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Original paper

New constructions and characterizations of semi-t-operators on bounded lattices

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Abstract

This paper aims to study new semi-t-operators on bounded lattices. We propose four new methods for constructing semi-t-operators on bounded lattices by using semi-t-norms, semi-t-conorms and order-preserving mappings. Comparative analysis reveals that the construction methods proposed in this paper, can be regarded as generalizations of various approaches in existing literature. Besides, we provide equivalent characterizations of the corresponding forms of these four semi-t-operators.

Keywords: Semi-t-operators, semi-t-norms, semi-t-conorms, bounded lattices, order-preserving mappings.

1 Introduction

The theory of aggregation operators has been proven to be an essential tool across various fields, such as fuzzy set theory [18, 25], functional equations [15, 16, 21, 38], decision making [14, 20] and so on. T-operators and nullnorms, as special aggregation operators, were introduced in [2, 3]. These aggregation operators are generalizations of the concepts of t-norms and t-conorms. Mas et al. [17] proved that they are actually equivalent, Drygaś [5] removed the commutativity from t-operators and nullnorms, deriving semi-t-operators and semi-nullnorms respectively. Much work has been done to study their structure and properties, such as the distributivity and conditional distributivity [5, 6, 10, 11, 19, 22, 24, 28, 36], the migrativity property [12, 23], the modularity condition [4, 30, 31, 35] and weak dominance [34]. Semi-t-operators and semi-nullnorms are specialized aggregation functions that also have wide applications in various fields, including fuzzy logic frameworks, neural networks, decision making and fuzzy quantifiers, [7, 8, 13].

It is noteworthy that, unlike the deterministic structure of binary aggregation operators on the unit interval of real numbers, the structure of aggregation operators on bounded lattices is characterized by uncertainty and diversity. In recent years, constructing or characterizing semi-t-operators on bounded lattices has been a widely studied research topic in the field of aggregation function studies. In [8], Fang and Hu proved the existence of semi-t-operators on bounded lattices and outlined multiple ways to construct them. However, their approaches are not completely correct. Later, Wang et al. [26] corrected the mistakes made by Fang and Hu and proposed new methods for constructing semi-t-operators and idempotent semi-t-operators on bounded lattices [27]. Subsequently, Wang and Zhu [29, 37] introduced new construction methods for semi-t-operators on bounded lattices, with order-preserving mappings as the basis. Hua and Fang [9] proposed semi-beam operations and further constructed semi-t-operators on bounded lattices using these operations. Zhang et al. [33] constructed semi-t-operators on bounded lattices via semi-t-(co)norms and order-preserving functions, while also providing their equivalent characterizations. Lately, Zhang et al. [32] studied two new classes of semi-t-operators on bounded lattices.

Although numerous studies have focused on the construction of semi-t-operators on bounded lattices, most existing construction methods impose constant assignment constraints on the domain consisting of incomparable elements, which

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limits the structural diversity of the resulting aggregation operators. In addition, there are few systematic analyses of the inherent logical differences among various construction approaches. However, specialized equivalent characterizations for determining whether a given binary operation can be represented by known constructions remain scarce. To reduce fixed constant assignments over incomparable regions and enhance the structural flexibility of the semi-t-operators, this paper constructs four types of semi-t-operators on bounded lattices by using order-preserving mappings, semi-t-norms and semi-t-conorms. Through comparative analysis, we find that the proposed constructions constitute more general forms than those reported in existing literature. Furthermore, to investigate the intrinsic structural properties of semi-t-operators and judge whether a binary mapping belongs to the target class of semi-t-operators, we derive equivalent characterizations for their corresponding representations.

The organization of this paper is outlined as follows. In Section 2, we recall the definitions related to bounded lattices, as well as semi-t-norms, semi-t-conorms, semi-nullnorms, and semi-t-operators. In Section 3, we put forward four methods for constructing semi-t-operators on bounded lattices, as well as their dual forms. In addition, we provide the equivalent characterizations for their corresponding representations. In Section 4, we present the conclusions of this paper and future work.

2 Preliminaries

In this section, we review the concepts and conclusions that are used throughout this paper.

Definition 2.1. [1] *A lattice $(L, \leq)$ is bounded if L has a top element 1_L and the bottom element 0_L, i.e., there exist two elements $0_L, 1_L \in L$ such that $0_L \leq k \leq 1_L$, for all $k \in L$.*

Let $a, b \in L$, the notation $a \parallel b$ means that a and b are incomparable. The notation $a \nparallel b$ denotes that a is comparable with b. We denote $I_a = \{x \in L \mid x \parallel a\}$, $I_a^b = \{x \in L \mid x \parallel a \text{ and } x \nparallel b\}$, $I_b^a = \{x \in L \mid x \parallel b \text{ and } x \nparallel a\}$and $I_{a,b} = \{x \in L \mid x \parallel a \text{ and } x \parallel b\}$. A subinterval $[m, n]$ of L is defined as $[m, n] = \{x \in L \mid m \leq x \leq n\}$. Similarly, $(m, n] = \{x \in L \mid m < x \leq n\}$, $[m, n) = \{x \in L \mid m \leq x < n\}$ and $(m, n) = \{x \in L \mid m < x < n\}$.

Definition 2.2. [8, 26] *A binary function, $T(S)$: $L^2 \to L$ is said to be a semi-t-norm (semi-t-conorm) if it is increasing in each variable, associative and has a neutral element 1_L (0_L), i.e., $T(1_L, x) = T(x, 1_L) = x$ (resp. $S(0_L, x) = S(x, 0_L) = x$), for all $x \in L$.*

Definition 2.3. [33] *A binary operation, $K : L^2 \to L$ is said to be a semi-t-operator if it is increasing in each variable, associative, satisfies $K(0_L, 0_L) = 0_L$, $K(1_L, 1_L) = 1_L$ and such that the functions $K_{0_L} \in \mathcal{M}(L, [0_L, a] \setminus I_b^a)$, $K^{0_L} \in \mathcal{M}(L, [0_L, b] \setminus I_a^b)$, $K_{1_L} \in \mathcal{M}(L, [b, 1_L] \setminus I_a^b)$, $K^{1_L} \in \mathcal{M}(L, [a, 1_L] \setminus I_b^a)$, where $\mathcal{M}(X, Y)$ is defined as the family of all surjective mappings from X to Y and $K_{0_L}(x) = K(0_L, x)$, $K_{1_L}(x) = K(1_L, x)$, $K^{0_L}(x) = K(x, 0_L)$, $K^{1_L}(x) = K(x, 1_L)$, $K(0_L, 1_L) = a$ and $K(1_L, 0_L) = b$.*

The symbol $\mathcal{K}_{a,b}$ is defined as the family of all semi-t-operators on L with $K(0_L, 1_L) = a$ and $K(1_L, 0_L) = b$.

Proposition 2.4. [8] *Let $a, b \in L \setminus \{0_L, 1_L\}$ and $K \in \mathcal{K}_{a,b}$. Then the following statements hold.*

(i) *$K(x, y) = a$ for $(x, y) \in [0_L, a] \times [a, 1_L]$.*

(ii) *$K(x, y) = b$ for $(x, y) \in [b, 1_L] \times [0_L, b]$.*

Proposition 2.5. [8] *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$ and $K \in \mathcal{K}_{a,b}$. Then $K(x, y) = x$ for $(x, y) \in [a, b] \times L$.*

Proposition 2.6. [8, 33] *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$ and $K \in \mathcal{K}_{a,b}$. Then the following statements hold.*

(i) *$S_K = K|_{[0_L,a]^2} : [0_L, a]^2 \to [0_L, a]$ is a semi-t-conorm on $[0_L, a]$.*

(ii) *$T_K = K|_{[b,1_L]^2} : [b, 1_L]^2 \to [b, 1_L]$ is a semi-t-norm on $[b, 1_L]$.*

Proposition 2.7. [26] *Let $a, b \in L \setminus \{0_L, 1_L\}$ and $a < b$.*

(i) *If $x \in I_a^b$, then $x < b$.*

(ii) *If $x \in I_b^a$, then $x > a$.*

Proposition 2.8. [27, 33] *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$ and $K \in \mathcal{K}_{a,b}$. Then the following statements hold.*

(i) *$K(x, y) = a$ for $(x, y) \in [0_L, a] \times I_b^a$.*

(ii) $K(x, y) = b$ *for* $(x, y) \in [b, 1_L] \times I_a^b$.

(iii) $a \le K(x, y) \le b$ *for* $(x, y) \in I_{a,b} \times [a, b]$.

(iv) $(x \wedge a) \vee (y \wedge a) \le K(x, y) \le a$ *for* $(x, y) \in [0_L, a] \times I_a^b \cup [0_L, a] \times I_{a,b}$.

(v) $b \le K(x, y) \le (x \vee b) \wedge (y \vee b)$ *for* $(x, y) \in [b, 1_L] \times I_b^a \cup [b, 1_L] \times I_{a,b}$.

(vi) $(x \wedge a) \vee (y \wedge a) \le K(x, y) \le (x \vee a)$ *for* $(x, y) \in I_a^b \times [0_L, a] \cup I_a^b \times I_a^b \cup I_a^b \times I_{a,b}$.

(vii) $a \le K(x, y) \le (x \vee a)$ *for* $(x, y) \in I_a^b \times [a, b] \cup I_a^b \times [b, 1_L] \cup I_a^b \times I_b^a$.

(viii) $(x \wedge b) \le K(x, y) \le b$ *for* $(x, y) \in I_b^a \times [0_L, a] \cup I_b^a \times [a, b] \cup I_b^a \times I_a^b$.

(ix) $(x \wedge b) \le K(x, y) \le (x \vee b) \wedge (y \vee b)$ *for* $(x, y) \in I_b^a \times [b, 1_L] \cup I_b^a \times I_b^a \cup I_b^a \times I_{a,b}$.

(x) $(x \wedge a) \vee (y \wedge a) \le K(x, y) \le b$ *for* $(x, y) \in I_{a,b} \times [0_L, a] \cup I_{a,b} \times I_a^b$.

(xi) $a \le K(x, y) \le (x \vee b) \wedge (y \vee b)$ *for* $(x, y) \in I_{a,b} \times [b, 1_L] \cup I_{a,b} \times I_b^a$.

Theorem 2.9. [27, 33] *The function* $K : L^2 \to L$ *is a semi-t-operator on* L *with* $a = K(0_L, 1_L)$ *and* $b = K(1_L, 0_L)$ *if and only if the function* $G : L^2 \to L$ *given by* $G(x, y) = K(y, x)$ *for all* $x, y \in L$ *is a semi-t-operator on* L *with* $b = G(0_L, 1_L)$ *and* $a = G(1_L, 0_L)$.

Semi-t-operators $\mathcal{K} \in \mathcal{K}_{a,b}$ defined on a lattice L fall into two broad categories: the comparable case $a \parallel b$ and the incomparable case $a \nparallel b$. We only study the construction methods of semi-t-operators on L in the case $a \le b$ satisfying $a, b \in L \setminus \{0_L, 1_L\}$. According to Theorem 2.9, the corresponding situation where $b \le a$ can be obtained immediately. In particular, when $a = b$, K is a semi-nullnorm. The case where a and b are incomparable will be addressed in future work.

3 Construction of semi-t-operators on bounded lattices

In this section, we introduce four new methods for constructing semi-t-operators on bounded lattices, via semi-t-(co)norms and order-preserving mappings. Analyzing the properties of the semi-t-operators constructed by the proposed constructions allows us to obtain equivalent characterizations that correspond to their respective semi-t-operators.

Theorem 3.1. *Let* $a, b \in L \setminus \{0_L, 1_L\}$ *with* $a \le b$*,* S *be a semi-t-conorm on* $[0_L, a]^2$*,* T *be a semi-t-norm on* $[b, 1_L]^2$*,* $p : L \to [0_L, a]$ *and* $q : L \to [b, 1_L]$*, be two order-preserving mappings such that* $p(x) = x$*, for* $x \in [0_L, a]$ *and* $q(x) = x$*, for* $x \in [b, 1_L]$*. Then the following four functions* K_1*,* K_2*,* K_3 *and* $K_4 : L^2 \to L$ *are semi-t-operators on L, respectively. For all* $x, y \in L$*,*

$$K_1(x, y) = \begin{cases} S(p(x), p(y)) & (x, y) \in ([0_L, a] \cup I_a^b \cup I_{a,b})^2, \\ T(q(x), q(y)) & (x, y) \in [b, 1_L]^2, \\ b & (x, y) \in [b, 1_L] \times (L \setminus [b, 1_L]), \\ h_1(x) & (x, y) \in ([a, b] \cup I_b^a) \times L \cup (I_a^b \cup I_{a,b}) \times [b, 1_L], \\ a & \textit{otherwise}, \end{cases} \tag{1}$$

where $h_1 : [a, b] \cup I_a^b \cup I_b^a \cup I_{a,b} \to [a, b]$ is order-preserving such that $h_1(x) = x$ for all $x \in (a, b)$.

$$K_2(x, y) = \begin{cases} S(p(x), p(y)) & (x, y) \in ([0_L, a] \cup I_a^b \cup I_{a,b})^2, \\ T(q(x), q(y)) & (x, y) \in ([b, 1_L] \cup I_b^a)^2, \\ b & (x, y) \in ([b, 1_L] \cup I_b^a) \times (L \setminus [b, 1_L] \cup I_b^a), \\ h_2(x) & (x, y) \in [a, b] \times L \cup (I_a^b \cup I_{a,b}) \times ([b, 1_L] \cup I_b^a), \\ a & \textit{otherwise}, \end{cases} \tag{2}$$

where $h_2 : [a, b] \cup I_a^b \cup I_{a,b} \to [a, b]$ is order-preserving such that $h_2(x) = x$ for all $x \in (a, b)$.

$$K_3(x,y)=\begin{cases} S(p(x),p(y)) & (x,y)\in([0_L,a]\cup I_a^b)^2,\\ T(q(x),q(y)) & (x,y)\in[b,1_L]^2,\\ b & (x,y)\in[b,1_L]\times(L\setminus[b,1_L]),\\ h_3(x) & (x,y)\in([a,b]\cup I_b^a\cup I_{a,b})\times L\cup I_a^b\times[b,1_L],\\ a & otherwise,\end{cases} \tag{3}$$

where $h_3:[a,b]\cup I_a^b\cup I_b^a\cup I_{a,b}\to[a,b]$ is order-preserving such that $h_3(x)=x$ for all $x\in(a,b)$.

$$K_4(x,y)=\begin{cases} S(p(x),p(y)) & (x,y)\in([0_L,a]\cup I_a^b)^2,\\ T(q(x),q(y)) & (x,y)\in([b,1_L]\cup I_b^a)^2,\\ b & (x,y)\in([b,1_L]\cup I_b^a)\times(L\setminus[b,1_L]\cup I_b^a),\\ h_4(x) & (x,y)\in([a,b]\cup I_{a,b})\times L\cup I_a^b\times([b,1_L]\cup I_b^a),\\ a & otherwise,\end{cases} \tag{4}$$

where $h_4:[a,b]\cup I_a^b\cup I_{a,b}\to[a,b]$ is order-preserving such that $h_4(x)=x$ for all $x\in(a,b)$. Figs. 1-4 show the structures of K_1-K_4, respectively.

Proof. We will prove that the function K_1 is a semi-t-operator on a bounded lattice, and the others are similar. It can be easily verified that $K_1(0_L,0_L)=0_L$ and $K_1(1_L,1_L)=1_L$. We have $(K_1)_{0_L}\in\mathcal{K}(L,[0_L,a]\setminus I_b^a)$, $(K_1)_{1_L}\in\mathcal{K}(L,[b,1_L]\setminus I_a^b)$, $(K_1)^{0_L}\in\mathcal{K}(L,[0_L,b]\setminus I_a^b)$, $(K_1)^{1_L}\in\mathcal{K}(L,[a,1_L]\setminus I_b^a)$. First of all, we need to prove that K_1 is increasing, that is, if $x\le y$ and $z\le t$, then $K_1(x,z)\le K_1(y,t)$. The following cases should be considered:

Case 1. $x,z\in[0_L,a]$.

1.1 If $y\in[0_L,a]$, $t\in L$, then $K_1(x,z)=S(p(x),p(z))\le K_1(y,t)$.

1.2 If $y\in[a,b]\cup I_b^a$, $t\in L$, then $K_1(x,z)=S(p(x),p(z))\le a\le h_1(y)=K_1(y,t)$.

1.3 If $y\in[b,1_L],t\in L$, then $K_1(x,z)=S(p(x),p(z))\le a\le b\le K_1(y,t)$.

1.4 $y\in I_a^b\cup I_{a,b}$.

1.4.1 Whenever $t\in[0_L,a]\cup I_a^b\cup I_{a,b}$, then $K_1(x,z)=S(p(x),p(z))\le S(p(y),p(t))=K_1(y,t)$.

1.4.2 Whenever $t\in[b,1_L]\cup(a,b)\cup I_b^a$, then $K_1(x,z)=S(p(x),p(z))\le a\le K_1(y,t)$.

Case 2. $x\in[0_L,a],z\in(a,b)$, it follows from $z\le t$ that $t\in(a,b)\cup[b,1_L]\cup I_b^a$. If $y\in L$, then $K_1(x,z)=a\trianglelefteq K_1(y,t)$.

Case 3. $x\in[0_L,a],z\in[b,1_L]$, it follows from $z\le t$ that $t\in[b,1]$. This case is similar to Case 2.

Case 4. $x\in[0_L,a],z\in I_a^b$, it follows from $z\le t$ that $t\notin[0_L,a]$.

4.1 If $y\in[0_L,a]$, $t\notin[0_L,a]$, then $K_1(x,z)=S(p(x),p(z))\le K_1(y,t)$.

4.2 If $y\in(a,b)\cup I_b^a$, $t\notin[0_L,a]$, then $K_1(x,z)=S(p(x),p(z))\le a\le h_1(y)=K_1(y,t)$.

4.3 If $y\in[b,1_L],t\notin[0_L,a]$, then $K_1(x,z)=S(p(x),p(z))\le a\le b\le K_1(y,t)$.

4.4 $y\in I_a^b\cup I_{a,b}$.

4.4.1 Whenever $t\in[b,1_L]\cup(a,b)\cup I_b^a$, then $K_1(x,z)=S(p(x),p(z))\le a\le K_1(y,t)$.

4.4.2 Whenever $t\in I_a^b\cup I_{a,b}$, then $K_1(x,z)=S(p(x),p(z))\le S(p(y),p(t))=K_1(y,t)$.

Case 5. $x\in[0_L,a],z\in I_b^a$, it follows from $z\le t$ that $t\in[b,1_L]\cup I_b^a$.

5.1 If $y\in[0_L,a]$, $t\in[b,1_L]\cup I_b^a$, then $K_1(x,z)=a=K_1(y,t)$.

5.2 If $y\in(a,b)\cup I_b^a$, $t\in[b,1_L]\cup I_b^a$, then $K_1(x,z)=a\le h_2(y)=K_1(y,t)$.

5.3 If $y\in[b,1_L],t\in[b,1_L]\cup I_b^a$, then $K_1(x,z)=a\le b\le K_1(y,t)$.

5.4 $y\in I_a^b\cup I_{a,b}$.

5.4.1 Whenever $t\in[b,1_L]$, then $K_1(x,z)=a\le h_1(x)=K_1(y,t)$.

5.4.2 Whenever $t\in I_b^a$, then $K_1(x,z)=a=K_1(y,t)$.

Case 6. $x\in[0_L,a],z\in I_{a,b}$, this case is similar to Case 4.

Case 7. $x\in(a,b),z\in L$.

7.1 If $y\in(a,b)\cup I_b^a,t\in L$, then $K_1(x,z)=h_1(x)\le h_1(y)=K_1(y,t)$.

7.2 If $y\in[b,1_L],t\in L$, then $K_1(x,z)=h_1(x)\le b\le K_1(y,t)$.

Case 8. $x\in[b,1_L],z\in[0_L,a]$.

8.1 If $y,t\in[b,1_L]$, then $K_1(x,z)=b\le T(q(y),q(t))=K_1(y,t)$.

8.2 If $y\in[b,1_L]$, $t\notin[b,1_L]$, then $K_1(x,z)=b=K_1(y,t)$.

Case 9. $x\in[b,1_L],z\in[b,1_L]$, we have $K_1(x,z)=T(q(x),q(z))\le T(q(y),q(t))=K_1(y,t)$.

Case 10. $x\in[b,1_L],z\in(a,b)\cup I_a^b\cup I_b^a\cup I_{a,b}$, this case is similar to Case 8.

Case 11. $x\in I_a^b\cup I_{a,b},z\in[0_L,a]$, this case is similar to Case 1.

Case 12. $x\in I_a^b\cup I_{a,b},z\in(a,b)$, this case is similar to Case 2.

Case 13. $x \in I_a^b, z \in [b, 1_L]$.

13.1 If $y \in (a, b) \cup I_b^a \cup I_a^b \cup I_{a,b}, t \in [b, 1_L]$, then $K_1(x, z) = h_1(x) \leq h_1(y) = K_1(y, t)$.

13.2 If $y \in [b, 1_L], t \in [b, 1]$, then $K_1(x, z) = h_1(x) \leq T(q(y), q(t)) = K_1(y, t)$.

Case 14. $x \in I_{a,b}, z \in [b, 1_L]$, this case is similar to Case 13.

Case 15 $x \in I_a^b \cup I_{a,b}, z \in I_a^b$, this case is similar to Case 4.

Case 16. $x \in I_a^b \cup I_{a,b}, z \in I_b^a$, this case is similar to Case 5.

Case 17. $x \in I_a^b \cup I_{a,b}, z \in I_{a,b}$, this case is similar to Case 6.

Case 18. $x \in I_b^a, z \in L$.

18.1 If $y \in [b, 1_L], t \in L$, then $K_1(x, z) = h_1(x) \leq b \leq K_1(y, t)$.

18.2 If $y \in I_b^a, t \in L$, then $K_1(x, z) = h_1(x) \leq h_1(y) = K_1(y, t)$.

Therefore, K_1 is increasing.

Next, we will prove that $K_1(x, K_1(y, z)) = K_1(K_1(x, y), z)$. The following cases should be considered:

1. $x \in [0_L, a]$.

1.1 $y \in [0_L, a]$.

1.1.1 Whenever $z \in [0_L, a] \cup I_a^b \cup I_{a,b}$, then $K_1(x, K_1(y, z)) = S(p(x), S(p(y), p(z))) = S(S(p(x), p(y)), p(z))$ $= K_1(K_1(x, y), z)$.

1.1.2 Whenever $z \in (a, b) \cup [b, 1_L] \cup I_b^a$, then $K_1(x, K_1(y, z)) = a = K_1(K_1(x, y), z)$.

1.2 If $y \in (a, b) \cup [b, 1_L] \cup I_b^a, z \in L$, then $K_1(x, K_1(y, z)) = a = K_1(a, z) = K_1(K_1(x, y), z)$.

1.3 $y \in I_a^b \cup I_{a,b}$.

1.3.1 Whenever $z \in [0_L, a] \cup I_a^b \cup I_{a,b}$, then $K_1(x, K_1(y, z)) = S(p(x), S(p(y), p(z))) = S(S(p(x), p(y)), p(z))$ $= K_1(K_1(x, y), z)$.

1.3.2 Whenever $z \in [b, 1_L]$, then $K_1(x, K_1(y, z)) = K_1(x, h_1(y)) = a = K_1(K_1(x, y), z)$.

1.3.3 Whenever $z \in (a, b) \cup I_b^a$, then $K_1(x, K_1(y, z)) = K_1(x, a) = a = K_1(K_1(x, y), z)$.

2. $x \in (a, b) \cup I_b^a$, $y, z \in L$, $K_1(x, K_1(y, z)) = h_1(x) = K_1(K_1(x, y), z)$.

3. $x \in [b, 1]$.

3.1 If $y \in [0_L, a], z \in L$, then $K_1(x, K_1(y, z)) = b = K_1(b, z) = K_1(K_1(x, y), z)$.

3.2 If $y \in (a, b) \cup I_b^a, z \in L$, then $K_1(x, K_1(y, z)) = K_1(x, h_1(y)) = b = K_1(b, z) = K_1(K_1(x, y), z)$.

3.3 $y \in [b, 1_L]$.

3.3.1 Whenever $z \in [b, 1_L]$, then $K_1(x, K_1(y, z)) = T(q(x), T(q(y), q(z))) = T(T(q(x), q(y)), q(z))$ $= K_1(K_1(x, y), z)$.

3.3.2 Whenever $z \notin [b, 1_L]$, then $K_1(x, K_1(y, z)) = K_1(x, b) = b = K_1(K_1(x, y), z)$.

3.4 If $y \in I_a^b \cup I_{a,b}, z \in L$, then $K_1(x, K_1(y, z)) = b = K_1(K_1(x, y), z)$.

4. $x \in I_a^b \cup I_{a,b}$.

4.1 $y \in [0_L, a]$, this case is similar to Case 1.1.

4.2 If $y \in (a, b) \cup I_b^a, z \in L$, then $K_1(x, K_1(y, z)) = K_1(x, h_1(y)) = a = K_1(a, z) = K_1(K_1(x, y), z)$.

4.3 If $y \in [b, 1_L], z \in L$, then $K_1(x, K_1(y, z)) = h_1(x) = K_1(K_1(x, y), z)$.

4.4 $y \in I_a^b \cup I_{a,b}$, this case is similar to Case 1.3.

Therefore, K_1 is a semi-t-operator on L .

□

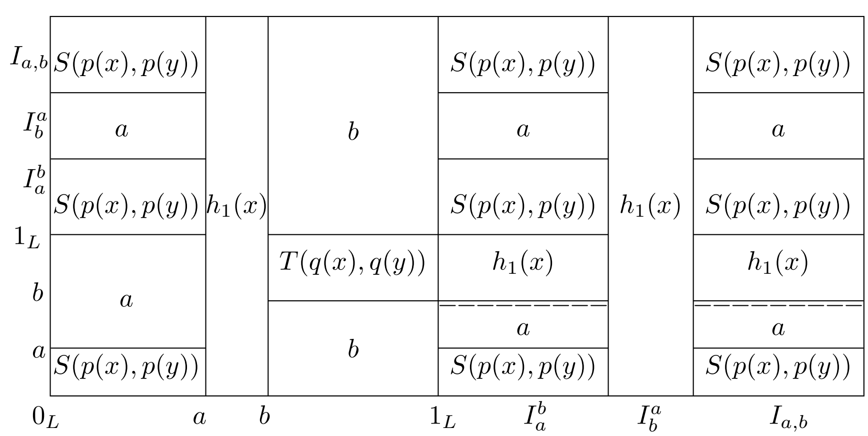

Figure 1: The structure of K_1 in Theorem 3.1

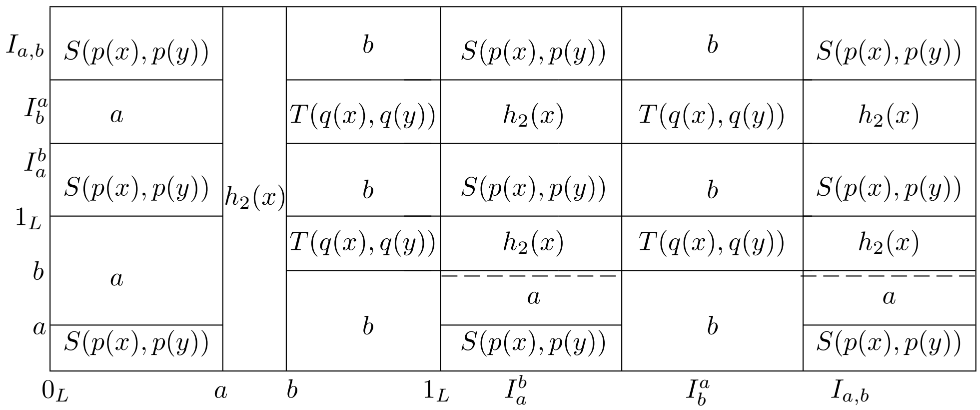

Figure 2: The structure of K_2 in Theorem 3.1

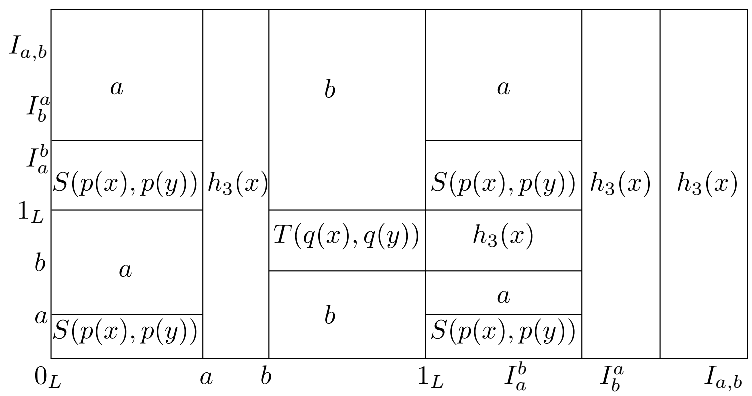

Figure 3: The structure of K_3 in Theorem 3.1

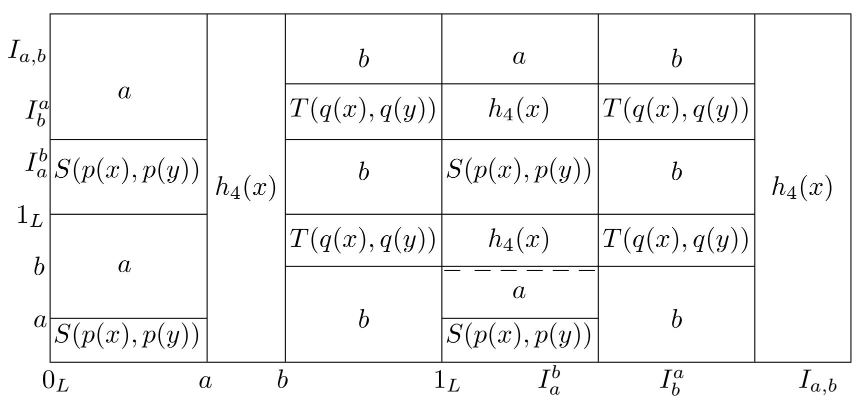

Figure 4: The structure of K_4 in Theorem 3.1

The following example illustrates the differences among the four methods of Theorem 3.1 on the same bounded lattice.

Example 3.2. *Let* $\mathbf{L} = (\{0_L, x_1, x_2, a, x_3, b, x_4, x_5, x_6, x_7, 1_L\}, \leq)$ *be a bounded lattice, and its lattice diagram is shown in Fig. 5. The semi-t-conorm* S_L *on* $[0_L, a]$ *and the semi-t-norm* T_L *on* $[b, 1_L]$ *are given in Table 1 . Consider an order-preserving mapping* $p(x)$ *in* K_1 *and* K_2 *as follows:*

$$p(x) = \begin{cases} x & x \in [0_L, a], \\ (x \vee a) \wedge x_1 & x \in I_a^b \cup I_{a,b}, \\ a & \textit{otherwise}. \end{cases}$$

Consider an order-preserving function $p(x)$ *in* K_3 *and* K_4 *as follows:*

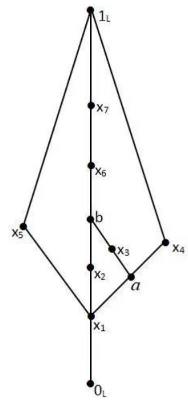

Figure 5: The lattice L in Example 3.2

$$p(x) = \begin{cases} x & x \in [0_L, a], \\ (x \vee a) \wedge x_1 & x \in I_a^b, \\ a & otherwise. \end{cases}$$

Consider an order-preserving function $q(x)$ in K_2 and K_4 as follows:

$$q(x) = \begin{cases} x & x \in [b, 1_L], \\ (x \vee b) \wedge x_6 & x \in I_b^a, \\ b & otherwise. \end{cases}$$

Consider an order-preserving function $h_1(x)$, $h_2(x)$, $h_3(x)$, $h_4(x)$ as follows:

$$h_1(x) = h_3(x) = \begin{cases} x & x \in (a, b), \\ (x \vee a) \wedge x_3 & x \in I_a^b \cup I_{a,b}, \\ (x \wedge b) \vee x_3 & x \in I_b^a, \end{cases} \quad and \quad h_2(x) = h_4(x) = \begin{cases} x & x \in (a, b), \\ (x \vee a) \wedge x_3 & x \in I_a^b \cup I_{a,b}. \end{cases}$$

Then by Theorem 3.1, we obtain the semi-t-operators K_1, K_2, K_3, K_4 presented in Tables 2, 3, 4 and 5, respectively.

Table 1: S_L, T_L in Example 3.2

S_L	0_L	x_1	a
0_L	0	x_1	a
x_1	x_1	x_1	a
a	a	a	a

T_L	b	x_6	x_7	1_L
b	b	b	b	b
x_6	b	x_6	x_6	x_6
x_7	b	b	x_7	x_7
1_L	b	x_6	x_7	1_L

Table 2: K_1 in Example 3.2

K_1	0_L	x_1	x_2	a	x_3	b	x_4	x_5	x_6	x_7	1_L
0_L	0	x_1	x_1	a	a	a	a	x_1	a	a	a
x_1	x_1	x_1	x_1	a	a	a	a	x_1	a	a	a
x_2	x_1	x_1	x_1	a	a	x_3	a	x_1	x_3	x_3	x_3
a	a	a	a	a	a	a	a	a	a	a	a
x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
b	b	b	b	b	b	b	b	b	b	b	b
x_4	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_5	x_1	x_1	x_1	a	a	x_3	a	x_1	x_3	x_3	x_3
x_6	b	b	b	b	b	b	b	b	x_6	x_6	x_6
x_7	b	b	b	b	b	b	b	b	x_6	x_7	x_7
1_L	b	b	b	b	b	b	b	b	x_6	x_7	1_L

Table 3: K_2 in Example 3.2

K_2	0_L	x_1	x_2	a	x_3	b	x_4	x_5	x_6	x_7	1_L
0_L	0	x_1	x_1	a	a	a	a	x_1	a	a	a
x_1	x_1	x_1	x_1	a	a	a	a	x_1	a	a	a
x_2	x_1	x_1	x_1	a	a	x_3	x_3	x_1	x_3	x_3	x_3
a	a	a	a	a	a	a	a	a	a	a	a
x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
b	b	b	b	b	b	b	b	b	b	b	b
x_4	b	b	b	b	b	b	x_6	b	x_6	x_6	x_6
x_5	x_1	x_1	x_1	a	a	x_3	x_3	x_1	x_3	x_3	x_3
x_6	b	b	b	b	b	b	x_6	b	x_6	x_6	x_6
x_7	b	b	b	b	b	b	x_6	b	x_6	x_7	x_7
1_L	b	b	b	b	b	b	x_6	b	x_6	x_7	1_L

Table 4: K_3 in Example 3.2

K_3	0_L	x_1	x_2	a	x_3	b	x_4	x_5	x_6	x_7	1_L
0_L	0	x_1	x_1	a	a	a	a	a	a	a	a
x_1	x_1	x_1	x_1	a	a	a	a	a	a	a	a
x_2	x_1	x_1	x_1	a	a	a	a	a	x_3	x_3	x_3
a	a	a	a	a	a	a	a	a	a	a	a
x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
b	b	b	b	b	b	b	b	b	b	b	b
x_4	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_5	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_6	b	b	b	b	b	b	b	b	x_6	x_6	x_6
x_7	b	b	b	b	b	b	b	b	x_6	x_7	x_7
1_L	b	b	b	b	b	b	b	b	x_6	x_7	1_L

Table 5: K_4 in Example 3.2

K_4	0_L	x_1	x_2	a	x_3	b	x_4	x_5	x_6	x_7	1_L
0_L	0	x_1	x_1	a	a	a	a	a	a	a	a
x_1	x_1	x_1	x_1	a	a	a	a	a	a	a	a
x_2	x_1	x_1	x_1	a	a	x_3	x_3	a	x_3	x_3	x_3
a	a	a	a	a	a	a	a	a	a	a	a
x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
b	b	b	b	b	b	b	b	b	b	b	b
x_4	b	b	b	b	b	b	x_6	b	x_6	x_6	x_6
x_5	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_6	b	b	b	b	b	b	x_6	b	x_6	x_6	x_6
x_7	b	b	b	b	b	b	x_6	b	x_6	x_7	x_7
1_L	b	b	b	b	b	b	x_6	b	x_6	x_7	1_L

Remark 3.3. (i) *Tables 2- 5 list the values of these four semi-t-operators defined on the same lattice, from which we can easily observe that the four semi-t-operators are distinct from each other.*

(ii) *If the value of K_1 on $(I_a^b \cup I_{a,b}) \times [b, 1_L]$ is a, then K_1 in Eq. (1) is identical to F_3 in Theorem 3.2 of* [29].

(iii) *If the value of K_2 on $I_a^b \times ([b, 1_L] \cup I_b^a)$ is a, then K_2 in Eq. (2) is identical to F in Theorem 3.1 of* [33].

(iv) *If the value of K_3 on $I_a^b \times [b, 1_L]$ is a, then K_3 in Eq. (3) is identical to F^2 in Theorem 3.2 of* [29].

(v) *If the value of K_4 on $I_a^b \times [b, 1_L]$ is a, then K_4 in Eq. (4) is identical to F_4 in Theorem 3.2 of* [29].

(vi) *The methods in Theorem 3.1 are different from those in* [32]. *In* [32], *the values on* $I_a^b \times L$ $(I_b^a \times L)$ *are* a (b), *whereas the values of the corresponding regions in Theorem 3.1 are not entirely* a (b).

(vii) *When* $a = b$, *the approaches presented in this paper are also applicable to the construction of semi-nullnorms on bounded lattices.*

The semi-t-operators constructed by Theorem 3.1 satisfy the following properties immediately.

Proposition 3.4. *Let* $a, b \in L \setminus \{0_L, 1_L\}$ *with* $a \leq b$. *The following properties hold for any semi-t-operator* K *derived from Eq. (1).*

(i) $K(x, y) \leq b$ *for all* $(x, y) \in [b, 1_L] \times (I_b^a \cup I_{a,b}) \cup (I_a^b \cup I_{a,b}) \times [b, 1_L]$.

(ii) $K(x, y) \leq a$ *for all* $(x, y) \in (I_a^b \cup I_{a,b}) \times (L \setminus [b, 1_L])$.

(iii) $K(x, 0_L) = K(x, 1_L) \in [a, b]$ *for* $x \in I_b^a$.

Every property listed in Proposition 3.4 is a direct structural consequence of the construction rule given in Theorem 3.1. Conversely, if a semi-t-operator satisfies all properties listed in Proposition 3.4, does it coincide with the form given in Eq. (1)? To answer this question, we establish the corresponding equivalent characterization theorem.

Theorem 3.5. *Let* $a, b \in L \setminus \{0_L, 1_L\}$ *with* $a \leq b$. *Then* $K \in \mathcal{K}_{a,b}$ *and* K *satisfies all the properties in Proposition 3.4 if and only if* K *can be constructed in the form of Eq. (1).*

Proof. Necessity: It can be obtained from Proposition 3.4.

Sufficiency: Let $K \in \mathcal{K}_{a,b}$ and K satisfies all the properties in Proposition 3.4. Define q, p, S, T as follows:
$q(x) = K(1_L, x)$ for all $x \in L$;
$p(x) = K(0_L, x)$ for all $x \in L$;
$S = K|_{[0_L,a]^2}, T = K|_{[b,1_L]^2}$.

It is easy to check that S is a semi-t-conorm and T is a semi-t-norm. Besides, $q(x) = K(1_L, x) = T(1_L, x) = x$ for $x \in [b, 1_L]$ and $p(x) = K(0_L, x) = S(0_L, x) = x$ for $x \in [0_L, a]$. From Proposition 2.8 (i), (iii), (vii), (xi) and Proposition 3.4, we have $K(x, y) = a$ for $(x, y) \in [0_L, a] \times I_b^a \cup (I_a^b \cup I_{a,b}) \times ((a, b) \cup I_b^a)$. Similarly, we have that $K(x, y) = b$ for $(x, y) \in [b, 1_L] \times (I_a^b \cup I_b^a \cup I_{a,b})$.

From Proposition 2.8 (iv) and Proposition 3.4, we have $K(x, y) \leq a$ for $(x, y) \in ([0_L, a] \cup I_a^b \cup I_{a,b})^2$. So we obtain that

$$\begin{aligned} K(x, y) &= K(0_L, K(x, y)) = K(K(0_L, x), y) = K(p(x), y) = K(K(p(x), 0_L), y) \\ &= K(p(x), K(0_L, y)) = K(p(x), p(y)) = S(p(x), p(y)). \end{aligned}$$

Define an order-preserving mapping $h_1 : [a, b] \cup I_a^b \cup I_b^a \cup I_{a,b} \to [a, b]$ by $h_1(x) = K(x, b)$ for $x \in [a, b] \cup I_a^b \cup I_b^a \cup I_{a,b}$ and we will prove $K(x, y) = h_1(x)$ for $(x, y) \in (I_a^b \cup I_{a,b}) \times [b, 1_L]$. From Proposition 2.8 (vii) and Proposition 3.4 (i), we obtain that $K(x, y) \in [a, b]$ for $(x, y) \in (I_a^b \cup I_{a,b}) \times [b, 1_L]$. Let $x \in I_a^b \cup I_{a,b}$, $y \in [b, 1_L]$ and $z \in [a, b]$. Then, $K(x, y) = K(K(x, y), z) = K(x, K(y, z)) = K(x, b) = h_1(x)$.

Finally, we prove that $K(x, y) = h_1(x)$, for $(x, y) \in I_b^a \times L$. From Proposition 3.4, we have $K(x, y) = K(x, 0_L) = K(x, 1_L)$, and $K(x, 0_L) \leq K(x, b) \leq K(x, 1_L)$. Thus, $K(x, b) = K(x, y) = h_1(x)$.

□

Similarly, we characterize semi-t-operators given by Eq. (2)-(4), respectively.

Proposition 3.6. *Let* $a, b \in L \setminus \{0_L, 1_L\}$ *with* $a \leq b$. *The following properties hold for any semi-t-operator* K *derived from Eq. (2).*

(i) $K(x, y) < b$ *for all* $(x, y) \in (I_a^b \cup I_{a,b}) \times [b, 1_L]$.

(ii) $K(x, y) \leq a$ *for all* $(x, y) \in (I_a^b \cup I_{a,b}) \times ([0_L, a] \cup (a, b) \cup I_a^b \cup I_{a,b})$.

(iii) $K(x, y) \geq b$ *for all* $(x, y) \in I_b^a \times L$.

Theorem 3.7. *Let* $a, b \in L \setminus \{0_L, 1_L\}$ *with* $a \leq b$. *Then* $K \in \mathcal{K}_{a,b}$ *and* K *satisfies all the properties in Proposition 3.6 if and only if* K *can be constructed in the form of Eq. (2).*

Proposition 3.8. *Let* $a, b \in L \setminus \{0_L, 1_L\}$ *with* $a \leq b$. *The following properties hold for any semi-t-operator* K *derived from Eq. (3).*

(i) $K(x,y) \le b$ *for all* $(x,y) \in [b,1_L] \times (I_b^a \cup I_{a,b}) \cup I_a^b \times [b,1_L]$.

(ii) $K(x,y) \le a$ *for all* $(x,y) \in I_a^b \times (L \setminus [b,1_L])$.

(iii) $K(x,0_L) = K(x,1_L) \in [a,b]$ *for* $x \in (I_b^a \cup I_{a,b})$.

Theorem 3.9. *Let* $a,b \in L \setminus \{0_L,1_L\}$ *with* $a \le b$. *Then* $K \in \mathcal{K}_{a,b}$ *and* K *satisfies all the properties in Proposition 3.8 if and only if* K *can be constructed in the form of Eq. (3).*

Proposition 3.10. *Let* $a,b \in L \setminus \{0_L,1_L\}$ *with* $a \le b$. *The following properties hold for any semi-t-operator* K *derived from Eq. (4).*

(i) $K(x,y) \ge b$ *for all* $(x,y) \in I_b^a \times L$.

(ii) $K(x,y) \le a$ *for all* $(x,y) \in I_a^b \times (L \setminus [b,1_L] \cup I_b^a)$.

(iii) $K(x,y) \le b$ *for all* $(x,y) \in I_a^b \times ([b,1_L] \cup I_b^a)$.

(iv) $K(x,0_L) = K(x,1_L) \in [a,b]$ *for* $x \in I_{a,b}$.

Theorem 3.11. *Let* $a,b \in L \setminus \{0_L,1_L\}$ *with* $a \le b$. *Then* $K \in \mathcal{K}_{a,b}$ *and* K *satisfies all the properties in Proposition 3.10 if and only if* K *can be constructed in the form of Eq. (4).*

Remark 3.12. *It should be clarified that the four semi-t-operators* K_1 *to* K_4 *are not trivial simple variants of a single base formula. On the contrary, they differ in the value ranges over regions consisting of incomparable elements, each satisfying a distinct set of structural constraints, and each possesses an independent characterization theorem.*

In the following theorem, we present the dual form of semi-t-operators in Theorem 3.1, where K_{11}, K_{21} and K_{31} are respectively dual to K_1, K_2 and K_3; the dual of K_4 is itself.

Theorem 3.13. *Let* $a,b \in L \setminus \{0_L,1_L\}$ *with* $a \le b$, S *be a semi-t-conorm on* $[0_L,a]^2$, T *be a semi-t-norm on* $[b,1_L]^2$, $p : L \to [0_L,a]$ *and* $q : L \to [b,1_L]$ *be two order-preserving mappings such that* $p(x) = x$ *for* $x \in [0_L,a]$ *and* $q(x) = x$ *for* $x \in [b,1_L]$. *Then the following three functions* K_{11}, K_{21} *and* K_{31} :$L^2 \to L$ *are semi-t-operators on L, respectively. For all* $x,y \in L$,

$$K_{11}(x,y) = \begin{cases} S(p(x),p(y)) & (x,y) \in [0_L,a]^2, \\ T(q(x),q(y)) & (x,y) \in ([b,1_L] \cup I_b^a \cup I_{a,b})^2, \\ a & (x,y) \in [0_L,a] \times (L \setminus [0_L,a]), \\ h_1(x) & (x,y) \in ([a,b] \cup I_a^b) \times L \cup (I_b^a \cup I_{a,b}) \times [0_L,a], \\ b & \textit{otherwise}, \end{cases} \tag{5}$$

where $h_1 : [a,b] \cup I_a^b \cup I_b^a \cup I_{a,b} \to [a,b]$ is order-preserving such that $h_1(x) = x$ for all $x \in (a,b)$.

$$K_{21}(x,y) = \begin{cases} S(p(x),p(y)) & (x,y) \in ([0_L,a] \cup I_a^b)^2, \\ T(q(x),q(y)) & (x,y) \in ([b,1_L] \cup I_b^a \cup I_{a,b})^2, \\ a & (x,y) \in ([0_L,a] \cup I_a^b) \times (L \setminus [0_L,a] \cup I_a^b), \\ h_2(x) & (x,y) \in [a,b] \times L \cup (I_b^a \cup I_{a,b}) \times ([0_L,a] \cup I_a^b), \\ b & \textit{otherwise}, \end{cases} \tag{6}$$

where $h_2 : [a,b] \cup I_b^a \cup I_{a,b} \to [a,b]$ is order-preserving such that $h_2(x) = x$ for all $x \in (a,b)$.

$$K_{31}(x,y) = \begin{cases} S(p(x),p(y)) & (x,y) \in [0_L,a]^2, \\ T(q(x),q(y)) & (x,y) \in ([b,1_L] \cup I_b^a)^2, \\ a & (x,y) \in [0_L,a] \times (L \setminus [0_L.a]), \\ h_3(x) & (x,y) \in ([a,b] \cup I_a^b \cup I_{a,b}) \times L \cup I_b^a \times [0_L,a], \\ b & \textit{otherwise}, \end{cases} \tag{7}$$

where $h_3 : [a,b] \cup I_a^b \cup I_b^a \cup I_{a,b} \to [a,b]$ is order-preserving such that $h_3(x) = x$ for all $x \in (a,b)$.

Remark 3.14. (i) *Let the value of K_{11} on $(I_b^a \cup I_{a,b}) \times [0_L, a]$ be b. Then the semi-t-operator K_{11} in Eq. (5) is identical to F_3 in Theorem 3.1 of* [29].

(ii) *Let the value of K_{21} on $I_b^a \times ([0_L, a] \cup I_a^b)$ be b. Then the semi-t-operator K_{21} in Eq. (6) is identical to F' in Theorem 3.5 of* [33].

(iii) *Let the value of K_{31} on $I_b^a \times [0_L, a]$ be b. Then the semi-t-operator K_{31} in Eq. (7) is identical to F_2 in Theorem 3.1 of* [29].

Similarly, we will provide the equivalent characterizations of the three semi-t-operators in Theorem 3.13 and the proofs are omitted.

Proposition 3.15. *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$. The following properties hold for any semi-t-operator K derived from Eq. (5),*

(i) *$K(x, y) \geq a$, for all $(x, y) \in [0_L, a] \times (I_a^b \cup I_{a,b}) \cup (I_a^b \cup I_{a,b}) \times [0_L, a]$.*

(ii) *$K(x, y) \geq b$, for all $(x, y) \in (I_b^a \cup I_{a,b}) \times (L \setminus [0_L, a])$.*

(iii) *$K(x, 0_L) = K(x, 1_L) \in [a, b]$ for all $x \in I_a^b$.*

Theorem 3.16. *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$. Then $K \in \mathcal{K}_{a,b}$ and K satisfies all the properties in Proposition 3.15 if and only if K can be constructed in the form of Eq. (5).*

Proposition 3.17. *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$. The following properties hold for any semi-t-operator K derived from Eq. (6).*

(i) *$K(x, y) > a$, for all $(x, y) \in (I_b^a \cup I_{a,b}) \times [0_L, a]$.*

(ii) *$K(x, y) \leq a$, for all $(x, y) \in I_a^b \times L$.*

(iii) *$K(x, y) \geq b$, for all $(x, y) \in (I_b^a \cup I_{a,b}) \times (L \setminus [0_L.a] \cup I_a^b)$.*

Theorem 3.18. *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$. Then $K \in \mathcal{K}_{a,b}$ and K satisfies all the properties in Proposition 3.17 if and only if K can be constructed in the form of Eq. (6).*

Proposition 3.19. *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$. The following properties hold for any semi-t-operator K derived from Eq. (7).*

(i) *$K(x, y) \geq b$, for all $(x, y) \in I_b^a \times (L \setminus [0_L, a])$.*

(ii) *$K(x, y) \geq a$, for all $(x, y) \in [0_L, a] \times (I_a^b \cup I_{a,b}) \cup I_b^a \times [0_L, a]$.*

(iii) *$K(x, 0_L) = K(x, 1_L) \in [a, b]$ for $x \in (I_a^b \cup I_{a,b})$.*

Theorem 3.20. *Let $a, b \in L \setminus \{0_L, 1_L\}$ with $a \leq b$. Then $K \in \mathcal{K}_{a,b}$ and K satisfies all the properties in Proposition 3.19 if and only if K can be constructed in the form of Eq. (7).*

4 Conclusions

In this paper, we presented four construction approaches for semi-t-operators on bounded lattices via semi-t-norms, semi-t-conorms and order-preserving mappings, along with their corresponding dual forms. It was shown that some of the construction methods in existing literature are special cases of those proposed in this paper. By analyzing the properties of the semi-t-operators in Theorems 3.1 and 3.13, we established their equivalent characterizations. This provided user-friendly criteria for determining whether a binary mapping belongs to the target class of semi-t-operators.

It is worth noting that the research methods in this article are only applicable to cases when a and b are comparable. In future research, we will study the semi-t-operators on bounded lattices when a and b are incomparable as well as the construction methods of semi-t-operators on bounded trellises.

Conflicts of interest

The authors declare that there is no conflict of interest.

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