = \{0, 3\}, \qquad h(p_3) = h(p_4) = \{0, 6\}.$$

Using formula (34)*, we compute the induced hesitant fuzzy equivalence E_h. For instance,*

$$\begin{aligned}
\{0,3\} \to \{0,6\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,6\} \textit{ and } c+3 \in \{0,6\}\} = \emptyset,\\
\{0,6\} \to \{0,3\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,3\} \textit{ and } c+6 \in \{0,3\}\} = \emptyset,\\
\{0,3\} \to \{0,3\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,3\} \textit{ and } c+3 \in \{0,3\}\} = \{0\},\\
\{0,6\} \to \{0,6\} &= \{c \in \mathbb{N}_0 \mid c \in \{0,6\} \textit{ and } c+6 \in \{0,6\}\} = \{0\}.
\end{aligned}$$

Hence,

$$E_h = \begin{bmatrix} 1_\otimes & 1_\otimes & \emptyset & \emptyset \\ 1_\otimes & 1_\otimes & \emptyset & \emptyset \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \end{bmatrix}.$$

Consequently, E_h produces exactly two equivalence classes, E_{p_1} and E_{p_3}, given by

$$E_{p_1} = \begin{bmatrix} 1_\otimes & 1_\otimes & \emptyset & \emptyset \end{bmatrix}, \qquad E_{p_3} = \begin{bmatrix} \emptyset & \emptyset & 1_\otimes & 1_\otimes \end{bmatrix},$$

which can naturally be interpreted as two clusters of projects with comparable budget-risk profiles: $\{p_1, p_2\}$ and $\{p_3, p_4\}$.

We now verify that the family of equivalence classes $\{E_{p_1}, E_{p_3}\}$ satisfies the condition (H1) of Corollary 5.22. Since $E_{p_1} \neq E_{p_3}$, condition (H1) requires $1_\otimes \nsubseteq \|E_{p_1} \otimes E_{p_3}\|$. Indeed,

$$\|E_{p_1} \otimes E_{p_3}\| = \bigcup_{z \in X} E_{p_1}(z) \otimes E_{p_3}(z) = \emptyset,$$

which strictly excludes $1_\otimes = \{0\}$, confirming that the two risk clusters are provably distinct with respect to the induced hesitant fuzzy equivalence relation, rather than merely numerically distant.

Condition (iii) of Theorem 5.23 allows us to verify the hesitant fuzzy partition property by checking conditions (H4)-(H5), instead of directly verifying that the relation induced by $\{E_{p_1}, E_{p_3}\}$ is reflexive, symmetric, and transitive, which is exactly what makes the complexity analysis following (47)*-*(48) *applicable in practice: for each $x \in X$, there exists a* unique *$h_x \in \{E_{p_1}, E_{p_3}\}$ with $1_\otimes \subseteq h_x(x)$ (condition (H4)), and one directly checks $h_x(y) = \|h_x \otimes h_y\|$ for every pair (condition (H5)), e.g.,*

$$E_{p_1}(p_1) = \{0\} = \|E_{p_1} \otimes E_{p_1}\|, \qquad E_{p_1}(p_3) = \emptyset = \|E_{p_1} \otimes E_{p_3}\|.$$

Both conditions hold, confirming that $\{E_{p_1}, E_{p_3}\}$ is indeed a hesitant fuzzy partition of X.

Now, we compare the obtained hesitant fuzzy clustering with the result that would be obtained after aggregating the hesitant information into single numerical values. If the two auditors' estimates were collapsed to a single averaged number (e.g., 1.5 and 3, in thousands of euros), a standard threshold-based clustering could plausibly merge projects whose averages lie close together, depending on the chosen threshold. In contrast, working directly with E_h over $\mathcal{P}(\mathbb{N}_0)$ yields a provable *$\emptyset$-separation between the two risk clusters, since no discrete estimate reported for one project ever coincides, even after accumulation, with an estimate reported for the other: this is a structural, not merely a numerical, guarantee.*

Having grouped the projects by their comparable budget-overrun profiles, we now turn to a related but distinct question: how a committee's pairwise preference judgments over the same portfolio induce a natural ranking structure.

Example 7.2 (Application to preference clustering via a hesitant fuzzy quasi-order)**.** *Let $X = \{p_1, p_2, p_3, p_4\}$ represent four candidate projects evaluated by several budget committee experts. Let F be a hesitant fuzzy relation on X over $\mathcal{Q} = \mathcal{P}(\mathbb{N}_0)$ representing the* margin of budgetary preference*: $F(x, y)$ is the set of additional funding values under which x is still preferred to (or tied with) y, given by the matrix*

$$F = \begin{bmatrix} 1_\otimes & 1_\otimes & \{0,6,9\} & \{0,6,9\} \\ 1_\otimes & 1_\otimes & \{0,6,9\} & \{0,6,9\} \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \\ \emptyset & \emptyset & 1_\otimes & 1_\otimes \end{bmatrix}.$$

It can be directly verified that F is reflexive and transitive (i.e., a hesitant fuzzy quasi-order on X): in particular, the only non-trivial products are of the form $1_\otimes \otimes \{0,6,9\} = \{0,6,9\}$ and $\{0,6,9\} \otimes 1_\otimes = \{0,6,9\}$, both of which are contained in the corresponding entry of F, while every product involving $\emptyset$ trivially satisfies transitivity. Note that the tied pairs are deliberately kept at the trivial margin $1_\otimes = \{0\}$: enriching a mutual *(two-directional) relation beyond the identity would force $F(p_1, p_1)$ to absorb the self-composition $F(p_1, p_2) \otimes F(p_2, p_1)$, which grows without bound under a non-idempotent operation such as addition; only the one-directional (strict dominance) entries can safely carry a richer margin.*

By Theorem 6.2(a), the hesitant fuzzy relation $E_F(x, y) = F(x, y) \cap F(y, x)$ is a hesitant fuzzy equivalence on X. Computing E_F yields

$$E_F(p_1, p_2) = 1_\otimes \cap 1_\otimes = 1_\otimes, \qquad E_F(p_3, p_4) = 1_\otimes \cap 1_\otimes = 1_\otimes,$$
$$E_F(p_1, p_3) = \{0,6,9\} \cap \emptyset = \emptyset, \qquad E_F(p_2, p_4) = \{0,6,9\} \cap \emptyset = \emptyset,$$

so E_F produces exactly two equivalence classes, $\{p_1, p_2\}$ and $\{p_3, p_4\}$, interpreted as two ranks of equally-preferred projects*: the committee considers p_1 and p_2 mutually substitutable (each is preferred to the other with only the trivial margin $1_\otimes = \{0\}$, i.e., no genuine advantage), and likewise for p_3, p_4, while both members of the first rank are strictly preferred to both members of the second, with a non-trivial margin $\{0,6,9\}$ (thousand euros) and no reverse preference ($\emptyset$).*

By Theorem 6.4(a), E_F is the greatest *hesitant fuzzy equivalence contained in F; hence no rougher tie can be justified without contradicting the observed preference $F(p_1, p_3) = \{0,6,9\} \neq \emptyset$, and no finer distinction can be justified within $\{p_1, p_2\}$, since $F(p_1, p_2) = F(p_2, p_1) = 1_\otimes = \{0\}$. This provides the budget committee with a provably optimal finitely-nested-possible grouping of projects into indistinguishable preference ranks, directly usable for multi-criteria funding decisions.*

When we combine hesitant fuzzy sets with the power set quantale of a general semigroup, we gain a framework capable of modeling multi-expert, highly complex budgeting systems, without being restricted to numerical aggregation.

8 Concluding remarks

The approach presented in this paper models hesitant fuzzy sets using the power set quantale $\mathcal{P}(\mathcal{S})$ of a monoid $\mathcal{S}$ as the value domain. This choice introduces a fundamental structural distinction from the usual approaches: while standard methods typically define join and meet of hesitant fuzzy sets using numeric aggregation functions such as maximum and minimum; in our framework these operations are defined using standard set-theoretic union and intersection. It gives a clear algebraic foundation based on quantales and monoids, providing a natural setting for axiomatic analysis. Also, since this approach does not rely on numeric aggregation, it can be applied in more abstract or symbolic contexts, not just in situations where precise numbers are available. This quantale-based framework admits the residuation property that allows solving hesitant fuzzy inequalities and equations, which will be the topic of our future research.

The results obtained in Sections 5 and 6 provide an algebraic framework for representing and analyzing relations between objects in situations where evaluations are uncertain, imprecise, or provided by multiple sources. One possible area of application is data clustering and pattern recognition. In many real-world applications, objects must be clustered based on similarity metrics that cannot be reduced to a single scalar value. Instead, several possible similarity degrees may arise due to disagreement among experts, measurement variability, or incomplete information. In such situations, hesitant fuzzy relations allow similarity assessments to be represented as sets of possible values. The characterization of hesitant fuzzy equivalence relations established in Section 5 then provides a mathematical basis for constructing hesitant fuzzy partitions, which can be interpreted as clusters of mutually similar objects. This framework can be useful in domains such as medical diagnosis, document classification, and image recognition, where similarity judgments are inherently uncertain.

Another important application concerns multi-expert decision making. In many decision processes, several experts evaluate alternatives with respect to different criteria. Their assessments often differ, producing sets of possible evaluation values rather than a single precise score. Hesitant fuzzy relations can model these evaluations without forcing premature aggregation. Hesitant fuzzy equivalence relations can then be used to identify groups of alternatives that experts consider essentially indistinguishable, while hesitant fuzzy quasi-orders can capture preference or dominance relations between alternatives. Such structures are particularly relevant in multi-criteria decision analysis, where alternatives must be ranked or grouped despite uncertainty in the evaluation process.

Overall, the developed theory of hesitant fuzzy sets, hesitant fuzzy relations, hesitant fuzzy equivalences and hesitant fuzzy quasi-orders provides a comprehensive mathematical framework for modeling similarity and preference relations in environments characterized by hesitation and uncertainty. The rigorous results obtained in this paper lay a theoretical foundation for future research on algorithmic methods and potential practical implementations in areas such as data analysis, decision support systems, and intelligent information processing.

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