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Optimization problems constrained by generalized fuzzy relational inequalities

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Abstract

The study of non-linear programming under fuzzy relational inequalities (FRI) defined by max-min and max-product t-norms have been studied in a wide variety of fields, both theoretically and practically. The paper presents a novel system of fuzzy relational inequalities (max-FRIs) and investigates a new general class of optimization models. This optimization model has a non-linear objective function defined by a combination of an arbitrary continuous s-norm and an arbitrary continuous t-norm. Additionally, the problem constraints are based on a combination of two arbitrary continuous t-norms formed by a system of max-FRIs. We begin by examining a few important special cases of the model and their applications. A complete characterization of the feasible region is then presented, as well as two necessary and sufficient conditions for determining the problem's feasibility. Subsequently, a general method is presented for determining the exact optimal solution. Using five rules, the algorithm is accelerated to find the best solution with less computational effort. To address some practical special cases of the main problem, a polynomial-time method is presented. To aid comprehension, we provide a numerical example, where the objective function is given by a combination of the Łukasiewicz t-norm and Dubois-Prade t-norm, and the constraints are defined by a combination of the product and Yager t-norms.

Keywords: Fuzzy relational inequalities, continuous t-norms, continuous s-norms, latticized linear programming, non-linear optimization.

1 Introduction

The first study of fuzzy relational equations (FREs) with max-min composition was conducted by Sanchez [37], and this approach has also been applied in medical diagnosis. In recent years, it has become evident that many issues related to body knowledge can be viewed as FRE problems [32]. The underlying result for FREs with max-product compositions was found by Pedrycz [33] and further studied in [1, 27]. Also in [100], Pedrycz generalized classical fuzzy relational equations by introducing s-t compositions based on s-norms (t-conorms) and t-norms instead of the traditional max–min operators. The paper established theoretical foundations for analyzing the existence and properties of solutions, providing a more flexible framework for fuzzy modeling and approximate reasoning. Following that, a large number of studies have been carried out on different FREs defined by various types of t-norm operators [9, 10, 14, 24, 29, 34, 35, 36, 39, 40, 42, 43, 46, 52]. Additionally, some researchers have introduced novel concepts and have improved some of the existing theoretical aspects and applications of fuzzy relational inequalities (FRI) [8, 11, 12, 13, 14, 15, 16, 19, 48].

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There are generally three significant difficulties associated with optimization problems subject to FRE or FRI constraints. First, to completely characterize FREs and FRIs, all minimal solutions must be identified, which is NP-hard [24, 30]. Second, a feasible region formed as a FRE or FRI is often a non-convex set determined by one maximum solution and a finite number of minimal solutions [8, 9, 10, 13, 34, 35, 36, 39, 40, 42, 52]. Third, FREs and FRIs as feasible regions lead to optimization problems with highly nonlinear constraints. Due to the above-mentioned difficulties, the optimization problem subject to FRE and FRI is one of the most interesting and ongoing research topics among similar problems [2, 5, 6, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 20, 25, 26, 38, 44, 45, 48].

Linear optimization problems with constraints defined by FRI systems have been widely addressed in the literature [8, 12, 13, 14, 15, 16, 19, 31, 47, 48, 49, 50]. Guo et al. [16] studied the linear programming problem with the max-min FRI constraint. Li and Yang [19] introduced the so-called addition-min FRI to characterize a peer-to-peer file sharing system. Based on the concept of pseudo-minimal index, Yang [47] developed a pseudominimal-index algorithm to minimize a linear objective function with addition-min FRI constraints defined as $a_{i1} \wedge x_1 + a_{i2} \wedge x_2 + \cdots + a_{in} \wedge x_n \geq b_i$ for $i = 1, \ldots, m$ and $a \wedge b = \min\{a, b\}$ [47]. To improve the results presented in [47], Yang et al. [49] proposed the min-max programming subject to addition-min fuzzy relational inequalities. They also studied the multi-level linear programming problem with an addition-min FRI constraint [50]. Drewniak and Matusiewicz studied max-* fuzzy relation equations and inequalities with the increasing operation * continuous on the second argument [31]. Ghodousian and Khorram [13] studied the linear optimization with constraints formed by $X\left(\boldsymbol{A}, \boldsymbol{D}, \boldsymbol{b}^1, \boldsymbol{b}^2\right) = \left\{\boldsymbol{x} \in [0,1]^n : \boldsymbol{A}\varphi\boldsymbol{x} \leq \boldsymbol{b}^1, \boldsymbol{D}\varphi\boldsymbol{x} \geq \boldsymbol{b}^2\right\}$ where φ represents an operator with convex solutions (e.g., non-decreasing or non-increasing operator). They showed that, the feasible region can be expressed as the union of a finite number of convex sets.

It has recently been introduced that there are many interesting generalizations of linear programming applied to fuzzy relation systems that have been developed based on composite operations used in FRE or FRI, fuzzy relations used in constraint definition, and some developments on the objective function of problems [3, 4, 7, 9, 10, 14, 22, 25, 43]. For example, Wu et al. represented an efficient method to optimize a linear fractional programming problem under FRE with a max-Archimedian t-norm composition [43]. Dempe and Ruziyeva generalized the fuzzy linear optimization problem by considering the fuzzy coefficients [3]. In addition, Dubey et al. studied linear programming problems involving interval uncertainty modeled using the intuitionistic fuzzy set [4].

The linear optimization of bipolar FRE was also the focus of a study carried out by some researchers, where FRE was defined with max-min composition [7] and max-Łukasiewicz composition [9, 22, 25]. For example, in [22], the authors introduced a linear optimization problem subjected to a system of bipolar FRE defined as $X\left(\boldsymbol{A}^+, \boldsymbol{A}^-, \boldsymbol{b}\right) = \left\{\boldsymbol{x} \in [0,1]^m : \boldsymbol{x} \circ \boldsymbol{A}^+ \vee \tilde{\boldsymbol{x}} \circ \boldsymbol{A}^- = \boldsymbol{b}\right\}$ where $\tilde{\boldsymbol{x}}_i = 1 - x_i$ for each component of $\tilde{\boldsymbol{x}} = (\tilde{x}_i)_{1\times m}$ and the notations "$\vee$" and "$\circ$" denote max operation and the max-Łukasiewicz composition, respectively. They translated the original problem into a 0-1 integer linear problem, which was then solved using well-developed techniques. In a separate study, the foregoing bipolar linear optimization problem was solved by an analytical method based on the resolution and some structural properties of the feasible region (using a necessary condition for characterizing an optimal solution and a simplification process for reducing the problem) [25]. In addition, in [14], the authors studied bipolar FREs defined by max-strict t-norms. Based on the constraints of the problem, they categorize them into four groups, each requiring a different approach to find the corresponding feasible solution set. Further, the article [82] considered a broad range of Archimedean t-norms. The basis of their analysis to resolve the problem was the use of the additive generator f_φ and pseudoinverse $f_\varphi^{(-1)}$ of each t-norm, and by utilizing the branch-and-bound method, they tried to prune non-optimal branches. Optimization problems with general nonlinear objective functions and FRE or FRI constraints were studied in the [9, 10, 11, 12, 28, 81]. In general, some heuristic algorithms are applied to deal with this kind of problem. However, some fuzzy relation nonlinear optimization problems, such as the geometric programming problem [51], could be solved by a specific method. Yang et al. [51] studied the single-variable term semi-latticized geometric programming subject to max-product fuzzy relation equations. The proposed problem was devised from a peer-to-peer network system, and the goal was to minimize the biggest dissatisfaction degrees of terminals in such a system.

There was an introduction to the latticized optimization problem in [41] where the conservative path method was proposed to find out all the minimal solutions of the max-min FRI. Then, the optimal solutions were obtained from the minimal solutions by pairwise comparison. The latticized linear programming problem subjected to max-min FRI was also investigated in the works [21, 23]. Li and Fang [21] obtained an optimal solution to the latticized linear programming problem. Furthermore, they studied some variants of the problem. In [23], based on the concept of semi-tensor product, a matrix approach was applied to handle the latticized linear programming problem subjected to max-min FRI. Also, Yang et al. [48] introduced the latticized programming problem defined by minimizing objective function $f(\boldsymbol{x}) = x_1 \vee x_2 \vee \ldots \vee x_n$ subject to feasible region $X(\boldsymbol{A}, \boldsymbol{b}) = \{\boldsymbol{x} \in [0,1]^n : \boldsymbol{A} \circ \boldsymbol{x} \geq \boldsymbol{b}\}$ where "$\circ$" denotes the fuzzy max-product composition. Also, in [12], the authors introduced a non-linear generalization of the latticized linear programming problem and investigated some related the solution algorithms.

In this paper, we investigate the following non-linear optimization model:

$$\begin{aligned} \min_{\boldsymbol{x}\in[0,1]^n} \quad & \tau\big(\varphi(d_1,x_1),\ldots,\varphi(d_n,x_n)\big) \\ \text{s.t.} \quad & \boldsymbol{A}\,\psi(\boldsymbol{C}\,\omega\boldsymbol{x}) \geq \boldsymbol{b}. \end{aligned} \tag{1}$$

where $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{C} = (c_{ij})_{m\times n}$ are two fuzzy matrices such that $0 \leq a_{ij} \leq 1$ and $0 \leq c_{ij} \leq 1$ for any $i \in I = \{1,2,\ldots,m\}$ and $j \in J = \{1,2,\ldots,n\}$; $\boldsymbol{x} = (x_1,\ldots,x_n) \in [0,1]^n$, $\boldsymbol{d} = (d_1,\ldots,d_n) \in [0,1]^n$ and $\boldsymbol{b} = [b_1,b_2,\ldots,b_m] \in [0,1]^m$. Also, $\tau : [0,1]^2 \to [0,1]$ is an arbitrary continuous t-conorm, and φ, ψ and ω are three arbitrary continuous t-norms from $[0,1]^2$ to $[0,1]$. Additionally, $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \geq \boldsymbol{b}$ denotes a system of generalized fuzzy relational inequalities defined as follows:

$$\max_{j=1}^{n}\{\psi\left(a_{ij},\omega\left(c_{ij},x_j\right)\right)\} \geq b_i, \quad i \in I. \tag{2}$$

So, if a_i and c_i $(i \in I)$ denote the i'th rows of matrices $\boldsymbol{A}$ and $\boldsymbol{C}$, respectively, then the i'th constraint $\max_{j=1}^{n}\{\psi(a_{ij}, \omega\left(c_{ij},x_j\right))\} \geq b_i$ can also expressed as $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\boldsymbol{x}\right) \geq b_i$ where $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\boldsymbol{x}\right) = \max_{j=1}^{n}\left\{\psi\left(a_{ij},\omega\left(c_{ij},x_j\right)\right)\right\}$.

It is worth noting that, for the sake of feasibility of the problem formulation, the operator ω may be endowed with alternative structural properties. However, to ensure consistency, of the proposed generalized model with fundamental special cases in the theory of the fuzzy relational inequalities, ω is assumed to be a continuous t-norm. The investigation of settings in which ω falls outside the class of t-norms and is equipped with other meaningful properties remains an interesting direction for future research.

The following is an overview of the remainder of the paper. Section 2 discusses some important special cases of Problem (1) and its applications. In Section 3, the solvability of the problem is studied, and two necessary and sufficient conditions are derived for the feasibility of the problem. The optimality condition of the problem is discussed in Section 4, where an algorithm is presented for solving the problem in a general case. Section 5 presents a polynomial-time algorithm that solves Problem (1) in some important special cases without finding all the minimal solutions to the feasible region. Section 6 introduces five techniques to reduce the size of the problem, and finally, Section 7 presents an illustration of the proposed methods.

2 Special cases and applications

This section briefly discusses several special cases of Problem (1) and their applications. For the first example, suppose that τ is the maximum t-conorm and φ is the minimum t-norm. Therefore, the objective function in (1) reduces to the following form:

$$f(\boldsymbol{x}) = \max\left\{\min\left\{d_1,x_1\right\},\ldots,\min\left\{d_n,x_n\right\}\right\}. \tag{3}$$

Moreover, let $\boldsymbol{C} = \mathbf{1}_{m\times n}$ where $\mathbf{1}_{m\times n}$ is the all-ones matrix (i.e., a matrix where every entry is equal to one). So, we have $c_{ij} = 1$ ($\forall i \in I$ and $\forall j \in J$), and therefore each constraint $\max_{j=1}^{n}\left\{\psi\left(a_{ij},\omega\left(c_{ij},x_j\right)\right)\right\} \geq b_i$ is reduced to $\max_{j=1}^{n}\left\{\psi\left(a_{ij},\omega\left(1,x_j\right)\right)\right\} \geq b_i$, $\forall i \in I$. Furthermore, by the identity law of ω, it follows that $\omega\left(1,x_j\right) = x_j$, $\forall j \in J$. Consequently, the constraints are converted into the following much simpler form:

$$\max_{j=1}^{n}\left\{\psi\left(a_{ij},x_j\right)\right\} \geq b_i. \tag{4}$$

Now, according to (3) and (4) where ψ is defined by the minimum t-norm, we obtain a special case of Problem (1) as the following non-linear model:

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) = \max\nolimits_{j=1}^{n}\left\{\min\{d_i,x_j\}\right\} & \\ & \max\nolimits_{j=1}^{n}\left\{\min\{a_{ij},x_j\}\right\} \geq b_i & i \in I \\ & x \in [0,1]. & j \in J \end{aligned} \tag{5}$$

Problem (5) was studied in [23] and used in designing the best fuzzy controller among all the optimal solutions. Additionally, by considering (3) and setting $\boldsymbol{d} = \mathbf{1}_{1\times n}$, where $\mathbf{1}_{1\times n}$ is the sum vector (i.e., a vector having each component equal to one), it follows that $\min\left\{d_j,x_j\right\} = \min\left\{1,x_j\right\} = x_j$, $\forall j \in J$, and therefore the objective function becomes $f(\boldsymbol{x}) = \max\left\{x_1,\ldots,x_n\right\}$. This fact, along with (4) where ψ is defined by the product t-norm, leads us to the following non-linear model:

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) = \max\nolimits_{j=1}^{n}\left\{x_j\right\} & \\ & \max\nolimits_{j=1}^{n}\left\{a_{ij}x_j\right\} \geq b_i & i \in I \\ & x_j \in [0,1] & j \in J \end{aligned} \tag{6}$$

Problem (6) was studied in [48] and used in the optimization management model of wireless communication emission base stations. For another example, consider a three-tier client-server network (depicted in Figure 1), where a server connects all users or clients in the network via a hub, and the middle-tier receives data with various quality levels from the server and sends them to m clients $P_1, P_2, \ldots, P_m$. The server S transfers the original data to m clients P_1 to P_m

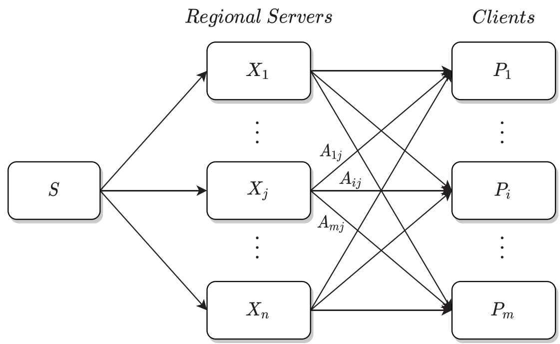

Figure 1: A three-tier client-server network

by the regional servers X_1 to X_n. Moreover, X_j sends out file data with quality level x_j and the client P_i receives file data from X_j via the network connection A_{ij} with the bandwidth a_{ij} (we refer interested readers to [70] where a more detailed description of client-server networking is provided). Then, when P_i downloads the file data from X_j, the download traffic is $\min(a_{ij}, x_j)$. In other words, if the transferred data from X_j to P_i has a quality level x_j less than the bandwidth a_{ij}, then the data is completely received. Otherwise, we will have difficulties with the delivered data. So, by assuming that the client P_i requires at least one data sender with at least quality level b_i, the requirements of all clients can be expressed as $\max j = 1^n \{\min(aij, x_j))\} \geq b_i$, where, $i \in I = \{1, \ldots, m\}$ and $j \in J = \{1, \ldots, n\}$. On the other hand, file data can be sent by regional servers with different quality levels. More precisely, suppose that the regional server X_j sends data to client P_i with the quality level $c_{ij}x_j$. Hence, considering these additional effects, the requirements of the clients are expressed by $\max j = 1^n \{\min(aij, c_{ij}x_j))\} \geq b_i$, where $i \in I = \{1, \ldots, m\}$ and $j \in J = \{1, \ldots, n\}$. Moreover, when many clients send their requests to the same server simultaneously, an overload occurs on the server. This overload may lead to connection failures or slowdowns. Therefore, it is necessary to decrease network congestion by minimizing the maximum quality level as follows:

$$\begin{array}{lll} \min & f(\boldsymbol{x}) = \max_{j=1}^{n} \{x_j\} & \\ & \max_{j=1}^{n} \{\min\{a_{ij}, c_{ij}x_j\}\} \geq b_i & i \in I \\ & x_j \in [0,1]. & j \in J \end{array} \tag{7}$$

Problem (7) is a form of (1) by defining $\tau = \max$, $\varphi = \min$, $\boldsymbol{d} = \boldsymbol{1}$, $\psi = \min$, and ω as the product t-norm.

3 Feasibility, solvability and fundamental results

This section describes the basic properties of the solutions to the system of generalized FRIs introduced in Problem (1). For simplicity, let S denote the feasible region of Problem (1), that is, $S = \{\boldsymbol{x} \in [0,1]^n : \boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \geq \boldsymbol{b}\}$. Also, for each constraint $\boldsymbol{a_i}\psi(\boldsymbol{c_i}\omega\boldsymbol{x}) \geq b_i$ $(i \in I)$, let $S_i = \{\boldsymbol{x} \in [0,1]^n : \boldsymbol{a_i}\psi(\boldsymbol{c_i}\omega\boldsymbol{x}) \geq b_i\}$. So, S_i denotes the feasible solution set of the i'th constraint, and therefore $S = \bigcap_{i \in l} S_i$.

3.1 Partial solution sets and their properties

Definition 3.1. *For each $i \in I$ and each $j \in J$, the partial solution set S_{ij} is defined as*

$$S_{ij} = \{x \in [0,1] : \psi(a_{ij}, \omega(c_{ij}, x)) \geq b_i\}.$$

The following lemma shows that any non-empty S_{ij} is a closed bounded interval.

Lemma 3.2. *Let $i \in I$ and $j \in J$, where ψ and ω are continuous t-norms. If $S_{ij} \neq \varnothing$, then $S_{ij} = \left[\underline{x}_{ij}, 1\right]$ where $\underline{x}_{ij} = \inf S_{ij}$.*

Proof. Since $S_{ij} \subseteq [0,1]$, from the least upper bound property of $\mathbb{R}$, it is clear that $\underline{x}_{ij} = \inf S_{ij}$ exists. Moreover, S_{ij} is a closed subset of $[0,1]$. To prove this, suppose x' is a limit point of S_{ij} and $\{x_k\}_{k\in\mathbb{N}}$ (where $\mathbb{N}$ denotes the set of natural numbers) is any convergent sequence of points x_k contained in S_{ij} with limit point x' (i.e., $x_k \in S_{ij}$ for each $k \in \mathbb{N}$ and $\lim_{k\to\infty} x_k = x'$). So, to prove that S_{ij} is closed, we shall show that $x' \in S_{ij}$. For this purpose, define $h(x) = \psi\left(a_{ij}, \omega\left(c_{ij}, x\right)\right) - b_i$. Since ψ and ω are continuous, $h(x)$ is a continuous function, and also for each $k \in \mathbb{N}$, we have $h\left(x_k\right) \geq 0$ (since $x_k \in S_{ij}$, $\forall k \in \mathbb{N}$). Therefore, $\psi\left(a_{ij}, \omega\left(c_{ij}, x'\right)\right) - b_i = h\left(x'\right) = h\left(\lim_{k\to\infty} x_k\right) = \lim_{k\to\infty} h\left(x_k\right) \geq 0$, that implies $\psi\left(a_{ij}, \omega\left(c_{ij}, x'\right)\right) - b_i \geq 0$ (i.e., $x' \in S_{ij}$). As a result, since S_{ij} is closed, we must have $\underline{x}_{ij} \in S_{ij}$ (or equivalently, $\psi(a_{ij}, \omega(c_{ij}, \underline{x}_{ij})) \geq b_i$). To complete the proof, it is sufficient to show that $x \in S_{ij}$, $\forall x \in \left[\underline{x}_{ij}, 1\right]$. But, given any $x \in \left[\underline{x}_{ij}, 1\right]$, the monotonicity property of t-norms implies $\psi\left(a_{ij}, \omega\left(c_{ij}, \underline{x}_{ij}\right)\right) \leq \psi\left(a_{ij}, \omega\left(c_{ij}, x\right)\right)$. Hence, $\psi\left(a_{ij}, \omega\left(c_{ij}, x\right)\right) \geq b_i$, which completes the proof. $\square$

The following lemma provides a necessary and sufficient condition for the feasibility of S_{ij}.

Lemma 3.3. *Let ψ and ω be continuous t-norms. Then, for each $i \in I$ and $j \in J$, $S_{ij} \neq \varnothing$ iff $\psi\left(a_{ij}, c_{ij}\right) \geq b_i$.*

Proof. Let $S_{ij} \neq \varnothing$ and $x \in S_{ij}$, i.e., $\psi\left(a_{ij}, \omega\left(c_{ij}, x\right)\right) \geq b_i$. So, from the identity law and monotonicity of ω, we have $b_i \leq \psi\left(a_{ij}, \omega\left(c_{ij}, x\right)\right) \leq \psi\left(a_{ij}, \omega\left(c_{ij}, 1\right)\right) = \psi\left(a_{ij}, c_{ij}\right)$. Conversely, let $\psi\left(a_{ij}, c_{ij}\right) \geq b_i$. So, from the identity law of ω, it follows that $\omega\left(c_{ij}, 1\right) = c_{ij}$ which together with $\psi\left(a_{ij}, c_{ij}\right) \geq b_i$ imply $b_i \leq \psi\left(a_{ij}, c_{ij}\right) = \psi\left(a_{ij}, \omega\left(c_{ij}, 1\right)\right)$. Hence, $\psi\left(a_{ij}, \omega\left(c_{ij}, 1\right)\right) \geq b_i$ that means $1 \in S_{ij}$. $\square$

3.2 Feasible solution set of S_i

This subsection describes the constraint $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\boldsymbol{x}\right) \geq b_i$ $(i \in I)$ and its feasible solution set (i.e., S_i). For each $i \in I$, it is shown that the fundamental properties of S_i can be derived based on the properties of the partial solution sets S_{ij} $(j \in J)$. The following corollary provides three necessary and sufficient conditions for the feasibility of set S_i.

Corollary 3.4. *Suppose that $i \in I$ and $\mathbf{1}$ denotes an n-dimensional vector with each component equal to one.*

(a) *$S_i \neq \varnothing$ iff there exists at least one $j \in J$ such that $S_{ij} \neq \varnothing$.*

(b) *$S_i \neq \varnothing$ iff $\mathbf{1} \in S_i$.*

(c) *$\boldsymbol{x} \in S_i$ iff $\boldsymbol{x} \in [0,1]^n$ and there exists at least one $j_0 \in J$ such that $x_{j_0} \in S_{ij_0}$.*

Proof. (a) Suppose that $S_i \neq \varnothing$ and let $\boldsymbol{x} \in S_i$. By contradiction, suppose that $S_{ij} = \varnothing$, $\forall j \in J$. So, Lemma 3.3 implies $\psi\left(a_{ij}, c_{ij}\right) < b_i$, $\forall j \in J$, and therefore we have

$$\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\boldsymbol{x}\right) = \max_{j=1}^{n}\left\{\psi\left(a_{ij}, \omega\left(c_{ij}, x_j\right)\right)\right\} \leq \max_{j=1}^{n}\left\{\psi\left(a_{ij}, \omega\left(c_{ij}, 1\right)\right)\right\} = \max_{j=1}^{n}\left\{\psi\left(a_{jj}, c_{ij}\right)\right\} < b_i,$$

that contradicts $\boldsymbol{x} \in S_i$. Conversely, suppose that $S_{ij_0} \neq \varnothing$ (equivalently, $\psi\left(a_{ij_0}, c_{ij_0}\right) \geq b_i$) for some $j_0 \in J$. So, $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\mathbf{1}\right) = \max_{j=1}^{n}\left\{\psi\left(a_{ij}, \omega\left(c_{ij}, 1\right)\right)\right\} \geq \psi\left(a_{ij_0}, \omega\left(c_{ij_0}, 1\right)\right) = \psi\left(a_{ij_0}, c_{ij_0}\right) \geq b_i$, that means $\mathbf{1} \in S_i$.

(b) If $\mathbf{1} \in S_i$, then obviously $S_i \neq \varnothing$. Conversely, let $S_i \neq \varnothing$. Hence, from (a), we have $S_{ij_0} \neq \varnothing$ (equivalently, $\psi\left(a_{ij_0}, c_{ij_0}\right) \geq b_i$) for some $j_0 \in J$, and therefore $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\mathbf{1}\right) = \max_{j=1}^{n}\left\{\psi\left(a_{ij}, \omega\left(c_{ij}, 1\right)\right)\right\} \geq \psi\left(a_{ij_0}, \omega\left(c_{ij_0}, 1\right)\right) = \psi\left(a_{ij_0}, c_{ij_0}\right) \geq b_i$, that means $\mathbf{1} \in S_i$.

(c) Suppose that $\boldsymbol{x} \in S_i$. Thus, $\boldsymbol{x} \in [0,1]^n$ and $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\boldsymbol{x}\right) = \max_{j=1}^{n}\left\{\psi\left(a_{ij}, \omega\left(c_{ij}, x_j\right)\right)\right\} \geq b_i$. Hence, there exist at least one $j_0 \in J$ such that $\psi\left(a_{ij_0}, \omega\left(c_{ij_0}, x_{j_0}\right)\right) \geq b_i$, i.e., $x_{j_0} \in S_{ij_0}$. Conversely, suppose that $\boldsymbol{x} \in [0,1]^n$ and $x_{j_0} \in S_{ij_0}$ for some $j_0 \in J$. So, $\psi\left(a_{ij_0}, \omega\left(c_{ij_0}, x_{j_0}\right)\right) \geq b_i$, and therefore $\boldsymbol{a_i}\psi\left(\boldsymbol{c_i}\omega\boldsymbol{x}\right) = \max_{j=1}^{n}\left\{\psi\left(a_{ij}, \omega\left(c_{ij}, x_j\right)\right)\right\} \geq \psi\left(a_{ij_0}, \omega\left(c_{ij_0}, x_{j_0}\right)\right) \geq b_i$, which implies $\boldsymbol{x} \in S_i$. $\square$

Definition 3.5. *Suppose that $i \in I$, $S_i \neq \varnothing$ and $J(i) = \left\{j \in J : S_{ij} \neq \varnothing\right\}$. For each $j \in J(i)$, let $\boldsymbol{x}(i,j) = (x(i,j)_1, x(i,j)_2\ , \ldots, x(i,j)_n)$ be a vector whose components are defined as follows:*

$$x(i,j)_k = \begin{cases} \underline{x}_{ij} & k = j \\ 0 & \textit{otherwise} \end{cases} \quad , \forall k \in J$$

Lemma 3.6 below shows that any S_i is formed as the union of a finite number of closed bounded sets.

Lemma 3.6. *Suppose that $i \in I$ and $S_i \neq \varnothing$.*

(a) *If $\boldsymbol{x} \in S_i$, then there exists some $j_0 \in J(i)$ such that $\boldsymbol{x}\left(i, j_0\right) \leq \boldsymbol{x}$.*

(b) $S_i = \bigcup_{j \in J(i)} [\boldsymbol{x}(i,j), \mathbf{1}]$.

Proof. (a) Since $\boldsymbol{x} \in S_i$, then from Corollary 3.4 (c), it follows that $x_{j_0} \in S_{ij_0}$ for some $j_0 \in J(i)$. So, $\underline{x}_{ij_0} \le x_{j_0} \le 1$ which together with Definition 3.5 imply $x(i,j_0)_{j_0} = \underline{x}_{ij_0} \le x_{j_0}$. On the other hand, $x(i,j_0)_j = 0$, $\forall j \in J - \{j_0\}$ (Definition 3.5). Hence, $x(i,j_0)_j \le x_j$, $\forall j \in J$.

(b) Let $\boldsymbol{x} \in S_i$. From (a), we have $\boldsymbol{x}(i,j_0) \le \boldsymbol{x}$ for some $j_0 \in J(i)$. Therefore, $\boldsymbol{x} \in [\boldsymbol{x}(i,j_0), \mathbf{1}]$. Conversely, let $\boldsymbol{x} \in [\boldsymbol{x}(i,j_0), \mathbf{1}]$ for some $j_0 \in J(i)$. Thus, from Definition 3.5 we have $\underline{x}_{ij_0} = x(i,j_0)_{j_0} \le x_{j_0} \le 1$ that implies $x_{j_0} \in S_{ij_0}$, and therefore $\boldsymbol{x} \in S_i$ from Corollary 3.4 (c). □

3.3 Feasible solution set of the max–$\psi\omega$ FRI

In this subsection, we investigate the structural properties of S, that is, the feasible solution set of the system $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \ge \boldsymbol{b}$ where $\boldsymbol{x} \in [0,1]^n$. First, two necessary and sufficient feasibility conditions are given which are directly result from $S = \bigcap_{i \in I} S_i$ and Corollary 3.4. Then, some definitions are presented, which are used to characterize the feasible region of Problem (1).

The following lemma presents two necessary and sufficient conditions for the consistency of the set S.

Lemma 3.7. (a) *$S \neq \varnothing$ iff $\mathbf{1} \in S$.*
(b) *$S \neq \varnothing$ iff for each $i \in I$ there exists at least one $j \in J$, such that $S_{ij} \neq \varnothing$.*

Proof. (a) If $\mathbf{1} \in S$, then we necessarily have $S \neq \varnothing$. Conversely, suppose that $S \neq \varnothing$. So, since $S = \bigcap_{i \in I} S_i$, it is concluded that $S_i \neq \varnothing$, $\forall i \in I$. Therefore, from Corollary 3.4 (b), we have $\mathbf{1} \in S_i$, $\forall i \in I$, which implies $\mathbf{1} \in S$.

(b) If $\boldsymbol{x} \in S$, then the equality $S = \bigcap_{i \in I} S_i$ requires that $\boldsymbol{x} \in S_i$, $\forall i \in I$. Thus, by Corollary 3.4 (a), for each $i \in I$ there must exist at least one $j \in J$ such that $S_{ij} \neq \varnothing$. Conversely, suppose that for each $i \in I$ there exists at least one $j \in J$ such that $S_{ij} \neq \varnothing$. So, Corollary 3.4 (a) implies $S_i \neq \varnothing$, $\forall i \in I$, and then by Corollary 3.4 (b) we have $\mathbf{1} \in S_i$, $\forall i \in I$. Hence, $\mathbf{1} \in S$ that means $S \neq \varnothing$. □

Definition 3.8. *Suppose that $S \neq \varnothing$. Let $e : I \to \bigcup_{i \in I} J(i)$ be a function from I to $\bigcup_{i \in I} J(i)$ such that $e(i) \in J(i)$, $\forall i \in I$, and let E denote the set of all such functions e. For the sake of convenience, each $e \in E$ is represented as an m-dimensional vector $e = [j_1, j_2, \ldots, j_m]$ in which $j_k = e(k)$, $k = 1, 2, \ldots, m$.*

Definition 3.9. *Suppose that $S \neq \varnothing$ and $e \in E$. Let $x(e) = (x(e)_1, x(e)_2, \ldots, x(e)_n)$ be an n-dimensional vector whose components are defined as $x(e)_j = \max_{i \in I} \{x(i, e(i))_j\}$, $\forall j \in J$.*

Remark 3.10. *Let $I(j) = \{i \in I : S_{ij} \neq \varnothing\}$, $\forall j \in J$. For each $e \in E$ we define $I_j(e) = \{i \in I(j) : e(i) = j\}$. So, based on Definitions 3 and 4 , each solution $\boldsymbol{x}(e)$ can be equivalently obtained as follows:*

$$x(e)_j = \begin{cases} \max_{i \in I_j(e)} \{\underline{x}_{ij}\} & I_j(e) \neq \varnothing \\ 0 & I_j(e) = \varnothing \end{cases} \quad , \forall j \in J \tag{8}$$

Based on the following theorem, the feasible region of Problem (1) is completely determined by the union of a finite number of closed convex sets $[\boldsymbol{x}(e), \mathbf{1}]$, $e \in E$.

Theorem 3.11. *If $S \neq \varnothing$, then $S = \bigcup_{e \in E} [\boldsymbol{x}(e), \mathbf{1}]$.*

Proof. From Lemma 3.6 (b) and the equality $S = \bigcap_{i \in I} S_i$, we have $S = \bigcap_{i \in I} \bigcup_{j \in J(i)} [\boldsymbol{x}(i,j), \mathbf{1}]$, or equivalently $S = \bigcup_{e \in E} \bigcap_{i \in I} [\boldsymbol{x}(i, e(i)), \mathbf{1}]$. Therefore, $S = \bigcup_{e \in E} [\max_{i \in I} \{\boldsymbol{x}(i, e(i))\}, \mathbf{1}]$. Now, the result follows from Definition 3.9. □

According to Theorem 3.11, solutions $\boldsymbol{x}(e)$ ($e \in E$) are the lower bounds of the feasible region. Generally, all the vectors $\boldsymbol{x}(e)$ may not necessarily the minimal solutions to S. In other words, there may exist $e_1, e_2 \in E$ such that $\boldsymbol{x}(e_1) \le \boldsymbol{x}(e_2)$. However, the following lemma shows that each minimal solution of S can be written in the form of $\boldsymbol{x}(e)$ for some $e \in E$.

Lemma 3.12. *Let $\underline{S}$ denote the set of all the minimal solutions of S and $S_E = \{\boldsymbol{x}(e) : e \in E\}$. Then, $\underline{S} \subseteq S_E$.*

Proof. Suppose that $\underline{\boldsymbol{x}} \in \underline{S}$. From Theorem 3.11, there exists some $e \in E$ such that $\underline{\boldsymbol{x}} \in [\boldsymbol{x}(e), \mathbf{1}]$. Since $\boldsymbol{x}(e) \in S$, $\boldsymbol{x}(e) \le \underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{x}}$ is a minimal solution, then we must have $\underline{\boldsymbol{x}} = \boldsymbol{x}(e)$, that is, $\underline{\boldsymbol{x}} \in S_E$. □

Corollary 3.13. *Suppose that $S \neq \varnothing$. Then, $S = \bigcup_{\underline{\boldsymbol{x}} \in \underline{S}} [\underline{\boldsymbol{x}}, \mathbf{1}]$.*

Proof. According to Lemma 3.12, for each $\boldsymbol{x}(e) \in S_E$ there exists some minimal solution $\underline{\boldsymbol{x}} \in \underline{S}$ such that $\underline{\boldsymbol{x}} \le \boldsymbol{x}(e)$, and therefore $[\boldsymbol{x}(e), \mathbf{1}] \subseteq [\underline{\boldsymbol{x}}, \mathbf{1}]$. Now, the result follows from Theorem 3.11. □

4 Optimality and a general algorithm for solving Problem (1)

The following theorem shows that one of the minimal solutions is always the optimal solution of Problem (1). Therefore, to find an optimal solution, it is sufficient to limit our search domain to the elements of $\underline{S}$.

Theorem 4.1. *Assume that φ is a continuous t-norm and τ is a continuous t-conorm. If $S \neq \varnothing$, then there exists a solution $\underline{\boldsymbol{x}}^* \in \underline{S}$ such that $\underline{\boldsymbol{x}}^*$ is an optimal solution of Problem (1).*

Proof. Suppose that $\boldsymbol{x} \in S$ is an arbitrary feasible solution and the minimal solution $\underline{\boldsymbol{x}}^*$ minimizes $f(\boldsymbol{x})$ among all minimal solutions. So, we have $f(\underline{\boldsymbol{x}}^*) \le f(\underline{\boldsymbol{x}})$, $\forall \underline{\boldsymbol{x}} \in \underline{S}$. From Corollary 3.13, there exists some $\underline{\boldsymbol{x}}' \in \underline{S}$ such that $\boldsymbol{x} \in [\underline{\boldsymbol{x}}', \mathbf{1}]$ (i.e., $\underline{x}'_j \le x_j$ for each $j \in J$). So, the monotonicity law of t-norms implies $\varphi\left(d_j, \underline{x}'_j\right) \le \varphi(d_j, x_j)$, $\forall j \in J$, and therefore from the monotonicity law of t-conorms, it is concluded that $f(\underline{\boldsymbol{x}}') = \tau\left(\varphi(d_1, \underline{x}'_1), \ldots, \varphi(d_n, \underline{x}'_n)\right) \le \tau\left(\varphi(d_1, x_1), \ldots, \varphi(d_n, x_n)\right) = f(\boldsymbol{x})$. But, since $f(\underline{\boldsymbol{x}}^*) \le f(\underline{\boldsymbol{x}}')$, it follows that $f(\underline{\boldsymbol{x}}^*) \le f(\boldsymbol{x})$. □

Now, we summarize the preceding discussion as an algorithm.

Algorithm 1: Find Optimal Solution for Problem (1)

Input : Given Problem (1)
Output: Find x^* (The Optimal Solution)

Compute S_{ij} for each $i \in I$ and $j \in J$ (Lemma 3.2).
if *there exists some $i \in I$ such that $S_{ij} = \varnothing$, $\forall j \in J$,* **then**
  stop; S is empty (Lemma 3.7 (b)).
end
else
  Compute $J(i)$, $\forall i \in I$ (Definition 3.5).
  Compute solutions $\boldsymbol{x}(e) \in S_E$, $\forall e \in E$ (Definitions 3 and 4).
  Find minimal solutions by pairwise comparison between vectors $\boldsymbol{x}(e)$ (Lemma 3.12).
  Select the optimal solution $\boldsymbol{x}^*$ from the set $\underline{S}$ (Lemma 3.12 and Theorem 4.1).
end

It should be noted that the feasible region of Problem (1) defined by the constraints $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \le \boldsymbol{b}$, can be easily attained in the form of $[\mathbf{0}, \overline{\boldsymbol{x}}]$, where $\mathbf{0}$ is the zero vector and $\bar{\boldsymbol{x}}$ is the unique maximum solution. On the other hand, the constraints $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) = \boldsymbol{b}$ can be equivalently expressed as $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \le \boldsymbol{b}$ and $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \ge \boldsymbol{b}$, and therefore the feasible region is then attained by $\bigcup_{e \in E}[\boldsymbol{x}(e), \mathbf{1}] \cap [\mathbf{0}, \bar{\boldsymbol{x}}] = \bigcup_{e \in E}[\boldsymbol{x}(e), \bar{\boldsymbol{x}}]$ (see Theorem 3.11). Consequently, since according to Theorem 4.1, the optimal solution will be one of the solutions of $\underline{S}$, Algorithm 1 can also be used to solve problems with the equality constraints. However, a major disadvantage of the above algorithm is related to the calculation of the elements of S_E. Indeed, S_E often contains many solutions $\boldsymbol{x}(e)$ that are not necessarily minimal. To accelerate the algorithm, we can initially remove some $e \in E$ that generate non-minimal solutions $\boldsymbol{x}(e)$. For this purpose, five simplification rules will be described in Section 6. Furthermore, in some special cases, we can find a fast optimal solution to Problem (1) by an efficient algorithm that is of polynomial complexity in the size of the problem. These special cases will be studied in the next section.

5 Special cases with polynomial-time algorithms

In this section, an algorithm is presented for solving a special class of Problem (1) without finding all the solutions $\boldsymbol{x}(e) \in S_E$. It is shown that the computational complexity of the algorithm is $O(mn)$.

Definition 5.1. *Define $L_i = \min_{j \in J(i)} \left\{\varphi\left(d_j, \underline{x}_{ij}\right)\right\}$ and $\underline{J}(i) = \left\{j \in J(i) : \varphi\left(d_j, \underline{x}_{ij}\right) = L_i\right\}$, $\forall i \in I$. Also, similar to Definition 3.8, let $\underline{E}$ be the set of all the functions $e : I \to \bigcup_{i \in I} \underline{J}(i)$.*

By Definition 5.1 , it is clear that $\underline{J}(i) \subseteq J(i)$ and $\underline{E} \subseteq E$. Also, for each $e \in \underline{E}$ we have $e(i) \in \underline{J}(i)$, $\forall i \in I$. Furthermore, since for each $e \in E$ and $e' \in \underline{E}$, we have $e(i) \in J(i)$ and $e'(i) \in \underline{J}(i)$, it follows from Definition 5.1 that

$$\varphi\left(d_{e'(i)}, \underline{x}_{ie'(i)}\right) = L_i \le \varphi\left(d_{e(i)}, \underline{x}_{ie(i)}\right), \quad \forall i \in I. \tag{9}$$

Lemma 5.2. *Assume φ is a continuous t-norm, and let $e' \in \underline{E}$ and $e \in E$. Then,*

$$\max_{j \in J}\{\varphi(d_j, x(e')_j)\} \le \max_{j \in J}\{\varphi(d_j, x(e)_j)\}.$$

Proof. From Remark 3.10, $\max_{j\in J}\{\varphi(d_j, x(e)_j)\} = \max_{j\in J}\varphi\left(d_j, \max_{i\in I_j(e)}\{\underline{x}_{ij}\}\right)$ that is equal to $\max_{i\in I}\{\varphi(d_{e(i)}, \underline{x}_{ie(i)})\}$. Also, from (9), we have $\max_{i\in I}\{\varphi(d_{e(i)}, \underline{x}_{ie(i)})\} \geq \max_{i\in I}\{\varphi(d_{e'(i)}, \underline{x}_{ie'(i)})\}$. consequently,

$$\max_{j\in J}\{\varphi(d_j, x(e)_j)\} \geq \max_{i\in I}\{\varphi(d_{e'(i)}, \underline{x}_{ie'(i)})\}.$$

On the other hand,

$$\max_{i\in I}\{\varphi(d_{e'(i)}, \underline{x}_{ie'(i)})\} = \max_{j\in J}\varphi(d_j, \max_{i\in I_j(e')}\{\underline{x}_{ij}\}) = \max_{j\in J}\{\varphi(d_j, x(e')_j)\},$$

and therefore $\max_{j\in J}\{\varphi(d_j, x(e')_j)\} \leq \max_{j\in J}\{\varphi(d_j, x(e)_j)\}$. □

Corollary 5.3. *For each* $e_1, e_2 \in \underline{E}$, $\max_{j\in J}\{\varphi(d_j, x(e_1)_j)\} = \max_{j\in J}\{\varphi(d_j, x(e_2)_j)\}$.

Proof. Since $\underline{E} \subseteq E$, we also have $e_1, e_2 \in E$. So, by considering $e_1 \in \underline{E}$ and $e_2 \in E$, Lemma 5.2 implies that

$$\max_{j\in J}\{\varphi(d_j, x(e_1)_j)\} \leq \max_{j\in J}\{\varphi(d_j, x(e_2)_j)\}.$$

By the same argument,

$$\max_{j\in J}\{\varphi(d_j, x(e_2)_j)\} \leq \max_{j\in J}\{\varphi(d_j,\ x(e_1)_j)\}.$$

Hence $\max_{j\in J}\{\varphi(d_j, x(e_1)_j)\} = \max_{j\in J}\{\varphi(d_j, x(e_2)_j)\}$. □

Theorem 5.4. *Suppose that* $S \neq \varnothing$, τ *is the maximum t-conorm and* φ *is a continuous t-norm. If* $e \in \underline{E}$, *then* $\boldsymbol{x}(e)$ *is an optimal solution.*

Proof. By assuming that τ is the maximum t-conorm, the objective function, $f(\boldsymbol{x}) = \tau(\varphi(d_1, x_1), \ldots, \varphi(d_n, x_n))$ is reduced to $f(\boldsymbol{x}) = \max\{\varphi(d_1, x_1), \ldots, \varphi(d_n, x_n)\} = \max_{j\in J}\{\varphi(d_j, x_j)\}$. From Theorem 4.1, for each optimal solution $\boldsymbol{x}^*$ we must have $\boldsymbol{x}^* \in \underline{S}$. On the other hand, Lemma 3.12 implies that $\underline{\boldsymbol{x}}^* \in S_E$. Now, the result follows from Lemma 5.2 and Corollary 5.3. □

Theorem 5.4, proposes an efficient polynomial-time algorithm for solving the special cases of Problem (1) where τ is the maximum t-conorm. Particularly, the classes of problems formed as (5), (6) and (7) can be solved by a polynomial-time algorithm. Algorithm 2 below shows the steps of this algorithm followed by the complete description of its complexity.

Algorithm 2: Find Optimal Solution for Problem (1)

Input : Given Problem (1) where τ is the maximum t-conorm
Output: Find x^* (The Optimal Solution)
1 Compute S_{ij} for each $i \in I$ and $j \in J$ (Lemma 3.2).
2 **if** *there exists some* $i \in I$ *such that* $S_{ij} = \varnothing$, $\forall j \in J$, **then**
3 | stop; S is empty (Lemma 3.7 (b)).
4 **end**
5 **else**
6 | Compute $\underline{J}(i)$, $\forall i \in I$ (Definition 5.1).
7 | Select an arbitrary $e \in \underline{E}$.
8 | Obtain the optimal solution $\boldsymbol{x}^*$ by computing $\boldsymbol{x}(e)$ (Theorem 5.4).
9 **end**

In Step 1, computing S_{ij} costs mn operations. In Step 2, checking the feasibility of the problem costs mn for pairwise comparisons. In Step 3, computing the index set costs $2mn$ operations, and finally, each of Steps 4 and 5 costs mn operations. Therefore, executing all steps of the algorithm requires a total of 6mn operations, yielding a computational complexity of $O(mn)$.

6 Simplification rules

In this section, five simplification rules are presented to accelerate the resolution of the problem. Throughout this section, $\boldsymbol{x}^*$ denotes an optimal solution for Problem (1) and $f(\boldsymbol{x})$ denotes the objective function of Problem (1); that is, $f(\boldsymbol{x}) = \tau\left(\varphi\left(d_1, x_1\right), \ldots, \varphi\left(d_n, x_n\right)\right)$.

Lemma 6.1. *Suppose that $J\left(i_0\right) = \{j_0\}$ for some $i_0 \in I$ and $j_0 \in J$. Also, Suppose that $\underline{x}_{i_0 j_0} \geq \underline{x}_{i j_0}$, $\forall i \in I\left(j_0\right)$. Then, $x(e)_{j_0} = \underline{x}_{i_0 j_0}, \forall e \in E$.*

Proof. Since $J\left(i_0\right)$ is a singleton set and $e\left(i_0\right) \in J\left(i_0\right)$, $\forall e \in E$, then $e\left(i_0\right) = j_0$, $\forall e \in E$. Therefore, $i_0 \in I_{j_0}(e), \forall e \in E$. Now, since $I_{j_0}(e) \neq \varnothing$, from relation (8) we obtain $x(e)j_0 = \max_{i \in I_{j_0}(e)}\left\{\underline{x}_{i j_0}\right\}$. The latter equality together with $I_{j_0}(e) \subseteq I\left(j_0\right)$ (Remark 3.10) and the assumption $\underline{x}_{i_0 j_0} \geq \underline{x}_{i j_0}$, $\forall i \in I\left(j_0\right)$, result in $x(e)_{j_0} = \underline{x}_{i_0 j_0}$. □

Based on Theorem 4.1 and Lemma 6.1, we know that $\boldsymbol{x}^* \in S_E$. Hence, under the assumptions of Lemma 6.1, we can set $\boldsymbol{x}_{j_0}^* = \underline{x}_{i_0 j_0}$. Therefore, since the j_0'th variable of the optimal solution is known, this variable can be removed from the problem by deleting its corresponding coefficients, i.e., the j_0'th column of $\boldsymbol{A}$. On the other hand, $x_{j_0}^* = \underline{x}_{i_0 j_0}$ means $x_{j_0}^* \in S_{i_0 j_0} = \left[\underline{\boldsymbol{x}}_{i_0 j_0}, \mathbf{1}\right]$ that together with Corollary 3.4 (c) imply $\boldsymbol{x}^* \in S_{i_0}$. Therefore, by assigning $x_{j_0}^* = \underline{x}_{i_0 j_0}$, the i_0'th constraint of the problem is always satisfied. Hence, we can remove this constraint from the problem by deleting the i_0'th row of $\boldsymbol{A}$ and b_{i_0}. Moreover, let $i' \in I\left(j_0\right)$ and $i' \neq i_0$. Since $\underline{x}_{i_0 j_0} \geq \underline{x}_{i j_0}, \forall i \in I\left(j_0\right)$, then $\underline{x}_{i_0 j_0} \geq \underline{x}_{i' j_0}$ that means $x_{j_0}^* \in S_{i' j_0} = \left[\underline{x}_{i' j_0}, \mathbf{1}\right]$. Therefore, Corollary 3.4 (c) implies $\boldsymbol{x}^* \in S_{i'}$, i.e., $\boldsymbol{x}^*$ also satisfies the i''th constraint of the problem. Hence, we can remove this constraint from the problem by deleting the i''th row of $\boldsymbol{A}$ and $b_{i'}$. The above discussions are summarized in the following rule.

Rule 1. *If there exist $i_0 \in I$ and $j_0 \in J$ such that $J\left(i_0\right) = \{j_0\}$ and $\underline{x}_{i_0 j_0} \geq \underline{x}_{i j_0}, \forall i \in I\left(j_0\right)$, then set $x_{j_0}^* = \underline{x}_{i_0 j_0}$ and delete the j_0'th column of $\boldsymbol{A}$. Moreover, for each $i \in I\left(j_0\right)$, delete the i'th row of $\boldsymbol{A}$ and component b_i.*

Definition 6.2. *Let $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$ be a system resulting from $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \geq \boldsymbol{b}$ by deleting the i_0'th constraint $\boldsymbol{a}_{\boldsymbol{i}_0}\psi\left(\boldsymbol{c}_{i_0}\omega\boldsymbol{x}\right) \geq b_{i_0}$. Also, we define S' as the feasible region of the reduced problem and $S_{E'} = \{\boldsymbol{x}(e) : e \in E'\}$ where E' denotes the set of all the functions $e : I - \{i_0\} \rightarrow \bigcup_{i \in I - (i_0\}} J(i)$ such that $e(i) \in J(i), \forall i \in I - \{i_0\}$. The constraint$\boldsymbol{a}_{\boldsymbol{i}_0}\psi\left(\boldsymbol{c}_{i_0}\omega\boldsymbol{x}\right) \geq b_{i_0}$ is called redundant if $S' = S$, that is, each feasible solution to the system $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$ also satisfies the constraint $\boldsymbol{a}_{\boldsymbol{i}_0}\psi\left(\boldsymbol{c}_{i_0}\omega\boldsymbol{x}\right) \geq b_{i_0}$.*

Lemma 6.3. *Suppose that $i_1, i_2 \in I$ such that $J\left(i_1\right) \subseteq J\left(i_2\right)$ and $\underline{x}_{i_2 j} \leq \underline{x}_{i_1 j}, \forall j \in J\left(i_1\right)$, where ψ and ω are continuous t-norms. Then, i_2'th constraint is redundant.*

Proof. Let $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$ be a system resulted from $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \geq \boldsymbol{b}$ by deleting the i_2'th constraint $\boldsymbol{a}_{\boldsymbol{i}_2}\psi\left(\boldsymbol{c}_{i_2}\omega\boldsymbol{x}\right) \geq b_{i_2}$. In the new system $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$, consider an arbitrary $e \in E'$ and suppose that $e\left(i_1\right) = j$. Therefore, $i_1 \in I_j(e)$ that means $I_j(e) \neq \varnothing$. Moreover, by Theorem 3.11, $\boldsymbol{x}(e)$ is a feasible solution to $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$. On the other side, since $j \in J\left(i_1\right)$ and $J\left(i_1\right) \subseteq J\left(i_2\right)$, then $j \in J\left(i_2\right)$ for the system $\boldsymbol{A}\psi(\boldsymbol{C}\omega\boldsymbol{x}) \geq \boldsymbol{b}$. Now, because $I_j(e) \neq \varnothing$, by relation (8) we obtain $x(e)_j = \max_{i \in I_j(e)}\left\{\underline{x}_{ij}\right\} \geq \underline{x}_{i_1 j} \geq \underline{x}_{i_2 j}$, where the last inequality is resulted from the assumption of the lemma. Therefore, $x(e)_j \geq \underline{x}_{i_2 j}$ which means $x(e)_j \in S_{i_2 j} = \left[\underline{x}_{i_2 j}, \mathbf{1}\right]$. Now, Corollary 3.4 (c) implies $\boldsymbol{x}(e) \in S_{i_2}$, that is, $\boldsymbol{x}(e)$ also satisfies the i_2'th constraint of the primal system. □

The following simplification rule then results from Lemma 6.3.

Rule 2. *If there exist $i_1, i_2 \in I$ such that $J\left(i_1\right) \subseteq J\left(i_2\right)$ and $\underline{x}_{i_2 j} \leq \underline{x}_{i_1 j}, \forall j \in J\left(i_1\right)$, then delete the i_2'th row of $\boldsymbol{A}$ and b_{i_2}.*

Lemma 6.4. *Let $j_1, j_2 \in J$ such that $I\left(j_2\right) \subseteq I\left(j_1\right)$. Assume that φ is a continuous t-norm and τ is a continuous t-conorm. If $\varphi\left(d_{j_1}, \underline{x}_{i j_1}\right) \leq \varphi\left(d_{j_2}, \underline{x}_{i j_2}\right)$, $\forall i \in I\left(j_2\right)$, then it follows that $x_{j_2}^* = 0$.*

Proof. Since each t-conorm τ is both commutative and associative, the objective function of Problem (1) can be equivalently rewritten as follows:

$$f(x) = \tau(\ldots\tau(\varphi(d_{j_1}, x_{j_1}), \varphi(d_{j_2}, x_{j_2})), \ldots, \varphi(d_n, x_n)). \tag{A1}$$

To prove the lemma, it is sufficient to show that a solution $\boldsymbol{x}(e)$ with $x(e)_{j_2} > 0$ cannot be an optimal solution. It should be noted that $x(e)_{j_2} > 0$ implies $I_{j_2}(e) \neq \varnothing$, that is, there exists at least one $i \in I$ such that $e(i) = j_2$. Based on this vector e, define $e' \in E$ as follows:

$$e'(i) = \begin{cases} j_1 & e(i) = j_2 \\ e(i) & e(i) \neq j_2 \end{cases} \quad \forall i \in I. \tag{A2}$$

So, according to (8), we have $x(e')_k = x(e)_k$, $\forall k \in J - \{j_1, j_2\}$, and $x(e')_{j_2} = 0 < x(e)_{j_2}$. Hence, from the assumption of the lemma and the fact that $I_{j_2}(e) \subseteq I(j_2)$ (Remark 3.10), it is concluded that $\varphi(d_{j_1}, \max_{i \in I_{j_2}(e)}\{\underline{x}_{ij_1}\}) \leq \varphi(d_{j_2}, \max_{i \in I_{j_2}(e)}\{\underline{x}_{ij_2}\})$. Now, from the latter inequality, the monotonicity property of φ and relation (8), we obtain

$$\begin{aligned}
\varphi(d_{j_1}, x(e')_{j_1}) &= \varphi(d_{j_1}, \max_{i \in I_{j_1}(e')}\{\underline{x}_{ij_1}\}) \\
&= \varphi(d_{j_1}, \max\{\max_{i \in I_{j_1}(e)}\{\underline{x}_{ij_1}\}, \max_{i \in I_{j_2}(e)}\{\underline{x}_{ij_1}\}\}) \\
&= \max\{\varphi(d_{j_1}, \max_{i \in I_{j_1}(e)}\{\underline{x}_{ij_1}\}), \varphi(d_{j_1}, \max_{i \in I_{j_2}(e)}\{\underline{x}_{ij_1}\})\} \\
&\leq \max\{\varphi(d_{j_1}, \max_{i \in I_{j_1}(e)}\{\underline{x}_{ij_1}\}), \varphi(d_{j_2}, \max_{i \in I_{j_2}(e)}\{\underline{x}_{ij_2}\})\} \\
&= \max\{\varphi(d_{j_1}, x(e)_{j_1}), \varphi(d_{j_2}, x(e)_{j_2}\}. \\
&\leq \tau(\varphi(d_{j_1}, x(e)_{j_1}), \varphi(d_{j_2}, x(e)_{j_2})),
\end{aligned} \tag{A3}$$

where the last inequality follows from the fact that $\max\{x, y\} \leq \tau(x, y)$ for any t-conorm τ. Consequently, from (A3) we have

$$\varphi\left(d_{j_1}, x(e')_{j_1}\right) \leq \tau\left(\varphi\left(d_{j_1}, x(e)_{j_1}\right), \varphi\left(d_{j_2}, x(e)_{j_2}\right)\right). \tag{A4}$$

Thus, by the equality $x(e')_{j_2} = 0$ and the identity law of s-norm τ and t-norm φ (i.e., $\tau(x, 0) = x$ and $\varphi(x, 0) = 0$, $\forall x \in [0, 1]$), it follows that

$$\varphi\left(d_{j_1}, x(e')_{j_1}\right) = \tau\left(\varphi\left(d_{j_1}, x(e')_{j_1}\right), 0\right) = \tau\left(\varphi\left(d_{j_1}, x(e')_{j_1}\right), \varphi\left(d_{j_2}, 0\right)\right) = \tau\left(\varphi\left(d_{j_1}, x(e')_{j_1}\right), \varphi\left(d_{j_2}, x(e')_{j_2}\right)\right),$$

which together with (A4) imply

$$\tau\left(\varphi\left(d_{j_1}, x(e')_{j_1}\right), \varphi\left(d_{j_2}, x(e')_{j_2}\right)\right) \leq \tau\left(\varphi\left(d_{j_1}, x(e)_{j_1}\right), \varphi\left(d_{j_2}, x(e)_{j_2}\right)\right). \tag{A5}$$

Now, since $x(e')_k = x(e)_k$ ($\forall k \in J - \{j_1, j_2\}$), from (A1) and (A5) and the monotonicity property of t-conorms, we obtain $f(\boldsymbol{x}(e')) \leq f(\boldsymbol{x}(e))$; that is, $\boldsymbol{x}(e)$ is not an optimal solution. $\square$

According to Lemma 6.4, we obtain the following simplification rule.

Rule 3. *If there exist $j_1, j_2 \in J$ such that $I(j_2) \subseteq I(j_1)$ and $\varphi\left(d_{j_1}, \underline{x}_{ij_1}\right) \leq \varphi\left(d_{j_2}, \underline{x}_{ij_2}\right), \forall i \in I(j_2)$, then set $x^*_{j_2} = 0$ and delete the j_2'th column of $\boldsymbol{A}$.*

Lemma 6.5. *Suppose that $\boldsymbol{x}_0$ is an arbitrary feasible solution to Problem (1) with the objective value $f(\boldsymbol{x}_0)$. Let φ be a continuous t-norm and let τ be a continuous t-conorm. Also, suppose $\varphi\left(d_{j_0}, \underline{x}_{i_0j_0}\right) \geq f(\boldsymbol{x}_0)$ for some $i_0 \in I$ and $j_0 \in J(i_0)$. Then, for each $e \in E$ such that $e(i_0) = j_0$, the corresponding solution $\boldsymbol{x}(e)$ is not an optimal solution.*

Proof. Let $e \in E$ such that $e(i_0) = j_0$ and suppose that $\boldsymbol{x}(e)$ is generated by relation (8). Define $\boldsymbol{x}' \in [0, 1]^n$ such that $x'_{j_0} = \underline{x}_{i_0j_0}$ and $x'_j = 0$, $\forall j \in J - \{j_0\}$. So, according to relation (8), we have $x'_j \leq x(e)_j, \forall j \in J$. Hence, based on the monotonicity law of t-norms and t-conorms, $f(\boldsymbol{x}') \leq f(\boldsymbol{x}(e))$. On the other hand, from the identity law of t-conorms, we obtain

$$f(\boldsymbol{x}') = \tau\left(0, \ldots, 0, \varphi\left(d_{j_0}, \underline{x}_{i_0j_0}\right), 0, \ldots 0\right) = \varphi\left(d_{j_0}, \underline{x}_{i_0j_0}\right) \geq f(\boldsymbol{x}_0),$$

that is, $f(\boldsymbol{x}') \geq f(\boldsymbol{x}_0)$. Consequently, the inequalities $f(\boldsymbol{x}') \geq f(\boldsymbol{x}_0)$ and $f(\boldsymbol{x}') \leq f(\boldsymbol{x}(e))$ imply $f(\boldsymbol{x}(e)) \geq f(\boldsymbol{x}_0)$, where $\boldsymbol{x}_0$ is a feasible solution to the problem. $\square$

Rule 4. *Let $\boldsymbol{x}_0$ be an arbitrary initial feasible solution of the problem. If $\varphi\left(d_{j_0}, \underline{x}_{i_0,j_0}\right) \geq f(\boldsymbol{x}_0)$ for some $i_0 \in I$ and $j_0 \in J(i_0)$, then delete j_0 from $J(i_0)$.*

Based on Rule 4, by deleting j_0 from $J(i_0)$, the cardinality of S_E is reduced from $\prod_{i \in I} |J(i)|$ to $|J(i_0) - 1| \cdot \prod_{i \in I - \{i_0\}} |J(i)|$. Moreover, according to Lemma 3.3 and Definition 3.5, the deletion of j_0 from $J(i_0)$ can be equivalently accomplished by assigning an arbitrary value from $[0, b_{i_0})$ to $a_{i_0j_0}$ or to $c_{i_0j_0}$ (e.g., $a_{i_0j_0} = 0$ or $c_{i_0j_0} = 0$); because, in this case we have $\psi\left(a_{i_0j_0}, c_{i_0j_0}\right) \leq \min\left\{a_{i_0j_0}, c_{i_0j_0}\right\} < b_{i_0}$ that implies $S_{i_0j_0} = \varnothing$ (Lemma 3.3). Furthermore, by applying Rule 4, we can consider $\boldsymbol{x}_0 = \boldsymbol{x}(e_0)$ as an initial feasible solution where $\boldsymbol{x}(e_0)$ is obtained by relation (8) for any arbitrary $e_0 \in \underline{E}$ as defined in Definition 5.1.

Lemma 6.6. *Suppose that $I(j_0) = \varnothing$ for some $j_0 \in J$. Then, $x(e)_j = 0, \forall \boldsymbol{x}(e) \in S_E$.*

Proof. Since $I_{j_0}(e) \subseteq I(j_0)$ (see Remark 3.10), $\forall e \in E$, then we have $I_{j_0}(e) = \varnothing$, $\forall e \in E$. Now, the result directly follows from relation (8). $\square$

Rule 5. *If there exists $j_0 \in J$ such that $I(j_0) = \varnothing$, then set $x^*_{j_0} = 0$ and delete the j_0'th column of $\boldsymbol{A}$.*

We now summarize the preceding discussion as an algorithm (see Algorithm 3).

Algorithm 3: Accelerated method

Input : Given Problem (1)
Output: Find $\underline{\boldsymbol{x}}^*$ (The Optimal Solution)
1 Compute S_{ij} for each $i \in I$ and $j \in J$ (Lemma 3.2).
2 **if** *there exists some $i \in I$ such that $S_{ij} = \varnothing$, $\forall j \in J$,* **then**
3 | stop; S is empty (Lemma 3.7 (b)).
4 **end**
5 **else**
6 | Compute $J(i), \forall i \in I$ (Definition 3.5).
7 | Apply the simplification rules 1-5 to determine the values of decision variables as many as possible. Denote the remaining problem by $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$.
8 | Compute solutions $\boldsymbol{x}(e) \in S_{E'}, \forall e \in E'$ (Definition 6.2).
9 | Find minimal solutions by pairwise comparison between vectors $\boldsymbol{x}$ (e) (Lemma 3.12).
10 | Select the optimal solution $\underline{\boldsymbol{x}}^*$ from the set $\underline{S}$ (Theorem 4.1).
11 **end**

Steps 5 and 6 provide all the minimal solutions to the feasible region, which is equivalent to the complete resolution of the feasible solution set (Theorem 3.11). However, as is well-known in FRE (FRI) theory, the resolution of FREs (FRIs) has high computational complexity in the sense that each minimal solution corresponds to an irredundant cover. Therefore, although the above algorithm accelerates the resolution of the problem by taking advantage of the five simplification rules and ignoring some non-minimal solutions, the number of minimal solutions may also grow exponentially. Consequently, finding other simplification rules or providing some methods (such as appropriate branch-and-bound techniques) that can reach the optimal solution by examining a small number of minimal (candidate) solutions is always in demand.

7 Numerical example

Example 7.1. *Consider the following optimization problem (1) in which the feasible region has been randomly generated as follows:*

$$\boldsymbol{A} = \begin{bmatrix} 0.64 & 0.29 & 0.47 & 0.85 & 0.52 \\ 0.91 & 0.97 & 0.91 & 0.97 & 0.99 \\ 0.03 & 0.09 & 0.03 & 0.38 & 0.98 \\ 0.97 & 0.98 & 0.76 & 0.94 & 0.91 \\ 0.94 & 0.90 & 0.91 & 0.94 & 0.81 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0.81 & 0.51 & 0.71 & 0.82 & 0.45 \\ 0.97 & 0.92 & 0.99 & 0.92 & 0.90 \\ 0.76 & 0.32 & 0.41 & 0.78 & 0.85 \\ 0.96 & 0.99 & 0.63 & 0.98 & 0.99 \\ 0.94 & 0.92 & 0.98 & 0.98 & 0.97 \end{bmatrix},$$

$$\boldsymbol{d} = [0.21, 0.15, 0.17, 0.74, 0.50],$$

$$\boldsymbol{b}^T = [0.70, 0.88, 0.13, 0.90, 0.79].$$

Also, τ is the Łukasiewicz t-conorm, φ is the Dubois-Prade t-norm (with $\gamma = 0.5$), ψ is the product t-norm and ω is the Yager t-norm (with $p = 2$); that is, $\tau(x, y) = \min\{x + y, 1\}$, $\varphi(x, y) = xy/\max\{x, y, 0.5\}$, $\psi(x, y) = xy$, and $\omega = \max\left\{0, 1 - \sqrt{(1-x)^2 + (1-y)^2}\right\}$. The steps of Algorithm 3 are as follows:

Step 1 *According to Lemma 2, $S_{ij} \neq \varnothing$ iff $\psi(a_{ij}, c_{ij}) \geq b_i$. For example, since $a_{11} = 0.64$, $c_{11} = 0.81$ and $b_1 = 0.7$, then we have $\psi(a_{11}, c_{11}) = a_{11}c_{11} = (0.64).(0.81) = 0.52 < 0.7 = b_1$ that implies $S_{11} = \varnothing$. On the other hand, if $S_{ij} \neq \varnothing$, then based on Lemma 3.2 we have $S_{ij} = \left[\underline{x}_{ij}, 1\right]$ where $\underline{x}_{ij}$ is easily obtained as follows:*

$$\underline{x}_{ij} = \begin{cases} 0 & , b_i = 0 \\ 1 - \sqrt{(1 - b_i/a_{ij})^2 - (1 - c_{ij})^2} & , b_i > 0 \end{cases}$$

The intervals S_{ij} are summarized in the following matrix
$\boldsymbol{\mathscr{S}} = (S_{ij})_{5x5}$ *where* $S_{ij} = [\underline{x}_{ij}, 1]$*:*

$$\boldsymbol{\mathscr{S}} = \begin{bmatrix} \varnothing & \varnothing & \varnothing & [0.95,1] & \varnothing \\ [0.98,1] & [0.94,1] & [0.97,1] & [0.95,1] & [0.96,1] \\ \varnothing & \varnothing & \varnothing & [0.39,1] & [0.15,1] \\ [0.93,1] & [0.92,1] & \varnothing & [0.96,1] & [0.99,1] \\ [0.85,1] & [0.89,1] & [0.87,1] & [0.84,1] & [0.98,1] \end{bmatrix}.$$

Step 2 *From Lemma 3.7 (b), the necessary and sufficient condition holds for the feasibility of the problem. Indeed, as shown in the matrix* $\boldsymbol{\mathscr{S}} = (S_{ij})_{5\times 5}$*, for each* $i \in I$ *there exists at least one* $j \in J$ *such that* $S_{ij} = [\underline{x}_{ij}, 1] \neq \varnothing$*. Also, we have*

$$\boldsymbol{A}\psi(\boldsymbol{C}\omega\mathbf{1}) = (0.70, 0.90, 0.83, 0.97, 0.92)^T \geq \boldsymbol{b}^T,$$

that means $\mathbf{1} \in S$ *(see Lemma 3.7 (a)).*

Step 3 *By Definition 3.5,* $J(1) = \{4\}$*,* $J(2) = J(5) = \{1, \ldots, 5\}$*,* $J(3) = \{4, 5\}$ *and* $J(4) = \{1, 2, 4, 5\}$*. So, we have* $|E| = \prod_{i=1}^{5} |J(i)| = 200$*. Therefore, according to Definitions 3 and 4, the number of all vectors* $\boldsymbol{x}(e) \in S_E$ $(e \in E)$ *is equal to* 200*. However, each feasible solution* $\boldsymbol{x}(e)$ $(e \in E)$ *is not necessarily a minimal solution for the problem. For instance, by selecting* $e' = [4, 3, 4, 1, 1]$*, the corresponding solution is obtained as* $\boldsymbol{x}(e') = (0.93, 0, 0.97, 0.95, 0)$*, which is not a minimal solution. To see this, let* $e'' = [4, 4, 4, 1, 1]$*. So, by relation (8) we obtain* $\boldsymbol{x}(e'') = (0.93, 0, 0, 0.95, 0)$*, and therefore* $\boldsymbol{x}(e'') \leq \boldsymbol{x}(e')$*, which shows that* $\boldsymbol{x}(e')$ *is not a minimal solution.*

Step 4 *By considering columns 2 and 3 of matrix* $\boldsymbol{A}$ *(and corresponding columns in matrix* $\mathscr{S}$ *), it follows that* $\{2, 5\} = I(3) \subseteq I(2) = \{2, 4, 5\}$*. Also, a simple calculation shows that*

$$\varphi(d_2, \underline{x}_{22}) = (d_2\ \underline{x}_{22}) / \max\{d_2, \underline{x}_{22}, 0.5\} = (0.15)(0.94)/\max\{0.15, 0.94, 0.5\} = 0.15,$$

and similarly $\varphi(d_3, \underline{x}_{23}) = 0.17$*,* $\varphi(d_2, \underline{x}_{52}) = 0.15$ *and* $\varphi(d_3, \underline{x}_{53}) = 0.17$*. So,* $\varphi(d_2, \underline{x}_{22}) \leq \varphi(d_3, \underline{x}_{23})$ *and* $\varphi(d_2, \underline{x}_{52}) \leq \varphi(d_3, \underline{x}_{53})$*, and therefore by applying Rule 3, we set* $x_3^* = 0$ *and delete column 3 from matrices* $\boldsymbol{A}$ *,*$\boldsymbol{C}$ *and* $\boldsymbol{\mathscr{S}}$*. Moreover, since* $I(1) = I(2) = \{2, 4, 5\}$*,* $\varphi(d_2, \underline{x}_{i2}) = 0.15$ *and* $\varphi(d_1, \underline{x}_{i1}) = 0.21, \forall i \in \{2, 4, 5\}$*, we have* $\varphi(d_2, \underline{x}_{i2}) \leq \varphi(d_1, \underline{x}_{i1}), \forall i \in \{2, 4, 5\}$ *. Hence, column 1 is also removed by Rule 3 and we set* $x_1^* = 0$*. After deletion, the reduced matrices* $\boldsymbol{A}'$*,* $\boldsymbol{C}'$ *and* $\boldsymbol{\mathscr{S}}'$ *are obtained as follows:*

$$\boldsymbol{A}' = \begin{bmatrix} 0.29 & 0.85 & 0.52 \\ 0.97 & 0.97 & 0.99 \\ 0.09 & 0.38 & 0.98 \\ 0.98 & 0.94 & 0.91 \\ 0.90 & 0.94 & 0.81 \end{bmatrix}, \ \boldsymbol{C}' = \begin{bmatrix} 0.51 & 0.82 & 0.45 \\ 0.92 & 0.92 & 0.90 \\ 0.32 & 0.78 & 0.85 \\ 0.99 & 0.98 & 0.99 \\ 0.92 & 0.98 & 0.97 \end{bmatrix}, \ \boldsymbol{\mathscr{S}}' = \begin{bmatrix} \varnothing & [0.95,1] & \varnothing \\ [0.94,1] & [0.95,1] & [0.96,1] \\ \varnothing & [0.39,1] & [0.15,1] \\ [0.92,1] & [0.96,1] & [0.99,1] \\ [0.89,1] & [0.84,1] & [0.98,1] \end{bmatrix}.$$

The reduced matrices $\boldsymbol{A}'$ *and* $\boldsymbol{C}'$ *are equivalent to five inequalities (constraints) with three variables. It should be noted that the columns of matrices* $\boldsymbol{A}'$*,* $\boldsymbol{C}'$ *and* $\boldsymbol{\mathscr{S}}'$ *correspond to the second, fourth and fifth columns of the primal matrices* $\boldsymbol{A}$*,* $\boldsymbol{C}$ *and* $\boldsymbol{\mathscr{S}}$*, respectively. So, by this simplification, in the current matrix* $\boldsymbol{\mathscr{S}}'$ *we have* $J(1) = \{2\}$*,* $J(2) = J(4) = J(5) = \{1, 2, 3\}$ *and* $J(3) = \{2, 3\}$*. As a result, it follows that* $|E|$ *is reduced from 200 to 54. Subsequently, in the current matrix* $\boldsymbol{\mathscr{S}}'$*, we have* $J(1) \subseteq J(2)$ *and* $\underline{x}_{22} = 0.95 \leq 0.95 = \underline{x}_{12}$*; that is,* $\underline{x}_{2j} \leq \underline{x}_{1j}$*,* $\forall j \in J(1)$*. Hence, the second constraint (i.e., the second row of the matrices* $\boldsymbol{A}', \boldsymbol{C}'$*, and* b_2*) is deleted by Rule 2. Similarly, since* $J(1) \subseteq J(3)$*,* $\underline{x}_{32} = 0.39 \leq 0.95 = \underline{x}_{12}$ *(i.e.,* $\underline{x}_{3j} \leq \underline{x}_{1j}$*,* $\forall j \in J(1)$*), and* $J(1) \subseteq J(5)$*,* $\underline{x}_{52} = 0.84 \leq 0.95 = \underline{x}_{12}$ *(i.e.,* $\underline{x}_{5j} \leq \underline{x}_{1j}$*,* $\forall j \in J(1)$*) then the third and fifth constraints are also deleted by Rule 2. So, we have the following reduced matrices:*

$$\boldsymbol{A}' = \begin{bmatrix} 0.29 & 0.85 & 0.52 \\ 0.98 & 0.94 & 0.91 \end{bmatrix}, \ \boldsymbol{C}' = \begin{bmatrix} 0.51 & 0.82 & 0.45 \\ 0.99 & 0.98 & 0.99 \end{bmatrix}, \ \boldsymbol{\mathscr{S}}' = \begin{bmatrix} \varnothing & [0.95,1] & \varnothing \\ [0.92,1] & [0.96,1] & [0.99,1] \end{bmatrix},$$

where $|E|$ *is reduced from 54 to 3. Finally, since in the matrices* $\boldsymbol{A}'$ *and* $\boldsymbol{\mathscr{S}}'$ *we have* $\{2\} = I(3) \subseteq I(2) = \{1, 2\}$ *and* $\varphi(d_2, \underline{x}_{22}) = 0.15 \leq 0.74 = \varphi(d_3, \underline{x}_{23})$ *(i.e.,* $\varphi(d_2, \underline{x}_{i2}) \leq \varphi(d_3, \underline{x}_{i3})$*,* $\forall i \in I(3)$*), then we can delete column 3 (i.e., the fifth column in primal matrices* $\boldsymbol{A}$*,* $\boldsymbol{C}$ *and* $\boldsymbol{\mathscr{S}}$*) by Rule 3 and set* $x_5^* = 0$ *to obtain the following new reduced matrices:*

$$\boldsymbol{A}' = \begin{bmatrix} 0.29 & 0.85 \\ 0.98 & 0.94 \end{bmatrix}, \ \boldsymbol{C}' = \begin{bmatrix} 0.51 & 0.82 \\ 0.99 & 0.98 \end{bmatrix}, \ \boldsymbol{\mathscr{S}}' = \begin{bmatrix} \varnothing & [0.95,1] \\ [0.92,1] & [0.96,1] \end{bmatrix}.$$

Thus, by the simplification rules, we set $x_1^* = x_3^* = x_5^* = 0$*. Also, based on the above matrices,* $|E|$ *is reduced further from 3 to 2.*

Step 5 *After applying the simplification rules, in the reduced problem $\boldsymbol{A}'\psi\left(\boldsymbol{C}'\omega\boldsymbol{x}\right) \geq \boldsymbol{b}'$, the set E' includes only two elements $e_1 = [2,2]$ and $e_2 = [2,1]$, that correspond to the solutions $\boldsymbol{x}(e_1) = (0, 0.96)$ and $\boldsymbol{x}(e_2) = (0.92, 0.95)$, respectively (i.e., $S_{E'} = \{\boldsymbol{x}(e_1), \boldsymbol{x}(e_2)\}$). It is easy to see that both solutions $\boldsymbol{x}(e_1)$ and $\boldsymbol{x}(e_2)$ are minimal. From this fact and the assignments $x_1^* = x_3^* = x_5^* = 0$ (resulted in Step 4), it is concluded that the main problem has two minimal solutions as follows:*

$$\underline{\boldsymbol{x}}_1 = (0, 0, 0, 0.963, 0),\ \underline{\boldsymbol{x}}_2 = (0, 0.92, 0, 0.951, 0).$$

Step 6 *By simple calculations, it follows that $f(\underline{\boldsymbol{x}}_1) = 0.74$ and $f(\underline{\boldsymbol{x}}_2) = 0.90$. Hence, by Lemma 3.12 and Theorem 4.1, the optimal solution of the primal problem is obtained as $\underline{\boldsymbol{x}}^* = \underline{\boldsymbol{x}}_1 = (0, 0, 0, 0.96, 0)$ with the objective value of $f(\underline{\boldsymbol{x}}_1) = 0.74$. Based on the activity analysis, by increasing the value of the fourth activity x_4 to 0.96 and keeping the other activity levels at zero, all constraints are satisfied and the objective function is minimized. On the other hand, in the objective function, the term $\varphi(d_j, x_j)$, $\forall j \in J$, expresses the contribution of the intersection of d_j and x_j to cost. The minimum operator is commonly used when an objective function requires conservative values in the sense that the goodness of one value, say d_j, cannot compensate the badness of another value, say x_j. However, some situations allow compensatability among the values. In this case, the minimum operator is not the best choice, and instead, product operator is preferred since it can yield better, or at least equivalent, results [ref71]. However, in this example, φ is the Dubois-Prade t-norm $\varphi(x,y) = xy/\max\{x, y, \gamma\}$ ($0 \leq \gamma \leq 1$), which covers the entire spectrum between two states "conservativity" and "compensatability"; because, $\varphi(x,y) = \min\{x,y\}$ for $\gamma = 1$, and $\varphi(x,y) = xy$ for $\gamma = 0$. In this example, the parameter γ is equal to 0.5; that is, the effect of two values d_j and x_j on each other is considered between these two states to determine their contribution to the cost. Finally, since τ is the Łukasiewicz t-conorm, we have $f(x) = \tau(\varphi(d_1, x_1), \cdots, \varphi(d_n, x_n)) = \min\{\varphi(d_1, x_1) + \cdots + \varphi(d_n, x_n), 1\}$, which means the total cost is the sum of the individual costs provided that it does not exceed the normalized upper limit.*

Moreover, from Definition 5.1, we obtain $L_1 = \varphi(d_4, \underline{x}_{14}) = 0.95$, $L_2 = \varphi(d_2, \underline{x}_{22}) = 0.94$, $L_3 = \varphi(d_5, \underline{x}_{35}) = 0.15, L_4 = \varphi(d_2, \underline{x}_{42}) = 0.92$ and $L_5 = \varphi(d_2, \underline{x}_{52}) = 0.89$. Hence, $\underline{J}(1) = \{4\}$, $\underline{J}(2) = \{2\}$, $\underline{J}(3) = \{5\}$ and $\underline{J}(4) = \underline{J}(5) = \{2\}$. So, $\underline{E}$ includes only one element $e = [4, 2, 5, 2, 2]$ whose corresponding solution is obtained by relation (8) as $\boldsymbol{x}(e) = (0, 0.94, 0, 0.95, 0.15)$. Therefore, if the objective function is defined by the maximum t-conorm, then from Theorem 5.4 (and Algorithm 2), $\boldsymbol{x}^* = (0, 0.9494, 0, 0.9512, 0.1526)$, is polynomially obtained as an optimal solution.

Conclusion

This paper presents a general class of optimization models with the objective function defined by a combination of an arbitrary continuous t-conorm and an arbitrary continuous t-norm, and the constraints are a system of generalized FRIs defined by a combination of two arbitrary continuous t-norms. In the general case, some special cases of the model and its applications were investigated, and the complete resolution of feasible solution sets for generalized FRIs was studied. It was also demonstrated that the feasibility of the problem could be determined by two necessary and sufficient conditions. In accordance with the proposed results, an algorithm was presented to find the exact optimal solution to the presented non-linear optimization model. Because finding all the minimal solutions to FRIs is an NP-complete problem, five simplification rules were derived to pre-assign values to as many decision variables as possible. Then, it has been proven that if the t-conorm is defined by the maximum in the objective function, it is always possible to solve the problem in polynomial time. As a numerical example, a special case of the non-linear model in which the objective function was defined by the combination of the Łukasiewicz t-conorm and the Dubois-Prade t-norm, and the feasible region was formed as a generalized FRI, with the product and Yager t-norms.

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