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Uncertainty propagation in linear fuzzy difference equations via Hamacher t-norms

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Abstract

This paper investigates linear fuzzy difference equations whose initial conditions are represented by fuzzy numbers. The classical arithmetic operations involved in recurrence relations are extended to the fuzzy t-norm-based extension principles, allowing the incorporation of interactivity between fuzzy variables. In particular, this paper deals with Hamacher t-norm, which provides a flexible interaction mechanism between different dependence regimes. The paper focuses on the dynamical behavior and uncertainty propagation of fuzzy sequences generated by interactive arithmetic. Moreover, theoretical results concerning the modal dynamics, support evolution, spectral stability, and sensitivity with respect to the interaction parameter are established. Computational aspects of the proposed algorithms are also discussed, including their asymptotic complexity. To illustrate the proposed framework, classical discrete dynamical systems, including fuzzy Fibonacci recurrences and simple population growth models, are analyzed under uncertain initial conditions. A comparative study between the Hamacher interaction and Zadeh's extension principle is presented, highlighting how different interaction mechanisms influence the propagation and concentration of uncertainty in fuzzy difference equations.

Keywords: Fuzzy difference equation, biomathematics, fuzzy interactivity, Hamacher t-norm.

1 Introduction

A difference equation is a mathematical relationship that expresses the connection between successive terms of a discrete sequence. In contrast to differential equations, which involve derivatives with respect to a continuous variable, difference equations deal with increments or differences between successive values at discrete points in time or space. These equations have wide applicability in mathematical modeling of discrete processes, algorithm analysis, discrete control theory, population dynamics, and many other contexts in which changes naturally occur in discrete steps [10].

Although difference equations are powerful tools from a modeling perspective, they generally do not incorporate the intrinsic uncertainties that are present in real-world phenomena. For instance, in ecological or agricultural contexts, determining the exact number of plants in a plantation, estimating the number of seeds produced by each plant, or measuring the impact of external agents such as insects can be extremely challenging. Insects, in particular, play a crucial role in both damaging crops and regulating ecological balances. Their populations fluctuate according to environmental conditions, predator-prey interactions, and reproductive cycles that can span different stages of development. These aspects make exact deterministic modeling not only difficult but also often unrealistic. Similarly, plant growth and reproduction are subject to environmental variability such as climate, soil quality, and nutrient availability. Thus, even when difference equations are capable of capturing the structure of such processes, a more flexible mathematical framework is needed to handle the uncertainty and imprecision inherently involved [5].

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An additional classical example of discrete dynamics appears in the Fibonacci sequence, which originally emerged from the study of rabbit population growth but has since been found to describe numerous natural patterns, such as phyllotaxis in plants, branching structures, and even shell spirals. The Fibonacci sequence illustrates how simple recurrence relations can capture complex natural phenomena, but it also highlights a limitation: in real ecosystems, growth rarely follows exact deterministic proportions. Environmental variability, mortality, and stochastic effects inevitably alter the idealized recurrence. In this sense, enriching the classical formulation with fuzzy set theory provides a way to bridge the gap between mathematical abstraction and biological reality.

Fuzzy set theory is an interdisciplinary area that aims to mathematically and computationally address uncertainties linked to natural phenomena [2]. Among its many tools, triangular norms (t-norms) play a central role. A t-norm is a function that combines two fuzzy values into a third fuzzy value, usually interpreted as their “intersection”. This operation is frequently used to model fuzzy logic, aggregation of imprecise information, and the conjunction of uncertain events. In the context of this article, t-norms are considered as a means to extend classical arithmetic operations into fuzzy arithmetic operations through the extension principle, as proposed in [19].

Several t-norms have been studied in the literature, such as the minimum, algebraic product, and Łukasiewicz t-norms, each of which exhibits distinct algebraic and modeling properties [11]. In particular, this work focuses on the Hamacher t-norm. The Hamacher family of operators is known for its flexibility, as it interpolates between the minimum and product t-norms depending on the parameter choice, making it adaptable to different contexts. This property is especially relevant in interactions between species, survival rates, or reproduction processes rarely conform to a single rigid mathematical rule. They often exhibit intermediate behaviors between complete independence and full dependence, which the Hamacher t-norm is well-suited to capture [14, 15, 16]. Thus, the Hamacher operator that is an appropriate candidate for modeling biological processes influenced by uncertainty and interactivity.

In addition to the use of fuzzy arithmetic and extension principles, another important line of research in fuzzy mathematics concerns fuzzy relational equations, which provide algebraic tools for modeling relationships between fuzzy sets and have been widely studied from both theoretical and applied perspectives. Classical and modern developments in this area address issues such as solvability, structure of solution sets, and computational methods, as discussed in [4, 12, 13, 18]. More recent works have also explored extensions of fuzzy relational models in connection with optimization, learning systems, and computational architectures, highlighting their relevance in contemporary applications [1].

These approaches focus primarily on structural and relational properties of fuzzy systems. In contrast, the present work is concerned with the temporal evolution of uncertainty through fuzzy difference equations. Rather than analyzing static relations between fuzzy quantities, we investigate how uncertainty propagates dynamically over discrete time under interactive arithmetic. In this sense, our approach complements existing studies by providing a dynamical perspective, which remains less explored in the context of interactive fuzzy models.

One of the central questions in the study of interactive fuzzy systems concerns the choice of the interaction mechanism. Classical approaches based on a fixed t-norm implicitly assume that the dependence between fuzzy quantities is known a priori. However, in many real-world applications, the actual degree of interaction between uncertain variables is rarely known with certainty and may vary according to the phenomenon under investigation.

From this perspective, the Hamacher family of t-norms offers an important advantage over the use of a single fixed t-norm. Rather than defining a unique interaction rule, it provides a continuous family of interaction mechanisms through the parameter r, allowing different dependence hypotheses to be explored. Consequently, the interaction itself becomes part of the modeling process instead of being a fixed assumption adopted before the analysis.

Motivated by this observation, the objective of the present work is not merely to investigate fuzzy difference equations under a particular t-norm, but to study how different interaction regimes influence the propagation of uncertainty in discrete dynamical systems. The Hamacher family is therefore employed as a mathematical tool that makes it possible to systematically analyze the sensitivity of fuzzy dynamics with respect to the interaction mechanism.

To illustrate the behavior of fuzzy sequences generated by this work, we investigate classical discrete models under uncertain initial conditions, including Fibonacci recurrences and simple population growth dynamics. These examples are intended primarily as illustrative case studies to analyze how interactive fuzzy arithmetic affects the evolution and spread of uncertainty over time. In particular, a comparative analysis with Zadeh's extension principle, based on the minimum t-norm, is presented in order to highlight how different interaction mechanisms influence the resulting fuzzy dynamics.

In summary, the proposed framework provides a mathematically tractable approach for incorporating interactivity into fuzzy difference equations while preserving the underlying deterministic dynamical structure. Theoretical results established in this work show that the main asymptotic properties of the system, such as modal stability and spectral behavior, remain governed by the deterministic core and are therefore independent of the chosen interaction mechanism. At the same time, the interaction process directly affects the concentration and propagation of uncertainty, allowing different fuzzy behaviors to emerge without altering the associated characteristic function.

Some structural properties established in this work, such as associativity and monotonicity of the interactive fuzzy sum, follow from standard properties of continuous t-norms and the sup-J extension principle. These results are included primarily to provide a self contained framework for the proposed fuzzy difference equations. Similarly, several stability properties follow from classical linear-recurrence theory once the deterministic core dynamics are identified. Thus, the main contribution of the paper is not the isolated proof of elementary structural properties, but rather the combination of interactive fuzzy arithmetic via Hamacher t-norms with the analysis of uncertainty dynamics under parametric interactivity.

In particular, the paper shows that the Hamacher parameter modifies the propagation and concentration of uncertainty without altering the spectral structure of the deterministic core. This produces a separation between deterministic stability and uncertainty dynamics, that is, while the asymptotic behavior remains governed by the classical function, the shape and spread of the fuzzy solutions depend strongly on the chosen interaction mechanism. Therefore, the proposed work provides a parametrized approach for controlling uncertainty propagation in fuzzy difference equations while preserving the underlying dynamical behavior.

From a computational perspective, the proposed approach preserves the same asymptotic complexity as classical implementations of the extension principle. In this sense, the use of interactive t-norms does not modify the intrinsic computational structure of the problem, differing only in the cost of the underlying aggregation operations. Nevertheless, for general continuous t-norms, the α-cuts of the interactive sum are defined only implicitly, and closed form expressions are not available for intermediate α-levels. Consequently, the proposed framework involves a trade off between modeling flexibility and computational simplicity.

We emphasize that the biological models considered in this paper are not intended to provide highly-detailed descriptions of real biological systems. Instead, they are used as simple and interpretable dynamical frameworks to investigate the effects of interactive fuzzy arithmetic on uncertainty propagation and long-term behavior.

This paper is organized as follows. Section 2 presents the preliminary definitions and results necessary for a better understanding of the work. Section 3 introduces the proposed approach and establishes auxiliary results. Section 4 presents an application of the proposed model along with the main results. Finally, Section 5 contains the concluding remarks.

2 Preliminaries

A real sequence is defined by a function of the type $x : \mathbb{N} \to \mathbb{R}$, denoted by $\{x_n\}_{n\in\mathbb{N}}$. For example, the well-known Fibonacci and Lucas sequences, which growths exponentially, are examples of this type of function, where $x_n = x_{n-1} + x_{n-2}$ and they differ from the initial conditions; that is, Fibonacci sequence considers $x_0 = x_1 = 1$ and Lucas sequence considers $x_0 = 1$ and $x_0 = 2$. In the context of Biomathematics, this sequence can be related to the growth of rabbits [5]. This means that at time n the number x_n of the population depends on the previous number of the population x_{n-1} and x_{n-2} at times $n-1$, and $n-2$, respectively.

Real sequences can be associated with the concept of difference equations, which is defined by an equation that relates the value x_n, at point n, with values at other points, such as $x_{n-1}, x_{n-2}, \ldots$ as in the example of Fibonacci and Lucas sequences. It is common to denote these equations by $\Delta x(n) = \psi(n)$, where $\Delta x(n) = x_n - x_{n-1}$ and ψ is a function that depends on the previous values of the sequence x_n. In the case of Fibonacci and Lucas sequences, the function ψ is given by $\psi(n) = x_{n-2}$, for $n > 2$.

Generalizing the type of difference equation presented above, we present the definition of linear difference equations, which will be the focus of this paper. A linear difference equation is defined by

$$a_0x_n + a_1x_{n-1} + a_2x_{n-2} + \ldots + a_mx_{n-m} = c(n),$$

where $a_0, \ldots, a_m$ are constant real values and $c(n)$ a given function.

Solving difference equations requires initial conditions. Once the initial value is provided, the other terms can be computed, for all $n \in \mathbb{N}$. Hence, this article is dedicated to studying difference equations with initial conditions. Moreover, the uncertainty of these initial conditions will be modeled using fuzzy set theory.

The fuzzy set theory is an extension of the classical set theory; in the following sense, a classical set is defined from the characteristic function, which assumes only two values: 0 if the statement is false and 1 if the statement is true. On the other hand, a fuzzy set is defined by a membership function that takes values in the interval $[0, 1]$, where values closer to 1 represent a higher degree of truth. So, the membership function incorporates the characteristic function, which means that every classical set is a fuzzy set as well.

To clarify the definition of a fuzzy set, we can define the set $A = \{x \in \mathbb{R} : x \text{ is small}\}$ from the membership function $\mu_A(x) = 1 - x$ if $x \in [0, 1]$, $\mu_A(x) = x + 1$ if $x \in [-1, 0]$ or $\mu_A(x) = 0$ otherwise. Note that $\mu_A(0) = 1$, which means that $x = 0$ completely satisfies the property of being small. On the other hand, $\mu_A(1) = \mu_A(-1) = 0$, which means that $x = 1$ and $x = -1$ do not satisfy the property of being small. Also, $\mu_A(0.25) = 0.75$, which means that $x = 0.25$ satisfies the property of being small with degree 0.75. So here we can "measure" the level of association of an element to a set. It is important to note that while this example only considers numbers in the interval $[-1, 1]$, a much larger interval may be appropriate depending on the context

A fuzzy number A is a fuzzy subset of $\mathbb{R}$ that is an extension of a real number, whose membership function is: 1) upper semi-continuous; 2) convex; 3) if there is r such that $\mu_A(r) = 1$, then μ_A is increasing in $(-\infty, r]$ and decreasing in $[r, \infty)$ [17].. Equivalently, a fuzzy number can also defined in terms of α-cuts $[A]_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$, that is, A is a fuzzy number if its α-cuts are non-empty, closed and bounded intervals with bounded support $supp(A) = \{x \in \mathbb{R} : \mu_A(x) > 0\}$ [2]. Since fuzzy numbers are widely used in modelling, here we will be only deal with the class of fuzzy numbers $\mathbb{R}_\mathcal{F}$. A fuzzy number A is contained in a fuzzy number B, $(A \subseteq B)$ if and only if $\mu_A(x) \leq \mu_B(x)$, for all $x \in \mathbb{R}$. A fuzzy number is said to be normal if $\sup_{x \in \mathbb{R}} \mu_A(x) = 1$. The core of a fuzzy number is defined by $core(A) = \{x \in \mathbb{A} : \mu_A(x) = 1\}$. While the support stands for the set of all elements that have some association with the set A, the core is the set defined by all elements that have total association with the set A.

There are several types of fuzzy numbers based on their shapes; one of the most used in literature is the triangular fuzzy number, which is defined by a triple $(a; u; b)$, whose membership function is given by the $\varphi(x) = \frac{x-a}{u-a}$ if $x \in [a, u]$, $\varphi(x) = \frac{b-x}{b-u}$ if $x \in [u, b]$, and $\varphi(x) = 0$, otherwise.

An operator $t : [0, 1] \times [0, 1] \to [0, 1]$ is called by t- norm if: 1) 1 acts as an identity, that is, $t(x, 1) = x$, for all $x \in [0, 1]$; 2) t is commutative; 3) t is associative; 4) t is increasing in both arguments, that is, $t(x, y) \leq t(u, v)$, for all $x, y, u, v \in [0, 1]$ such that $x \leq u$ and $y \leq v$. The minimum operator, given by $t_\wedge(x, y) = \min\{x, y\}$, is an example of t-norm. Also, $t_p = xy$ is a t-norm called a product t-norm. This second t-norm can be generalized by the Hamacher operator, which is given by

$$t_H(x, y) = \frac{xy}{r + (1 - r)(x + y - xy)},$$

where t_H is also a t-norm with $r > 0$ [17]. Note that, if $r = 1$ then $t_H = t_p$.

A fuzzy relation R in the universe $X \times Y$ is defined by any fuzzy subset of $X \times Y$, that is, the membership function that defines the fuzzy relation R is given by $\mu_R : X \times Y \to [0, 1]$, where $\mu_R(x, y) \in [0, 1]$ stands for the degree of association between x and y in terms of the relation R. Since the domain is $X \times Y$, the relation R is also called a binary fuzzy relation. One example of a binary fuzzy relation is a joint possibility distribution J (JPD), which is a relation between two fuzzy numbers A and B that satisfies $\mu_A(x) = \sup_{u \in \mathbb{R}} \mu_J(x, u)$ and $\mu_B(y) = \sup_{v \in \mathbb{R}} \mu_J(v, y)$, simultaneously [9].

The JPDs, in combination with extension principles, have been used to extend classical arithmetic operations to fuzzy arithmetic operations [8]. There are mainly two types of extension principles,Zadeh's extension principle gives rise to standard fuzzy arithmetic, whereas the sup-J extension principle yields interactive arithmetic, providing powerful tools for mathematical modelling [21].

There are several JPDs proposed in the literature, each of them, with a different application purpose. Here, we focus on the JPDs that are based on t-norms in order to avoid bad properties of arithmetic operations [7], such non defined operations, propagation of increasing uncertainty, and so on. In this context, the sup-J extension principle as becomes

$$\begin{aligned}
\mu_{(A \otimes B)}(z) &= \sup_{x \otimes y = z} \min\{\mu_A(x), \mu_B(y)\} \quad \text{(Sup-}J\text{ extension for } t = t_\wedge\text{)}, && (1) \\
\mu_{(A \otimes_{J_t} B)}(z) &= \sup_{x \otimes y = z} t_H(\mu_A(x), \mu_B(y)) \quad \text{(Sup-}J\text{ extension for } t = t_H\text{)} \\
&= \sup_{x \otimes y = z} \frac{\mu_A(x)\mu_B(y)}{r + (1 - r)(\mu_A(x) + \mu_B(y) - \mu_A(x)\mu_B(y))}, && (2) \\
\mu_{(A \otimes_{J_t} B)}(z) &= \sup_{x \otimes y = z} t_p(\mu_A(x), \mu_B(y)) \quad \text{(Sup-}J\text{ extension for } t = t_p\text{)} \\
&= \sup_{x \otimes y = z} \mu_A(x)\mu_B(y), && (3)
\end{aligned}$$

where $\otimes$ is an arithmetic operation.

The minimum t-norm gives rise to the concept of non-interactivity, proposed by Zadeh [22], which is similar to independence for random variables. The precise definition is given as follows. Let A and B be two fuzzy numbers. If the joint possibility distribution (JPD) of A and B is defined by the minimum t-norm, then A and B are non-interactive.

For any other t-norm, the fuzzy numbers A and B are called interactive, which is associated with dependence relation. Thus, Zadeh's extension principle is a mechanism to deal with "independence" for fuzzy variables, whereas the sup-J extension principle deals with "dependence".

Several papers have shown that interactivity produces responses with less uncertainty than the usual approaches associated with non-interactivity [20, 21]. Here, the uncertainty will be modeled by the support of a fuzzy number supp $(\mu_A) = \{x \in \mathbb{R} : \mu_A(x) > 0\}$, that is, the set of all real numbers that have some association with the set A. So, the larger the support, more uncertainty A incorporates. The following theorem shows that interactive arithmetic operations always produce the fuzzy numbers with smaller support than the usual fuzzy arithmetic operators.

Theorem 2.1. [20] *Let $A, B, C, D \in \mathbb{R}_{\mathcal{F}}$. For all JPD J such that $J \neq J_{\wedge}$, given by (1), and $\otimes$ be an arithmetic operator, the following statements hold:*

a) $\mu_{A \otimes_J B}(z) \leq \mu_{A \otimes B}(z)$, *for all* $z \in \mathbb{R}$*;*

b) $\lambda\mu_{A \otimes_J B}(z) \leq \lambda\mu_{A \otimes B}(z)$, *for all* $z, \lambda \in \mathbb{R}$*;*

c) *If* $\mu_A(x) \leq \mu_B(x)$, *for all* $x \in \mathbb{R}$, *then* $\mu_{A \otimes_J C}(z) \leq \mu_{B \otimes C}(z)$, *for all* $z \in \mathbb{R}$*;*

d) *If* $\mu_A(x) \leq \mu_B(x)$ *and* $\mu_C(x) \leq \mu_D(x)$, *for all* $x \in \mathbb{R}$, *then* $\mu_{A \otimes_J C}(z) \leq \mu_{B \otimes D}(z)$, *for all* $z \in \mathbb{R}$.

The above theorem will be used to prove that solutions to fuzzy difference equations via interactive arithmetic have smaller support than the solution to fuzzy difference equations via usual fuzzy arithmetic, as it will be illustrated in the next section.

3 Linear fuzzy difference equation via interactive arithmetic

As presented before, this paper is dedicated to studying difference equations, with initial conditions given by fuzzy numbers. Since there are operations involved in the classical difference equations, these operations will be extended by Zadeh's extension principle and the sup-J extension principle, comparing these problems from the perspective of interactivity here.

The fuzzy sequences studied here are defined by functions of the form $X : \mathbb{N} \to \mathbb{R}_{\mathcal{F}}$, which we denoted by $\{X_n\}_{n\in\mathbb{N}}$. For example, the extension of the Fibonacci and Lucas sequences to fuzzy values is given by $X_n = X_{n-1} +_J X_{n-2}$, where $+_J$ is a fuzzy arithmetic sum. Since the arithmetic operations involved in the fuzzy sequences depend on distributions J, to compute their values, we will denote a fuzzy sequence by $\{X_n\}^J$.

Given a fuzzy sequence $\{X_n\}^J$, a fuzzy difference equation is defined by $\Delta_J X(n) = \Psi(n)$, where $\Delta_J X(n) = X(n) -_J X(n-1)$ and Ψ is a fuzzy function that depends on the previous values of the fuzzy sequence $\{X_n\}^J$. Moreover, a linear fuzzy difference equation is defined by

$$a_0 X(n) +_J a_1 X(n-1) +_J a_2 X(n-2) +_J \ldots +_J a_m X(n-m) = C(n), \tag{4}$$

where $a_0, \ldots, a_m$ are constant real values and $C(n)$ is a given fuzzy function.

Note that Equation (4) can be written in the form of $\Delta_J X(n) = \Psi(n)$, considering $a_0 = 1$ and $a_1 = -1$, that is

$$\Delta_J X(n) = X(n) -_J X(n-1) = \Psi(n),$$

where $\Psi(n) = C(n) -_J a_2 X(n-2) -_J \ldots -_J a_m X(n-m)$.

The next example illustrates these sequences in the fuzzy context.

Example 3.1. *Insects go through several stages in their life cycle, which may last from days to years. Since these stages have different durations, it is customary to adopt a generation as the unit of time, thereby simplifying the modeling of population growth* [5]. *Since insects have multiple stages, they can be represented by difference equations, condensing a large amount of information into just a few equations.*

As an example, we consider the reproduction of aphids. All offspring (progeny) of an aphid are contained within a gall, which consists of plant tissues induced by substances secreted by aphids. These are spherical, closed structures within which aphids feed, protected from predators and weather conditions.

The ability to survive to adulthood and to produce offspring depends on climate, food quality, population size, among other factors. For the sake of simplification, we assume these effects to be constant. Now, consider the following variables:

$a_n =$ *number of adult female aphids in the n-th generation.*

$p_n =$ *number of progenitors (parents) in the n-th generation.*

$m =$ *fractional mortality of young aphids.*

$f =$ *number of progenitors (parents) per female aphid.*

$r =$ *proportion of female aphids relative to the total adult aphid population.*

The following equation describes the successive aphid populations and is used to determine the number of adult females in the n-th generation, assuming a_0 initial females, with each female producing f progenitors. Thus:

$$\begin{aligned} p_{n+1} &= \textit{number of progenitors (parents) in the } (n+1)\textit{-th generation (next generation).} \\ f &= \textit{number of progenitors per female aphid.} \\ a_n &= \textit{number of adult female aphids in the } n\textit{-th generation (previous generation).} \end{aligned}$$

Hence, we obtain

$$p_{n+1} = f \cdot a_n. \tag{5}$$

Of these progenitors, only a fraction $(1-m)$ survive to adulthood, producing a final proportion r of females. Therefore,

$$a_{n+1} = r(1-m)p_{n+1}. \tag{6}$$

From a combination of Equations (5) and (6), it follows that

$$a_{n+1} = f \cdot r(1-m)a_n, \tag{7}$$

where the constant $fr(1-m)$ represents the number of adult females produced by each mother aphid.

Now, we aim to extend this sequence to the fuzzy case. Since a_{n+1} depends only one variable a_n then all the extension principles coincide, and consequently, A_{n+1} is given by

$$A_{n+1} = fr(1-m)A_n,$$

where $fr(1-m) \in \mathbb{R}$ and $A_n \in \mathbb{R}_{\mathcal{F}_C}$, for all $n \in \mathbb{N}$.

From the initial condition $A_0 = (25; 50; 75)$ and parameters $f = 10$, $r = 0.5$ and $m = 0.2$, Figures 1, 2 and 3 depict some values of this fuzzy sequence.

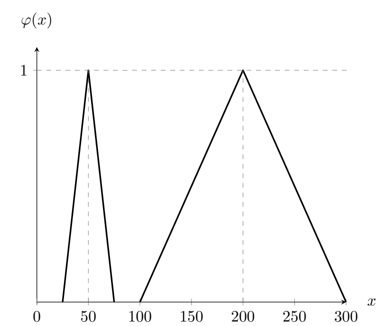

Figure 1: Fuzzy sequence of insects A_n considering parameters $A_0 = (25; 50; 75)$, $f = 10$, $r = 0.5$ and $m = 0.2$, for the second generation

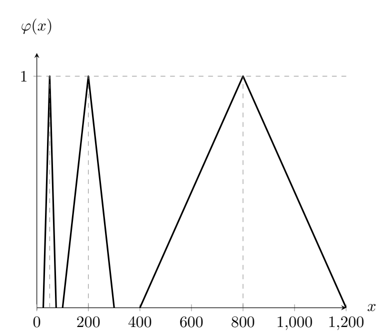

Figure 2: Fuzzy sequence of insects A_n considering parameters $A_0 = (25; 50; 75)$, $f = 10$, $r = 0.5$ and $m = 0.2$, for the third generation

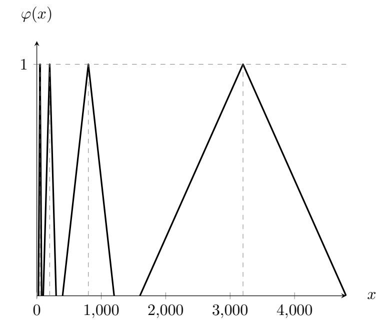

Figure 3: Fuzzy sequence of insects A_n considering parameters $A_0 = (25; 50; 75)$, $f = 10$, $r = 0.5$ and $m = 0.2$, for the fourth generation

Note that the uncertainty at each fuzzy number A_n increases as its diameter expands. Thus, one of the concerns is to control the uncertainty of each fuzzy number A_n. Also, note that since A_{n+1} depends only on A_n, the influence of the choice of the t-norm does not change the behavior of this sequence at all, since it does not appear explicitly in the calculation to obtain the term A_{n+1}. The following Fibonacci sequence shows that the choice of the t-norm influences the values obtained by the sequence.

Example 3.2. *Leonardo Pisano, well known as Leonardo Fibonacci published a paper called "Liber abaci", where the sequence $x_{n+2} = x_{n+1} + x_n$ was first introduced. This sequence has the first elements given by*

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\},$$

where the first initial conditions were $x_0 = 1$ and $x_1 = 1$. Some interesting facts are this sequence appears in studies involving nature, flowers, animals, galaxies and many other phenomena.

From a practical perspective, this application could start with values larger than $x_0 = 1$ and $x_1 = 1$. For example, one could consider that at the initial instant there are around 50 pairs of rabbits. The linguistic variable around incorporates an uncertainty in the counting of the rabbits. This uncertainty is then represented by fuzzy numbers. Moreover, since the value x_{n+2} depends on x_{n+1} and x_n, this dependence will be described, in the fuzzy context, by the interactivity associated with t-norms.

By extending this sequence to the fuzzy values, we obtain the following

$$X_{n+2} = X_{n+1} +_J X_n,$$

where $+_J$ is a sum based on JPDs.

If one considers the minimum t-norm, then $J_t = J_\wedge$ and $+_J$ boils down to the fuzzy standard sum. Figure 4 depicts the first elements of this sequence, considering $+_J = +$. On the other hand, if one considers the Hamacher t-norm, then $+_J$ is given by (2) for $\otimes = +$. Figure 5 depicts the first elements of this sequence, considering $+_J = +_{J_H}$.

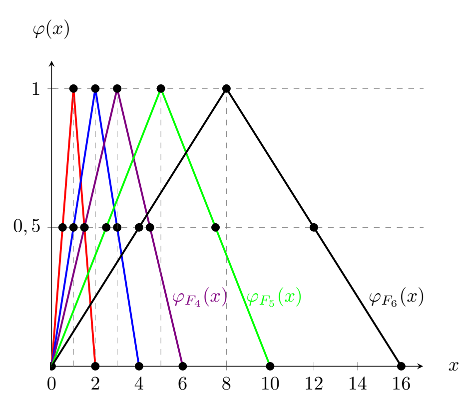

Figure 4: Fibonacci fuzzy sequence considering $X_0 = X_1 = (0; 1; 2)$ and $+_J = +$, for $n = 6$.

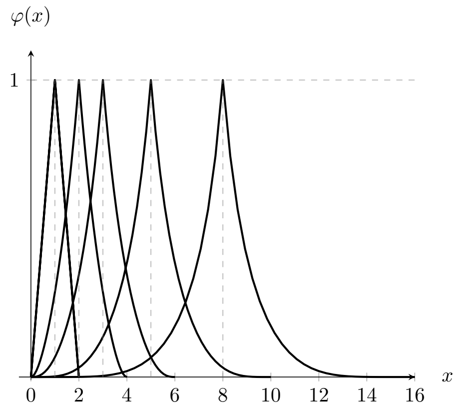

Figure 5: Fibonacci fuzzy sequence considering $X_0 = X_1 = (0;1;2)$ and $+_J = +_{J_H}$, for $n = 6$.

Algorithms 1 and 2 provide pseudocode to reproduce the Fibonacci sequences for the fuzzy standard sum and Hamacher t-norm, respectively.

Algorithm 1 Fuzzy Fibonacci Sequence with Standard Fuzzy Sum $(+)$

Input: Initial fuzzy numbers $X_0, X_1 \in \mathbb{R}_{\mathcal{F}}$, number of terms N, grid of values Z
Output: Sequence: $\{X_n\}_{n=0}^{N-1}$ of fuzzy numbers

Initialize sequence: $X[0] := X_0$, $X[1] := X_1$
for $n = 2$ **to** $N - 1$ **do**
  foreach $z \in Z$ **do**
    Initialize maxVal $\leftarrow 0$ **foreach** $x \in Z$ **do**
      $y \leftarrow z - x$ **if** $y \in Z$ **then**
        val $\leftarrow \min(\mu_{X[n-1]}(x), \mu_{X[n-2]}(y))$ maxVal $\leftarrow \max(\text{maxVal}, \text{val})$
    Set $\mu_{X[n]}(z) \leftarrow$ maxVal
return $\{X_n\}_{n=0}^{N-1}$

Algorithm 1 consists of three nested loops. Inside the innermost loop, the operations consist of constant-time. Therefore, its total computational cost is $\mathcal{O}(N \cdot M \cdot M) = \mathcal{O}(NM^2)$.

Algorithm 2 Fuzzy Fibonacci Sequence with Interactive Fuzzy Sum $(+_J)$

Input: Initial fuzzy numbers $X_0, X_1 \in \mathbb{R}_{\mathcal{F}}$, number of terms N, grid of values Z, parameter $r \in (0, 1]$
Output: Sequence $\{X_n\}_{n=0}^{N-1}$ of fuzzy numbers

Initialize sequence: $X[0] := X_0$, $X[1] := X_1$
for $n = 2$ **to** $N - 1$ **do**
  foreach $z \in Z$ **do**
    Initialize maxVal $\leftarrow 0$ **foreach** $x \in Z$ **do**
      $y \leftarrow z - x$ **if** $y \in Z$ **then**
        val $\leftarrow \dfrac{\mu_{X[n-1]}(x)\,\mu_{X[n-2]}(y)}{r + (1-r)\,(\mu_{X[n-1]}(x) + \mu_{X[n-2]}(y) - \mu_{X[n-1]}(x)\mu_{X[n-2]}(y))}$ maxVal $\leftarrow \max(\text{maxVal}, \text{val})$
    Set $\mu_{X[n]}(z) \leftarrow$ maxVal
return $\{X_n\}_{n=0}^{N-1}$

The structure of Algorithm 2 is very similar to that of Algorithm 1, which difference lies in the evaluation of the membership value, which is given by

$$\frac{\mu_{X[n-1]}(x)\,\mu_{X[n-2]}(y)}{r + (1-r)\,(\mu_{X[n-1]}(x) + \mu_{X[n-2]}(y) - \mu_{X[n-1]}(x)\mu_{X[n-2]}(y))}. \tag{8}$$

This expression involves a fixed number of arithmetic operations in constant time. Therefore, despite the increased algebraic complexity of the t-norm, the asymptotic computational complexity remains $\mathcal{O}(NM^2)$.

It is important to observe that both the standard fuzzy sum and the interactive fuzzy sum algorithms have the same asymptotic complexity. However, the interactive approach has a higher computational constant, due to the evaluation of a rational function at each iteration. Moreover, the quadratic dependence on M reflects the need to evaluate the supremum over all decompositions $z = x + y$, which is inherent to the extension principle. Therefore, the computational cost is dominated by the discretization of the universe rather than by the specific choice of t-norm.

The computational analysis presented in this work is intended only to provide a basic estimate of the asymptotic cost associated with the direct implementation of the sup-J extension principle. Since the algorithms are based on exhaustive evaluation of all decompositions z = x+y over a discretized universe, the resulting complexity is quadratic with respect to the grid size.

We emphasize that the present work is not focused on numerical optimization or high-performance implementations. More sophisticated computational issues, such as convergence of the discretization process, α-cut based implementations, fast convolution techniques, or optimized numerical schemes, remain open problems and constitute-interesting directions for future research.

Note that the above figures indicate the Fibonacci sequence obtained by Hamacher t-norm is contained in the Fibonacci sequence obtained by minimum t-norm. The next theorem proves that this result always holds.

Theorem 3.3. *Let $\{X_n\}^J$ and $\{X_n\}^{J_\wedge}$ be the fuzzy sequences that are associated with the interactive arithmetic and the usual fuzzy arithmetic, respectively. If $A_1, \ldots, A_m \in \mathbb{R}_\mathcal{F}$ are the fuzzy initial conditions associated with the linear fuzzy difference equations $\Delta_J X_n = \Psi_1(n)$ and $\Delta_{J_\wedge} X_n = \Psi_2(n)$, then $\{X_n\}^J \subseteq \{X_n\}^{J_\wedge}$.*

Proof. Let J and $J_\wedge$ be joint possibility distributions among $A_1, \ldots, A_m \in \mathbb{R}_\mathcal{F}$. From item a) of Theorem 1, it follows that $\Psi_1(n) \subseteq \Psi_2(n)$, where $\Psi_1(n) = C(n) -_J a_2 X(n-2) -_J \ldots -_J a_m X(n-m)$ and $\Psi_2(n) = C(n) -_{J_\wedge} a_2 X(n-2) -_{J_\wedge} \ldots -_{J_\wedge} a_m X(n-m)$. Consequently, from item c) of Theorem 1, it follows that

$$X(n+1)^{(J)} = X(n) +_J \Psi_1(n) \subseteq X(n) +_{J_\wedge} \Psi_2(n) = X(n+1)^{(J_\wedge)}, \quad \forall n \in \mathbb{N}.$$

Therefore, $\{X_n\}^J \subseteq \{X_n\}^{J_\wedge}$. □

Theorem 3.3 shows that fuzzy sequences that are constructed based on interactive arithmetics produce more specific fuzzy values that is, the diameter of the support of $\{X_n\}^J$ always smaller than $\{X_n\}^{J_\wedge}$.

Remark 3.4. *Some observations about Theorem 3.3 are needed. Given a set of the fuzzy numbers $A_1, \ldots, A_m \in \mathbb{R}_\mathcal{F}$ and a joint possibility distribution J, it is not guaranteed that the fuzzy numbers $A_1, \ldots, A_m$ satisfy the condition imposed by the distribution J. For example, if A_1 is a triangular fuzzy number, A_2 is a trapezoidal fuzzy number and $J = J_L$ is a linear joint possibility distribution, as proposed in* [3]*, then the pair (A_1, A_2) can not be correlated by the J_L distribution, and therefore, it is not possible to compute $A_1 +_{J_L} A_2$. In this case, a generalization of J_L, denoted by J_0, can be computed* [8]*. To compare the two types of fuzzy sequences (interactive and non-interactive), we will assume that $\{X_n\}^J$ is always well defined, that is, it is always possible to compute its elements. For t-norms, that is, $J = J_t$ these operations can always be computed.*

Remark 3.5. *In view of Remark 3.4, the application that will be provided in this paper is based on Hamacher t-norm. So the joint possibility distribution will be given by, $J = J_{t_H}$, which can be applied to any pair of the fuzzy numbers; that is, it is always possible to compute $A_1 +_{J_{t_H}} A_2$. Consequently, $\{X_n\}^{J_{t_H}}$ is well defined.*

Remark 3.6. *Similar to the Hamacher case, the product t-norm induces a well-defined sup-J extension for any pair of fuzzy numbers in $\mathbb{R}_F$. Since $t_P(x, y) = xy$ is continuous, associative, and defined on $[0, 1] \times [0, 1]$, the corresponding fuzzy arithmetic operations can always be computed. Consequently, fuzzy sequences constructed via the product t-norm are well defined for all initial conditions in $\mathbb{R}_F$.*

It is also important to observe that the fuzzy addition $+_J$ defined through the sup-J extension principle, based on a continuous t-norm t, is associative, that is, $(A +_J B) +_J C = A +_J (B +_J C)$, for all $A, B, C \in \mathbb{R}_F$ and $J = J_t$. This property follows from the fact that, since the t-norm is associative, we obtain

$$\begin{aligned}
\mu_{(A+_{J_t}B)+_{J_t}C}(z) &= \sup_{a+b+c=z} t(t(\mu_A(a), \mu_B(b)), \mu_C(c)) \\
&= \sup_{a+b+c=z} t(\mu_A(a), t(\mu_B(b), \mu_C(c))) \\
&= \mu_{A+_{J_t}(B+_{J_t}C)}(z),
\end{aligned}$$

for all $z \in \mathbb{R}$.

Next, we compare the fuzzy sequences $\{X_n\}^{J_{t_H}}$ and $\{X_n\}^{J_{t_P}}$ in terms of inclusion. More precisely, the next proposition shows that the comparison between these sequences depends on the parameter r in the definition of the Hamacher t-norm. To this end, we will denote the fuzzy sequence obtained by the Hamacher t-norm by $\{X_n\}^{(J_{t_H},r)}$.

Proposition 3.7. *Let $\{X_n\}^{J_{t_P}}$ be the fuzzy sequence generated by the product t-norm and $\{X_n\}^{(J_{t_H},r)}$ be the fuzzy sequence generated by the Hamacher t-norm with parameter r. Thus, one of the following conditions occurs:*

1. *If $0 < r < 1$, then $\{X_n\}^{(J_{t_H},r)} \supseteq \{X_n\}^{J_{t_P}}$, for all n.*
2. *If $r > 1$, then $\{X_n\}^{(J_{t_H},r)} \subseteq \{X_n\}^{J_{t_P}}$, for all n.*
3. *If $r = 1$, the sequences coincide.*

Proof. Let $t_P(x,y) = xy$ be the product t-norm, and for simplification of notation, let $D = r + (1-r)(x+y-xy)$ be the denominator of the Hamacher t-norm $t_H^{(r)}(x,y) = \frac{xy}{r+(1-r)(x+y-xy)}$ with parameter $r > 0$. First, note that since $x, y \in [0,1]$, we have $0 \le x + y - xy \le 1$.

Now, if $0 < r < 1$, then $1-r > 0$ and therefore $(1-r)(x+y-xy) \le (1-r)$. This implies that $D = r+(1-r)(x+y-xy) \le r+(1-r) = 1$. Hence, $t_H^{(r)}(x,y) = \frac{xy}{D} \ge xy = t_p(x,y)$. Consequently, from the sup-J extension principle, we have

$$\mu_{(A \otimes_{J_{t_H}} B)}(z) = \sup_{x \otimes y = z} t_H^{(r)}(\mu_A(x), \mu_B(y)) \ge \sup_{x \otimes y = z} t_p(\mu_A(x), \mu_B(y)) = \mu_{(A \otimes_{J_t} B)}(z),$$

for all $z \in \mathbb{R}$ such that $x \otimes y = z$. Thus, we have $\{X_n\}^{(J_{t_H},r)} \supseteq \{X_n\}^{J_{t_P}}$ for all n.

If $r > 1$, then $1 - r < 0$ and in a similar way, we can prove that $D \ge 1$. Hence, $t_H^{(r)}(x,y) = \frac{xy}{D} \le xy = t_p(x,y)$, implying that

$$\mu_{(A \otimes_{J_{t_H}} B)}(z) = \sup_{x \otimes y = z} t_H^{(r)}(\mu_A(x), \mu_B(y)) \le \sup_{x \otimes y = z} t_p(\mu_A(x), \mu_B(y)) = \mu_{(A \otimes_{J_t} B)}(z),$$

for all $z \in \mathbb{R}$ such that $x \otimes y = z$. Consequently, $\{X_n\}^{(J_{t_H},r)} \subseteq \{X_n\}^{J_{t_P}}$ for all n.

If $r = 1$, then $D = 1 + 0(x+y-xy) = 1$. Hence, $t_H^{(1)}(x,y) = xy = t_p(x,y)$ and the equality follows immediately. $\square$

Notably, although the deterministic dynamics are completely characterized by the underlying recurrence equation and remain independent of the Hamacher parameter r, the uncertainty dynamics exhibit fundamentally different behavior. In particular, the parameter r continuously modifies the propagation and concentration of uncertainty without affecting the asymptotic stability of the modal sequence. Consequently, we have two distinct aspects of the dynamics. First, the deterministic evolution is governed exclusively by the classical recurrence relation, preserving its properties. Second, the fuzzy component of the solution is directly influenced by the adopted interaction mechanism. Hence, the choice of the t-norm should not be regarded merely as an alternative arithmetic operation, but rather as a modeling decision that determines how uncertainty propagates throughout the dynamical process.

From this perspective, the Hamacher family provides a significant advantage over fixed interaction models, since different values of r correspond to different interaction regimes. Instead of comparing only isolated interaction mechanisms, such as the minimum and product t-norms, one can investigate an entire family of fuzzy dynamical systems and analyze the sensitivity of uncertainty propagation with respect to the interaction intensity.

We emphasize that the Hamacher family is not the only possible choice for this purpose. Other parameterized interaction frameworks, such as families of joint possibility distributions connecting the minimum t-norm with correlation-based models, could also be considered [20]. However, besides providing a mathematically tractable parameterization of the interaction mechanism, the Hamacher family has been successfully employed in several applied studies, including recent biomathematical models such as the COVID-19 framework proposed by Lopes et al. [14, 16, 15]. Since the motivating applications considered in the present work belong to the same general area, the Hamacher interaction constitutes a natural choice for investigating how different interaction assumptions influence uncertainty propagation in biomathematical inspired discrete dynamical systems.

In the next section we provide an application of fuzzy sequences in a plant growth dynamic, in order to show that interactive arithmetic-operations provide more precise results, in contrast to the usual fuzzy arithmetic.

4 Application in a plant growth dynamic

Annually the plants produce flowers and seeds. At the end of summer the flowers wither and die, while some of the seeds survive the winter to give rise to a new generation of plants for the next spring. Some seeds can survive for a year or more before reviving. Some of these seeds are lost due to predation, disease or the effects of climate. Therefore, for a plant to survive as a species, a large population must be renewed every year [5].

To formulate a mathematical model that describes the propagation of these plants, one must take into account the fact that some seeds end up remaining dormant for a year or more before germinating. Therefore, we must pay attention to the reserves of seeds of different ages that remain dormant.

1. **The problem:**

   Plants produce seeds at the end of the growing season (mid-August in the Northern Hemisphere) and then die. It is known that a portion of these seeds survives the winter and germinates at the beginning of the following season (mid-May), giving rise to a new generation, with the fraction that germinates depending on the age of the seeds.

2. **Definitions and assumptions:**

   Plants produce a γ amount of seeds each summer. Seeds can germinate in up to two years, so a fraction α of seeds from one year ago and a fraction β from two years germinate the following year, and over the winter some seeds end up dying. So, we have:

$$\begin{aligned}\gamma &= \text{number of seeds produced per plant,}\\ \alpha &= \text{fraction of seeds from one year ago,}\\ \beta &= \text{fraction of seeds from two years ago,}\\ \sigma &= \text{fraction of seeds that survive the winter.}\end{aligned}$$

   To find the quantity of seeds that germinate in a given year, it is necessary to take into account some important points, namely: the quantity of new seeds produced; the amount that actually germinates; the aging and partial mortality of seeds. This last point leads us to a simplification of the model, as seeds that are two years old are not viable for producing new seeds, and will therefore be discarded from the model. So we have:

$$\begin{aligned}p_n &= \text{number of plants in generation } n,\\ S_n^0 &= \text{number of new seeds produced,}\\ S_n^1 &= \text{number of seeds from one year ago (April),}\\ S_n^2 &= \text{number of seeds from two years ago (April),}\\ \overline{S}_n^1 &= \text{number of seeds from one year ago (May),}\\ \overline{S}_n^2 &= \text{number of seeds from one year ago (May).}\end{aligned}$$

   Initially, all these variables are considered significant; however, some will subsequently be excluded to simply the model. Note that the superscript denotes the age of the seed, and the subscript denotes the year.

3. **Equations:**

   In May, a fraction α of seeds from one year ago and a fraction β from two years ago will produce new plants, thus

$$p_n = \alpha S_n^1 + \beta S_n^2. \tag{9}$$

   The number of seeds for next year will be reduced due to germination. Therefore, for seeds from different years, the quantity that did not germinate and the quantity of seeds from April are taken into account. Thus,

$$\overline{S}_n^1 = (1-\alpha) S_n^1, \tag{10}$$

$$\overline{S}_n^2 = (1-\beta) S_n^2. \tag{11}$$

In August, new seeds (from the same year) in quantity γ are produced per plant. So,

$$S_n^0 = \gamma p_n. \tag{12}$$

After winter the number of seeds for the next generation changes due to mortality and aging. New seeds in generation n, will have one year in the next generation, that is, generation $n+1$. Hence,

$$S_{n+1}^1 = \sigma S_n^0, \tag{13}$$

$$S_{n+1}^2 = \sigma \overline{S}_n^1. \tag{14}$$

4. Simplification of equations

From Equations (12) and (13) and from Equations (10) and (14), we obtain

$$S_{n+1}^1 = \sigma(\gamma p_n) \quad \text{and} \quad S_{n+1}^2 = \sigma(1-\alpha)S_n^1, \tag{15}$$

respectively.

Rewriting Equation (9) for generation $n+1$ and combining with Equations (15), we have

$$p_{n+1} = \alpha S_{n+1}^1 + \beta S_{n+1}^2. \tag{16}$$

From Equations (15) and (16), we obtain a system of two equations that relates plants and one-year seeds,

$$p_{n+1} = \alpha\sigma\gamma p_n + \beta\sigma(1-\alpha)S_n^1, \tag{17}$$

$$S_{n+1}^1 = \sigma\gamma p_n. \tag{18}$$

Rewriting Equation (18) by $S_n^1 = \sigma\gamma p_{n-1}$ and replacing it in Equation (17), we have

$$p_{n+1} = \alpha\sigma\gamma p_n + \beta\sigma^2(1-\alpha)\gamma p_{n-1}. \tag{19}$$

5. The mathematical model

For Generation n, we have

$$p_n = \alpha\sigma\gamma p_{n-1} + \beta\sigma^2(1-\alpha)\gamma p_{n-2}, \tag{20}$$

where the parameters can be interpreted as

$$\begin{aligned}
\gamma p_{n-2} &= \text{number of seeds produced two years ago;}\\
\sigma\gamma p_{n-2} &= \text{number of seeds that survived the first winter;}\\
(1-\alpha)\sigma\gamma p_{n-2} &= \text{number of seeds that did not germinate in the last year;}\\
\sigma(1-\alpha)\sigma\gamma p_{n-2} &= \text{number of seeds that did not germinate last winter;}\\
\beta\sigma(1-\alpha)\sigma\gamma p_{n-2} &= \text{number of two-year-old seeds that germinated.}\\
\gamma p_{n-1} &= \text{number of seeds produced one year ago.}\\
\sigma\gamma p_{n-1} &= \text{number of seeds that survived the winter.}\\
\alpha\sigma\gamma p_{n-1} &= \text{number of one-year-old seeds that germinated.}
\end{aligned}$$

Equation (20) can be rewritten as follows,

$$p_{n+2} = sp_{n+1} + qp_n, \tag{21}$$

where $s = \alpha\sigma\gamma$ is the number of two-year-old seeds that germinated and $q = \beta\sigma^2(1-\alpha)\gamma$ is the number of one-year-old seeds that germinated.

In the next subsection we discuss theoretical results and some simulations of the fuzzy case of Equation (21), that is the case where the initial conditions, are given by fuzzy numbers and the sum, is given by an interactive sum via t-norm.

4.1 Analysis of the fuzzy sequence in the plant growth dynamic

First we will prove that the fuzzy sequence generalizes the classical sequence; in the following sense: we show that the core and support of the fuzzy number evolve in the same manner as the corresponding classical sequence.

Theorem 4.1. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a fuzzy sequence defined by*

$$P_{n+2} = sP_{n+1} +_{J_t} qP_n, \tag{22}$$

where $s, q > 0$, the scalar multiplications are defined via Zadeh's extension and $+_J$ is the interactive fuzzy sum via sup-J extension based on a continuous t-norm t. If each $P_n \in \mathbb{R}_F$ is normal, convex, and has a unique modal value $m_n \in \mathbb{R}$ satisfying $\mu_{P_n}(m_n) = 1$, then the sequence of modal values $\{m_n\}$ satisfies, the classical linear recurrence

$$m_{n+2} = sm_{n+1} + qm_n. \tag{23}$$

Proof. Since each P_n is normal and has a unique modal value, there exists m_n such that $\mu_{P_n}(m_n) = 1$. Since the scalar multiplication is calculated in terms of Zadeh's extension; the membership function of λP_n is given by

$$\mu_{\lambda P_n}(z) = \mu_{P_n}\left(\frac{z}{\lambda}\right),$$

where $\lambda > 0$. Thus, we have

$$\mu_{sP_{n+1}}(sm_{n+1}) = \mu_{P_{n+1}}(m_{n+1}) = 1,$$

and

$$\mu_{qP_n}(qm_n) = \mu_{P_n}(m_n) = 1.$$

Now consider the sup-J extension for the sum given by

$$\mu_{P_{n+2}}(z) = \sup_{x+y=z} t\big(\mu_{sP_{n+1}}(x),\, \mu_{qP_n}(y)\big),$$

where t is a continuous t-norm.

Now, let z be given by $z = sm_{n+1} + qm_n$. Since t is a t-norm, it follows that $t(1,1) = 1$ and we have

$$\mu_{P_{n+2}}(z) = \sup_{sm_{n+1}+qm_n=z} t\big(\mu_{sP_{n+1}}(sm_{n+1}),\, \mu_{qP_n}(qm_n)\big) \geq t(1,1) = 1,$$

which implies that $\mu_{P_{n+2}}(z) = 1$, since membership functions take values in $[0,1]$.

It remains to show that no other point $z \neq sm_{n+1} + qm_n$ yields a membership value equal to 1. Since each P_n has a unique modal value and scalar multiplication preserves uniqueness, the only points where $\mu_{sP_{n+1}}(x) = 1$ and $\mu_{qP_n}(y) = 1$ are $x = sm_{n+1}$ and $y = qm_n$, respectively. Because a t-norm satisfies $t(a,b) = 1$ if and only if $a = b = 1$, the supremum is equal to 1 only when $x = sm_{n+1}$ and $y = qm_n$. Hence the unique modal value of P_{n+2} is $m_{n+2} = sm_{n+1} + qm_n$, satisfying the classical deterministic recurrence. □

Theorem 4.2. *Let $P_n \in \mathbb{R}_F$ be normal, convex fuzzy numbers, with compact support given by* $\mathrm{supp}(P_n) = (\underline{p}_n(0), \overline{p}_n(0))$, *for all $n \in \mathbb{N}$. If the interactive sum $+_J$ is defined via $J = J_t$, where t is a continuous t-norm satisfying, $t(u,v) > 0$, whenever $u > 0$ and $v > 0$, then*

$$\underline{p}_{n+2}(0) = s\underline{p}_{n+1}(0) + q\underline{p}_n(0) \quad \text{and} \quad \overline{p}_{n+2}(0) = s\overline{p}_{n+1}(0) + q\overline{p}_n(0),$$

where $s, q > 0$.

Proof. To show that $\underline{p}_{n+2}(0) = s\underline{p}_{n+1}(0) + q\underline{p}_n(0)$ and $\overline{p}_{n+2}(0) = s\overline{p}_{n+1}(0) + q\overline{p}_n(0)$, we must prove that $\mathrm{supp}(A +_J B) = \mathrm{supp}(A) + \mathrm{supp}(B)$.

Let $z \in \mathrm{supp}(A) + \mathrm{supp}(B)$; that is, let $z = x + y$ with $x \in \mathrm{supp}(A)$ and $y \in \mathrm{supp}(B)$. Thus $\mu_A(x) > 0$ and $\mu_B(y) > 0$. By hypothesis on the t-norm, we have $t(\mu_A(x), \mu_B(y)) > 0$, which implies that $\mu_{A+_JB}(z) > 0$. Hence $\mathrm{supp}(A) + \mathrm{supp}(B) \subseteq \mathrm{supp}(A +_J B)$.

Conversely, if $z \in \mathrm{supp}(A +_J B)$, then there exists $x + y = z$ such that $t(\mu_A(x), \mu_B(y)) > 0$. By the hypothesis, this implies $\mu_A(x) > 0$ and $\mu_B(y) > 0$, so $x \in \mathrm{supp}(A)$ and $y \in \mathrm{supp}(B)$. Thus $\mathrm{supp}(A +_J B) \subseteq \mathrm{supp}(A) + \mathrm{supp}(B)$.

Therefore,

$$\mathrm{supp}(A +_J B) = \mathrm{supp}(A) + \mathrm{supp}(B).$$

Since scalar multiplication by a positive constant preserves supports $\mathrm{supp}(sP_n) = s\,\mathrm{supp}(P_n)$ and $\mathrm{supp}(qP_n) = q\,\mathrm{supp}(P_n)$, we obtain

$$\mathrm{supp}(P_{n+2}) = s\,\mathrm{supp}(P_{n+1}) + q\,\mathrm{supp}(P_n),$$

and the endpoints satisfy

$$\underline{p}_{n+2}(0) = s\underline{p}_{n+1}(0) + q\underline{p}_n(0) \quad \text{and} \quad \overline{p}_{n+2}(0) = s\overline{p}_{n+1}(0) + q\overline{p}_n(0).$$

□

Since the product and Hamacher t-norms satisfy the hypotheses of Theorems 4.1 and 4.2, the results hold for the fuzzy sequences computed by these t-norms.

As a consequence of Theorems 4.1 and 4.2, the stability of the deterministic core of the fuzzy sequence $\{P_n\}_{n\in\mathbb{N}}$ follows from the properties of the associated characteristic polynomial, as stated in the corollary below.

Corollary 4.3. *The fuzzy recurrence*

$$P_{n+2} = sP_{n+1} +_J qP_n,$$

is asymptotically stable if and only if the roots of $\lambda^2 - s\lambda - q$ lie inside the unit disk. Moreover, under $s > 0$ and $q > 0$, this is equivalent to $0 < q < 1$ and $s + q < 1$.

Proof. From Theorems 4.1 and 4.2, the modal sequence and the support diameter satisfy the deterministic recurrence, $p_{n+2} = sp_{n+1} + qp_n$. By the Jury stability criterion (see [6]), all solutions converge to zero if and only if

$$|q| < 1, \quad 1 - s - q > 0, \quad 1 + s - q > 0.$$

If $s, q > 0$, then we obtain that $0 < q < 1$ and $s + q < 1$. Since every α-cut is contained in the support, the convergence is uniform in α. □

Since the modal values satisfy the deterministic recurrence, the spectral stability of the crisp core is completely characterized by the region $s + q < 1$.

Now we will investigate the evolution of the uncertainty of these fuzzy sequences.

Proposition 4.4. *Let $\{P_n\}$ be the fuzzy sequence defined by*

$$P_{n+2} = sP_{n+1} +_J qP_n,$$

with $s, q > 0$, scalar multiplication defined via Zadeh's extension, and $+_J$ defined through a continuous t-norm. Let $a_n = \inf(P_n)$, $b_n = \sup(P_n)$, and the diameter $d_n = b_n - a_n$.

If each P_n is a normal, convex fuzzy number with compact support, then the sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_{n+2} = sa_{n+1} + qa_n$ and $b_{n+2} = sb_{n+1} + qb_n$, respectively.

Moreover, it follows that $d_{n+2} = sd_{n+1} + qd_n$ and

$$\limsup_{n\to\infty} \frac{d_{n+1}}{d_n} = \rho,$$

where $\rho = \max\{|\lambda_1|, |\lambda_2|\}$ denotes the spectral radius of the characteristic polynomial $\lambda^2 - s\lambda - q = 0$ with roots λ_1 and λ_2.

Proof. Since scalar multiplication by positive constants preserves order and the sup-J sum based on a t-norm is increasing in each argument, the recurrence is monotone with respect to the endpoints. Hence the minimal value of P_{n+2} is obtained by combining the minimal values of P_{n+1} and P_n, yielding $a_{n+2} = sa_{n+1} + qa_n$. Similarly, the maximal value satisfies $b_{n+2} = sb_{n+1} + qb_n$. Consequently, subtracting the two relations we have, $d_{n+2} = sd_{n+1} + qd_n$.

Let λ_1, λ_2 be the roots of $\lambda^2 - s\lambda - q$. Thus, the general solution of the recurrence is, $d_n = A\lambda_1^n + B\lambda_2^n$. If $|\lambda_1| > |\lambda_2|$, then

$$d_n = \lambda_1^n \left(A + B\left(\frac{\lambda_2}{\lambda_1}\right)^n \right),$$

and since $\left|\frac{\lambda_2}{\lambda_1}\right| < 1$, we obtain

$$\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \lambda_1.$$

The case where $|\lambda_1| > |\lambda_2|$ is analogous. Therefore,

$$\limsup_{n\to\infty} \frac{d_{n+1}}{d_n} = \rho,$$

where $\rho = \max\{|\lambda_1|, |\lambda_2|\}$ is the spectral radius. □

The support of P_n contains all α-cuts; consequently the diameter of any intermediate α-cut cannot exceed the diameter of the support. Since the support endpoints satisfy the same deterministic linear recurrence as the modal sequence, their asymptotic behavior is governed; by the spectral radius ρ. Therefore, the support provides a uniform bound for the growth of all α-cuts. This result is states in the corollary below.

Corollary 4.5. *Let ρ be the spectral radius of $\lambda^2 - s\lambda - q = 0$. Thus, the support diameter $d_n = b_n(0) - a_n(0)$ satisfies*

$$d_{n+2} = sd_{n+1} + qd_n,$$

and therefore

$$\limsup_{n\to\infty} \frac{d_{n+1}}{d_n} = \rho.$$

Moreover, for every $\alpha \in [0,1]$, $0 \le b_n(\alpha) - a_n(\alpha) \le d_n$, so that

$$\limsup_{n\to\infty} \frac{b_{n+1}(\alpha) - a_{n+1}(\alpha)}{b_n(\alpha) - a_n(\alpha)} \le \rho.$$

In particular, if $\rho < 1$, all α-cuts converge to zero uniformly.

We now focus on the analysis of the specific fuzzy sequence $\{P_n\}^{(J_{t_H},r)}$ generated by the Hamacher t-norm. To analyze the propagation of uncertainty of this sequence, let us first prove that the sup-J sum based on a t-norm is monotone with respect to the inclusion of fuzzy numbers.

Proposition 4.6. *Let $A_1, A_2, B \in \mathbb{R}_F$ and suppose that $A_1 \subseteq A_2$. Let $+_J$ denote the fuzzy addition defined through the sup-J extension principle based on a t-norm t. Then*

$$A_1 +_J B \subseteq A_2 +_J B.$$

An analogous property holds in the second argument.

Proof. Since t is increasing in each argument, for every $x, y \in \mathbb{R}$ such that $x + y = z$, we have

$$\mu_{A_1}(x) \le \mu_{A_2}(x) \quad \Longrightarrow \quad t\big(\mu_{A_1}(x), \mu_B(y)\big) \le t\big(\mu_{A_2}(x), \mu_B(y)\big),\ \forall B \in \mathbb{R}_{\mathcal{F}_C}.$$

Taking the supremum over $z = x + y$, it follows that

$$\mu_{A_1 +_J B}(z) = \sup_{x+y=z} t\big(\mu_{A_1}(x), \mu_B(y)\big) \le \sup_{x+y=z} t\big(\mu_{A_2}(x), \mu_B(y)\big) \le \mu_{A_2 +_J B}(z), \quad \forall z \in \mathbb{R}.$$

Hence, $A_1 +_J B \subseteq A_2 +_J B$. □

Note that for fixed $x, y \in (0,1)$, we have that $t_H^{(r)}(x,y)$ is decreasing with respect to r, and since the inclusion of fuzzy sequences follows from the monotonicity of the sup-J extension, another interesting property of this fuzzy sequence can be obtained, that is, $P_n^{(r_1)} \supseteq P_n^{(r_2)}$, for $r_1 < r_2$ and for all $n \in \mathbb{N}$. This result is enunciated in the below proposition.

Proposition 4.7. *Consider the fuzzy recurrence*

$$P_{n+2}^{(r)} = sP_{n+1}^{(r)} +_{J_{H_r}} qP_n^{(r)},$$

where $s, q > 0$ and $+_{J_{H_r}}$ is the interactive sum via the Hamacher t-norm with parameter r. If $r_1 < r_2$, then

$$P_n^{(r_1)} \supseteq P_n^{(r_2)} \quad \textit{for all } n.$$

Recall that according to Theorem 4.2, the endpoints of the fuzzy sequence support via Hamacher t-norm are given by, $a_{n+2}^{(r)}(0) = sa_{n+1}^{(r)}(0) + qa_n^{(r)}(0)$, and $b_{n+2}^{(r)}(0) = sb_{n+1}^{(r)}(0) + qb_n^{(r)}(0)$. This implies that the diameter of the support is given by $d_{n+2}^{(r)} = sd_{n+1}^{(r)} + qd_n^{(r)}$. Also, note that the characteristic polynomial of this linear recurrence is $\lambda^2 - s\lambda - q$, which does not depend on r. Hence

$$\limsup_{n\to\infty} \frac{d_{n+1}^{(r)}}{d_n^{(r)}} = \max(|\lambda_1|, |\lambda_2|) = \rho,$$

independently of r.

The next theorem enunciates this result.

Theorem 4.8. *Let $\{P_n\}^{(J_{t_H},r)}$ be the fuzzy sequence obtained via Hamacher t-norm with the diameter of the support given by $d_n^{(r)} = b_n^{(r)}(0) - a_n^{(r)}(0)$. Thus, it follows that*

$$d_{n+2}^{(r)} = sd_{n+1}^{(r)} + qd_n^{(r)},$$

and

$$\limsup_{n\to\infty} \frac{d_{n+1}^{(r)}}{d_n^{(r)}} = \rho,$$

where ρ is the spectral radius of $\lambda^2 - s\lambda - q$. In particular, the exponential growth rate of the support diameter is independent of r.

So, we conclude that the parameter r affects only the initial distribution of uncertainty, but not the linear recurrence satisfied by the support endpoints. Hence, the characteristic polynomial and its spectral radius are independent of r, which implies that the exponential growth rate of the support diameter is the same for all $r > 0$.

4.2 Numerical simulations of the fuzzy sequence in the plant growth dynamic

In this application, we will consider that the initial condition will be given by fuzzy numbers because the initial population of plants represents an estimated value. The next subsections provide two approaches, the first one is based on the minimum t-norm and the second is based on the Hamacher t-norm.

Figures 6, 7 and 8 depict the simulation of the plant dynamics for the minimum t-norm, from the fuzzy initial conditions $P_1 = (5; 10; 15)$ and $P_2 = (32; 37; 42)$.

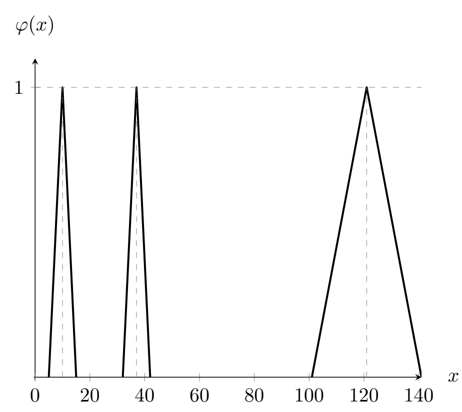

Figure 6: Plant reproduction fuzzy sequence considering $P_1 = (5; 10; 15)$ and $P_2 = (32; 37; 42)$ and $+_J = +$, for $n = 3$.

It is worth noting that, in the fuzzy sequence generated by the minimum t-norm, although the behavior is qualitatively similar to the classical case, in the sense that it contains the classical values of the sequence, the resulting terms exhibit increasingly larger diameters, which is a characteristic behavior of uncertainty propagation.

In Figure 9, the fuzzy sequence for Hamacher t-norm is presented. In contrast, for the Hamacher t-norm, the diameters of the solutions are narrower than those obtained with the minimum t-norm, providing more precise results and corroborating the theoretical results provided in Theorem 3.3.

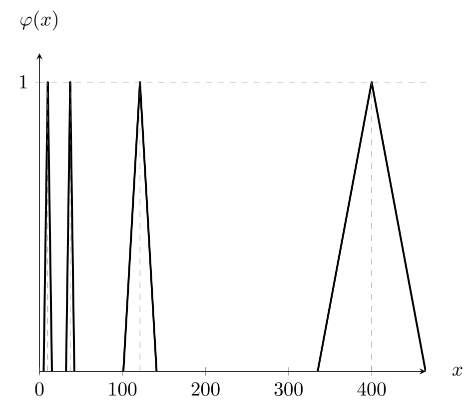

Figure 7: Plant reproduction fuzzy sequence considering $P_1 = (5; 10; 15)$ and $P_2 = (32; 37; 42)$ and $+_J = +$, for $n = 4$.

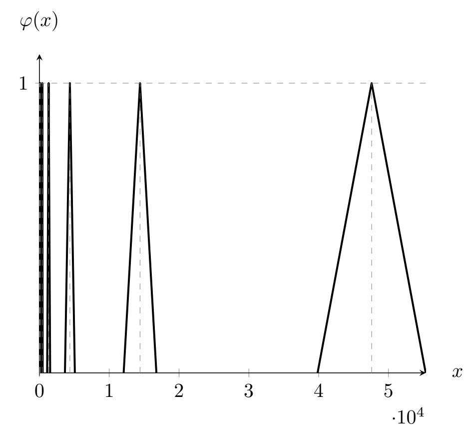

Figure 8: Plant reproduction fuzzy sequence considering $P_1 = (5; 10; 15)$ and $P_2 = (32; 37; 42)$ and $+_J = +$, for $n = 8$.

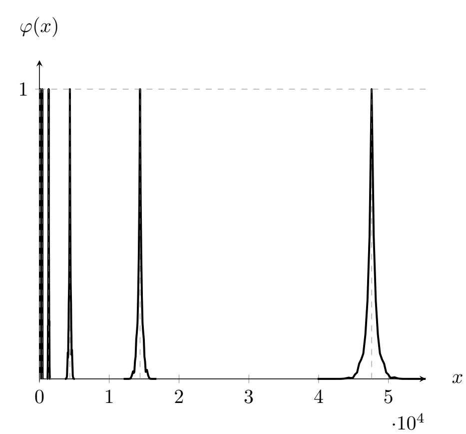

Figure 9: Plant reproduction fuzzy sequence considering $P_1 = (5; 10; 15)$ and $P_2 = (32; 37; 42)$ and $+_J = +_{t_H}$, for $n = 8$.

Since the Hamacher t-norm is a generalization of the product t-norm, Figure 10 illustrates the fuzzy sequence obtained by the product t-norm. A comparison of the three cases shows that the Hamacher t-norm produces intermediate results with respect to those obtained with the product and the minimum t-norm.

Algorithms 3, 4 and 5, provide pseudocode to reproduce the Plant growth sequences for the fuzzy standard sum, Hamacher and product t-norms, respectively.

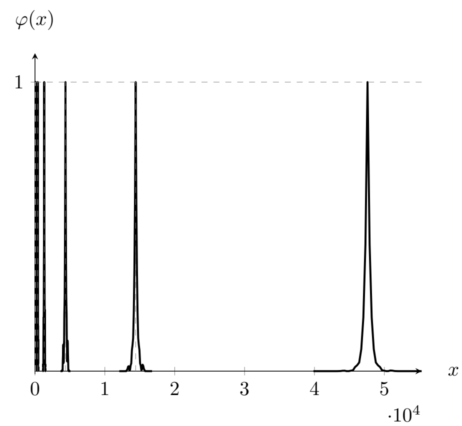

Figure 10: Plant reproduction fuzzy sequence considering $P_1 = (5; 10; 15)$,$P_2 = (32; 37; 42)$ and $+_J = +_{t_p}$, for $n = 8$.

Algorithm 3 Plant Growth Fuzzy Sequence with Standard Fuzzy Sum $(sP_{n+1} + qP_n)$

Input: Initial fuzzy numbers $P_0, P_1 \in \mathbb{R}_{\mathcal{F}}$, coefficients $s, q > 0$, number of terms N, grid of values Z
Output: Sequence: $\{P_n\}_{n=0}^{N-1}$ of fuzzy numbers

```
Initialize sequence: P[0] := P_0, P[1] := P_1
for n = 2 to N − 1 do
    foreach z ∈ Z do
        Initialize maxVal ← 0  foreach x ∈ Z do
            y ← z − x  if y ∈ Z then
                val ← min(s μ_{P[n−1]}(x), q μ_{P[n−2]}(y))  maxVal ← max(maxVal, val)
        Set μ_{P[n]}(z) ← maxVal
return {P_n}_{n=0}^{N−1}
```

Algorithm 4 Plant Growth Fuzzy Sequence with Interactive Fuzzy Sum $(sP_{n+1} +_{J_{t_H}} P_n)$

Input: Initial fuzzy numbers $P_0, P_1 \in \mathbb{R}_{\mathcal{F}}$, coefficients $s, q > 0$, number of terms N, grid of values Z, parameter $r \in (0, 1]$
Output: Sequence: $\{P_n\}_{n=0}^{N-1}$ of fuzzy numbers

```
Initialize sequence: P[0] := P_0, P[1] := P_1
for n = 2 to N − 1 do
    foreach z ∈ Z do
        Initialize maxVal ← 0  foreach x ∈ Z do
            y ← z − x  if y ∈ Z then
                val ← (s μ_{P[n−1]}(x) q μ_{P[n−2]}(y)) / (r + (1 − r)(s μ_{P[n−1]}(x) + q μ_{P[n−2]}(y) − s μ_{P[n−1]}(x) q μ_{P[n−2]}(y)))  maxVal ← max(maxVal, val)
        Set μ_{P[n]}(z) ← maxVal
return {P_n}_{n=0}^{N−1}
```

$$\text{val} \leftarrow \frac{s\mu_{P[n-1]}(x)\, q\mu_{P[n-2]}(y)}{r + (1-r)\left(s\mu_{P[n-1]}(x) + q\mu_{P[n-2]}(y) - s\mu_{P[n-1]}(x)\, q\mu_{P[n-2]}(y)\right)}$$

Algorithm 5 Plant Growth Fuzzy Sequence with Interactive Fuzzy Sum $(sP_{n+1} +_{J_{t_p}} qP_n)$

Input: Initial fuzzy numbers $P_0, P_1 \in \mathbb{R}_{\mathcal{F}}$, coefficients $s, q > 0$, number of terms N, grid of values Z, parameter $r \in (0, 1]$

Output: Sequence: $\{P_n\}_{n=0}^{N-1}$ of fuzzy numbers

Initialize sequence: $P[0] := P_0$, $P[1] := P_1$

for $n = 2$ **to** $N - 1$ **do**

  foreach $z \in Z$ **do**

    Initialize maxVal $\leftarrow 0$ **foreach** $x \in Z$ **do**

      $y \leftarrow z - x$ **if** $y \in Z$ **then**

        val $\leftarrow s\mu_{P[n-1]}(x) \cdot q\mu_{P[n-2]}(y)$ maxVal $\leftarrow \max(\text{maxVal}, \text{val})$

    Set $\mu_{P[n]}(z) \leftarrow$ maxVal

return $\{P_n\}_{n=0}^{N-1}$

Algorithms 3-5 follow analogously to Algorithms 1 and 2 and consequently they exhibit the same asymptotic complexity of $\mathcal{O}(NM^2)$.

The choice of a t-norm has a direct impact on both the theoretical properties of the model and its practical applications. In this context, the Hamacher t-norm offers several advantages over the extreme cases of the minimum and product t-norms, since it can be regarded as an intermediate operator that interpolates between them. While the minimum t-norm is highly conservative, amplifying the propagation of uncertainty, and the product t-norm is smoother, often contracting uncertainties, the Hamacher t-norm provides a balanced intermediate behavior.

In biological phenomena, such as plant growth or insect population dynamics, uncertainty does not propagate as drastically as in the minimum case, nor does it contract as much as in the product case. The Hamacher t-norm captures this intermediate behavior more faithfully. The diameters of the fuzzy solutions obtained with Hamacher are narrower than those with the minimum t-norm, leading to more precise estimates without neglecting uncertainty. Previous studies [14, 15, 16] have shown that the Hamacher t-norm adapts well to models in Biology and Ecology, precisely because it balances the "expansive" behavior of the minimum with the "contractive" behavior of the product. Since fuzzy interactivity is related to the dependence between variables, the Hamacher t-norm provides a gradual representation of dependence, less restrictive than the minimum and less relaxed than the product, which is crucial for partially interactive phenomena.

Therefore, the Hamacher t-norm combines solid theoretical foundations with practical modeling advantages, making it particularly suitable for applications where uncertainty propagation must be controlled without being either overestimated or underestimated.

5 Final remarks

This paper presented an investigation of fuzzy linear difference equations constructed under two distinct approaches the interactive arithmetic, associated with a joint possibility distribution J and the sup-J extension principle, and the non-interactive arithmetic, associated with Zadeh's extension principle, which coincides with the standard approach to fuzzy arithmetic.

The comparison between these two perspectives revealed an important theoretical result: fuzzy sequences arising from interactive difference equations are always contained within those obtained from non-interactive equations (see Theorem 3.3). This implies that interactive sequences propagate less uncertainty, leading to fuzzy values with narrower diameters. From a modeling perspective, this property is highly relevant, as it provides a more precise yet consistent representation of the inherent uncertainties in discrete dynamical systems.

To illustrate the theoretical developments, two applications were analyzed: the Fibonacci sequence under fuzzy initial conditions and a recurrence model for insect population dynamics. Also, an application in a growth dynamic problem was presented from the perspective of minimum, Hamacher and product t-norms. A comparison among them was presented as well.

Furthermore, the computational analysis of the proposed algorithms, shows that all implementations based on the sup-J extension principle exhibits quadratic complexity with respect to the discretization size, given by $\mathcal{O}(NM^2)$, where N is the number of iterations and M is the number of grid points. Consequently, the interactive fuzzy models considered in this work provides an alternative modeling without altering the fundamental computational order, that is, the proposed model shares the same computational order as the usual arithmetic, differing only in the cost of the underlying aggregation operations.

The results obtained in this work reinforce the importance of incorporating interactive fuzzy arithmetic into the study of difference equations. Moreover, while the deterministic dynamics remain completely characterized by the underlying recurrence relation, the interaction mechanism governs how uncertainty propagates throughout the system.

Within this perspective, the Hamacher t-norm plays a relevant role, since its parameterized structure makes it possible to investigate a continuum of interaction regimes, allowing the sensitivity of uncertainty propagation to be analyzed without altering the deterministic dynamics. Consequently, the proposed approach provides not only an alternative fuzzy arithmetic, but also a systematic way of studying the influence of interaction assumptions on the behavior of fuzzy dynamical systems.

Finally, although the Hamacher family was adopted in this work because of its flexibility and successful applications in biomathematical problems, the proposed framework is not restricted to this particular family. Future research may investigate other parameterized interaction models, extend the analysis to nonlinear fuzzy difference equations, and explore additional applications in ecology, epidemiology, and other areas involving uncertain discrete dynamics.

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