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Original paper

Pseudo-Fredholm operators in generator-based $*$-fuzzy measure spaces: Stability and Nyström approximation

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Abstract

In this paper, we study pseudo-Fredholm equations in generator-based $*$–fuzzy measure spaces. A control radius is defined from the generator and the $*$-fuzzy integral, and a Boyd-Wong-type fixed-point theorem is proved. We obtain residual, inexact-iteration, and data-stability estimates. For the Hamacher triangular norm, the pseudo-Nyström scheme is second-order convergent. Changes of generator, perturbations of the $*$-fuzzy measure, and a bi-scale L^1–L^2 generator error construction are also considered.

Keywords: $*$-fuzzy measure, generator-based pseudo-analysis, pseudo-Fredholm operator, Hamacher triangular norm, Nyström approximation.

1 Introduction

We use a Boyd–Wong argument in the $*$–fuzzy control geometry; see [7, 21]. Fuzzy and probabilistic metric counterparts are discussed in [13, 14, 18, 22].

Pseudo-analysis uses generator-induced algebraic operations; see [19, 20]. Pseudo-linear operators are treated in [5, 4], pseudo-fractional models in [24, 25], and pseudo-integrals in [17, 26, 27]. Here pseudo-linearity is verified in the $\mathcal{G}$-coordinate; cf. [2].

For the $*$–fuzzy measure and integral we use [11, 12]; related differentiation and martingale results, are in [1, 10]. The numerical part uses Nyström and quadrature results from [3, 8, 9, 16, 23]; see also [6].

2 Pseudo-linear generator structure in $*$–fuzzy measure spaces

Let $\mathcal{G} : \mathcal{J} \to [0, \infty)$ be a continuous strictly increasing bijection. The generated pseudo-addition and pseudo-multiplication are

$$\xi \oplus \eta = \mathcal{G}^{-1}\big(\mathcal{G}(\xi) + \mathcal{G}(\eta)\big), \qquad \xi \odot \eta = \mathcal{G}^{-1}\big(\mathcal{G}(\xi)\mathcal{G}(\eta)\big),$$

with $0_\oplus = \mathcal{G}^{-1}(0)$ and $1_\odot = \mathcal{G}^{-1}(1)$; see [5, 15, 19, 20]. Related pseudo-integral and discrete pseudo-integral constructions are developed in [17, 26, 27]. On a class $\mathbb{X}$ of $\mathcal{J}$-valued measurable functions, we use the pointwise operations

$$(f \oplus g)(\omega) = f(\omega) \oplus g(\omega), \qquad (\lambda \odot f)(\omega) = \lambda \odot f(\omega).$$

Assume that $\mathbb{X}$ is closed under these operations. A mapping $\mathtt{P} : \mathbb{X} \to \mathbb{X}$ is pseudo-linear if

$$\mathtt{P}\big((\lambda \odot f) \oplus (\mu \odot g)\big) = (\lambda \odot \mathtt{P}f) \oplus (\mu \odot \mathtt{P}g), \qquad f, g \in \mathbb{X},$$

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for scalars λ, μ for which the pseudo-combination is defined; cf. [5]. In particular,

$$\mathcal{G}(f \oplus g) = \mathcal{G}(f) + \mathcal{G}(g), \qquad \mathcal{G}(\lambda \odot f) = \mathcal{G}(\lambda)\mathcal{G}(f),$$

pointwise.

The symbols $\oplus, \odot$ refer to the pseudo-algebra generated by $\mathcal{G}$, whereas $*$ denotes the triangular norm of the fuzzy measure. Let $(I, \mathfrak{N})$ be a measurable space. A continuous Archimedean triangular norm $*$ and a mapping $\rho : \mathfrak{N} \times (0,\infty) \to [0,1]$ form a $*$–fuzzy measure space if $\rho(\emptyset, \zeta) = 1$, $\zeta \mapsto \rho(D, \zeta)$ is left-continuous and nondecreasing with limit 1 at infinity, and

$$\rho\left(\bigcup_{j=1}^{\infty} D_j, \zeta\right) = \mathop{*}_{j=1}^{\infty} \rho(D_j, \zeta),$$

for pairwise disjoint $D_j \in \mathfrak{N}$; see [10, 12, 22]. For a nonnegative simple function $s = \sum_{j=1}^{m} a_j \chi_{E_j}$,

$$\int_D s \,\mathrm{d}\rho(\cdot, \zeta) = \mathop{*}_{j=1}^{m} \rho\left(D \cap E_j, \frac{\zeta}{a_j}\right),$$

and for measurable $h : I \to [0, \infty)$,

$$\int_D h \,\mathrm{d}\rho(\cdot, \zeta) = \inf_{\substack{0 \le s \le h \\ s \text{ simple}}} \int_D s \,\mathrm{d}\rho(\cdot, \zeta).$$

The following relations are the only integral properties used below.

Lemma 2.1. [11, 12] *For measurable $h_1, h_2 \ge 0$, $D \in \mathfrak{N}$, $q > 0$, and $\zeta > 0$,*

$$h_1 \le h_2 \quad \Longrightarrow \quad \int_D h_1 \,\mathrm{d}\rho(\cdot, \zeta) \ge \int_D h_2 \,\mathrm{d}\rho(\cdot, \zeta),$$

and

$$\int_D (h_1 + h_2) \,\mathrm{d}\rho(\cdot, \zeta) \ge \left(\int_D h_1 \,\mathrm{d}\rho(\cdot, \zeta)\right) * \left(\int_D h_2 \,\mathrm{d}\rho(\cdot, \zeta)\right),$$

and

$$\int_D q h_1 \,\mathrm{d}\rho(\cdot, \zeta) = \int_D h_1 \,\mathrm{d}\rho\left(\cdot, \frac{\zeta}{q}\right).$$

Let $\phi : (0, \infty) \to (0, 1]$ be increasing and continuous, $\phi(\eta) \to 1$ as $\eta \to \infty$, and

$$\phi\left(\frac{\eta}{r+s}\right) \le \phi\left(\frac{\eta}{r}\right) * \phi\left(\frac{\eta}{s}\right), \qquad r, s, \eta > 0.$$

For a strict Archimedean $*$ with additive generator τ,

$$\phi_c(\eta) = \tau^{-1}\left(\frac{c}{\eta}\right), \qquad c > 0,$$

gives

$$\phi_c\left(\frac{\eta}{r+s}\right) = \phi_c\left(\frac{\eta}{r}\right) * \phi_c\left(\frac{\eta}{s}\right).$$

For the product and Hamacher norms, $\phi_c(\eta) = e^{-c/\eta}$ and $\phi_c(\eta) = \eta/(\eta + c)$, respectively.

Definition 2.2. *For $f, g \in \mathbb{X}$, let*

$$\mathscr{R}(f, g) := \left\{\delta > 0 : \int_I |\mathcal{G}(f(\omega)) - \mathcal{G}(g(\omega))| \,\mathrm{d}\rho(\omega, \zeta) \ge \phi\left(\frac{\zeta}{\delta}\right), \ \zeta > 0\right\},$$

and define

$$\mathtt{d}_{\mathcal{G}}(f, g) = \mathcal{G}^{-1}(\inf \mathscr{R}(f, g)). \tag{1}$$

We call $\mathscr{R}(f, g)$ the fuzzy control-radius set of (f, g).

Theorem 2.3. *Assume that $\mathscr{R}(f,g) \neq \varnothing$ for every $f, g \in \mathbb{X}$. Then $\mathtt{d}_{\mathcal{G}}$ is a $\mathcal{G}$–distance. Moreover, for $f, g, h \in \mathbb{X}$ and $\lambda \in \mathcal{J}$ with $\mathcal{G}(\lambda) > 0$,*

$$\mathtt{d}_{\mathcal{G}}(f \oplus h, g \oplus h) = \mathtt{d}_{\mathcal{G}}(f,g), \qquad \mathtt{d}_{\mathcal{G}}(\lambda \odot f, \lambda \odot g) = \lambda \odot \mathtt{d}_{\mathcal{G}}(f,g).$$

At the integral level,

$$\begin{aligned}\int_I |\mathcal{G}(f \oplus h) - \mathcal{G}(g \oplus h)| \, \mathrm{d}\rho(\cdot,\zeta) &= \int_I |\mathcal{G}(f) - \mathcal{G}(g)| \, \mathrm{d}\rho(\cdot,\zeta), \\ \int_I |\mathcal{G}(\lambda \odot f) - \mathcal{G}(\lambda \odot g)| \, \mathrm{d}\rho(\cdot,\zeta) &= \int_I |\mathcal{G}(f) - \mathcal{G}(g)| \, \mathrm{d}\rho\left(\cdot, \frac{\zeta}{\mathcal{G}(\lambda)}\right). \end{aligned} \tag{2}$$

Proof. If $\mathtt{d}_{\mathcal{G}}(f,g) = 0_{\oplus}$, choose $\delta_n \in \mathscr{R}(f,g)$ with $\delta_n \downarrow 0$. Then

$$\int_I |\mathcal{G}(f) - \mathcal{G}(g)| \, \mathrm{d}\rho(\cdot,\zeta) \geq \phi\left(\frac{\zeta}{\delta_n}\right) \longrightarrow 1,$$

so $f = g$ ρ-a.e. by the unit-value property of the ∗–fuzzy integral; see [12, Theorem 4.7]. Let $\delta_1 \in \mathscr{R}(f,h)$ and $\delta_2 \in \mathscr{R}(h,g)$. Since

$$|\mathcal{G}(f) - \mathcal{G}(g)| \leq |\mathcal{G}(f) - \mathcal{G}(h)| + |\mathcal{G}(h) - \mathcal{G}(g)|,$$

Lemma 2.1 gives

$$\begin{aligned} &\int_I |\mathcal{G}(f) - \mathcal{G}(g)| \, \mathrm{d}\rho(\cdot,\zeta) \\ &\quad \geq \left(\int_I |\mathcal{G}(f) - \mathcal{G}(h)| \, \mathrm{d}\rho(\cdot,\zeta)\right) * \left(\int_I |\mathcal{G}(h) - \mathcal{G}(g)| \, \mathrm{d}\rho(\cdot,\zeta)\right) \\ &\quad \geq \phi\left(\frac{\zeta}{\delta_1}\right) * \phi\left(\frac{\zeta}{\delta_2}\right) \\ &\quad \geq \phi\left(\frac{\zeta}{\delta_1 + \delta_2}\right). \end{aligned}$$

Thus $\delta_1 + \delta_2 \in \mathscr{R}(f,g)$, and infima give the $\mathcal{G}$–triangle inequality. For pseudo-translation,

$$\mathcal{G}(f \oplus h) - \mathcal{G}(g \oplus h) = \mathcal{G}(f) - \mathcal{G}(g).$$

For pseudo-scaling,

$$|\mathcal{G}(\lambda \odot f) - \mathcal{G}(\lambda \odot g)| = \mathcal{G}(\lambda)|\mathcal{G}(f) - \mathcal{G}(g)|.$$

Lemma 2.1 now gives (2). Hence, $\delta \in \mathscr{R}(f,g)$ exactly when $\mathcal{G}(\lambda)\delta \in \mathscr{R}(\lambda \odot f, \lambda \odot g)$. ☐

3 Fixed points for pseudo-operators in the ∗–fuzzy generator geometry

We consider the pseudo-affine map

$$\mathtt{T}f = b \oplus \mathtt{P}(\mathtt{N}f),$$

where $\mathtt{P} : \mathbb{X} \to \mathbb{X}$ is pseudo-linear and $\mathtt{N} : \mathbb{X} \to \mathbb{X}$ is nonlinear.

Lemma 3.1. *Let $\mathtt{P}_1, \ldots, \mathtt{P}_m : \mathbb{X} \to \mathbb{X}$ be pseudo-linear and suppose that, for $j = 1, \ldots, m$,*

$$\int_I |\mathcal{G}(\mathtt{P}_j u) - \mathcal{G}(\mathtt{P}_j v)| \, \mathrm{d}\rho(\cdot,\zeta) \geq \int_I |\mathcal{G}(u) - \mathcal{G}(v)| \, \mathrm{d}\rho\left(\cdot, \frac{\zeta}{c_j}\right),$$

with $c_j > 0$. Then the composition $\mathtt{P}_m \circ \cdots \circ \mathtt{P}_1$ satisfies the same inequality with constant $\prod_{j=1}^m c_j$. Moreover, if

$$\mathtt{S}u = \mathtt{P}_1 u \oplus \cdots \oplus \mathtt{P}_m u,$$

then every $\delta \in \mathscr{R}(u,v)$ satisfies $(c_1 + \cdots + c_m)\delta \in \mathscr{R}(\mathtt{S}u, \mathtt{S}v)$.

Proof. Iterating the assumed estimate, we obtain

$$\begin{aligned}&\int_I |\mathcal{G}((\mathtt{P}_m \circ \cdots \circ \mathtt{P}_1)u) - \mathcal{G}((\mathtt{P}_m \circ \cdots \circ \mathtt{P}_1)v)|\, \mathrm{d}\rho(\cdot,\zeta)\\ &\quad \geq \int_I |\mathcal{G}(u)-\mathcal{G}(v)|\, \mathrm{d}\rho\left(\cdot, \frac{\zeta}{\prod_{j=1}^m c_j}\right).\end{aligned}$$

For $\mathtt{S}$, we have

$$|\mathcal{G}(\mathtt{S}u) - \mathcal{G}(\mathtt{S}v)| \leq \sum_{j=1}^m |\mathcal{G}(\mathtt{P}_j u) - \mathcal{G}(\mathtt{P}_j v)|.$$

Lemma 2.1 gives

$$\begin{aligned}&\int_I |\mathcal{G}(\mathtt{S}u) - \mathcal{G}(\mathtt{S}v)|\, \mathrm{d}\rho(\cdot,\zeta)\\ &\quad \geq \mathop{*}_{j=1}^{m} \int_I |\mathcal{G}(\mathtt{P}_j u) - \mathcal{G}(\mathtt{P}_j v)|\, \mathrm{d}\rho(\cdot,\zeta)\\ &\quad \geq \mathop{*}_{j=1}^{m} \phi\left(\frac{\zeta}{c_j\delta}\right) \geq \phi\left(\frac{\zeta}{(c_1+\cdots+c_m)\delta}\right).\end{aligned}$$

□

Theorem 3.2. *Let $(\mathbb{X}, \mathtt{d}_{\mathcal{G}})$ be complete and let $\mathtt{T}f = b \oplus \mathtt{P}(\mathtt{N}f)$. Assume that $\mathtt{P}$ is pseudo-linear and that, for some $c > 0$,*

$$\int_I |\mathcal{G}(\mathtt{P}u) - \mathcal{G}(\mathtt{P}v)|\, \mathrm{d}\rho(\cdot,\zeta) \geq \int_I |\mathcal{G}(u)-\mathcal{G}(v)|\, \mathrm{d}\rho\left(\cdot, \frac{\zeta}{c}\right), \tag{3}$$

for all $u, v \in \mathbb{X}$ and $\zeta > 0$. Let $\vartheta : [0,\infty) \to [0,\infty)$ be nondecreasing and continuous, $\vartheta(0) = 0$, and suppose that, whenever $\delta \in \mathscr{R}(f,g)$,

$$\int_I |\mathcal{G}(\mathtt{N}f) - \mathcal{G}(\mathtt{N}g)|\, \mathrm{d}\rho(\cdot,\zeta) \geq \phi\left(\frac{\zeta}{\vartheta(\delta)}\right), \qquad \zeta > 0. \tag{4}$$

Set $\psi(r) = c\vartheta(r)$. Assume that ψ is right-continuous on $(0,\infty)$ and $0 < \psi(r) < r$ for $r > 0$. Then $\mathtt{T}$ has a unique fixed point f^*. For any $f_0 \in \mathbb{X}$, the iterates $f_{n+1} = \mathtt{T}f_n$ satisfy

$$\int_I |\mathcal{G}(f_n) - \mathcal{G}(f^*)|\, \mathrm{d}\rho(\cdot,\zeta) \longrightarrow 1, \qquad \zeta > 0. \tag{5}$$

Proof. For every $\delta \in \mathscr{R}(f,g)$,

$$\begin{aligned}&\int_I |\mathcal{G}(\mathtt{T}f) - \mathcal{G}(\mathtt{T}g)|\, \mathrm{d}\rho(\cdot,\zeta)\\ &\quad = \int_I |\mathcal{G}(\mathtt{PN}f) - \mathcal{G}(\mathtt{PN}g)|\, \mathrm{d}\rho(\cdot,\zeta)\\ &\quad \geq \int_I |\mathcal{G}(\mathtt{N}f) - \mathcal{G}(\mathtt{N}g)|\, \mathrm{d}\rho\left(\cdot, \frac{\zeta}{c}\right) \geq \phi\left(\frac{\zeta}{\psi(\delta)}\right).\end{aligned} \tag{6}$$

Hence, $\psi(\delta) \in \mathscr{R}(\mathtt{T}f, \mathtt{T}g)$. If $r \in \mathscr{R}(u,v)$ and $s \in \mathscr{R}(v,w)$, then

$$\int_I |\mathcal{G}(u) - \mathcal{G}(w)|\, \mathrm{d}\rho(\cdot,\zeta) \geq \phi\left(\frac{\zeta}{r}\right) * \phi\left(\frac{\zeta}{s}\right) \geq \phi\left(\frac{\zeta}{r+s}\right).$$

Thus $r + s \in \mathscr{R}(u,w)$. Take $a_0 \in \mathscr{R}(f_0, f_1)$ and put $a_{n+1} = \psi(a_n)$. Then $a_n \in \mathscr{R}(f_n, f_{n+1})$ and $a_n \downarrow a \geq 0$. Right-continuity rules out $a > 0$, because it would give $a = \psi(a) < a$. Hence $a = 0$. Assume now that (f_n) is not Cauchy. Then there are $\varepsilon > 0$ and $n_k < m_k$, $n_k \to \infty$, with m_k minimal such that $\varepsilon \notin \mathscr{R}(f_{n_k}, f_{m_k})$. For large k, we have,

$$\varepsilon \in \mathscr{R}(f_{n_k}, f_{m_k-1}), \qquad \varepsilon \notin \mathscr{R}(f_{n_k}, f_{m_k}).$$

Hence, $\varepsilon + a_{m_k-1} \in \mathscr{R}(f_{n_k}, f_{m_k})$, and (6) gives $\psi(\varepsilon + a_{m_k-1}) \in \mathscr{R}(f_{n_k+1}, f_{m_k+1})$. Thus $r_k := a_{n_k} + \psi(\varepsilon + a_{m_k-1}) + a_{m_k}$ belongs to $\mathscr{R}(f_{n_k}, f_{m_k})$. Since $r_k \longrightarrow \psi(\varepsilon) < \varepsilon$, we have $r_k < \varepsilon$ for large k, and therefore $\varepsilon \in \mathscr{R}(f_{n_k}, f_{m_k})$, a contradiction. Thus (f_n) is Cauchy. Let $f_n \to f^*$. For every $\eta > 0$, eventually $\eta \in \mathscr{R}(f_n, f^*)$, thus:

$$\int_I |\mathcal{G}(f_n) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\eta}\right).$$

First let $n \to \infty$ and then $\eta \downarrow 0$; this gives (5). Also, (6) gives $\mathtt{T} f_n \to \mathtt{T} f^*$, while $\mathtt{T} f_n = f_{n+1} \to f^*$. Thus $\mathtt{T} f^* = f^*$. If g^* is another fixed point and $\delta \in \mathscr{R}(f^*, g^*)$, then $\psi^n(\delta) \in \mathscr{R}(f^*, g^*)$ and

$$\int_I |\mathcal{G}(f^*) - \mathcal{G}(g^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\psi^n(\delta)}\right) \longrightarrow 1.$$

Hence $f^* = g^*$. $\square$

Proposition 3.3. *Let the hypotheses of Theorem 3.2 hold. If $\delta_0 \in \mathscr{R}(f_0, f_1)$ and $\sum_{j \geq 0} \psi^j(\delta_0) < \infty$, then*

$$\int_I |\mathcal{G}(f_n) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\sum_{j=n}^{\infty} \psi^j(\delta_0)}\right). \tag{7}$$

In the linear case $\vartheta(r) = Lr$ and $q = cL < 1$,

$$\int_I |\mathcal{G}(f_n) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{(1-q)\zeta}{q^n \delta_0}\right). \tag{8}$$

If $y \in \mathbb{X}$ satisfies

$$\int_I |\mathcal{G}(y) - \mathcal{G}(\mathtt{T} y)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\varepsilon}\right),$$

then

$$\int_I |\mathcal{G}(y) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{(1-q)\zeta}{\varepsilon}\right). \tag{9}$$

If an inexact orbit satisfies

$$\int_I |\mathcal{G}(y_{n+1}) - \mathcal{G}(\mathtt{T} y_n)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\varepsilon_n}\right),$$

and $r_0 \in \mathscr{R}(y_0, f^*)$, then

$$r_n = q^n r_0 + \sum_{j=0}^{n-1} q^{n-1-j} \varepsilon_j, \tag{10}$$

belongs to $\mathscr{R}(y_n, f^*)$. Finally, let $\widetilde{\mathtt{T}}$ satisfy the same factor q, with fixed point $\widetilde{f}^*$, and suppose that, uniformly in u,

$$\int_I |\mathcal{G}(\mathtt{T} u) - \mathcal{G}(\widetilde{\mathtt{T}} u)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\varepsilon}\right).$$

Then

$$\int_I |\mathcal{G}(f^*) - \mathcal{G}(\widetilde{f}^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{(1-q)\zeta}{\varepsilon}\right). \tag{11}$$

Proof. Put $\delta_j = \psi^j(\delta_0)$. For $p \geq 1$,

$$\int_I |\mathcal{G}(f_n) - \mathcal{G}(f_{n+p})| \, \mathrm{d}\rho(\cdot, \zeta) \geq \mathop{*}_{j=n}^{n+p-1} \phi\left(\frac{\zeta}{\delta_j}\right) \geq \phi\left(\frac{\zeta}{\sum_{j=n}^{n+p-1} \delta_j}\right).$$

Let $p \to \infty$ to obtain (7). For $\psi(r) = qr$, the tail is $q^n\delta_0/(1-q)$, hence (8). Let $r \in \mathscr{R}(y, f^*)$. The residual assumption and (6) give

$$\int_I |\mathcal{G}(y) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\varepsilon}\right) * \phi\left(\frac{\zeta}{qr}\right) \geq \phi\left(\frac{\zeta}{\varepsilon + qr}\right).$$

Thus $\varepsilon + qr \in \mathscr{R}(y, f^*)$. If $r_0 \in \mathscr{R}(y, f^*)$ and $r_0 \geq \varepsilon/(1-q)$, then $r_k = q^k r_0 + (1-q^k)\varepsilon/(1-q) \to \varepsilon/(1-q)$; this gives (9). If $r_n \in \mathscr{R}(y_n, f^*)$, then

$$\int_I |\mathcal{G}(y_{n+1}) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{\varepsilon_n}\right) * \phi\left(\frac{\zeta}{qr_n}\right) \geq \phi\left(\frac{\zeta}{\varepsilon_n + qr_n}\right),$$

so $r_{n+1} = \varepsilon_n + qr_n \in \mathscr{R}(y_{n+1}, f^*)$, and (10) follows by iteration. For $r \in \mathscr{R}(f^*, \widetilde{f}^*)$,

$$\int_I |\mathcal{G}(f^*) - \mathcal{G}(\widetilde{f}^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{\zeta}{qr}\right) * \phi\left(\frac{\zeta}{\varepsilon}\right) \geq \phi\left(\frac{\zeta}{qr + \varepsilon}\right).$$

The limit $\varepsilon/(1-q)$ gives (11). □

Proposition 3.4. *Let*

$$\mathtt{T}f = b \oplus \bigoplus_{\nu=1}^{m} \mathtt{P}_\nu(\mathtt{N}_\nu f),$$

where $\mathtt{P}_\nu$ *satisfies* (3) *with* $c = c_\nu$ *and* $\mathtt{N}_\nu$ *satisfies* (4) *with* $\vartheta = \vartheta_\nu$. *Define*

$$\Psi(r) = \sum_{\nu=1}^{m} c_\nu \vartheta_\nu(r). \tag{12}$$

If Ψ *is right-continuous and* $0 < \Psi(r) < r$ *for* $r > 0$, *the conclusion of Theorem 3.2 holds with* Ψ *in place of* ψ.

Proof. For every $\delta \in \mathscr{R}(f, g)$, we have

$$\int_I |\mathcal{G}(\mathtt{T}f) - \mathcal{G}(\mathtt{T}g)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \mathop{\ast}_{\nu=1}^{m} \phi\left(\frac{\zeta}{c_\nu \vartheta_\nu(\delta)}\right) \geq \phi\left(\frac{\zeta}{\Psi(\delta)}\right).$$

Hence $\Psi(\delta) \in \mathscr{R}(\mathtt{T}f, \mathtt{T}g)$, and Theorem 3.2 applies. □

If

$$|\mathcal{G}(\mathtt{P}u) - \mathcal{G}(\mathtt{P}v)| \leq \mathcal{G}(\lambda)|\mathcal{G}(u) - \mathcal{G}(v)|,$$

then Lemma 2.1 gives (3) with $c = \mathcal{G}(\lambda)$.

Theorem 3.5. *Let* $\mathcal{J}_0 \subset \mathcal{J}$ *be compact,* $\mathcal{G} \in C^1(\mathcal{J}_0)$, *and* $\mathcal{G}' > 0$. *Let* $\mathtt{H} : I \times \mathcal{J}_0 \to \mathcal{J}_0$ *be measurable in* ω *and continuously differentiable in* x, *and define* $\mathtt{N}f(\omega) = \mathtt{H}(\omega, f(\omega))$. *Assume that the pseudo-linear operator* $\mathtt{P}$ *satisfies* (3) *with constant* c *and that*

$$L_{\mathcal{G}} := \operatorname*{ess\,sup}_{\omega \in I} \sup_{x \in \mathcal{J}_0} \frac{\mathcal{G}'(\mathtt{H}(\omega, x))|\partial_x \mathtt{H}(\omega, x)|}{\mathcal{G}'(x)} < \frac{1}{c}.$$

Then $\mathtt{T}f = b \oplus \mathtt{P}(\mathtt{N}f)$ *has a unique fixed point* f^*. *With* $q = cL_{\mathcal{G}}$, *every* $\delta_0 \in \mathscr{R}(f_0, f_1)$ *satisfies*

$$\int_I |\mathcal{G}(f_n) - \mathcal{G}(f^*)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \phi\left(\frac{(1-q)\zeta}{q^n \delta_0}\right). \tag{13}$$

Proof. For a.e. ω and $x, y \in \mathcal{J}_0$, the mean-value theorem in the $\mathcal{G}$-coordinate gives

$$|\mathcal{G}(\mathtt{H}(\omega, x)) - \mathcal{G}(\mathtt{H}(\omega, y))| \leq L_{\mathcal{G}}|\mathcal{G}(x) - \mathcal{G}(y)|.$$

Lemma 2.1 gives

$$\int_I |\mathcal{G}(\mathtt{N}f) - \mathcal{G}(\mathtt{N}g)| \, \mathrm{d}\rho(\cdot, \zeta) \geq \int_I |\mathcal{G}(f) - \mathcal{G}(g)| \, \mathrm{d}\rho\left(\cdot, \frac{\zeta}{L_{\mathcal{G}}}\right).$$

Thus $\vartheta(r) = L_{\mathcal{G}} r$ in (4), and Theorem 3.2 gives (13). □

4 Pseudo-Fredholm equations in ∗–fuzzy spaces

For measurable $h : I \to \mathcal{J}$, set

$$\int_I^{\oplus} h(\mathtt{s}) \odot \mathrm{d}\mathtt{s} = \mathcal{G}^{-1}\left(\int_I \mathcal{G}(h(\mathtt{s}))\,\mathrm{d}\mathtt{s}\right),$$

and, for a measurable kernel $\mathtt{K} : I \times I \to \mathcal{J}$, define

$$(\mathtt{P}_{\mathtt{K}} f)(\mathtt{t}) = \int_I^{\oplus} \mathtt{K}(\mathtt{t},\mathtt{s}) \odot f(\mathtt{s}) \odot \mathrm{d}\mathtt{s}.$$

Whenever the right-hand side is finite, $\mathtt{P}_{\mathtt{K}}$ is pseudo-linear after applying $\mathcal{G}$. We use this direct form because pseudo-linearity is not valid for arbitrary pseudo-integrals without additional assumptions [2].

Assume that there exists $M_\rho > 0$ such that, for every measurable $h : I \to \mathcal{J}$ and $\zeta > 0$,

$$\int_I \mathcal{G}(\mathtt{P}_{\mathtt{K}} h)\,\mathrm{d}\rho(\cdot,\zeta) \geq \int_I \mathcal{G}(h)\,\mathrm{d}\rho\left(\cdot, \frac{\zeta}{M_\rho}\right). \tag{14}$$

For $\mathtt{a} : I \to \mathcal{J}$ and $\mathtt{F} : I \times \mathcal{J} \to \mathcal{J}$, consider

$$\mathtt{u}(\mathtt{t}) = \mathtt{a}(\mathtt{t}) \oplus \int_I^{\oplus} \mathtt{K}(\mathtt{t},\mathtt{s}) \odot \mathtt{F}(\mathtt{s},\mathtt{u}(\mathtt{s})) \odot \mathrm{d}\mathtt{s}. \tag{15}$$

Theorem 4.1. *Let $\mathbb{X}$ be complete with respect to Definition 2.2, and assume that $\mathtt{P}_{\mathtt{K}} : \mathbb{X} \to \mathbb{X}$ satisfies* (14). *Suppose that $\vartheta : [0,\infty) \to [0,\infty)$ is nondecreasing and continuous, $\vartheta(0) = 0$, and that the following inequality holds whenever $\delta \in \mathscr{R}(\mathtt{u},\mathtt{v})$:*

$$\int_I |\mathcal{G}(\mathtt{F}(\mathtt{s},\mathtt{u}(\mathtt{s}))) - \mathcal{G}(\mathtt{F}(\mathtt{s},\mathtt{v}(\mathtt{s})))|\,\mathrm{d}\rho(\mathtt{s},\zeta) \geq \phi\left(\frac{\zeta}{\vartheta(\delta)}\right).$$

If $\psi(r) = M_\rho \vartheta(r)$ is right-continuous on $(0,\infty)$ and $0 < \psi(r) < r$ for $r > 0$, then (15) *has a unique solution $\mathtt{u}^* \in \mathbb{X}$. If $\delta_0 \in \mathscr{R}(\mathtt{u}_0,\mathtt{u}_1)$ and $\sum_{j\geq 0} \psi^j(\delta_0) < \infty$, then*

$$\int_I |\mathcal{G}(\mathtt{u}_n) - \mathcal{G}(\mathtt{u}^*)|\,\mathrm{d}\rho(\cdot,\zeta) \geq \phi\left(\frac{\zeta}{\displaystyle\sum_{j=n}^{\infty} \psi^j(\delta_0)}\right). \tag{16}$$

Assume, in addition, that a compact pseudo-order interval $\mathcal{J}_0 \subset \mathcal{J}$ is invariant, $\mathcal{G} \in C^1(\mathcal{J}_0)$, $\mathcal{G}' > 0$, and

$$L_{\mathcal{G}} := \operatorname*{ess\,sup}_{\mathtt{s}\in I} \sup_{x\in\mathcal{J}_0} \frac{\mathcal{G}'(\mathtt{F}(\mathtt{s},x))|\partial_x \mathtt{F}(\mathtt{s},x)|}{\mathcal{G}'(x)} < \frac{1}{M_\rho}.$$

Then $q = M_\rho L_{\mathcal{G}} < 1$ and

$$\int_I |\mathcal{G}(\mathtt{u}_n) - \mathcal{G}(\mathtt{u}^*)|\,\mathrm{d}\rho(\cdot,\zeta) \geq \phi\left(\frac{(1-q)\zeta}{q^n \delta_0}\right). \tag{17}$$

Proof. Put $(\mathtt{Nu})(\mathtt{s}) = \mathtt{F}(\mathtt{s},\mathtt{u}(\mathtt{s}))$ and

$$\mathtt{h}_{\mathtt{u},\mathtt{v}}(\mathtt{s}) = \mathcal{G}^{-1}(|\mathcal{G}(\mathtt{Nu}(\mathtt{s})) - \mathcal{G}(\mathtt{Nv}(\mathtt{s}))|).$$

Since $\mathcal{G}(\mathtt{K}) \geq 0$,

$$|\mathcal{G}(\mathtt{P}_{\mathtt{K}}\mathtt{Nu}) - \mathcal{G}(\mathtt{P}_{\mathtt{K}}\mathtt{Nv})| \leq \mathcal{G}(\mathtt{P}_{\mathtt{K}}\mathtt{h}_{\mathtt{u},\mathtt{v}}).$$

Using (14),

$$\int_I |\mathcal{G}(\mathtt{P}_{\mathtt{K}}\mathtt{Nu}) - \mathcal{G}(\mathtt{P}_{\mathtt{K}}\mathtt{Nv})|\,\mathrm{d}\rho(\cdot,\zeta) \geq \int_I |\mathcal{G}(\mathtt{Nu}) - \mathcal{G}(\mathtt{Nv})|\,\mathrm{d}\rho\left(\cdot, \frac{\zeta}{M_\rho}\right).$$

Thus (3) holds with $c = M_\rho$. Theorem 3.2 gives existence, uniqueness, and (16). The mean-value estimate from Theorem 3.5 gives

$$|\mathcal{G}(\mathtt{F}(\mathtt{s},x)) - \mathcal{G}(\mathtt{F}(\mathtt{s},y))| \leq L_{\mathcal{G}}|\mathcal{G}(x) - \mathcal{G}(y)|.$$

Thus $\vartheta(r) = L_{\mathcal{G}} r$, and (17) follows from (8). □

If

$$\mathcal{G}(\mathtt{a}_0) \le \mathcal{G}(\mathtt{a}(\mathtt{t})) \le \mathcal{G}(\mathtt{a}_1), \qquad 0 \le \mathcal{G}(\mathtt{F}(\mathtt{s}, x)) \le \mathcal{G}(\mathtt{b}(\mathtt{s})),$$

then positivity of $\mathtt{P_K}$ gives

$$\mathcal{G}(\mathtt{a}_0) \le \mathcal{G}(\mathtt{Tu}(\mathtt{t})) \le \mathcal{G}(\mathtt{a}_1) + \mathcal{G}((\mathtt{P_K b})(\mathtt{t})),$$

so $L_{\mathcal{G}}$ may be computed on this invariant interval.

Theorem 4.2. *Let* $\mathtt{Tu} = \mathtt{a} \oplus \mathtt{P_K}(\mathtt{Nu})$ *and* $\widetilde{\mathtt{T}}\mathtt{u} = \widetilde{\mathtt{a}} \oplus \mathtt{P}_{\widetilde{\mathtt{K}}}(\widetilde{\mathtt{N}}\mathtt{u})$ *satisfy the linear case of Theorem 4.1 with a common* $q < 1$. *Suppose that, uniformly in* $\mathtt{u}$ *and for all* $\zeta > 0$,

$$\int_I |\mathcal{G}(\mathtt{a}) - \mathcal{G}(\widetilde{\mathtt{a}})| \,\mathrm{d}\rho(\cdot, \zeta) \ge \phi\left(\frac{\zeta}{\varepsilon_a}\right),$$

$$\int_I |\mathcal{G}(\mathtt{P_K Nu}) - \mathcal{G}(\mathtt{P}_{\widetilde{\mathtt{K}}}\mathtt{Nu})| \,\mathrm{d}\rho(\cdot, \zeta) \ge \phi\left(\frac{\zeta}{\varepsilon_K}\right),$$

and

$$\int_I |\mathcal{G}(\mathtt{P}_{\widetilde{\mathtt{K}}}\mathtt{Nu}) - \mathcal{G}(\mathtt{P}_{\widetilde{\mathtt{K}}}\widetilde{\mathtt{N}}\mathtt{u})| \,\mathrm{d}\rho(\cdot, \zeta) \ge \phi\left(\frac{\zeta}{\varepsilon_F}\right).$$

If $\mathtt{u}^*$ *and* $\widetilde{\mathtt{u}}^*$ *are the fixed points, then*

$$\int_I |\mathcal{G}(\mathtt{u}^*) - \mathcal{G}(\widetilde{\mathtt{u}}^*)| \,\mathrm{d}\rho(\cdot, \zeta) \ge \phi\left(\frac{(1-q)\zeta}{\varepsilon_a + \varepsilon_K + \varepsilon_F}\right). \tag{18}$$

Proof. For fixed $\mathtt{u}$, the three assumptions and $*$–subadditivity give

$$\int_I |\mathcal{G}(\mathtt{Tu}) - \mathcal{G}(\widetilde{\mathtt{T}}\mathtt{u})| \,\mathrm{d}\rho(\cdot, \zeta) \ge \phi\left(\frac{\zeta}{\varepsilon_a + \varepsilon_K + \varepsilon_F}\right).$$

Now (11) gives (18). □

Lemma 4.3. *Let* (I, κ) *be a finite measure space and put*

$$M := \operatorname*{ess\,sup}_{\mathtt{s} \in I} \int_I \mathcal{G}(\mathtt{K}(\mathtt{t}, \mathtt{s})) \,\mathrm{d}\kappa(\mathtt{t}) < \infty.$$

For the Hamacher fuzzy measure

$$\rho_H(D, \zeta) = \frac{\zeta}{\zeta + \kappa(D)},$$

and for the product fuzzy measure

$$\rho_P(D, \zeta) = \exp\left(-\frac{\kappa(D)}{\zeta}\right),$$

condition (14) *holds with* $M_\rho = M$.

Proof. The definitions of the corresponding $*$–fuzzy integrals give, for measurable $r \ge 0$,

$$\int_I r \,\mathrm{d}\rho_H(\cdot, \zeta) = \frac{\zeta}{\zeta + \int_I r \,\mathrm{d}\kappa}, \qquad \int_I r \,\mathrm{d}\rho_P(\cdot, \zeta) = \exp\left(-\frac{1}{\zeta} \int_I r \,\mathrm{d}\kappa\right).$$

Tonelli's theorem gives

$$\begin{aligned} \int_I \mathcal{G}(\mathtt{P_K}h)(\mathtt{t}) \,\mathrm{d}\kappa(\mathtt{t}) &= \int_I \int_I \mathcal{G}(\mathtt{K}(\mathtt{t}, \mathtt{s}))\mathcal{G}(h(\mathtt{s})) \,\mathrm{d}\kappa(\mathtt{s}) \,\mathrm{d}\kappa(\mathtt{t}) \\ &\le M \int_I \mathcal{G}(h(\mathtt{s})) \,\mathrm{d}\kappa(\mathtt{s}). \end{aligned}$$

Hence

$$\int_I \mathcal{G}(\mathtt{P_K}h) \,\mathrm{d}\rho_H(\cdot, \zeta) \ge \frac{\zeta}{\zeta + M \displaystyle\int_I \mathcal{G}(h) \,\mathrm{d}\kappa} = \int_I \mathcal{G}(h) \,\mathrm{d}\rho_H\left(\cdot, \frac{\zeta}{M}\right),$$

$$\int_I \mathcal{G}(\mathtt{P_K}h) \,\mathrm{d}\rho_P(\cdot, \zeta) \ge \exp\left(-\frac{M}{\zeta} \int_I \mathcal{G}(h) \,\mathrm{d}\kappa\right) = \int_I \mathcal{G}(h) \,\mathrm{d}\rho_P\left(\cdot, \frac{\zeta}{M}\right).$$

□

For the numerical test, take $\mathcal{G}(r) = \ln(1+r)$, put $\mathtt{v} = \mathcal{G} \circ \mathtt{u}$, and set

$$\mathcal{G}(\mathtt{a}_\lambda(\mathtt{t})) = 0.01 + 0.01\lambda + 0.04\mathtt{t} + 0.01\sin^2(2\pi\mathtt{t}),$$

$$\mathcal{G}(\mathtt{K}(\mathtt{t},\mathtt{s})) = \mathtt{c}e^{-|\mathtt{t}-\mathtt{s}|}, \qquad \mathtt{c} = (2 - 2e^{-1/2})^{-1},$$

and

$$\mathcal{G}(\mathtt{F}_\lambda(\mathtt{s}, x)) = \big(0.65 + 0.02\lambda + 0.35\sin^2(2\pi\mathtt{s})\big)\,\frac{\mathcal{G}(x)}{1+\mathcal{G}(x)}.$$

In the generator coordinate $\mathtt{v} = \mathcal{G}(\mathtt{u})$, equation (15) becomes

$$\mathtt{v}(\mathtt{t}) = 0.01 + 0.01\lambda + 0.04\mathtt{t} + 0.01\sin^2(2\pi\mathtt{t}) + \mathtt{c}\int_0^1 e^{-|\mathtt{t}-\mathtt{s}|}\big(0.65 + 0.02\lambda + 0.35\sin^2(2\pi\mathtt{s})\big)\,\frac{\mathtt{v}(\mathtt{s})}{1+\mathtt{v}(\mathtt{s})}\,\mathrm{d}\mathtt{s}. \tag{19}$$

Proposition 4.4. *For every $\lambda \geq 0$, model* (19) *has a unique nonnegative solution. After the first Picard step,*

$$\mathtt{v}_n(\mathtt{t}) \geq m_\lambda := 0.01 + 0.01\lambda, \qquad n \geq 1,$$

and the Hamacher contraction factor may be chosen as

$$q_\lambda = \frac{1 + 0.02\lambda}{(1.01 + 0.01\lambda)^2} < 1.$$

If $\delta_1 \in \mathscr{R}(\mathtt{u}_1, \mathtt{u}_2)$, then

$$\int_I |\mathcal{G}(\mathtt{u}_n) - \mathcal{G}(\mathtt{u}^*)|\,\mathrm{d}\rho_H(\cdot,\zeta) \geq \frac{(1-q_\lambda)\zeta}{(1-q_\lambda)\zeta + q_\lambda^{n-1}\delta_1}, \qquad n \geq 1.$$

Proof. The kernel is normalized by

$$\sup_{\mathtt{s}\in[0,1]} \int_0^1 \mathtt{c}e^{-|\mathtt{t}-\mathtt{s}|}\,\mathrm{d}\mathtt{t} = 1,$$

hence $M_H = 1$ by Lemma 4.3. After one Picard step, positivity gives $\mathtt{v}_n \geq m_\lambda$. Put $B_\lambda(\mathtt{s}) = 0.65 + 0.02\lambda + 0.35\sin^2(2\pi\mathtt{s})$,

$$\left| B_\lambda(\mathtt{s})\frac{x}{1+x} - B_\lambda(\mathtt{s})\frac{y}{1+y} \right| \leq \frac{1+0.02\lambda}{(1+m_\lambda)^2}|x-y|.$$

Also,

$$(1.01 + 0.01\lambda)^2 - (1 + 0.02\lambda) = 0.0201 + 0.0002\lambda + 0.0001\lambda^2 > 0.$$

Now apply Theorem 4.1 with $\phi(\eta) = \eta/(\eta+1)$. □

5 Hamacher ∗–fuzzy pseudo-Nyström scheme

For Nyström discretizations and quadrature error estimates, see [3, 6, 8, 9, 16, 23].

Let $0 = \mathtt{t}_0 < \cdots < \mathtt{t}_N = 1$, $h = 1/N$, and let $\omega_0 = \omega_N = h/2$, $\omega_j = h$ for $1 \leq j \leq N-1$. Put $\widehat{\omega}_j = \mathcal{G}^{-1}(\omega_j)$. The trapezoidal pseudo-Nyström discretization of (15) is

$$(\mathtt{T}_N U)_i = \mathtt{a}_\lambda(\mathtt{t}_i) \oplus \bigoplus_{j=0}^{N} \widehat{\omega}_j \odot \mathtt{K}(\mathtt{t}_i, \mathtt{t}_j) \odot \mathtt{F}_\lambda(\mathtt{t}_j, U_j), \qquad 0 \leq i \leq N.$$

Application of $\mathcal{G}$ gives

$$V_i^{(k+1)} = A_i + \sum_{j=0}^{N} \omega_j k_{ij} B_j \frac{V_j^{(k)}}{1 + V_j^{(k)}},$$

where

$$V_i^{(k)} = \mathcal{G}(U_i^{(k)}), \qquad A_i = \mathcal{G}(\mathtt{a}_\lambda(\mathtt{t}_i)), \qquad k_{ij} = \mathcal{G}(\mathtt{K}(\mathtt{t}_i, \mathtt{t}_j)),$$

and $B_j = 0.65 + 0.02\lambda + 0.35\sin^2(2\pi\mathtt{t}_j)$. For grid vectors U, V, set $e_j^{\mathcal{G}}(U,V) = |\mathcal{G}(U_j) - \mathcal{G}(V_j)|$. The full-grid Hamacher $*$–fuzzy error integral is

$$I_{H,N}^{\mathcal{G}}(U,V;\zeta) = \frac{\zeta}{\zeta + \sum_{j=0}^{N} \omega_j e_j^{\mathcal{G}}(U,V)}. \tag{20}$$

A number $r > 0$ is called a Hamacher control radius for (U,V) if

$$I_{H,N}^{\mathcal{G}}(U,V;\zeta) \geq \frac{\zeta}{\zeta + r}, \qquad \zeta > 0.$$

For $D_i = [0, \mathtt{t}_i]$, $i \geq 1$, set

$$Q_i(U,V) = h\left(\frac{1}{2}e_0^{\mathcal{G}} + \sum_{j=1}^{i-1} e_j^{\mathcal{G}} + \frac{1}{2}e_i^{\mathcal{G}}\right),$$

The corresponding Hamacher value is $\zeta/(\zeta + Q_i(U,V))$; for $i = 0$ it is 1. Define

$$(\mathtt{P}_{\mathtt{K},N}Z)_i = \bigoplus_{j=0}^{N} \widehat{\omega}_j \odot \mathtt{K}(\mathtt{t}_i, \mathtt{t}_j) \odot Z_j.$$

The map $\mathtt{P}_{\mathtt{K},N}$ is pseudo-linear. With $M_{H,N} = \max_{0\leq j\leq N} \sum_{i=0}^{N} \omega_i k_{ij}$ we have

$$\sum_{i=0}^{N} \omega_i |\mathcal{G}((\mathtt{P}_{\mathtt{K},N}Z)_i) - \mathcal{G}((\mathtt{P}_{\mathtt{K},N}W)_i)| \leq M_{H,N} \sum_{j=0}^{N} \omega_j |\mathcal{G}(Z_j) - \mathcal{G}(W_j)|,$$

so

$$I_{H,N}^{\mathcal{G}}(\mathtt{P}_{\mathtt{K},N}Z, \mathtt{P}_{\mathtt{K},N}W;\zeta) \geq I_{H,N}^{\mathcal{G}}\left(Z, W; \frac{\zeta}{M_{H,N}}\right).$$

Theorem 5.1. *Assume that $\mathtt{T}_N$ leaves invariant a closed region on which $\mathcal{G}(U_j) \geq m_0 > 0$ for every component, and put*

$$q_N = \max_{0\leq j\leq N}\left\{\frac{B_j}{(1+m_0)^2}\sum_{i=0}^{N}\omega_i k_{ij}\right\}.$$

If $q_N < 1$, equation $U = \mathtt{T}_N U$ has a unique fixed point U_N^* and

$$I_{H,N}^{\mathcal{G}}(\mathtt{T}_N U, \mathtt{T}_N V;\zeta) \geq I_{H,N}^{\mathcal{G}}\left(U, V; \frac{\zeta}{q_N}\right). \tag{21}$$

If $\delta_k = \sum_{j=0}^{N} \omega_j |V_j^{(k+1)} - V_j^{(k)}|$, then

$$I_{H,N}^{\mathcal{G}}(U^{(k)}, U_N^*;\zeta) \geq \frac{\zeta}{\zeta + \frac{\delta_k}{1-q_N}}. \tag{22}$$

More generally, if an inexact step satisfies

$$I_{H,N}^{\mathcal{G}}(U^{(k+1)}, \mathtt{T}_N U^{(k)};\zeta) \geq \frac{\zeta}{\zeta + \varepsilon_k},$$

then every Hamacher control radius r_k for $(U^{(k)}, U_N^*)$ generates the control radius $r_{k+1} = q_N r_k + \varepsilon_k$. Hence

$$r_k \leq q_N^k r_0 + \sum_{j=0}^{k-1} q_N^{k-1-j}\varepsilon_j. \tag{23}$$

Proof. Since $r \mapsto r/(1+r)$ has derivative $(1+r)^{-2}$, for U, V in the invariant region,

$$\begin{aligned} I^{\mathcal{G}}_{H,N}(\mathtt{T}_N U, \mathtt{T}_N V; \zeta) &= \frac{\zeta}{\zeta + \sum_{i=0}^{N} \omega_i |\mathcal{G}((\mathtt{T}_N U)_i) - \mathcal{G}((\mathtt{T}_N V)_i)|} \\ &\geq \frac{\zeta}{\zeta + \sum_{i=0}^{N} \omega_i \sum_{j=0}^{N} \omega_j k_{ij} B_j \frac{|\mathcal{G}(U_j) - \mathcal{G}(V_j)|}{(1+\mathcal{G}(U_j))(1+\mathcal{G}(V_j))}} \\ &\geq \frac{\zeta}{\zeta + q_N \sum_{j=0}^{N} \omega_j |\mathcal{G}(U_j) - \mathcal{G}(V_j)|} \\ &= I^{\mathcal{G}}_{H,N}\left(U, V; \frac{\zeta}{q_N}\right). \end{aligned}$$

The fixed point follows from (21) and Theorem 3.2. For $j \geq 0$,

$$I^{\mathcal{G}}_{H,N}(U^{(k+j)}, U^{(k+j+1)}; \zeta) \geq I^{\mathcal{G}}_{H,N}\left(U^{(k)}, U^{(k+1)}; \frac{\zeta}{q_N^j}\right) = \frac{\zeta}{\zeta + q_N^j \delta_k}.$$

Applying $*_H$–subadditivity,

$$I^{\mathcal{G}}_{H,N}(U^{(k)}, U^{(k+p)}; \zeta) \geq \mathop{*_H}_{j=0}^{p-1} \frac{\zeta}{\zeta + q_N^j \delta_k} = \frac{\zeta}{\zeta + \delta_k \sum_{j=0}^{p-1} q_N^j}.$$

Let $p \to \infty$ and use $U^{(k+p)} \to U_N^*$ to obtain (22). For an inexact step, let r_k be a Hamacher control radius. Then (21) and $*_H$–subadditivity give

$$\begin{aligned} I^{\mathcal{G}}_{H,N}(U^{(k+1)}, U_N^*; \zeta) &\geq I^{\mathcal{G}}_{H,N}(U^{(k+1)}, \mathtt{T}_N U^{(k)}; \zeta) *_H I^{\mathcal{G}}_{H,N}(\mathtt{T}_N U^{(k)}, \mathtt{T}_N U_N^*; \zeta) \\ &\geq \frac{\zeta}{\zeta + \varepsilon_k} *_H \frac{\zeta}{\zeta + q_N r_k} \\ &= \frac{\zeta}{\zeta + \varepsilon_k + q_N r_k}. \end{aligned}$$

Hence $q_N r_k + \varepsilon_k$ is the next control radius, and iteration gives (23). ☐

Theorem 5.2. *Let $U^N = (\mathtt{u}(\mathtt{t}_0), \ldots, \mathtt{u}(\mathtt{t}_N))$ be the exact nodal vector of (15) and define*

$$R_N = \sum_{i=0}^{N} \omega_i |\mathcal{G}(U_i^N) - \mathcal{G}((\mathtt{T}_N U^N)_i)|.$$

Assume $q_N \leq q < 1$ for all sufficiently large N. Then

$$I^{\mathcal{G}}_{H,N}(U^N, U_N^*; \zeta) \geq \frac{\zeta}{\zeta + \frac{R_N}{1-q}}. \tag{24}$$

Consequently, $R_N \to 0$ implies $I^{\mathcal{G}}_{H,N}(U^N, U_N^*; \zeta) \to 1$ for every $\zeta > 0$.

Proof. Set

$$E_N = \sum_{i=0}^{N} \omega_i |\mathcal{G}(U_i^N) - \mathcal{G}((U_N^*)_i)|.$$

Use $U_N^* = \mathtt{T}_N U_N^*$ in (21) and apply Hamacher $*_H$–subadditivity:

$$\begin{aligned}\frac{\zeta}{\zeta+E_N} &= I^{\mathcal{G}}_{H,N}(U^N, U_N^*;\zeta)\\ &\geq I^{\mathcal{G}}_{H,N}(U^N, \mathtt{T}_N U^N;\zeta) *_H I^{\mathcal{G}}_{H,N}(\mathtt{T}_N U^N, \mathtt{T}_N U_N^*;\zeta)\\ &\geq \frac{\zeta}{\zeta+R_N} *_H I^{\mathcal{G}}_{H,N}\left(U^N, U_N^*; \frac{\zeta}{q_N}\right)\\ &= \frac{\zeta}{\zeta+R_N+q_N E_N}.\end{aligned}$$

Thus $E_N \leq R_N + q_N E_N$, and

$$E_N \leq \frac{R_N}{1-q_N} \leq \frac{R_N}{1-q}.$$

Together with (20), this proves (24). The last assertion is immediate when $R_N \to 0$. □

Lemma 5.3. *Let U^N be the exact nodal vector of (15), with $U_i^N = \mathtt{u}(\mathtt{t}_i)$. Write $\mathtt{v} = \mathcal{G}\circ\mathtt{u}$. For each node $\mathtt{t}_i$, set*

$$\mathtt{g}_i(\mathtt{s}) = \mathcal{G}(\mathtt{K}(\mathtt{t}_i,\mathtt{s}))\mathcal{G}(\mathtt{F}_\lambda(\mathtt{s},\mathtt{u}(\mathtt{s}))).$$

Assume that $\mathtt{g}_i \in C^2([0,\mathtt{t}_i]) \cap C^2([\mathtt{t}_i,1])$ and

$$\sup_{N\geq N_0}\max_{0\leq i\leq N}\left(\|\mathtt{g}_i''\|_{L^\infty(0,\mathtt{t}_i)} + \|\mathtt{g}_i''\|_{L^\infty(\mathtt{t}_i,1)}\right) \leq M_2,$$

where the term over a degenerate interval is omitted. Then, with $C_0 = M_2/12$, we have for every $\zeta > 0$

$$I^{\mathcal{G}}_{H,N}(U^N, \mathtt{T}_N U^N;\zeta) \geq \frac{\zeta}{\zeta + C_0 h^2}.$$

Proof. At the node $\mathtt{t}_i$, subtracting the discrete equation from the continuous one gives

$$\left|\mathcal{G}(U_i^N) - \mathcal{G}((\mathtt{T}_N U^N)_i)\right| = \left|\int_0^1 \mathtt{g}_i(\mathtt{s})\,\mathrm{d}\mathtt{s} - Q_N \mathtt{g}_i\right|,$$

where Q_N denotes the composite trapezoidal rule. Since $\mathtt{t}_i$ is a grid point, apply the remainder estimate on $[0,\mathtt{t}_i]$ and $[\mathtt{t}_i,1]$:

$$\left|\int_0^1 \mathtt{g}_i(\mathtt{s})\,\mathrm{d}\mathtt{s} - Q_N\mathtt{g}_i\right| \leq \frac{h^2}{12}\left(\mathtt{t}_i\|\mathtt{g}_i''\|_{L^\infty(0,\mathtt{t}_i)} + (1-\mathtt{t}_i)\|\mathtt{g}_i''\|_{L^\infty(\mathtt{t}_i,1)}\right) \leq C_0 h^2.$$

Therefore

$$\sum_{i=0}^N \omega_i\left|\mathcal{G}(U_i^N) - \mathcal{G}((\mathtt{T}_N U^N)_i)\right| \leq C_0 h^2 \sum_{i=0}^N \omega_i = C_0 h^2.$$

Now use (20). □

Theorem 5.4. *Assume the hypotheses of Lemma 5.3 and Theorem 5.1. Suppose that $0 < q_N \leq q < 1$ for all $N \geq N_0$. Let U_N^* be the discrete pseudo-Nyström fixed point. Then, for every $\zeta > 0$ and $N \geq N_0$,*

$$I^{\mathcal{G}}_{H,N}(U^N, U_N^*;\zeta) \geq \frac{\zeta}{\zeta + \dfrac{C_0}{1-q}h^2}.$$

Also,

$$1 - I^{\mathcal{G}}_{H,N}(U^N, U_N^*;\zeta) \leq \frac{C_0}{\zeta(1-q)}h^2.$$

Proof. Set

$$E_N = \sum_{i=0}^{N} \omega_i \left| \mathcal{G}(U_i^N) - \mathcal{G}((U_N^*)_i) \right|.$$

With $U_N^* = \mathtt{T}_N U_N^*$, the triangle inequality and Hamacher $*$–subadditivity yield

$$\begin{aligned} I_{H,N}^{\mathcal{G}}(U^N, U_N^*; \zeta) &\geq I_{H,N}^{\mathcal{G}}(U^N, \mathtt{T}_N U^N; \zeta) *_H I_{H,N}^{\mathcal{G}}(\mathtt{T}_N U^N, \mathtt{T}_N U_N^*; \zeta) \\ &\geq \frac{\zeta}{\zeta + C_0 h^2} *_H I_{H,N}^{\mathcal{G}}\left(U^N, U_N^*; \frac{\zeta}{q_N}\right), \end{aligned}$$

where Lemma 5.3 and (21) were used. Moreover,

$$I_{H,N}^{\mathcal{G}}\left(U^N, U_N^*; \frac{\zeta}{q_N}\right) = \frac{\zeta}{\zeta + q_N E_N}.$$

For $a, b \geq 0$,

$$\frac{\zeta}{\zeta + a} *_H \frac{\zeta}{\zeta + b} = \frac{\zeta}{\zeta + a + b}, \qquad a, b \geq 0.$$

Hence

$$I_{H,N}^{\mathcal{G}}(U^N, U_N^*; \zeta) \geq \frac{\zeta}{\zeta + C_0 h^2 + q_N E_N}.$$

Also, (20) gives

$$I_{H,N}^{\mathcal{G}}(U^N, U_N^*; \zeta) = \frac{\zeta}{\zeta + E_N}.$$

Thus $E_N \leq C_0 h^2 + q_N E_N$, so $E_N \leq C_0 h^2/(1 - q_N) \leq C_0 h^2/(1 - q)$. With (20),

$$1 - I_{H,N}^{\mathcal{G}}(U^N, U_N^*; \zeta) = \frac{E_N}{\zeta + E_N} \leq \frac{E_N}{\zeta} \leq \frac{C_0}{\zeta(1 - q)} h^2.$$

□

For the computations, we take $\lambda = 0.4$, $\mathcal{G}(r) = \ln(1 + r)$, $N_{\text{ref}} = 1600$, $\text{Tol}_H = 10^{-12}$, and $\zeta = 10^{-6}$. The iteration is stopped when

$$I_{H,N}^{\mathcal{G}}(U^{(k+1)}, U^{(k)}; \zeta) > \frac{\zeta}{\zeta + \text{Tol}_H}.$$

For the reference grid,

$$0.0923553 \leq \mathtt{u}(0.4, \mathtt{t}) \leq 0.158300.$$

Selected values are given in Table 1.

Table 1: Selected values of the pseudo-Fredholm solution at $\lambda = 0.4$.

$\mathtt{t}$	$\mathcal{G}(\mathtt{u}(0.4, \mathtt{t}))$	$\mathtt{u}(0.4, \mathtt{t})$
0.00	0.088336	0.092355
0.25	0.123844	0.131840
0.50	0.129489	0.138247
0.75	0.146937	0.158281
1.00	0.132128	0.141254

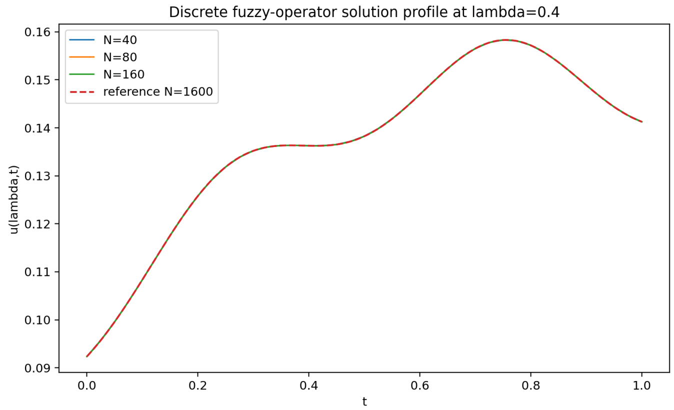

Figure 1: Pseudo-Nyström approximations of $\mathtt{u}(0.4, \mathtt{t})$.

For a grid solution U_N and the restricted reference vector U_N^{ref}, Figure 2 displays the prefix Hamacher integrals of the generator error.

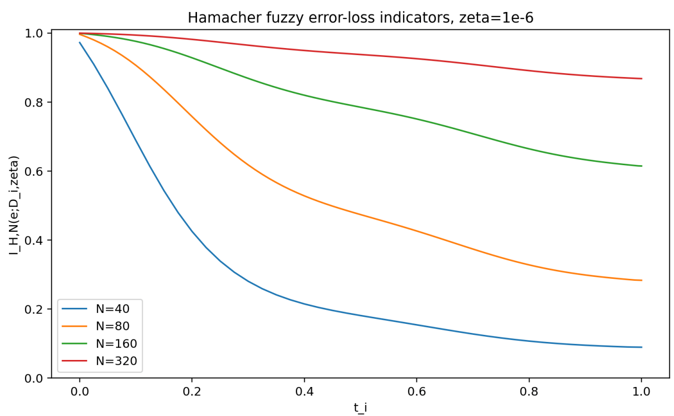

Figure 2: Hamacher $*$–fuzzy prefix error integrals for the generator error.

At $N = 80$, starting from $\mathcal{G}(U_j^{(0)}) = 5$, the Picard indicator satisfies

$$I_{H,N}^{\mathcal{G}}(U^{(k+1)}, U^{(k)}; 10^{-6}) \longrightarrow 1.$$

Table 2: Picard iteration at $N = 80$ with $\mathcal{G}(U_j^{(0)}) = 5$.

k	δ_k	$I_{H,N}^{\mathcal{G}}(U^{(k+1)}, U^{(k)}; 10^{-6})$
1	4.310508×10^{0}	0.000000232
2	3.309545×10^{-1}	0.000003022
3	1.126168×10^{-1}	0.000008880
6	1.540520×10^{-2}	0.000064909
11	1.202269×10^{-3}	0.000831070
21	9.285287×10^{-6}	0.097226258
31	7.253862×10^{-8}	0.932367355
55	6.357965×10^{-13}	0.999999364

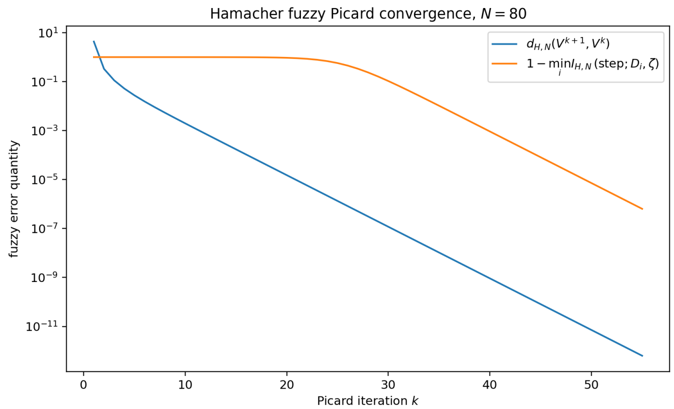

Figure 3: Picard convergence in the Hamacher ∗–fuzzy generator error.

Table 3: Hamacher-fuzzy refinement of the pseudo-Nyström scheme.

N	Iter.	Picard step	Generator error	Order	q_N	$I^{\mathcal{G}}_{H,N}$
40	53	7.78×10^{-13}	1.01×10^{-5}	—	0.933988	0.089698
80	53	7.77×10^{-13}	2.53×10^{-6}	2.006	0.935544	0.283616
160	53	7.77×10^{-13}	6.26×10^{-7}	2.012	0.935535	0.614883
320	53	7.77×10^{-13}	1.52×10^{-7}	2.045	0.935533	0.868197
800	53	7.77×10^{-13}	1.90×10^{-8}	2.269	0.935563	0.981378

The rates 2.006, 2.012, and 2.045 agree with Theorem 5.4. The last entry uses a finite reference grid. With $N_{\text{ref}} = 1600$, the usual h^2 reference correction gives 2.2694 for $(N_1, N_2) = (320, 800)$.

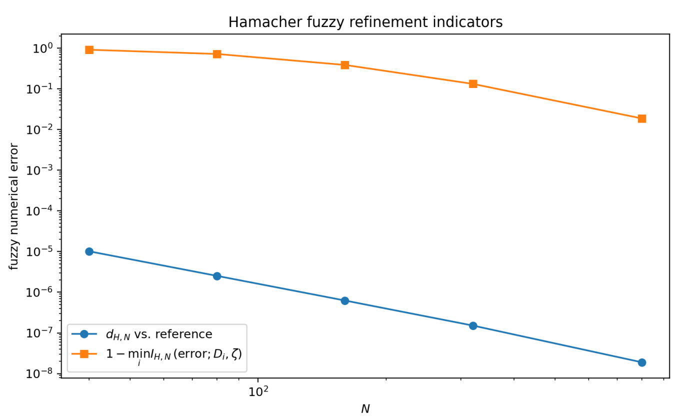

Figure 4: Generator error and Hamacher ∗–fuzzy refinement indicator.

6 Generator changes and ∗–fuzzy measure perturbations

Proposition 6.1. *Let $\mathcal{G} : \mathcal{J} \to [0, \infty)$ and $\widetilde{\mathcal{G}} : \widetilde{\mathcal{J}} \to [0, \infty)$ be continuous, strictly increasing bijections and define $\Xi = \widetilde{\mathcal{G}}^{-1} \circ \mathcal{G}$. Then, for $x, y \in \mathcal{J}$, we have*

$$\Xi(x \oplus_{\mathcal{G}} y) = \Xi(x) \oplus_{\widetilde{\mathcal{G}}} \Xi(y), \qquad \Xi(x \odot_{\mathcal{G}} y) = \Xi(x) \odot_{\widetilde{\mathcal{G}}} \Xi(y).$$

and whenever the pseudo-integrals are finite,

$$\Xi\left(\int_I^{\oplus_{\mathcal{G}}} h(s) \odot_{\mathcal{G}} \mathrm{d}s \right) = \int_I^{\oplus_{\widetilde{\mathcal{G}}}} \Xi(h(s)) \odot_{\widetilde{\mathcal{G}}} \mathrm{d}s.$$

For the pseudo-Fredholm data $(\mathtt{a},\mathtt{K},\mathtt{F})$*, put*

$$\widetilde{\mathtt{a}}=\Xi\circ\mathtt{a},\qquad \widetilde{\mathtt{K}}=\Xi\circ\mathtt{K},\qquad \widetilde{\mathtt{F}}(s,y)=\Xi\big(\mathtt{F}(s,\Xi^{-1}y)\big)\,.$$

If $\mathtt{T}_{\mathcal{G}}$ *and* $\widetilde{\mathtt{T}}$ *denote the corresponding pseudo-Fredholm maps, then*

$$\Xi\circ\mathtt{T}_{\mathcal{G}}=\widetilde{\mathtt{T}}\circ\Xi,$$
$$\widetilde{\mathcal{G}}\big(\widetilde{\mathtt{T}}(\Xi u)\big)=\mathcal{G}(\mathtt{T}_{\mathcal{G}}u).$$

For a fixed $*$*–fuzzy measure* ρ *and control function* ϕ*,*

$$\mathscr{R}_{\mathcal{G},\rho}(f,g)=\mathscr{R}_{\widetilde{\mathcal{G}},\rho}(\Xi f,\Xi g), \tag{25}$$
$$\mathtt{d}_{\widetilde{\mathcal{G}},\rho}(\Xi f,\Xi g)=\Xi(\mathtt{d}_{\mathcal{G},\rho}(f,g))\,.$$

Thus

$$\mathtt{T}_{\mathcal{G}}u^*=u^*\quad\Longleftrightarrow\quad\widetilde{\mathtt{T}}(\Xi u^*)=\Xi u^*,$$

Equation (25) *identifies the two control-radius geometries. In particular, convergence, completeness, and linear contraction factors are unchanged. If the generators and nonlinearities are* C^1 *on the corresponding intervals, then*

$$\frac{\widetilde{\mathcal{G}}'(\widetilde{\mathtt{F}}(s,\Xi x))\,\big|\partial_y\widetilde{\mathtt{F}}(s,\Xi x)\big|}{\widetilde{\mathcal{G}}'(\Xi x)}=\frac{\mathcal{G}'(\mathtt{F}(s,x))|\partial_x\mathtt{F}(s,x)|}{\mathcal{G}'(x)}.$$

The coefficient in Theorem 4.1 is invariant as well.

Proof. The identity $\widetilde{\mathcal{G}}(\Xi x)=\mathcal{G}(x)$ gives

$$\widetilde{\mathcal{G}}(\Xi(x\oplus_{\mathcal{G}}y))=\mathcal{G}(x)+\mathcal{G}(y)=\widetilde{\mathcal{G}}\big(\Xi x\oplus_{\widetilde{\mathcal{G}}}\Xi y\big)\,,$$
$$\widetilde{\mathcal{G}}(\Xi(x\odot_{\mathcal{G}}y))=\mathcal{G}(x)\mathcal{G}(y)=\widetilde{\mathcal{G}}\big(\Xi x\odot_{\widetilde{\mathcal{G}}}\Xi y\big)\,.$$

Injectivity gives the two identities. Moreover,

$$\Xi\left(\int_I^{\oplus_{\mathcal{G}}}h\odot_{\mathcal{G}}\mathrm{d}s\right)=\widetilde{\mathcal{G}}^{-1}\left(\int_I\mathcal{G}(h(s))\,\mathrm{d}s\right)=\widetilde{\mathcal{G}}^{-1}\left(\int_I\widetilde{\mathcal{G}}(\Xi h(s))\,\mathrm{d}s\right).$$

Also,

$$\widetilde{\mathcal{G}}\big(\widetilde{\mathtt{T}}(\Xi u)(t)\big)=\mathcal{G}(\mathtt{a}(t))+\int_I\mathcal{G}(\mathtt{K}(t,s))\mathcal{G}(\mathtt{F}(s,u(s)))\,\mathrm{d}s=\mathcal{G}(\mathtt{T}_{\mathcal{G}}u(t)),$$

and

$$\Big|\widetilde{\mathcal{G}}(\Xi f)-\widetilde{\mathcal{G}}(\Xi g)\Big|=|\mathcal{G}(f)-\mathcal{G}(g)|,$$

hence the two radius sets coincide and (25) follows. Also,

$$\mathtt{d}_{\widetilde{\mathcal{G}},\rho}(\Xi f,\Xi g)=\widetilde{\mathcal{G}}^{-1}(\inf\mathscr{R}_{\mathcal{G},\rho}(f,g))=\Xi(\mathtt{d}_{\mathcal{G},\rho}(f,g))\,.$$

Differentiating

$$\widetilde{\mathcal{G}}(\widetilde{\mathtt{F}}(s,\Xi x))=\mathcal{G}(\mathtt{F}(s,x)),\qquad\widetilde{\mathcal{G}}(\Xi x)=\mathcal{G}(x),$$

gives the differential identity. ☐

Next keep $\mathcal{G}$ fixed and compare ρ with $\widetilde{\rho}$.

Lemma 6.2. *Let* ρ *and* $\widetilde{\rho}$ *be* $*$*–fuzzy measures generated by the same continuous Archimedean triangular norm and controlled by the same* ϕ*. Assume that, for some* $\alpha,\beta>0$*,*

$$\widetilde{\rho}(D,\zeta)\ge\rho\left(D,\frac{\zeta}{\alpha}\right),\qquad\rho(D,\zeta)\ge\widetilde{\rho}\left(D,\frac{\zeta}{\beta}\right), \tag{26}$$

for $D \in \mathfrak{N}$ and $\zeta > 0$. Then, for every measurable $h \geq 0$,

$$\int_I h\,\mathrm{d}\rho\Big(\cdot, \frac{\zeta}{\alpha}\Big) \leq \int_I h\,\mathrm{d}\widetilde{\rho}(\cdot,\zeta) \leq \int_I h\,\mathrm{d}\rho(\cdot,\beta\zeta).$$

Moreover,

$$\delta \in \mathscr{R}_{\mathcal{G},\rho}(f,g) \Longrightarrow \alpha\delta \in \mathscr{R}_{\mathcal{G},\widetilde{\rho}}(f,g), \qquad \delta \in \mathscr{R}_{\mathcal{G},\widetilde{\rho}}(f,g) \Longrightarrow \beta\delta \in \mathscr{R}_{\mathcal{G},\rho}(f,g), \tag{27}$$

and hence

$$\frac{1}{\beta}\inf \mathscr{R}_{\mathcal{G},\rho}(f,g) \leq \inf \mathscr{R}_{\mathcal{G},\widetilde{\rho}}(f,g) \leq \alpha \inf \mathscr{R}_{\mathcal{G},\rho}(f,g). \tag{28}$$

Equation (28) gives the same convergent and Cauchy sequences for both geometries; completeness is therefore equivalent. If ψ controls $\mathtt{T}$ for ρ, then $\widetilde{\psi}(r) = \alpha\psi(\beta r)$ controls $\mathtt{T}$ in the $\widetilde{\rho}$–geometry; in particular, $\psi(r) = qr$ gives $\widetilde{q} = \alpha\beta q$.

Proof. For $s = \sum_{j=1}^m a_j\chi_{E_j} \geq 0$, monotonicity of $*$ and (26) give

$$\int_I s\,\mathrm{d}\widetilde{\rho}(\cdot,\zeta) = \mathop{*}_{j=1}^{m} \widetilde{\rho}\Big(E_j, \frac{\zeta}{a_j}\Big) \geq \mathop{*}_{j=1}^{m} \rho\Big(E_j, \frac{\zeta}{\alpha a_j}\Big) = \int_I s\,\mathrm{d}\rho\Big(\cdot, \frac{\zeta}{\alpha}\Big).$$

Taking the infimum over $0 \leq s \leq h$ gives the first bound; the second is (26) at $\beta\zeta$. If $\delta \in \mathscr{R}_{\mathcal{G},\rho}(f,g)$, then

$$\int_I |\mathcal{G}(f) - \mathcal{G}(g)|\,\mathrm{d}\widetilde{\rho}(\cdot,\zeta) \geq \int_I |\mathcal{G}(f) - \mathcal{G}(g)|\,\mathrm{d}\rho\Big(\cdot, \frac{\zeta}{\alpha}\Big) \geq \phi\Big(\frac{\zeta}{\alpha\delta}\Big),$$

so $\alpha\delta \in \mathscr{R}_{\mathcal{G},\widetilde{\rho}}(f,g)$. Interchanging the two measures gives the second implication in (27); taking infima gives (28). The same estimate applies to pairs (f_n, f_m), so completeness is equivalent. For the comparison function,

$$\begin{aligned} r \in \mathscr{R}_{\mathcal{G},\widetilde{\rho}}(f,g) &\Longrightarrow \beta r \in \mathscr{R}_{\mathcal{G},\rho}(f,g) \\ &\Longrightarrow \psi(\beta r) \in \mathscr{R}_{\mathcal{G},\rho}(\mathtt{T}f, \mathtt{T}g) \\ &\Longrightarrow \alpha\psi(\beta r) \in \mathscr{R}_{\mathcal{G},\widetilde{\rho}}(\mathtt{T}f, \mathtt{T}g). \end{aligned}$$

Hence $\widetilde{\psi}(r) = \alpha\psi(\beta r)$; for $\psi(r) = qr$, $\widetilde{q} = \alpha\beta q$. In the Boyd–Wong case one also assumes $0 < \widetilde{\psi}(r) < r$. □

Theorem 6.3. *Let $\mathbb{X}$ be complete in the $(\mathcal{G},\rho)$ geometry, let $\widetilde{\mathbb{X}} = \Xi(\mathbb{X})$, and assume (26). Let $\mathtt{T} : \mathbb{X} \to \mathbb{X}$ and $\widetilde{\mathtt{T}} : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$ be pseudo-Fredholm maps with*

$$\begin{aligned} \delta \in \mathscr{R}_{\mathcal{G},\rho}(u,v) &\Longrightarrow q\delta \in \mathscr{R}_{\mathcal{G},\rho}(\mathtt{T}u, \mathtt{T}v), \\ r \in \mathscr{R}_{\widetilde{\mathcal{G}},\widetilde{\rho}}(y,z) &\Longrightarrow \widetilde{q}r \in \mathscr{R}_{\widetilde{\mathcal{G}},\widetilde{\rho}}(\widetilde{\mathtt{T}}y, \widetilde{\mathtt{T}}z), \end{aligned}$$

where $0 < q, \widetilde{q} < 1$. Set $\mathtt{S} = \Xi^{-1} \circ \widetilde{\mathtt{T}} \circ \Xi$ and $Q = \max\{q, \alpha\beta\widetilde{q}\} < 1$. Assume, uniformly in $u \in \mathbb{X}$,

$$\int_I |\mathcal{G}(\mathtt{T}u) - \mathcal{G}(\mathtt{S}u)|\,\mathrm{d}\rho(\cdot,\zeta) \geq \phi\Big(\frac{\zeta}{\varepsilon}\Big), \qquad \zeta > 0.$$

If u^* and $\widetilde{u}^*$ are the corresponding fixed points, then

$$\begin{aligned} \int_I |\mathcal{G}(u^*) - \widetilde{\mathcal{G}}(\widetilde{u}^*)|\,\mathrm{d}\rho(\cdot,\zeta) &\geq \phi\Big(\frac{(1-Q)\zeta}{\varepsilon}\Big), \\ \int_I \Big|\widetilde{\mathcal{G}}(\Xi u^*) - \widetilde{\mathcal{G}}(\widetilde{u}^*)\Big|\,\mathrm{d}\widetilde{\rho}(\cdot,\zeta) &\geq \phi\Big(\frac{(1-Q)\zeta}{\alpha\varepsilon}\Big). \end{aligned} \tag{29}$$

Proof. For $r \in \mathscr{R}_{\mathcal{G},\rho}(u,v)$, Proposition 6.1 and Lemma 6.2 give

$$\begin{aligned} r \in \mathscr{R}_{\mathcal{G},\rho}(u,v) &\Longleftrightarrow r \in \mathscr{R}_{\widetilde{\mathcal{G}},\rho}(\Xi u, \Xi v) \\ &\Longrightarrow \alpha r \in \mathscr{R}_{\widetilde{\mathcal{G}},\widetilde{\rho}}(\Xi u, \Xi v) \\ &\Longrightarrow \alpha\widetilde{q}r \in \mathscr{R}_{\widetilde{\mathcal{G}},\widetilde{\rho}}(\widetilde{\mathtt{T}}\Xi u, \widetilde{\mathtt{T}}\Xi v) \\ &\Longrightarrow \alpha\beta\widetilde{q}r \in \mathscr{R}_{\mathcal{G},\rho}(\mathtt{S}u, \mathtt{S}v). \end{aligned}$$

Both maps have factor Q, and completeness follows from (25)–(28). Therefore

$$\mathtt{T}u^* = u^*, \qquad \mathtt{S}v^* = v^*, \qquad v^* = \Xi^{-1}(\widetilde{u}^*), \qquad \mathcal{G}(v^*) = \widetilde{\mathcal{G}}(\widetilde{u}^*).$$

From (11), $\frac{\varepsilon}{1-Q} \in \mathscr{R}_{\mathcal{G},\rho}(u^*, v^*)$, giving (29). Then (25) and (27) give $\frac{\alpha\varepsilon}{1-Q} \in \mathscr{R}_{\widetilde{\mathcal{G}},\widetilde{\rho}}(\Xi u^*, \widetilde{u}^*)$ and the second estimate. □

Proposition 6.4. *Let κ and $\widetilde{\kappa}$ be finite measures satisfying*

$$\underline{\gamma}\,\kappa(D) \le \widetilde{\kappa}(D) \le \overline{\gamma}\,\kappa(D),$$

with $D \in \mathfrak{N}$ and $0 < \underline{\gamma} \le \overline{\gamma} < \infty$. For $\sigma \in \{H, P\}$, where

$$\rho_H^\kappa(D,\zeta) = \frac{\zeta}{\zeta + \kappa(D)}, \qquad \rho_P^\kappa(D,\zeta) = \exp\left(-\frac{\kappa(D)}{\zeta}\right),$$

and analogously for $\widetilde{\kappa}$, one has

$$\int_I h\,\mathrm{d}\rho_\sigma^\kappa\left(\cdot, \frac{\zeta}{\overline{\gamma}}\right) \le \int_I h\,\mathrm{d}\rho_\sigma^{\widetilde{\kappa}}(\cdot,\zeta) \le \int_I h\,\mathrm{d}\rho_\sigma^\kappa\left(\cdot, \frac{\zeta}{\underline{\gamma}}\right),$$

for every measurable $h \ge 0$. Hence (26) holds with $\alpha = \overline{\gamma}$, $\beta = \frac{1}{\underline{\gamma}}$, and

$$\underline{\gamma}\inf \mathscr{R}_{\mathcal{G},\rho_\sigma^\kappa}(f,g) \le \inf \mathscr{R}_{\mathcal{G},\rho_\sigma^{\widetilde{\kappa}}}(f,g) \le \overline{\gamma}\inf \mathscr{R}_{\mathcal{G},\rho_\sigma^\kappa}(f,g).$$

Thus a linear factor becomes $q_{\text{pert}} = \frac{\overline{\gamma}}{\underline{\gamma}}\,q$. For $\widetilde{\kappa} = c\kappa$ one has the exact relations

$$\int_I h\,\mathrm{d}\rho_\sigma^{c\kappa}(\cdot,\zeta) = \int_I h\,\mathrm{d}\rho_\sigma^\kappa\left(\cdot, \frac{\zeta}{c}\right),$$
$$\mathscr{R}_{\mathcal{G},\rho_\sigma^{c\kappa}}(f,g) = c\,\mathscr{R}_{\mathcal{G},\rho_\sigma^\kappa}(f,g), \qquad q_{\text{pert}} = q.$$

Proof. For the Hamacher realization,

$$\int_I h\,\mathrm{d}\rho_H^\kappa(\cdot,\zeta) = \frac{\zeta}{\zeta + \int_I h\,\mathrm{d}\kappa},$$

whereas for the product realization,

$$\int_I h\,\mathrm{d}\rho_P^\kappa(\cdot,\zeta) = \exp\left(-\frac{1}{\zeta}\int_I h\,\mathrm{d}\kappa\right).$$

Since

$$\underline{\gamma}\int_I h\,\mathrm{d}\kappa \le \int_I h\,\mathrm{d}\widetilde{\kappa} \le \overline{\gamma}\int_I h\,\mathrm{d}\kappa,$$

the two integral bounds follow. The radius bounds and q_{pert} now follow from Lemma 6.2. For $\widetilde{\kappa} = c\kappa$, direct substitution gives the two scaling identities and $q_{\text{pert}} = q$. □

7 Bi-scale $*$–fuzzy control of L^1 and L^2 generator errors

Let $*$ be strict Archimedean with additive generator τ; see [12, 15]. Then

$$a * b = \tau^{-1}(\tau(a) + \tau(b)), \qquad \tau(1) = 0.$$

Proposition 7.1. *Let κ be a finite measure and*

$$\rho_\kappa(D,\zeta) = \tau^{-1}\left(\frac{\kappa(D)}{\zeta}\right).$$

For measurable $h \ge 0$ with $h \in L^1(I,\kappa)$,

$$\int_I h\,\mathrm{d}\rho_\kappa(\cdot,\zeta) = \tau^{-1}\left(\frac{\int_I h\,\mathrm{d}\kappa}{\zeta}\right).$$

If $\phi_c(\eta) = \tau^{-1}(c/\eta)$, $c > 0$, then the corresponding control-radius set satisfies

$$\mathscr{R}_{\kappa,c}(f,g) = \left[\frac{1}{c}\int_I |\mathcal{G}(f) - \mathcal{G}(g)|\,\mathrm{d}\kappa, \infty\right).$$

Proof. For $s = \sum_{j=1}^n a_j\chi_{E_j}$ in canonical disjoint form,

$$\int_I s\,\mathrm{d}\rho_\kappa(\cdot,\zeta) = \mathop{*}_{j=1}^{n} \tau^{-1}\left(\frac{a_j\kappa(E_j)}{\zeta}\right) = \tau^{-1}\left(\frac{1}{\zeta}\sum_{j=1}^n a_j\kappa(E_j)\right) = \tau^{-1}\left(\frac{\int_I s\,\mathrm{d}\kappa}{\zeta}\right).$$

Let $s_n \uparrow h$ and pass to the limit. For $e = |\mathcal{G}(f) - \mathcal{G}(g)|$,

$$\int_I e\,\mathrm{d}\rho_\kappa(\cdot,\zeta) \geq \phi_c\left(\frac{\zeta}{\delta}\right) \Longleftrightarrow \tau^{-1}\left(\frac{\int_I e\,\mathrm{d}\kappa}{\zeta}\right) \geq \tau^{-1}\left(\frac{c\delta}{\zeta}\right) \Longleftrightarrow \int_I e\,\mathrm{d}\kappa \leq c\delta,$$

This gives the radius formula. □

Let μ_1 and μ_2 be finite positive measures on $(I,\mathfrak{N})$. For $D \in \mathfrak{N}$ and $\zeta > 0$, define

$$\rho_{12}(D,\zeta) = \tau^{-1}\left(\frac{\mu_1(D)}{\zeta} + \frac{\mu_2(D)}{\zeta^2}\right). \tag{30}$$

Theorem 7.2. *The mapping ρ_{12} in (30) is a ∗–fuzzy measure. If $h \geq 0$, $h \in L^1(I,\mu_1)$ and $h \in L^2(I,\mu_2)$, then*

$$\int_I h\,\mathrm{d}\rho_{12}(\cdot,\zeta) = \tau^{-1}\left[\frac{\int_I h\,\mathrm{d}\mu_1}{\zeta} + \frac{\int_I h^2\,\mathrm{d}\mu_2}{\zeta^2}\right]. \tag{31}$$

Proof. Clearly $\rho_{12}(\varnothing,\zeta) = 1$, while $\rho_{12}(D,\zeta)$ is continuous and nondecreasing in ζ and tends to 1. For pairwise disjoint (D_j),

$$\tau\left(\rho_{12}\left(\bigcup_{j=1}^\infty D_j,\zeta\right)\right) = \frac{1}{\zeta}\sum_{j=1}^\infty \mu_1(D_j) + \frac{1}{\zeta^2}\sum_{j=1}^\infty \mu_2(D_j) = \sum_{j=1}^\infty \tau(\rho_{12}(D_j,\zeta)),$$

thus

$$\rho_{12}\left(\bigcup_{j=1}^\infty D_j,\zeta\right) = \mathop{*}_{j=1}^{\infty} \rho_{12}(D_j,\zeta).$$

For $s = \sum_{j=1}^n a_j\chi_{E_j}$ in canonical disjoint form,

$$\begin{aligned}\tau\left(\int_I s\,\mathrm{d}\rho_{12}(\cdot,\zeta)\right) &= \sum_{j=1}^n \tau\left(\rho_{12}\left(E_j,\frac{\zeta}{a_j}\right)\right)\\ &= \frac{1}{\zeta}\sum_{j=1}^n a_j\mu_1(E_j) + \frac{1}{\zeta^2}\sum_{j=1}^n a_j^2\mu_2(E_j)\\ &= \frac{\int_I s\,\mathrm{d}\mu_1}{\zeta} + \frac{\int_I s^2\,\mathrm{d}\mu_2}{\zeta^2}.\end{aligned}$$

Take $s_n \uparrow h$. Then

$$\int_I s_n\,\mathrm{d}\mu_1 \uparrow \int_I h\,\mathrm{d}\mu_1, \qquad \int_I s_n^2\,\mathrm{d}\mu_2 \uparrow \int_I h^2\,\mathrm{d}\mu_2.$$

Hence (31). □

For f, g in the natural domain of (31), put

$$E_1(f,g) = \int_I |\mathcal{G}(f) - \mathcal{G}(g)| \,\mathrm{d}\mu_1, \qquad E_2(f,g) = \left(\int_I |\mathcal{G}(f) - \mathcal{G}(g)|^2 \,\mathrm{d}\mu_2\right)^{1/2},$$

and

$$D_{12}(f,g) = \max\{E_1(f,g), E_2(f,g)\}, \qquad \phi_{12}(\eta) = \tau^{-1}\left(\frac{1}{\eta} + \frac{1}{\eta^2}\right).$$

Theorem 7.3. *For any $r, s, \eta > 0$, the function ϕ_{12} satisfies*

$$\phi_{12}\left(\frac{\eta}{r+s}\right) \leq \phi_{12}\left(\frac{\eta}{r}\right) * \phi_{12}\left(\frac{\eta}{s}\right).$$

For the control-radius set defined by ρ_{12} and ϕ_{12},

$$\mathscr{R}_{12}(f,g) = [D_{12}(f,g), \infty). \tag{32}$$

If $I^{\mathcal{G}}_{12}(f,g;\zeta) = \int_I |\mathcal{G}(f) - \mathcal{G}(g)| \,\mathrm{d}\rho_{12}(\cdot,\zeta)$, then $I^{\mathcal{G}}_{12}(f,g;\zeta_1)$ and $I^{\mathcal{G}}_{12}(f,g;\zeta_2)$ determine $E_1(f,g)$ and $E_2(f,g)$ whenever $\zeta_1 \neq \zeta_2$.

Proof. Applying τ,

$$\begin{aligned}
\tau\left(\phi_{12}\left(\frac{\eta}{r+s}\right)\right) &= \frac{r+s}{\eta} + \frac{(r+s)^2}{\eta^2} \\
&\geq \frac{r}{\eta} + \frac{r^2}{\eta^2} + \frac{s}{\eta} + \frac{s^2}{\eta^2} \\
&= \tau\left(\phi_{12}\left(\frac{\eta}{r}\right) * \phi_{12}\left(\frac{\eta}{s}\right)\right).
\end{aligned}$$

Because τ^{-1} is decreasing, this proves the first assertion. From (31),

$$\begin{aligned}
\delta \in \mathscr{R}_{12}(f,g) &\iff \frac{E_1(f,g)}{\zeta} + \frac{E_2(f,g)^2}{\zeta^2} \leq \frac{\delta}{\zeta} + \frac{\delta^2}{\zeta^2} \\
&\iff (E_1(f,g) - \delta)\zeta + E_2(f,g)^2 - \delta^2 \leq 0.
\end{aligned}$$

Let $\zeta \to \infty$ and then $\zeta \downarrow 0$. This yields $E_1(f,g) \leq \delta$ and $E_2(f,g) \leq \delta$; the converse is immediate, proving (32). For $\zeta_1 \neq \zeta_2$, set $Y_i = \tau(I^{\mathcal{G}}_{12}(f,g;\zeta_i))$. Then

$$\zeta_i^2 Y_i = E_1(f,g)\zeta_i + E_2(f,g)^2,$$

for $i = 1, 2$, and hence

$$E_1(f,g) = \frac{\zeta_1^2 Y_1 - \zeta_2^2 Y_2}{\zeta_1 - \zeta_2}, \qquad E_2(f,g)^2 = \frac{\zeta_1 \zeta_2^2 Y_2 - \zeta_2 \zeta_1^2 Y_1}{\zeta_1 - \zeta_2}.$$

□

Proposition 7.4. *If μ_1 and μ_2 are not proportional, there are no finite measure κ and positive function $a(\zeta)$ such that*

$$\rho_{12}(D,\zeta) = \tau^{-1}(a(\zeta)\kappa(D)) \qquad (D \in \mathfrak{N},\ \zeta > 0).$$

Moreover, if $\mathcal{G}(\lambda) > 0$, then $D_{12}(\lambda \odot f, \lambda \odot g) = \mathcal{G}(\lambda) D_{12}(f,g)$, and

$$I^{\mathcal{G}}_{12}(\lambda \odot f, \lambda \odot g; \zeta) = I^{\mathcal{G}}_{12}\left(f, g; \frac{\zeta}{\mathcal{G}(\lambda)}\right). \tag{33}$$

Pseudo-translation leaves both D_{12} and $I^{\mathcal{G}}_{12}$ invariant.

Proof. Suppose

$$\frac{\mu_1(D)}{\zeta} + \frac{\mu_2(D)}{\zeta^2} = a(\zeta)\kappa(D),$$

for all D and ζ. Choosing $\zeta_1 \neq \zeta_2$ gives

$$\begin{pmatrix} \zeta_1^{-1} & \zeta_1^{-2} \\ \zeta_2^{-1} & \zeta_2^{-2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} a(\zeta_1) \\ a(\zeta_2) \end{pmatrix} \kappa.$$

The matrix is nonsingular, so $\mu_1 = c_1\kappa$ and $\mu_2 = c_2\kappa$, contrary to the hypothesis. Moreover,

$$|\mathcal{G}(\lambda \odot f) - \mathcal{G}(\lambda \odot g)| = \mathcal{G}(\lambda)|\mathcal{G}(f) - \mathcal{G}(g)|.$$

Thus $E_j(\lambda \odot f, \lambda \odot g) = \mathcal{G}(\lambda)E_j(f,g)$ for $j = 1,2$. Equation (31) gives (33), and the same cancellation gives pseudo-translation invariance. □

Let $\nu = \mu_1 + \mu_2$ and define

$$\mathbb{X}_{12} = \left\{ f : I \to \mathcal{J} : \mathcal{G}(f) \in L^1(I,\mu_1) \cap L^2(I,\mu_2) \right\} / \sim_\nu .$$

Theorem 7.5. *The space $(\mathbb{X}_{12}, D_{12})$ is complete. By (32), the associated bi-scale control geometry is complete as well.*

Proof. Let (f_n) be D_{12}–Cauchy and put $v_n = \mathcal{G}(f_n)$. Choose a subsequence (v_{n_k}) such that

$$\|v_{n_{k+1}} - v_{n_k}\|_{L^1(\mu_1)} + \|v_{n_{k+1}} - v_{n_k}\|_{L^2(\mu_2)} \le 2^{-k}.$$

Then

$$\sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\|_{L^1(\mu_1)} < \infty, \qquad \sum_{k=1}^{\infty} \|v_{n_{k+1}} - v_{n_k}\|_{L^2(\mu_2)} < \infty.$$

Thus

$$\sum_{k=1}^{\infty} |v_{n_{k+1}} - v_{n_k}| < \infty,$$

μ_1–a.e. and μ_2–a.e., hence ν–a.e. The subsequence has a measurable limit $v \ge 0$, and completeness of the two spaces gives

$$\|v_{n_k} - v\|_{L^1(\mu_1)} \to 0, \qquad \|v_{n_k} - v\|_{L^2(\mu_2)} \to 0.$$

Since (v_n) is Cauchy, the same limits hold for the full sequence. Put $f = \mathcal{G}^{-1}(v)$. Then

$$D_{12}(f_n, f) = \max\left\{ \|v_n - v\|_{L^1(\mu_1)}, \|v_n - v\|_{L^2(\mu_2)} \right\} \longrightarrow 0.$$

Thus $f \in \mathbb{X}_{12}$ and $(\mathbb{X}_{12}, D_{12})$ is complete. □

For $j = 1,2$, assume $\mathrm{d}\mu_j(t) = w_j(t)\,\mathrm{d}t$, $w_j(t) > 0$ a.e., and write $k_{\mathcal{G}}(t,s) = \mathcal{G}(\mathtt{K}(t,s))$. Set

$$M_1 = \operatorname*{ess\,sup}_{s \in I} \frac{\displaystyle\int_I k_{\mathcal{G}}(t,s) w_1(t)\,\mathrm{d}t}{w_1(s)},$$

$$M_2^2 = \int_I \int_I k_{\mathcal{G}}(t,s)^2 \frac{w_2(t)}{w_2(s)}\,\mathrm{d}s\,\mathrm{d}t.$$

Lemma 7.6. *If $M_1, M_2 < \infty$, then the generator-coordinate kernel operator*

$$(\mathcal{P}v)(t) = \int_I k_{\mathcal{G}}(t,s) v(s)\,\mathrm{d}s$$

satisfies

$$\|\mathcal{P}v\|_{L^1(\mu_1)} \le M_1 \|v\|_{L^1(\mu_1)}, \qquad \|\mathcal{P}v\|_{L^2(\mu_2)} \le M_2 \|v\|_{L^2(\mu_2)}.$$

Proof. For the $L^1(\mu_1)$ estimate, Tonelli gives

$$\begin{aligned}\|\mathcal{P}v\|_{L^1(\mu_1)} &\le \int_I |v(s)| \left(\int_I k_{\mathcal{G}}(t,s) w_1(t)\,\mathrm{d}t\right) \mathrm{d}s \\ &\le M_1 \int_I |v(s)| w_1(s)\,\mathrm{d}s.\end{aligned}$$

For $L^2(\mu_2)$, put $z(s) = v(s)\sqrt{w_2(s)}$. Then

$$\sqrt{w_2(t)}\,(\mathcal{P}v)(t) = \int_I k_{\mathcal{G}}(t,s) \sqrt{\frac{w_2(t)}{w_2(s)}}\, z(s)\,\mathrm{d}s.$$

The Hilbert–Schmidt bound gives

$$\|\mathcal{P}v\|^2_{L^2(\mu_2)} \le M_2^2 \|v\|^2_{L^2(\mu_2)}.$$

□

Theorem 7.7. *Let $\mathbb{X}_{12,0} \subset \mathbb{X}_{12}$ be a closed invariant set for the pseudo-Fredholm map $\mathtt{T}$ in (15). Assume $M_1, M_2 < \infty$ and, on the corresponding generator interval,*

$$|\mathcal{G}(\mathtt{F}(s,x)) - \mathcal{G}(\mathtt{F}(s,y))| \le L\,|\mathcal{G}(x) - \mathcal{G}(y)|.$$

Put $q_1 = M_1 L$ and $q_2 = M_2 L$ with $q = \max\{q_1, q_2\} < 1$. Then $\mathtt{T}$ has a unique fixed point $u^* \in \mathbb{X}_{12,0}$ and

$$I_{12}^{\mathcal{G}}(\mathtt{T}u, \mathtt{T}v; \zeta) \ge I_{12}^{\mathcal{G}}\left(u, v; \frac{\zeta}{q}\right). \tag{34}$$

For the Picard orbit $u_{n+1} = \mathtt{T}u_n$,

$$I_{12}^{\mathcal{G}}(u_n, u^*; \zeta) \ge \tau^{-1}\left[\frac{q_1^n E_1(u_1,u_0)}{(1-q_1)\zeta} + \frac{q_2^{2n} E_2(u_1,u_0)^2}{(1-q_2)^2\zeta^2}\right]. \tag{35}$$

Proof. For $j = 1, 2$, Lemma 7.6 and the generator-Lipschitz estimate give $E_j(\mathtt{T}u, \mathtt{T}v) \le q_j E_j(u,v)$. Therefore, $D_{12}(\mathtt{T}u, \mathtt{T}v) \le qD_{12}(u,v)$. Since $q < 1$, Banach's theorem gives the fixed point. From (31),

$$\begin{aligned}I_{12}^{\mathcal{G}}(\mathtt{T}u, \mathtt{T}v; \zeta) &= \tau^{-1}\left[\frac{E_1(\mathtt{T}u,\mathtt{T}v)}{\zeta} + \frac{E_2(\mathtt{T}u,\mathtt{T}v)^2}{\zeta^2}\right] \\ &\ge \tau^{-1}\left[\frac{qE_1(u,v)}{\zeta} + \frac{q^2 E_2(u,v)^2}{\zeta^2}\right] \\ &= I_{12}^{\mathcal{G}}\left(u, v; \frac{\zeta}{q}\right),\end{aligned}$$

which proves (34). Also,

$$E_j(u_{n+1}, u_n) \le q_j^n E_j(u_1, u_0),$$

so

$$E_j(u_n, u^*) \le \sum_{k=n}^{\infty} E_j(u_k, u_{k+1}) \le \frac{q_j^n}{1-q_j} E_j(u_1, u_0).$$

Equation (31) now gives (35). □

Proposition 7.8. *Under the hypotheses of Theorem 7.7, for $j = 1, 2$, let $r_j = E_j(y, \mathtt{T}y)$. Then $E_j(y, u^*) \le \frac{r_j}{1-q_j}$, and therefore*

$$I_{12}^{\mathcal{G}}(y, u^*; \zeta) \ge \tau^{-1}\left[\frac{r_1}{(1-q_1)\zeta} + \frac{r_2^2}{(1-q_2)^2\zeta^2}\right].$$

If an inexact orbit satisfies

$$E_j(y_{n+1}, \mathtt{T}y_n) \le \varepsilon_{j,n}, \qquad j = 1, 2,$$

then

$$E_j(y_n, u^*) \le q_j^n E_j(y_0, u^*) + \sum_{k=0}^{n-1} q_j^{n-1-k} \varepsilon_{j,k}, \qquad j = 1, 2. \tag{36}$$

Finally, if $\widetilde{\mathtt{T}}$ *has the same componentwise factors* q_j *and*

$$E_j(\mathtt{T}u, \widetilde{\mathtt{T}}u) \le \varepsilon_j \quad \textit{uniformly in } u,$$

then its fixed point $\widetilde{u}^*$ *satisfies*

$$E_j(u^*, \widetilde{u}^*) \le \frac{\varepsilon_j}{1 - q_j}, \qquad j = 1, 2. \tag{37}$$

Proof. For the residual,

$$E_j(y, u^*) \le E_j(y, \mathtt{T}y) + E_j(\mathtt{T}y, \mathtt{T}u^*) \le r_j + q_j E_j(y, u^*),$$

Rearranging gives the first estimate, and (31) gives its fuzzy form. For the inexact orbit,

$$E_j(y_{n+1}, u^*) \le \varepsilon_{j,n} + q_j E_j(y_n, u^*),$$

and iteration yields (36). Also,

$$E_j(u^*, \widetilde{u}^*) \le E_j(\mathtt{T}u^*, \mathtt{T}\widetilde{u}^*) + E_j(\mathtt{T}\widetilde{u}^*, \widetilde{\mathtt{T}}\widetilde{u}^*) \le q_j E_j(u^*, \widetilde{u}^*) + \varepsilon_j,$$

which gives (37). By (31),

$$I_{12}^{\mathcal{G}}(u^*, \widetilde{u}^*; \zeta) \ge \tau^{-1}\left[\frac{\varepsilon_1}{(1 - q_1)\zeta} + \frac{\varepsilon_2^2}{(1 - q_2)^2 \zeta^2}\right].$$

□

For the product and Hamacher triangular norms, respectively, the bi-scale integral takes the explicit forms

$$I_{12}^{\mathcal{G}}(f, g; \zeta) = \exp\left(-\frac{E_1(f, g)}{\zeta} - \frac{E_2(f, g)^2}{\zeta^2}\right),$$

$$I_{12}^{\mathcal{G}}(f, g; \zeta) = \frac{\zeta^2}{\zeta^2 + \zeta E_1(f, g) + E_2(f, g)^2}.$$

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