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Original paper

Fuzzy fractional framework under granular ψ-Caputo fractional derivative with applications to supply chain dynamics

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Abstract

This article develops a mathematical framework for fuzzy fractional differential system using the concept of granular Caputo fractional derivative with respect to another function ψ within the horizontal membership function (HMF)-based granular differentiability framework. The propose technique addresses the need for a unified approach that simultaneously incorporates the function ψ as memory kernels for flexible modeling, multi-dimensional representation of uncertainty through HMFs, and analytical fuzzy solution using granular fuzzy Laplace transform with respect to another function (LTWR-AF). First, we introduce the granular ψ-fuzzy fractional integral (ψ-FFI), granular ψ-Caputo fractional derivative (ψ-CFD) and granular fuzzy LTWR-AF. Fundamental properties, results, and relationships among these properties, are established in HMF representation. Based on this foundation, closed-form analytical fuzzy solution (AFS) of granular Caputo fuzzy fractional differential equations (GCFFDEs), involving a granular ψ-CFD, are determined and expressed compactly in terms of the Mittag-Leffler function (M-LF). Moreover, the existence and uniqueness (E&U) of the fuzzy solutions for the abstract granular Caputo fuzzy fractional differential system (GCFFDS) with fuzzy parameters and initial data are examined under the mild conditions. To illustrate the applicability of the proposed framework, the proposed methodology is applied to the granular Caputo fuzzy fractional sustainable, impact-intensive, quarantined, and recycled/remanufactured (SIQR) supply-chain model under environmental disruptions. Numerical simulations and graphical representations demonstrate that the proposed model effectively describe the uncertainty propagation and memory-dependent behavior in supply-chain dynamics.

Keywords: Fuzzy fractional calculus, granular ψ-Caputo fractional derivative, horizontal membership functions, existence and uniqueness, supply chain resilience.

1 Introduction

Fractional calculus has attracted considerable attention in modeling complex systems with memory effects, non-local characteristics and hereditary properties among different engineering and scientific domains. Fractional derivatives encapsulate the historical evolution of system. This fundamental memory characteristic has led to successful applications in science and engineering [28, 41]. Various well-established definitions of fractional integral and derivative operators include the Riemann-Liouville (RL), a Hadamard, and Erdelyi-Kober integrals and derivatives, extending classical concepts into non-integer orders. Researchers have studied the general formulations of these definitions where certain choices of the kernel functions and differentiation operators lead to the classical fractional derivatives. For instance, using the kernel function $k(z,p) = z - p$ and $k(z,p) = \ln z/p$ with the differential operator d/dz and zd/dz leads to the Riemann-Liouville and the Hadamard fractional derivatives. However, these classical fractional operators impose

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restrictions on the kernel function that may not adequately represent the diverse physical processes encountered in real-world applications. To overcome these limitations and provide greater flexibility in modeling, another approach has been developed where the kernel function is of the form $k(z,p) = \psi(z) - \psi(p)$ and the derivative operator is of the form $1/\psi'(z)d/dz$. This generalized approach, which we term fractional calculus with respect to another function (AF), enables to fit the kernel function ψ according to the system under consideration, which leads us not only theoretical generality but also to practical applicability. The fundamental concepts and properties of the fractional integrals and derivatives concerning another nonnegative and continuous function ψ have been defined in [26, 35]. Almeida [4] studied the ψ-Caputo fractional derivative (ψ-CFD), examined its properties and proposed an approximate numerical technique. It demonstrated the method's efficiency and applied it to a population growth model to achieve higher accuracy with the different kernels. Almeida [5] has explored generalized fractional calculus using variable-order operators. Moreover, he established the relationship between concepts and verified the proposed method through simulations. Jajarmi [21] investigated the dynamics, stability, and optimal chaos control of a ψ-Caputo cancer–immune system. Agarwal et al. [1] studied the strict stability of fractional-order nonlinear differential systems with respect to another function under ψ-CFD. They derived sufficient stability conditions using Lyapunov functions. Taqbibt et al. [40] have studied the fractional differential inequalities using ψ-Caputo fractional operator. They demonstrate the application of σ-convex functions using ψ-Caputo derivative. Mali et al. [30] developed the idea of generalized k-fractional derivatives, including (k,Φ)-RL, (k,Φ)-Caputo and (k,Φ)-Hilfer fractional derivatives. They discussed the key results, particular cases and significant insights into the theory. Jr et al. [25] analyzed the theory of Lie groups to analyze the symmetries of the fractional order differential equations with the variable order. The key findings have been applied to solve Harry Dym-type fractional differential equation. For a comprehensive overview of recent advancements on fractional derivatives with respect to AF, the reader is referred to [3, 10, 28, 41]. Alanazi et al. [3] introduce a novel control system that uses s ψ-CFD to ensure the practical stability of nonlinear systems. Fractional-order modeling and control frameworks have also been developed for tumor–immune surveillance, chaotic oscillators, and epidemiological optimal-control problems [8, 14, 22]. Amir et al. [6] have developed the relationship between Hadamard-type and tempered fractional calculus under a fuzzy environment. They have unified various fractional operators using the Hadamard-Caputo tempered fractional derivative. They also determined E&U using fixed-point theorems and demonstrated the applicability of the proposed technique. Real-world phenomena are inherently affected by the uncertainty of various factors including subjective judgments, measurement errors, and incomplete information. Traditional deterministic fractional differential equations (FDEs) are inadequate to capture such uncertainties. Fuzzy set theory, proposed by Zadeh, offers a natural mathematical tool for expressing and manipulating vague and imprecise information. Combining fuzzy theory with fractional calculus, one can obtain the fuzzy fractional differential equations (FFDEs), which capture the memory phenomena and uncertainty through fractional operator and fuzzy sets [2]. Fuzzy Laplace transform (FLT) is an important tool to analyze FFDEs, due to its versatile ability of solving complex problems with uncertain knowledge. This approach has gained popularity in recent decades because of its applications [42]. This work improves the reader's capability to comprehend, simulate, investigate and control systems subjected to uncertainties. More general operator structures were attempted to propose using generalized fuzzy fractional-derivative operators including Caputo, Caputo-Fabrizio and Atangana-Baleanu kernels. Johnson et al. [24] investigated the optimal control strategies of integro-differential system using resolvent operator. Recent studies have further advanced fuzzy fractional systems involving generalized Caputo derivatives, fuzzy fractional delay equations, multi–order fractional derivatives, and piecewise constant arguments [15, 19, 20, 27]. These fuzzy derivatives based on the fuzzy standard interval arithmetic (for short FSIA) have been used for integer and FFDEs; however, they present some limitations that restrict their applicability in real-world scenario. These drawbacks include the catastrophe of physics law violation (CPLV), non-existence due to missing H- or gH-differences, the doubling property, unnatural behavior in modeling (UBM), inability to factorize and multiplicity of solutions. To solve these problems, a novel granular derivative (gr-derivative) was proposed by, Mazandarani [31] based on relative distance measure fuzzy interval arithmetic (RDM-FIA) and HMF [36]. In contrast to methodologies based on FSIA, the gr-derivative provides a unique, physically consistent without doubling the system dimension. Son et al. [38] developed the solution of delay-uncertain FDEs using the concepts of gr-differentiability. They established some fundamental properties and then studied the E&U of the proposed model. Dong et al. [13] analyzed the finite-time stability of delay FFFDEs under gr-differentiability. Dong et al. [12] conducted the study on the optimal control for the SEIR fractional mathematical model using granular differentiability. They also determined the AFS of the fuzzy SEIR mathematical model using fuzzy Laplace transform (FLT). For further developments on granular fractional calculus and gr-differentiability, the reader is referred to the [29, 33, 39, 43].

1.1 Research gap

Despite significant advances in fuzzy fractional calculus, several critical gaps remain:

1. Fuzzy approaches use the HMF framework to provide a more comprehensive granular representation by incorporating both vertical (membership level α) and horizontal (cover variable $\alpha_{\varkappa}$) dimensions of uncertainty. This granular approach has not been systematically integrated with fractional calculus with respect to another function (ψ).

2. While ψ-Caputo fractional derivatives have been studied in deterministic settings, their extension to fuzzy-valued functions under gr-differentiability remains unexplored. The gr-differentiability concept offers computational advantages and better uncertainty propagation compared to H-differentiability [31], yet its application to fractional systems with arbitrary kernel functions ψ has not been rigorously developed.

3. The generalized LTWR-AF has been studied in crisp settings [16, 23], but its granular fuzzy extension has not been formulated. Consequently, there is no systematic analytical methodology to solve GCFFDEs involving granular ψ-CFD.

4. The E&U of solutions for GCFFDS under granular uncertainty has not been rigorously established. Fixed-point theorems suitable for granular metric spaces need to be applied to guarantee well-posedness of fuzzy initial value problems (FIVPs) in this generalized framework.

5. While fuzzy fractional models use have been applied to epidemiology, control systems [12, 24], and economics [7], their advantages over that classical approaches are often asserted but not convincingly implemented in supply chain models to capture the environmental interactions under fractional memory and granular fuzzy uncertainty.

These gaps motivate the development of a rigorous mathematical framework for solving GCFFDEs under granular ψ-CFD and determine their analytical fuzzy solution.

1.2 Novelty of the proposed framework

The present study integrates fuzzy set theory, fractional calculus with respect to another function, and an HMF-based granular uncertainty modeling. Fuzzy set theory introduced by Zadeh, provides the fundamental mathematical basis for representing vagueness and imprecision. Classical uncertain fractional-order models usually extend Caputo-type derivatives to fuzzy-valued functions using α-cut representations, H-differentiability, or gH-differentiability. Although these approaches are useful for describing uncertainty, they may suffer from several limitations in fuzzy dynamical modeling, such as non-existence of differences, dimensional doubling, and unnatural uncertainty propagation [31]. The ψ-CFD provides a flexible memory structure by replacing the time variable with a strictly increasing function ψ. It is more general than the standard Caputo derivative and allows several classical fractional models to be recovered through suitable choices of ψ [4]. Recent studies have also extended this idea to fuzzy fractional models. Ghazouani et al. [17] investigated fuzzy fractional delay integro-differential equations involving the generalized Hukuhara ψ-Caputo fractional derivative. Their work mainly focuses on gH-differentiability and fuzzy interval-based representations. In contrast, the proposed granular ψ-Caputo fuzzy fractional derivative combines the memory flexibility of the ψ-Caputo operator with the HMF-based granular representation of fuzzy-valued functions. This provides a more detailed representation of fuzzy uncertainty than conventional α-cut, H-differentiability, and gH-differentiability based formulations. The proposed framework also differs from fuzzy Caputo–Fabrizio and Atangana–Baleanu type models. Caputo–Fabrizio operators use a nonsingular exponential kernel, whereas Atangana–Baleanu operators employ a Mittag–Leffler kernel. These operators modify the memory kernel, but they do not directly incorporate the HMF-based granular structure of fuzzy-valued functions. The present work develops an HMF-based granular ψ-Caputo fuzzy fractional framework in which the memory effect is governed by the arbitrary function ψ, while uncertainty is quantified through HMF-based granular variables. Furthermore, the granular fuzzy LTWR-AF is constructed as an analytical transform tool adapted to the granular ψ-Caputo setting. Existing generalized Laplace transforms with respect to another function are mainly formulated for crisp functions or non-granular fuzzy settings. In this work, the transform is developed for HMF representation, and its existence is first established under suitable measurability and granular ψ-exponential-order conditions. It is then used to derive analytical fuzzy solutions of GCFFDEs. Therefore, the novelty of the proposed work lies in the integrated construction of a granular ψ-Caputo fuzzy fractional framework, its associated granular fuzzy LTWR-AF, analytical solution representation, existence–uniqueness analysis and application to supply-chain dynamics under memory effects and granular fuzzy uncertainty.

1.3 Contribution of the study

Based on the above novelty comparison, this paper makes the following fundamental contributions. We present the granular fuzzy fractional differential systems under the concept of granular ψ-CFD. A granular ψ-Caputo fuzzy fractional framework is developed by combining the flexibility of fractional differentiation with respect to another function ψ and the HMF-based granular representation of fuzzy-valued functions. This distinguishes the proposed model from existing fuzzy Caputo, Caputo–Fabrizio, Atangana–Baleanu, and conventional ψ-Caputo fuzzy fractional models. This operator unifies three key concepts: (i) Caputo fractional differentiation for better initial condition handling, (ii) generalized kernel ψ for flexible memory representation, and (iii) HMFs for granular uncertainty propagation. This extends significantly beyond classical fuzzy Caputo derivatives and provides a more general framework than existing granular fuzzy fractional operators that are limited to standard kernels. We present the concept granular fuzzy LTWR-AF with respect to nonnegative increasing function ψ and establish its HMF. Furthermore, we derive AFSs of GCFFDEs involving ψ-CFD and express via generalized M-LF. We rigorously determine the E&U of mild solutions for the abstract GCFFDS with fuzzy parameters and initial data using Banach contraction principle. Beyond its theoretical relevance, the proposed methodology is applied them to fuzzy fractional cerebrospinal fluid model to handle the non-locality and uncertainty, as well as the granular Caputo fuzzy fractional SIQR a model that captures the feedback of the supply chain resilience under environmental disruptions. Numerical simulations and graphical studies demonstrate the intricate behavioral patterns capture the uncertainty propagation and memory effects in real-world supply chain dynamics.

1.4 Structure of the article

The rest of the article is structured as follows: Section 2 presents an overview of the fundamental concepts of that granular representation, fractional integral, granular fuzzy fractional integral (GFFI), granular ψ-CFD, and granular fuzzy LTWR-AF with respect to another function. Section 3 presents analytical fuzzy solutions of GCFFDEs with their numerical examples. Section 4 presents a rigorous mathematical investigation of the E&U of the AFS of the GCFFDS with the fuzzy parameters and initial conditions. A granular fuzzy fractional SIQR model which addresses the supply chain environmental impact is investigated in Section 5. Finally, Section 6 is devoted to the conclusion, limitations and possible directions for future research.

2 Preliminaries

In this section, we recall some basic concepts of the fuzzy fractional calculus and its HMF representation.

Definition 2.1. *A fuzzy set $\mathfrak{w} : [\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{R} \longrightarrow [0,1]$ is said to be a fuzzy number (FN) if it is satisfied the four properties including normality, fuzzy-convexity, upper semi-continuity and compact support. The level set (α-cut) of $\mathfrak{w}$, is denoted by $[\mathfrak{w}]^\alpha$, and defined by $[\mathfrak{w}^-_\alpha, \mathfrak{w}^+_\alpha]$, for all $\alpha \in [0,1]$.*

Throughout this paper, $\Xi_{\mathfrak{R}}$ denotes the class of FNs represented through HMF forms together with their relative distance measure (RDM) coordinate structure. The inverse HMF transformation $\mathscr{H}^{-1}$ reconstructs the corresponding type-1 FN from its HMF representation. Triangular fuzzy numbers (TFNs) form an important subclass of $\Xi_{\mathfrak{R}}$ and are used to describe the fuzzy parameters and illustrative examples in this study.

Remark 2.2. *Let $\mathcal{S} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$ be the finite set of primitive uncertainty sources involved in the granular formulation, and let $\alpha_\varkappa = (\alpha_{\varkappa_1}, \ldots, \alpha_{\varkappa_m}) \in [0,1]^m$ be the associated RDM coordinate vector. If the HMF representation of a fuzzy quantity depends only on a subset of $\mathcal{S}$, then it is regarded as a function on the full coordinate domain $[0,1]^m$ by keeping it constant with respect to the remaining RDM coordinates. This convention allows all granular operations to be written in a common RDM coordinate system. Here, m denotes the number of uncertainty sources, while n is reserved for the dimension of the state space $\Xi^n_{\mathfrak{R}}$.*

Definition 2.3. [31] *Let $[a,b] \subseteq \mathfrak{R}$ be a compact set and let $\mathfrak{w} \in \Xi_{\mathfrak{R}}$ be an FN with α-cut $[\mathfrak{w}]^\alpha = [\mathfrak{w}^-_\alpha, \mathfrak{w}^+_\alpha]$, $\alpha \in [0,1]$. The HMF of $\mathfrak{w}$ is the mapping $\mathfrak{w}^{gr} : [0,1] \times [0,1] \longrightarrow [a,b]$, defined by $\mathfrak{w}^{gr}(\alpha, \alpha_\varkappa) = \mathfrak{w}^-_\alpha + \big(\mathfrak{w}^+_\alpha - \mathfrak{w}^-_\alpha\big)\alpha_\varkappa$. The symbol gr denotes the granular form of $\mathfrak{w}$ over $x \in [a,b]$ for $\alpha, \alpha_\varkappa \in [0,1]$. The HMF of $\mathfrak{w}$ is denoted by $\mathscr{H}(\mathfrak{w}) \triangleq \mathfrak{w}^{gr}(\alpha, \alpha_\varkappa)$, where α denotes the membership level and $\alpha_\varkappa$ is the RDM variable.*
After granular arithmetic operations or propagation through a fuzzy-valued function, the resulting HMF may depend on several RDM coordinates. In this case, we write the full HMF representation as

$$\mathfrak{w}^{gr}(\alpha, \alpha_\varkappa), \qquad \alpha_\varkappa = (\alpha_{\varkappa_1}, \ldots, \alpha_{\varkappa_m}) \in [0,1]^m,$$

where m is the number of uncertainty sources involved in the granular formulation.

Proposition 2.4. *The transformation property of the α-cuts of FN $\mathfrak{w} \in \Xi_{\mathfrak{R}}$* [31] *is defined as*

$$\mathscr{H}^{-1}(\mathfrak{w}^{gr}(\alpha, \alpha_\varkappa)) := [\mathfrak{w}]^\alpha := \left[\inf_{\nu \geq \alpha} \min_{\alpha_\varkappa} \mathfrak{w}^{gr}(\nu, \alpha_\varkappa), \sup_{\nu \geq \alpha} \max_{\alpha_\varkappa} \mathfrak{w}^{gr}(\nu, \alpha_\varkappa)\right]. \tag{1}$$

Definition 2.5. [31] *Let $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$ be two FNs with HMF representations $\mathscr{H}(\mathfrak{w}_1) = \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_1}})$ and $\mathscr{H}(\mathfrak{w}_2) = \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}})$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the uncertainty-source sets associated with $\mathfrak{w}_1$ and $\mathfrak{w}_2$, respectively. The corresponding RDM coordinate vectors are denoted by $\alpha_{\varkappa_{\mathcal{S}_i}} = (\alpha_{\varkappa_s})_{\mathfrak{s}\in\mathcal{S}_i}, \quad i = 1, 2$. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. Then the two HMFs are considered in the common RDM coordinate system $\alpha_{\varkappa_{\mathcal{S}}} = (\alpha_{\varkappa_s})_{\mathfrak{s}\in\mathcal{S}}$. Each unique uncertainty source is assigned a distinct RDM coordinate, whereas a source shared between S1 and S2 retains the same RDM coordinate across both HMFs. Let $*$ denote one of the ordinary real arithmetic operations on $\mathfrak{R}$, namely addition, subtraction, multiplication, or division, and let $\diamond^{gr}$ denote the corresponding HMF-based granular operation on $\Xi_{\mathfrak{R}}$. The granular arithmetic operation is defined by*

$$\mathscr{H}\left(\mathfrak{w}_1 \diamond^{gr} \mathfrak{w}_2\right) \triangleq \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_1}}) * \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}}), \tag{2}$$

where $\mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}}) \neq 0$ and for all $(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}}) \in [0,1]^2$, where $*$ is the division operator. The right-hand side of Equation (2) is the resulting HMF granule and corresponding FN is reconstructed through the inverse HMF transformation $\mathscr{H}^{-1}$.

Remark 2.6. *In HMF-based granular arithmetic, each uncertainty source is associated with an RDM variable. Therefore, distinct uncertainty sources are represented by distinct RDM variables, whereas repeated occurrences or algebraic transformations of the same uncertainty source contain the same RDM variable. More generally, if two granular quantities depend on partially overlapping sets of uncertainty sources, then they are represented in the common RDM coordinate system obtained from the union of these sources. For every common source, the same RDM coordinate is used in both HMF representations. In particular, if $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are associated with two distinct uncertainty sources, their HMF representations can be written as*

$$\mathscr{H}(\mathfrak{w}_1) = \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_1}), \qquad \mathscr{H}(\mathfrak{w}_2) = \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_2}), \tag{3}$$

where $\alpha_{\varkappa_1}$ and $\alpha_{\varkappa_2}$ are distinct RDM variables. Hence, the corresponding granular arithmetic is performed in the two–dimensional RDM coordinate system $(\alpha_{\varkappa_1}, \alpha_{\varkappa_2}) \in [0,1]^2$. On the other hand, if $\mathfrak{w}_2$ is obtained from $\mathfrak{w}_1$ by a granular algebraic operation, for instance, we take

$$\mathfrak{w}_2 = \lambda \odot^{gr} \mathfrak{w}_1, \qquad \lambda \in \mathfrak{R}, \tag{4}$$

then no new uncertainty source is introduced. In this case,

$$\mathscr{H}(\mathfrak{w}_2) = \lambda \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_1}), \tag{5}$$

and the same RDM variable $\alpha_{\varkappa_1}$ is retained. In particular, suppose

$$\mathfrak{v} = (-1) \odot^{gr} \mathfrak{w}, \tag{6}$$

we have

$$\mathscr{H}(\mathfrak{v}) = -\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}), \tag{7}$$

and consequently

$$\mathscr{H}\left(\mathfrak{w} \oplus^{gr} \mathfrak{v}\right) = \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) - \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) = 0. \tag{8}$$

Thus,

$$\mathfrak{w} \oplus^{gr} \mathfrak{v} = \hat{0}. \tag{9}$$

This source-preserving convention is consistent with multidimensional RDM granular operations.

Definition 2.7. *Let $\varkappa : [a,b] \subseteq \mathfrak{R} \to \Xi_{\mathfrak{R}}$ be a fuzzy-valued function (FVF)* [31] *whose uncertainty is obtained by the finite source set $\mathcal{S} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$. The HMF representation of $\varkappa(z)$ is denoted by $\mathscr{H}(\varkappa(z)) \triangleq \varkappa^{gr}(z, \alpha, \alpha_{\varkappa})$, and is defined by*

$$\varkappa^{gr} : [a,b] \times [0,1] \times [0,1]^m \longrightarrow \mathfrak{R}, \tag{10}$$

where $\alpha_{\varkappa} = (\alpha_{\varkappa_1}, \alpha_{\varkappa_2}, \ldots, \alpha_{\varkappa_n}) \in [0,1]^m$ is the RDM variables associated with the uncertainty source in $\mathcal{S}$, $z \in [a,b]$, and $\alpha \in [0,1]$.

Definition 2.8. *Let $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$ be two FNs with uncertainty-source sets $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and let $\alpha_{\varkappa_{\mathcal{S}}} = (\alpha_{\varkappa_s})_{\mathfrak{s}\in\mathcal{S}} \in [0,1]^{|\mathcal{S}|}$ be the common RDM coordinate vector constructed according to the RDM-source dependency convention in Remark 2.2. The granular metric on $\Xi_{\mathfrak{R}}$ is the mapping $\rho^{gr} : \Xi_{\mathfrak{R}} \times \Xi_{\mathfrak{R}} \longrightarrow \mathfrak{R}^+ \cup \{0\}$, defined by*

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_{\mathcal{S}}}\in[0,1]^{|\mathcal{S}|}} \left|\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_1}}) - \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}})\right|. \tag{11}$$

In this formulation, the coordinates corresponding to distinct uncertainty sources vary independently, whereas a source common to both FNs is represented by the same RDM variable across both HMF representations. In particular, if $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$, then $\mathfrak{w}_1$ and $\mathfrak{w}_2$ involve distinct uncertainty sources, and Equation (11) becomes

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_1},\alpha_{\varkappa_2}\in[0,1]} \left|\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_1}) - \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_2})\right|. \tag{12}$$

If the FN $\mathfrak{w}_1$ and $\mathfrak{w}_2$ depend on the same uncertainty source, that is, $\mathcal{S}_1 = \mathcal{S}_2 = \{\mathfrak{s}\}$, then the same RDM variable is used in both HMFs, and Equation (11) reduces to

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_s}\in[0,1]} \left|\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_s}) - \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_s})\right|. \tag{13}$$

Remark 2.9. *In Definition 2.8, the maximum is taken with respect to the RDM coordinates associated with the uncertainty sources of the fuzzy numbers under comparison. In particular, when a fuzzy number is compared with itself, no new uncertainty is introduced and hence the same RDM variable is used in both HMFs, and*

$$\rho^{gr}(\mathfrak{w}, \mathfrak{w}) = \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa}\in[0,1]} \left|\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) - \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa})\right| = 0. \tag{14}$$

Moreover, if $\mathfrak{v} = (-1) \odot^{gr} \mathfrak{w}$, then $\mathscr{H}(\mathfrak{v}) = -\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa})$, and hence

$$\mathscr{H}\left(\mathfrak{w} \oplus^{gr} \mathfrak{v}\right) = \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) - \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) = 0. \tag{15}$$

Thus, $\mathfrak{w} \oplus^{gr} \mathfrak{v} = \hat{0}$. On the other hand, distinct uncertainty sources are represented by distinct RDM variables and are varied independently in the granular metric as stated in Remark 2.6. For metric properties, let $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \in \Xi_{\mathfrak{R}}$ be FNs with the uncertainty-source sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, respectively, and let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, then the HMFs are considered in the common RDM coordinate $\alpha_{\varkappa_{\mathcal{S}}} = (\alpha_{\varkappa_s})_{\mathfrak{s}\in\mathcal{S}}$. The nonnegativity and symmetry of the granular metric ρ^{gr} follow directly from the absolute value. Moreover, if $\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = 0$, then $\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_1}}) = \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}})$ for every source-consistent RDM coordinate. Hence, by the inverse HMF transformation, we have $\mathfrak{w}_1 = \mathfrak{w}_2$. Moreover, using the pointwise triangle inequality in $\mathfrak{R}$, we have

$$\left|\mathfrak{w}_1^{gr} - \mathfrak{w}_3^{gr}\right| \leq \left|\mathfrak{w}_1^{gr} - \mathfrak{w}_2^{gr}\right| + \left|\mathfrak{w}_2^{gr} - \mathfrak{w}_3^{gr}\right|. \tag{16}$$

Taking the maximum over the RDM variables and then the supremum over $\alpha \in [0,1]$ gives

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_3) \leq \rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) + \rho^{gr}(\mathfrak{w}_2, \mathfrak{w}_3). \tag{17}$$

Therefore, ρ^{gr} satisfies the metric properties on $\Xi_{\mathfrak{R}}$.

Algebraic convention. In the following results, each FN $\mathfrak{w} \in \Xi_{\mathfrak{R}}$ is considered together with its HMF representation and its RDM coordinates, as described in Remark 2.6. When several FNs appear in the same algebraic identity, their HMFs are written on the common RDM coordinate and determined by the union of their uncertainty-source sets. Equality of two FNs means the equality of their HMF representations on the common RDM coordinate. Applying the inverse HMF transformation $\mathscr{H}^{-1}$ then gives equality of the corresponding reconstructed FNs.

Proposition 2.10. *The class $\Xi_{\mathfrak{R}}$, with the HMF representation and RDM coordinates, is a linear space under granular addition $\oplus^{gr}$ and granular scalar multiplication $\odot^{gr}$ defined in Equations (18) and (19), respectively.*

Proof. Let $\mathfrak{w}_i \in \Xi_{\mathfrak{R}}$ $i = 1, 2, 3$, and suppose $\mathcal{S}_i$ is the corresponding uncertainty-source set; let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. All the HMFs can represented in a common RDM coordinate system $\varkappa_{\mathcal{S}} = \varkappa_s, s \in \mathcal{S}$, when it is necessary. The granular addition and scalar multiplication are defined by:

For $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$ and $\lambda \in \mathfrak{R}$, the granular addition and granular scalar multiplication are defined by:

$$\mathscr{H}\left(\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2\right) \triangleq \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{S_1}}) + \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{S_2}}), \tag{18}$$

$$\mathscr{H}\left(\lambda \odot^{gr} \mathfrak{w}_1\right) \triangleq \lambda \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{S_1}}). \tag{19}$$

According to the HMF-based granular arithmetic, the expressions on the left-hand sides of Equations (18) and (19) denote the FNs reconstructed from the real-valued HMFs on the right-hand sides. Therefore, $\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$, and $\lambda \odot^{gr} \mathfrak{w}_1 \in \Xi_{\mathfrak{R}}$. Hence, $\Xi_{\mathfrak{R}}$ is closed under granular addition and granular scalar multiplication. We now verify the algebraic properties. Let $\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \in \Xi_{\mathfrak{R}}$. Since the operations are performed pointwise in the real-valued HMF representation, we have

$$\mathscr{H}\left(\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2\right) = \mathfrak{w}_1^{gr} + \mathfrak{w}_2^{gr} = \mathfrak{w}_2^{gr} + \mathfrak{w}_1^{gr} = \mathscr{H}\left(\mathfrak{w}_2 \oplus^{gr} \mathfrak{w}_1\right). \tag{20}$$

Thus, $\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2 = \mathfrak{w}_2 \oplus^{gr} \mathfrak{w}_1$. Therefore, granular addition is commutative. Similarly,

$$\mathscr{H}\left((\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2) \oplus^{gr} \mathfrak{w}_3\right) = \mathfrak{w}_1^{gr} + \mathfrak{w}_2^{gr} + \mathfrak{w}_3^{gr} = \mathscr{H}\left(\mathfrak{w}_1 \oplus^{gr} (\mathfrak{w}_2 \oplus^{gr} \mathfrak{w}_3)\right). \tag{21}$$

Hence, $(\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2) \oplus^{gr} \mathfrak{w}_3 = \mathfrak{w}_1 \oplus^{gr} (\mathfrak{w}_2 \oplus^{gr} \mathfrak{w}_3)$. Therefore, granular addition is associative. The additive identity is the zero fuzzy number $\hat{0}$, whose HMF is $\hat{0}^{gr}(\alpha, \alpha_{\varkappa}) = 0$. Thus, $\mathscr{H}\left(\mathfrak{w}_1 \oplus^{gr} \hat{0}\right) = \mathfrak{w}_1^{gr} + 0 = \mathfrak{w}_1^{gr}$. By the inverse HMF transformation, we have $\mathfrak{w}_1 \oplus^{gr} \hat{0} = \mathfrak{w}_1$. Moreover, the zero element is unique. Indeed, if $\mathfrak{e} \in \Xi_{\mathfrak{R}}$ also satisfies $\mathfrak{w}_1 \oplus^{gr} \mathfrak{e} = \mathfrak{w}_1$ for every $\mathfrak{w}_1 \in \Xi_{\mathfrak{R}}$, then $\mathscr{H}(\mathfrak{w}_1) + \mathscr{H}(\mathfrak{e}) = \mathscr{H}(\mathfrak{w}_1)$. Therefore, $\mathscr{H}(\mathfrak{e}) = 0$, and hence $\mathfrak{e} = \hat{0}$. Thus, the additive identity is unique. Next, let $\mathfrak{w}_1 \in \Xi_{\mathfrak{R}}$. Define

$$\mathfrak{v}_1 := (-1) \odot^{gr} \mathfrak{w}_1. \tag{22}$$

Since granular scalar multiplication does not introduce a new uncertainty source, the same RDM coordinate is used. Hence, by Equation (19), we have

$$\mathscr{H}(\mathfrak{v}_1) = \mathscr{H}\left((-1) \odot^{gr} \mathfrak{w}_1\right) = -\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa}). \tag{23}$$

Therefore,

$$\mathscr{H}\left(\mathfrak{w}_1 \oplus^{gr} \mathfrak{v}_1\right) = \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa}) + \mathfrak{v}_1^{gr}(\alpha, \alpha_{\varkappa}). \tag{24}$$

Using the expression for $\mathscr{H}(\mathfrak{v}_1)$, we obtain

$$\mathscr{H}\left(\mathfrak{w}_1 \oplus^{gr} \mathfrak{v}_1\right) = \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa}) - \mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa}) = 0. \tag{25}$$

Since $\mathscr{H}^{-1}(0) = [0, 0]$ for every $\alpha \in [0, 1]$, it follows that

$$\mathfrak{w}_1 \oplus^{gr} \mathfrak{v}_1 = \hat{0}. \tag{26}$$

The additive inverse is unique. Suppose that $\mathfrak{v} \in \Xi_{\mathfrak{R}}$ also satisfies

$$\mathfrak{w}_1 \oplus^{gr} \mathfrak{v} = \hat{0}. \tag{27}$$

Taking HMFs on the common RDM coordinate gives

$$\mathfrak{w}_1^{gr} + \mathfrak{v}^{gr} = 0. \tag{28}$$

Therefore,

$$\mathfrak{v}^{gr} = -\mathfrak{w}_1^{gr} = \mathscr{H}\left((-1) \odot^{gr} \mathfrak{w}_1\right). \tag{29}$$

Applying the inverse HMF transform, we obtain

$$\mathfrak{v} = (-1) \odot^{gr} \mathfrak{w}_1 = \mathfrak{v}_1. \tag{30}$$

Hence, the additive inverse is unique. Now let $\lambda, \eta \in \mathfrak{R}$. The scalar identity property follows from

$$\mathscr{H}\left(1 \odot^{gr} \mathfrak{w}_1\right) = 1\mathfrak{w}_1^{gr} = \mathfrak{w}_1^{gr}. \tag{31}$$

Thus, $1 \odot^{gr} \mathfrak{w}_1 = \mathfrak{w}_1$. Scalar associativity follows from

$$\mathscr{H}\left((\lambda\eta) \odot^{gr} \mathfrak{w}_1\right) = (\lambda\eta)\mathfrak{w}_1^{gr} = \lambda\left(\eta\mathfrak{w}_1^{gr}\right) = \mathscr{H}\left(\lambda \odot^{gr} (\eta \odot^{gr} \mathfrak{w}_1)\right). \tag{32}$$

Hence, $(\lambda\eta) \odot^{gr} \mathfrak{w}_1 = \lambda \odot^{gr} (\eta \odot^{gr} \mathfrak{w}_1)$. The distributive law with respect to scalar addition follows from

$$\mathscr{H}\left((\lambda+\eta) \odot^{gr} \mathfrak{w}_1\right) = (\lambda+\eta)\mathfrak{w}_1^{gr} = \lambda\mathfrak{w}_1^{gr} + \eta\mathfrak{w}_1^{gr} = \mathscr{H}\left((\lambda \odot^{gr} \mathfrak{w}) \oplus^{gr} (\eta \odot^{gr} \mathfrak{w})\right). \tag{33}$$

Therefore, $(\lambda+\eta) \odot^{gr} \mathfrak{w}_1 = (\lambda \odot^{gr} \mathfrak{w}_1) \oplus^{gr} (\eta \odot^{gr} \mathfrak{w}_1)$. The distributive law with respect to vector addition follows from

$$\mathscr{H}\left(\lambda \odot^{gr} (\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2)\right) = \lambda\left(\mathfrak{w}_1^{gr} + \mathfrak{w}_2^{gr}\right) = \lambda\mathfrak{w}_1^{gr} + \lambda\mathfrak{w}_2^{gr} = \mathscr{H}\left((\lambda \odot^{gr} \mathfrak{w}_1) \oplus^{gr} (\lambda \odot^{gr} \mathfrak{w}_2)\right). \tag{34}$$

Hence, $\lambda \odot^{gr} (\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2) = (\lambda \odot^{gr} \mathfrak{w}_1) \oplus^{gr} (\lambda \odot^{gr} \mathfrak{w}_2)$.
Therefore, all vector space axioms over the scalar field $\mathfrak{R}$ are satisfied. Consequently, $\Xi_{\mathfrak{R}}$ is a real linear space under the HMF-based granular operations $\oplus^{gr}$ and $\odot^{gr}$. □

Remark 2.11. *It is important to distinguish the HMF-based granular operations from the classical fuzzy arithmetic operations. Although the two operations may coincide in some cases, they are not identical in general. For instance, let* $\mathfrak{w}_1 = (a_1, a_2, a_3)$ *and* $-\mathfrak{w}_1 = (-a_3, -a_2, -a_1)$ *be the two triangular FNs. Under the classical fuzzy addition, we have*

$$\mathfrak{w}_1 + (-\mathfrak{w}_1) = (a_1 - a_3, 0, a_3 - a_1), \tag{35}$$

which is not equal to $\hat{0}$. *Hence, classical fuzzy addition does not provide the additive inverse property on* $\Xi_{\mathfrak{R}}$. *On the other hand, under the HMF-based granular addition, the FN* $-\mathfrak{w}_1$ *is obtained from* $\mathfrak{w}_1$ *by granular scalar multiplication, namely*

$$-\mathfrak{m}_1 := (-1) \odot^{gr} \mathfrak{w}_1. \tag{36}$$

By the definition of granular scalar multiplication in Equation (19)*, we obtain*

$$\mathscr{H}(-\mathfrak{w}_1) = \mathscr{H}\left((-1) \odot^{gr} \mathfrak{w}_1\right) = -\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa}). \tag{37}$$

The same RDM variable is used because $-\mathfrak{w}$ *is obtained algebraically from* $\mathfrak{w}$. *Therefore,*

$$\mathscr{H}\left(\mathfrak{w} \oplus^{gr} (-\mathfrak{w})\right) = \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) - \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa}) = 0. \tag{38}$$

Applying the inverse HMF transformation gives

$$\mathfrak{w} \oplus^{gr} (-\mathfrak{w}) = \hat{0}. \tag{39}$$

Therefore $-\mathfrak{w}_1 = (-a_3, -a_2, -a_1)$ *is the additive inverse of* $\mathfrak{w}_1$ *with respect to the granular addition* $\oplus^{gr}$*, but it is not generally the additive inverse of* $\mathfrak{w}_1$ *with respect to classical fuzzy addition. This shows that classical fuzzy addition and HMF-based granular addition are different operations. Thus, Proposition 2.10 establishes a vector-space structure on* $\Xi_{\mathfrak{R}}$ *with respect to the HMF-based granular operations* $\oplus^{gr}$ *and* $\odot^{gr}$.

Proposition 2.12. *Let* $\Xi_{\mathfrak{R}}$ *be equipped with the granular addition* $\oplus^{gr}$ *and granular scalar multiplication* $\odot^{gr}$ *defined in Equations* (18) *and* (19)*. Then the mapping*

$$\|\mathfrak{w}\|_1 := \rho^{gr}(\mathfrak{w}, \hat{0}), \qquad \mathfrak{w} \in \Xi_{\mathfrak{R}}, \tag{40}$$

defines a norm on $\Xi_{\mathfrak{R}}$.

Proof. Let $\mathfrak{w} \in \Xi_{\mathfrak{R}}$ be an FN having uncertainty source set $\mathcal{S}_{\mathfrak{w}}$ and HMF representation $\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}})$. The zero fuzzy number $\hat{0}$ has the HMF representation $\hat{0}^{gr}(\alpha, \alpha_{\varkappa}) = 0$. Therefore, by the definition of the granular metric,

$$\|\mathfrak{w}\|_1 = \rho^{gr}(\mathfrak{w}, \hat{0}) = \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}} \in[0,1]^{|\mathcal{S}_{\mathfrak{w}}|}} \left|\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}})\right|. \tag{41}$$

Clearly, $\|\mathfrak{w}\|_1 \geq 0$. Moreover, $\|\mathfrak{w}\|_1 = 0$ if and only if $\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}}) = 0$ for all $\alpha \in [0,1]$ and all RDM coordinates $\alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}}$. Thus, $\mathscr{H}(\mathfrak{w}) = \mathscr{H}(\hat{0})$, and by the inverse HMF transformation, we have $\mathfrak{w} = \hat{0}$. Conversely, if $\mathfrak{w} = \hat{0}$, then $\|\mathfrak{w}\|_1 = 0$. Hence,

$$\|\mathfrak{w}\|_1 = 0 \iff \mathfrak{w} = \hat{0}. \tag{42}$$

For $\lambda \in \mathfrak{R}$, using the definition of granular scalar multiplication, we have $\mathscr{H}(\lambda \odot^{gr} \mathfrak{w}) = \lambda \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}})$. Therefore,

$$\begin{aligned}\|\lambda \odot^{gr} \mathfrak{w}\|_1 &= \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}}\in[0,1]^{|\mathcal{S}_{\mathfrak{w}}|}} \left|\lambda \mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}})\right| \\ &= |\lambda| \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}}\in[0,1]^{|\mathcal{S}_{\mathfrak{w}}|}} \left|\mathfrak{w}^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}_{\mathfrak{w}}}})\right| \\ &= |\lambda|\|\mathfrak{w}\|_1. \end{aligned} \tag{43}$$

Hence, the norm is homogeneous with respect to scalar multiplication. Now let $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$ with uncertainty-source sets $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. Both HMFs are considered on the common RDM coordinate domain associated with $\mathcal{S}$. Pointwise, we have

$$|\mathfrak{w}_1^{gr} + \mathfrak{w}_2^{gr}| \leq |\mathfrak{w}_1^{gr}| + |\mathfrak{w}_2^{gr}| \,. \tag{44}$$

Taking the maximum over the RDM coordinates and the supremum over $\alpha \in [0,1]$, we obtain

$$\|\mathfrak{w}_1 \oplus^{gr} \mathfrak{w}_2\|_1 \leq \|\mathfrak{w}_1\|_1 + \|\mathfrak{w}_2\|_1. \tag{45}$$

Thus, the triangle inequality holds. Therefore, $\|\mathfrak{w}\|_1 = \rho^{gr}(\mathfrak{w}, \hat{0})$ defines a norm on $\Xi_{\mathfrak{R}}$. □

Proposition 2.13. *For all* $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$,

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \|\mathfrak{w}_1 \ominus^{gr} \mathfrak{w}_2\|_1 \,. \tag{46}$$

Proof. Let $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi_{\mathfrak{R}}$ be two FNs having uncertainty-source sets $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. By the granular difference in the RDM coordinate domain, we have

$$\mathscr{H}(\mathfrak{w}_1 \ominus^{gr} \mathfrak{w}_2) = \mathfrak{w}_1^{gr}\left(\alpha, \alpha_{\varkappa_{\mathcal{S}_1}}\right) - \mathfrak{w}_2^{gr}\left(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}}\right). \tag{47}$$

Therefore, by the definition of $\|\cdot\|_1$,

$$\|\mathfrak{w}_1 \ominus^{gr} \mathfrak{w}_2\|_1 = \sup_{\alpha\in[0,1]} \max_{\alpha_{\varkappa_{\mathcal{S}}}\in[0,1]^{|\mathcal{S}|}} \left|\mathfrak{w}_1^{gr}\left(\alpha, \alpha_{\varkappa_{\mathcal{S}_1}}\right) - \mathfrak{w}_2^{gr}\left(\alpha, \alpha_{\varkappa_{\mathcal{S}_2}}\right)\right| = \rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2). \tag{48}$$

Hence,

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \|\mathfrak{w}_1 \ominus^{gr} \mathfrak{w}_2\|_1 \,. \tag{49}$$

The identity holds for independent, common and partially overlapping uncertainty sources, because both the granular difference and granular metric are evaluated on the same common RDM coordinate domain. □

Theorem 2.14. *The metric space* $(\Xi_{\mathfrak{R}}, \rho^{gr})$ *is complete.*

Proof. Let $\mathcal{S} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$ be the finite set of uncertainty sources involved in the granular representation, and set $K_{\mathcal{S}} = [0,1] \times [0,1]^m$. Every HMF whose uncertainty-source set is contained in $\mathcal{S}$ is written on $K_{\mathcal{S}}$ by keeping it constant with respect to the RDM coordinates on which it does not depend. Under this convention, the granular metric can be written as

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \sup_{(\alpha, \alpha_{\varkappa_{\mathcal{S}}})\in K_{\mathcal{S}}} |\mathfrak{w}_1^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}}}) - \mathfrak{w}_2^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}}})| \,. \tag{50}$$

Let $\{\mathfrak{w}_n\}_{n\geq 1} \subseteq \Xi_{\mathfrak{R}}$ be a Cauchy sequence with respect to ρ^{gr}, and write $G_n(\alpha, \alpha_{\varkappa_{\mathcal{S}}}) = \mathfrak{w}_n^{gr}(\alpha, \alpha_{\varkappa_{\mathcal{S}}})$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{(\alpha, \alpha_{\varkappa_{\mathcal{S}}})\in K_{\mathcal{S}}} |G_n(\alpha, \alpha_{\varkappa_{\mathcal{S}}}) - G_k(\alpha, \alpha_{\varkappa_{\mathcal{S}}})| < \varepsilon, \qquad n, k \geq N. \tag{51}$$

Hence, $\{G_n\}$ is uniformly Cauchy on $K_{\mathcal{S}}$. Since $\mathfrak{R}$ is complete, for every fixed $(\alpha, \alpha_{\varkappa_{\mathcal{S}}}) \in K_{\mathcal{S}}$, the sequence $\{G_n(\alpha, \alpha_{\varkappa_{\mathcal{S}}})\}$ converges in $\mathfrak{R}$. Define

$$G(\alpha, \alpha_{\varkappa_{\mathcal{S}}}) = \lim_{n\to\infty} G_n(\alpha, \alpha_{\varkappa_{\mathcal{S}}}). \tag{52}$$

Taking $k \to \infty$ in the Cauchy inequality gives

$$\sup_{(\alpha,\alpha_{\varkappa_{\mathcal{S}}})\in K_{\mathcal{S}}} |G_n(\alpha,\alpha_{\varkappa_{\mathcal{S}}}) - G(\alpha,\alpha_{\varkappa_{\mathcal{S}}})| \le \varepsilon, \qquad n \ge N. \tag{53}$$

Therefore, $G_n \to G$ uniformly on $K_{\mathcal{S}}$. We now show that the limit G determines a fuzzy number. For $\alpha \in [0,1]$, define

$$\underline{G}_\alpha = \inf_{\nu\ge\alpha} \min_{\alpha_{\varkappa_{\mathcal{S}}}\in[0,1]^m} G(\nu,\alpha_{\varkappa_{\mathcal{S}}}), \qquad \overline{G}_\alpha = \sup_{\nu\ge\alpha} \max_{\alpha_{\varkappa_{\mathcal{S}}}\in[0,1]^m} G(\nu,\alpha_{\varkappa_{\mathcal{S}}}). \tag{54}$$

Then $\mathscr{H}^{-1}(G) = [\underline{G}_\alpha, \overline{G}_\alpha]$. Similarly, for each n, write

$$\mathscr{H}^{-1}(G_n) = [\underline{G}_{n,\alpha}, \overline{G}_{n,\alpha}]. \tag{55}$$

If $\|G_n - G\|_\infty \le \varepsilon$, then $G - \varepsilon \le G_n \le G + \varepsilon$ on $K_{\mathcal{S}}$. Taking the corresponding infimum and minimum for the lower endpoint, and the supremum and maximum for the upper endpoint, gives

$$|\underline{G}_{n,\alpha} - \underline{G}_\alpha| \le \varepsilon, \qquad |\overline{G}_{n,\alpha} - \overline{G}_\alpha| \le \varepsilon. \tag{56}$$

Thus, the lower and upper endpoint functions of the reconstructed level sets converge uniformly. Since each G_n corresponds to an FN, the ordering, monotonicity, boundedness, and one-sided continuity properties of the level-set endpoints, are preserved in the limit. Moreover, the support remains bounded and the level set at $\alpha = 1$ is nonempty. Hence, $\mathscr{H}^{-1}(G)$ is an FN in $\Xi_{\mathfrak{R}}$. Let $\mathfrak{w} = \mathscr{H}^{-1}(G)$, and G is the HMF representation of $\mathfrak{w}$. Then

$$\rho^{gr}(\mathfrak{w}_n, \mathfrak{w}) = \sup_{(\alpha,\alpha_{\varkappa_{\mathcal{S}}})\in K_{\mathcal{S}}} |G_n(\alpha,\alpha_{\varkappa_{\mathcal{S}}}) - G(\alpha,\alpha_{\varkappa_{\mathcal{S}}})| \longrightarrow 0, \tag{57}$$

as $n \to \infty$. Therefore, every Cauchy sequence in $(\Xi_{\mathfrak{R}}, \rho^{gr})$ converges to an element of $\Xi_{\mathfrak{R}}$. Hence, $(\Xi_{\mathfrak{R}}, \rho^{gr})$ is complete. □

Corollary 2.15. *The normed linear space* $(\Xi_{\mathfrak{R}}, \|\cdot\|_1)$ *is a Banach space. Moreover, the finite product space* $(\Xi^n_{\mathfrak{R}}, \|\cdot\|)$ *is also a Banach space, where*

$$\|\mathfrak{w}\| = \mathcal{D}^{gr}(\mathfrak{w}, \overline{0}), \qquad \overline{0} = (\hat{0}, \ldots, \hat{0}),$$

and

$$\mathcal{D}^{gr}(\mathfrak{w}, \mathfrak{y}) = \sum_{i=1}^{n} \rho^{gr}(\mathfrak{w}_i, \mathfrak{y}_i),$$

for $\mathfrak{w} = (\mathfrak{w}_1, \ldots, \mathfrak{w}_n), \mathfrak{y} = (\mathfrak{y}_1, \ldots, \mathfrak{y}_n) \in \Xi^n_{\mathfrak{R}}$.

Proof. By Proposition 2.10, $\Xi_{\mathfrak{R}}$ is a real linear space under the HMF-based granular operations $\oplus^{gr}$ and $\odot^{gr}$. By Proposition 2.12, $\|\cdot\|_1$ defines a norm on $\Xi_{\mathfrak{R}}$. Furthermore, by Proposition 2.13, the granular metric and the norm are compatible in the sense that

$$\rho^{gr}(\mathfrak{w}_1, \mathfrak{w}_2) = \|\mathfrak{w}_1 \ominus^{gr} \mathfrak{w}_2\|_1 . \tag{58}$$

Therefore, Cauchy convergence with respect to $\|\cdot\|_1$ is equivalent to Cauchy convergence with respect to ρ^{gr}. Since $(\Xi_{\mathfrak{R}}, \rho^{gr})$ is complete by Theorem 2.14, every Cauchy sequence in $(\Xi_{\mathfrak{R}}, \|\cdot\|_1)$ converges to an element of $\Xi_{\mathfrak{R}}$. Hence, $(\Xi_{\mathfrak{R}}, \|\cdot\|_1)$ is a Banach space. For the finite product space $\Xi^n_{\mathfrak{R}}$, the metric $\mathcal{D}^{gr}$ is the finite sum of the complete component metrics ρ^{gr}. Hence, every Cauchy sequence in $(\Xi^n_{\mathfrak{R}}, \mathcal{D}^{gr})$ is componentwise Cauchy in $(\Xi_{\mathfrak{R}}, \rho^{gr})$, and each component converges in $\Xi_{\mathfrak{R}}$. Therefore, the sequence converges in $\Xi^n_{\mathfrak{R}}$. Thus, $(\Xi^n_{\mathfrak{R}}, \|\cdot\|)$ is a Banach space. □

Definition 2.16. [12] *Consider* $\Xi^n_{\mathfrak{R}} = \underbrace{\Xi_{\mathfrak{R}} \times \Xi_{\mathfrak{R}} \times \cdots \times \Xi_{\mathfrak{R}}}_{n-times}$. $\mathcal{D}^{gr}$ *represents the metric on* $\Xi^n_{\mathfrak{R}}$*, which may be expressed as*

$$\mathcal{D}^{gr}(\mathfrak{w}, \mathfrak{y}) = \sum_{i=1}^{n} \rho^{gr}(\mathfrak{w}_i, \mathfrak{y}_i), \tag{59}$$

for every $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3, \cdots, \mathfrak{w}_n), \mathfrak{y} = (\mathfrak{y}_1, \mathfrak{y}_2, \mathfrak{y}_3, \cdots, \mathfrak{y}_n) \in \Xi^n_{\mathfrak{R}}$*. Moreover, it is a Banach space with the norm* $\|\mathfrak{w}\| = \mathcal{D}^{gr}(\mathfrak{w}, \overline{0})$*, where* $\mathfrak{w} \in \Xi^n_{\mathfrak{R}}$ *and* $\overline{0} = (\hat{0}, \cdots, \hat{0})$ *is the zero fuzzy vector.*

Definition 2.17. *An FVF $\varkappa : (a,b) \longrightarrow \Xi_{\mathfrak{R}}$ is gr-differentiable* [31] *at $z_0 \in (a,b)$ if the limit*

$$\varkappa'_{gr}(z_0) = \lim_{\epsilon \to 0} \frac{\varkappa(z_0+\epsilon) \ominus^{gr} \varkappa(z_0)}{\epsilon}, \tag{60}$$

exists. A function $\varkappa$ is gr-differentiable if its gr-derivative $\varkappa'_{gr}(z)$ exists for all $z \in (a,b)$.

Let $\mathcal{C}^1((a,b),\ \Xi_{\mathfrak{R}})$ represent the collection of all continuous gr-differentiable FVF on (a,b).

Definition 2.18. *A FVF $\varkappa$ is said to be gr-differentiable* [31] *if and only if its corresponding HMF is differentiable and moreover,*

$$\mathscr{H}\left(\varkappa'_{gr}(z_0)\right) = \frac{\partial \varkappa^{gr}(z_0,\alpha,\alpha_{\varkappa})}{\partial z}. \tag{61}$$

Denote $L^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$ and $\varkappa \in C^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$ as the classes all Lebesgue integrable and continuously differentiable functions on $[\mathfrak{a},\mathfrak{b}]$, respectively.

Definition 2.19. *Let $\varkappa \in L^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$, the RL integral of fractional-order $\vartheta \in \mathbb{C}$, $Re(\vartheta) > 0$ of $\varkappa$, is defined as:*

$$[\mathfrak{a},\mathfrak{b}]^{\vartheta}_{a^+}\varkappa(z) := \frac{1}{\Gamma(\vartheta)} \int_a^z \frac{\varkappa(p)}{(z-p)^{1-\vartheta}} dp, \quad z \in [\mathfrak{a},\mathfrak{b}]. \tag{62}$$

The Caputo fractional derivative (CFD) of order $\vartheta \in \mathbb{C}$, $Re(\vartheta) > 0$ of $\varkappa(z)$, denoted by ${}^{C}_{a^+}\mathcal{D}^{\vartheta}\varkappa(z)$, is defined as:

$${}^{C}_{a^+}\mathcal{D}^{\vartheta}\varkappa(z) = \frac{1}{\Gamma(\mathfrak{n}-\vartheta)} \int_a^z (z-p)^{\mathfrak{n}-\vartheta-1}\varkappa^{(\mathfrak{n})}(p)dp, \tag{63}$$

where $\mathfrak{n} \in \mathbb{N}$, $\mathfrak{n}-1 < \vartheta < \mathfrak{n}$, and $z \in [\mathfrak{a},\mathfrak{b}]$.

Definition 2.20. *Let $\varkappa : [\mathfrak{a},\mathfrak{b}] \subseteq \mathfrak{R} \longrightarrow \Xi_{\mathfrak{R}}$; the right-sided and left-sided (RS&LS)* [34] *granular fuzzy fractional order (GFFIs) of order $0 < \vartheta \leq 1$ are defined by*

$${}^{gr}\mathfrak{I}^{\vartheta}_{a^+}\varkappa(z) = \frac{1}{\Gamma(\vartheta)} \int_a^z \frac{\varkappa(p)}{(z-p)^{1-\vartheta}} dp,\ z > a, \quad \text{and} \quad {}^{gr}\mathfrak{I}^{\vartheta}_{b^-}\varkappa(z) = \frac{1}{\Gamma(\vartheta)} \int_z^b \frac{\varkappa(p)}{(p-z)^{1-\vartheta}} dp,\ z < b,$$

respectively.

Definition 2.21. *Let $\varkappa : [\mathfrak{a},\mathfrak{b}] \subseteq \mathfrak{R} \longrightarrow \Xi_{\mathfrak{R}}$, the RS&LS granular Caputo derivative* [34] *of fractional-order $0 < \vartheta \leq 1$ are defined by*

$${}^{gr}_{a^+}\mathfrak{D}^{\vartheta}\varkappa(z) = \frac{1}{\Gamma(1-\vartheta)} \int_a^z \frac{\varkappa'_{gr}(p)}{(z-p)^{\vartheta}} dp = {}^{gr}\mathfrak{I}^{1-\vartheta}_{a^+}(\varkappa'_{gr}(z)), \tag{64}$$

and

$${}^{gr}_{b^-}\mathfrak{D}^{\vartheta}\varkappa(z) = \frac{-1}{\Gamma(1-\vartheta)} \int_z^b \frac{\varkappa'_{gr}(p)}{(z-p)^{\vartheta}} dp = -{}^{gr}\mathfrak{I}^{1-\vartheta}_{b^-}(\varkappa'_{gr}(z)). \tag{65}$$

Denote by $\psi \in C^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R}_+)$ a strictly increasing function (IF) such that $\psi'(z) > 0$ for all $z \in [\mathfrak{a},\mathfrak{b}]$. For the LTWR-AF formulation, we additionally assume $\psi(0) = 0$.

Definition 2.22. *Let $\varkappa \in L^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$, the RS&LS ψ-RL fractional integral* [26, 35] *of order $\vartheta > 0$ are defined by*

$$[\mathfrak{a},\mathfrak{b}]^{\vartheta,\psi}_{a^+}\ \varkappa(z) = \frac{1}{\Gamma(\vartheta)} \int_{\mathfrak{a}}^z \psi'(p)(\psi(z)-\psi(p))^{\vartheta-1}\varkappa(p)dp,\ z > a, \tag{66}$$

and

$$[\mathfrak{a},\mathfrak{b}]^{\vartheta,\psi}_{b^-}\varkappa(z) = \frac{1}{\Gamma(\vartheta)} \int_z^b \psi'(p)(\psi(p)-\psi(z))^{\vartheta-1}\varkappa(p)\,dp, \qquad z < b. \tag{67}$$

respectively.

Definition 2.23. *Let $\varkappa \in L^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$, the right- and left-sided ψ-fractional derivatives* [26, 35] *of order $\vartheta > 0$ are defined by*

$$_{a^+}\mathcal{D}^{\vartheta;\psi}\varkappa(z) = \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \mathfrak{I}_{a^+}^{n-\vartheta,\psi}\ \varkappa(z) = \frac{1}{\Gamma(n-\vartheta)}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \int_a^z \psi^{'}(p)(\psi(z)-\psi(p))^{n-\vartheta-1}\varkappa(p)dp, \tag{68}$$

and

$$_{b^-}\mathcal{D}^{\vartheta;\psi}\varkappa(z) = \left(-\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \mathfrak{I}_{b^-}^{n-\vartheta,\psi}\ \varkappa(z) = \frac{1}{\Gamma(n-\vartheta)}\left(-\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \int_b^z \psi^{'}(p)(\psi(p)-\psi(z))^{n-\vartheta-1}\varkappa(p)dp, \tag{69}$$

respectively, where $n \in \mathbb{N}$ and $z \in [\mathfrak{a},\mathfrak{b}]$.

Definition 2.24. *Let $\varkappa \in L^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$; the right- and left-sided ψ-CFD* [4, 26, 35] *of order $\vartheta > 0$ are defined by*

$$^{C}_{a^+}\mathcal{D}^{\vartheta;\psi}\varkappa(z) = [\mathfrak{a},\mathfrak{b}]_{a^+}^{n-\vartheta,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \varkappa(z) = \frac{1}{\Gamma(\mathfrak{n}-\vartheta)}\int_{\mathfrak{a}}^z \psi^{'}(p)(\psi(z)-\psi(p))^{\mathfrak{n}-\vartheta-1}\varkappa_\psi^{[\mathfrak{n}]}(p)dp, \tag{70}$$

and

$$^{C}_{b^-}\mathcal{D}^{\vartheta;\psi}\varkappa(z) = [\mathfrak{a},\mathfrak{b}]_{b^-}^{n-\vartheta,\psi}\left(-\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \varkappa(z) = \frac{1}{\Gamma(n-\vartheta)}\int_z^b \psi^{'}(p)(\psi(p)-\psi(z))^{n-\vartheta-1}(-1)^n\varkappa_\psi^{[n]}(p)dp, \tag{71}$$

respectively, where, $n-1 < Re(\vartheta) < n$, $n \in \mathbb{N}$. We use $\varkappa_\psi^{[\mathfrak{n}]}(z) := \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^{\mathfrak{n}}\varkappa(z)$ to simplify the notation.

2.1 Generalized Laplace transform for FVF

Now we discuss the generalized integral transform that was earlier introduced by Kyrgyz mathematicians [9]. This transformation is the generalization of Laplace transform [16, 23]. First, we review some fundamental concepts of the LTWR-AF; then, we present the idea of the granular fuzzy LTWR-AF and its HMF.

Definition 2.25. [9, 16, 23] *Suppose $\varkappa : [0,\infty) \longrightarrow \mathfrak{R}$ and ψ is nonnegative, monotonically IF satisfying $\psi(0) = 0$. The LTWR-AF ψ, is defined by*

$$\mathcal{L}_\psi\big[\varkappa(z)\big]{=}\int_0^\infty e^{-p(\psi(z))}\psi^{'}(z)\varkappa(z)dz, \quad p \in \mathbb{C}, \tag{72}$$

exists and converges.

The relation between the LTWR-AF and classical Laplace transform is defined by:

Definition 2.26. [16, 23] *Suppose Definition 2.25 exists; then*

$$\mathcal{L}_\psi\big[\varkappa(z)\big] = \mathcal{L}\big[\varkappa(\psi^{-1}(z))\big], \tag{73}$$

where $\mathcal{L}_\psi$ is the LTWR-AF, and $\mathcal{L}$ is the classical Laplace transform.

Theorem 2.27. *Let $\varkappa$ and ψ be piecewise continuous functions on the interval $\mathcal{T} = [0, T]$ with ψ-exponential order. The ψ-convolution* [16, 23] *of $\varkappa$ and ψ is as follows:*

$$(\varkappa \circ_\psi \psi)(z){=}\int_0^z \varkappa(q)\psi\big(\psi^{-1}(\psi(z)-\psi(q))\big)\psi^{'}(q)dq. \tag{74}$$

The LTWR-AF of the convolution is defined by

$$\mathcal{L}_\psi\big[(\varkappa \circ_\psi \psi)(z)\big] = \mathcal{L}_\psi\big[\varkappa(z)\big]\mathcal{L}_\psi\big[\psi(z)\big]. \tag{75}$$

Remark 2.28. [16, 23] *For all $\mathfrak{n}-1 < \vartheta \leq \mathfrak{n}$, $\mathfrak{n} \in \mathbb{N}$, we have*

$$\mathcal{L}_\psi\big[^{C}_{a^+}\mathcal{D}^{\vartheta;\psi}\varkappa(z)\big] = p^\vartheta\mathcal{L}_\psi\big[\varkappa(z)\big] - \sum_{k=0}^{\mathfrak{n}-1} p^{\vartheta-k-1}\varkappa_\psi^{[k]}(0), \tag{76}$$

where $\varkappa_\psi^{[k]}(0)$ represents the k-th derivative of $\varkappa$ evaluated at 0, as defined in Definition 2.24.

Dong et al. [12] introduced a granular fuzzy Laplace transform for standard Caputo fractional derivatives (with $\psi(z) = z$) using HMF. However, their approach: (i) is restricted to power-law kernels, (ii) does not establish the fundamental relationship $\mathscr{H}(\mathfrak{L}[\varkappa]) = \mathcal{L}[\mathscr{H}(\varkappa)]$ for generalized kernels, and (iii) lacks a systematic framework for arbitrary function ψ. The granular LTWR-AF provides a powerful analytical framework for solving GCFFDEs involving granular ψ-CFD under uncertainty. By combining the flexible kernel $\psi(z)$ with HMF-based granular representation, this transform preserves uncertainty throughout the solution process while enabling closed-form analytical solutions via M-LF functions.

Definition 2.29. *A fuzzy-valued function $\varkappa : [0,\infty) \to \Xi_{\mathfrak{R}}$ is said to be of granular ψ-exponential order $c > 0$ if there exist constants $M > 0$ and $z_0 \geq 0$ such that*

$$\mathcal{D}^{gr}(\varkappa(z), \hat{0}) \leq M e^{c\psi(z)}, \qquad z \geq z_0,$$

where ψ is the IF.

Definition 2.30. *Let $\varkappa : [0,\infty) \to \Xi_{\mathfrak{R}}$ be a granular measurable fuzzy-valued function. The granular fuzzy LTWR-AF of $\varkappa$ with respect to the IF ψ is defined by*

$$\mathfrak{L}_{\psi}\big[\varkappa(z)\big] = \int_0^{\infty} e^{-p\psi(z)} \varkappa(z) \psi^{'}(z)\, dz, \tag{77}$$

provided that the above integral exists in $\Xi_{\mathfrak{R}}$.

Lemma 2.31. *Let $\varkappa : [0,\infty) \to \Xi_{\mathfrak{R}}$ be granular measurable and piecewise continuous in the HMF sense. If $\varkappa$ is of granular ψ-exponential order $c > 0$, then its granular fuzzy LTWR-AF exists for all $p > c$.*

Proof. By the HMF representation, it is sufficient to verify the convergence of the corresponding real-valued granular components. Since $\varkappa$ is of granular ψ-exponential order c, there exist $M > 0$ and $z_0 \geq 0$ such that

$$|\varkappa^{gr}(z, \alpha, \alpha_{\varkappa})| \leq M e^{c\psi(z)},$$

for all $z \geq z_0$, $\alpha \in [0,1]$ and admissible $\alpha_{\varkappa} \in [0,1]$. Therefore, for $p > c$,

$$\int_{z_0}^{\infty} e^{-p\psi(z)} \left|\varkappa^{gr}(z, \alpha, \alpha_{\varkappa})\right| \psi'(z)\, dz \leq M \int_{z_0}^{\infty} e^{-(p-c)\psi(z)} \psi'(z)\, dz.$$

Using the change of variable $y = \psi(z)$, we obtain

$$M \int_{\psi(z_0)}^{\infty} e^{-(p-c)y}\, dy = \frac{M}{p-c} e^{-(p-c)\psi(z_0)} < \infty.$$

On the finite interval $[0, z_0]$, piecewise continuity implies integrability. Hence the granular components of $\mathfrak{L}_{\psi}[\varkappa(z)]$ exist for all $\alpha \in [0,1]$ and admissible $\alpha_{\varkappa}$. Consequently, the granular fuzzy LTWR-AF exists, for all $p > c$. □

Remark 2.32. *The granular representation of fuzzy LTWR-AF given in Definition 2.30 is represented as follows:*

$$\mathscr{H}\bigg(\mathfrak{L}_{\psi}\big[\varkappa(z)\big]\bigg) = \int_0^{\infty} \mathscr{H}\big(e^{-p(\psi(z))}\psi^{'}(z)\varkappa(z)\big)dz = \int_0^{\infty} e^{-p(\psi(z))}\psi^{'}(z)\mathscr{H}\big(\varkappa(z)\big)dz = \mathcal{L}_{\psi}\bigg[\mathscr{H}\bigg(\varkappa(z)\bigg)\bigg]. \tag{78}$$

The Mittag-Leffler function assumes a critical role in the study of fractional calculus [18]; its representations with one parameter and variable is defined as:

$$\mathbb{E}_{\vartheta_1}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\vartheta_1 + 1)}, \quad \mathbb{E}_{\vartheta_1,\vartheta_2}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\vartheta_1 + \vartheta_2)}, \quad \mathbb{E}^{\delta}_{\vartheta_1,\vartheta_2}(z) = \sum_{i=0}^{\infty} \frac{(\delta)_i z^i}{\Gamma(i\vartheta_1 + \vartheta_2) i!}, \tag{79}$$

where $(\delta)_i := \dfrac{\Gamma(\delta + i)}{\Gamma(\delta)}$, $Re(\vartheta_1) > 0$, $\delta, \vartheta_2 \in \mathbb{C}$.

Theorem 2.33. [23, 37] *Let $\vartheta_1, \vartheta_2 > 0$ and let A be a square fuzzy matrix. Then*

$$\mathfrak{L}_{\psi}\left[(\psi(z))^{\vartheta_2 - 1}\mathbb{E}_{\vartheta_1,\vartheta_2}\left(A(\psi(z))^{\vartheta_1}\right)\right] = \frac{p^{\vartheta_1 - \vartheta_2}}{p^{\vartheta_1} I - A}, \tag{80}$$

whenever the transform exists and $p > \|A\|^{1/\vartheta_1}$.

Proof. It is sufficient to prove the identity at the HMF level. For fixed α and α_A, let $A_{gr} = A_{gr}(\alpha, \alpha_A)$. Using the series representation of the two-parameter Mittag–Leffler function, we have

$$(\psi(z))^{\vartheta_2-1}\mathbb{E}_{\vartheta_1,\vartheta_2}\left(A_{gr}(\psi(z))^{\vartheta_1}\right) = \sum_{k=0}^{\infty} \frac{A_{gr}^k(\psi(z))^{\vartheta_1 k+\vartheta_2-1}}{\Gamma(\vartheta_1 k+\vartheta_2)}. \tag{81}$$

Applying the LTWR-AF term by term gives

$$\begin{aligned}\mathcal{L}_\psi\left[(\psi(z))^{\vartheta_2-1}\mathbb{E}_{\vartheta_1,\vartheta_2}\left(A_{gr}(\psi(z))^{\vartheta_1}\right)\right] &= \sum_{k=0}^{\infty} \frac{A_{gr}^k}{\Gamma(\vartheta_1 k+\vartheta_2)}\mathcal{L}_\psi\left[(\psi(z))^{\vartheta_1 k+\vartheta_2-1}\right] \\ &= \sum_{k=0}^{\infty} \frac{A_{gr}^k}{\Gamma(\vartheta_1 k+\vartheta_2)}\frac{\Gamma(\vartheta_1 k+\vartheta_2)}{p^{\vartheta_1 k+\vartheta_2}} = \frac{1}{p^{\vartheta_2}}\sum_{k=0}^{\infty}\left(\frac{A_{gr}}{p^{\vartheta_1}}\right)^k. \end{aligned} \tag{82}$$

Since $p > \|A_{gr}\|^{1/\vartheta_1}$, we have $\|A_{gr}/p^{\vartheta_1}\| < 1$, and hence the Neumann series converges:

$$\sum_{k=0}^{\infty}\left(\frac{A_{gr}}{p^{\vartheta_1}}\right)^k = \left(I - \frac{A_{gr}}{p^{\vartheta_1}}\right)^{-1}.$$

Therefore,

$$\mathcal{L}_\psi\left[(\psi(z))^{\vartheta_2-1}\mathbb{E}_{\vartheta_1,\vartheta_2}\left(A_{gr}(\psi(z))^{\vartheta_1}\right)\right] = \frac{1}{p^{\vartheta_2}}\left(I - \frac{A_{gr}}{p^{\vartheta_1}}\right)^{-1} = \frac{p^{\vartheta_1-\vartheta_2}}{p^{\vartheta_1}I - A_{gr}}. \tag{83}$$

Finally, using Proposition 2.4, we obtain the desired result. This completes the proof. □

3 Analytical fuzzy solution of GCFFDEs involving granular ψ-CFD

This section outlines the methodology for AFS of the following GCFFDEs involving granular ψ-CFD

$$\begin{cases} \left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(z) := \lambda\varkappa(z) + f(z,\varkappa(z)), \\ \quad \varkappa(0) = \varkappa_0, \quad \left(\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(0) = \varkappa_\psi^{[k]}(0), \end{cases} \tag{84}$$

where $n-1 < Re(\vartheta) < n, n \in \mathbb{N}$, $\lambda \in \mathfrak{R}$, $k = 1,2,3,\cdots,\mathfrak{n}-1$, $f : [0,T]\times\Xi^n_{\mathfrak{R}} \longrightarrow \Xi^n_{\mathfrak{R}}$, and $\left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(z)$ is the granular ψ-CFD, where $\psi(z)$ is a monotonically increasing function such that $\psi^{'}(z) \neq 0$ over the domain. Moreover, $\varkappa(0)$, $\varkappa_\psi^{[k]}(0) \in C([0,T],\Xi^n_{\mathfrak{R}})$.

Based on Definitions 2.22 and 2.24, we introduce the concept of granular ψ-FFI and granular ψ-CFD of order $\vartheta \in (0,1]$ in the following:

Definition 3.1. [4, 26] *Suppose $\psi(z)$ is an increasing function and $\psi^{'}(z) \neq 0$ for all $z \in [\mathfrak{a},\mathfrak{b}]$, then the RS&LS granular ψ-FFI of order $\vartheta > 0$ for $\varkappa \in \Xi^n_{\mathfrak{R}}$ are defined as:*

$${}^{gr}\mathfrak{I}^{\vartheta,\psi}_{a^+}\varkappa(z) := \frac{1}{\Gamma(\vartheta)}\int_a^z \psi^{'}(p)(\psi(z)-\psi(p))^{\vartheta-1}\varkappa(p)dp, \quad z > a, \tag{85}$$

and

$${}^{gr}\mathfrak{I}^{\vartheta,\psi}_{b^-}\varkappa(z) = \frac{1}{\Gamma(\vartheta)}\int_z^b \psi'(p)(\psi(p)-\psi(z))^{\vartheta-1}\varkappa(p)\,dp, \qquad z < b, \tag{86}$$

respectively. Furthermore, in the HMF form, Equations (85) and (86) may be expressed as

$$\mathscr{H}\left({}^{gr}\mathfrak{I}^{\vartheta,\psi}_{a^+}\varkappa(z)\right) = [\mathfrak{a},\mathfrak{b}]^{\vartheta,\psi}_{a^+}\mathscr{H}\big(\varkappa(z)\big) \text{ and } \mathscr{H}\left({}^{gr}\mathfrak{I}^{\vartheta,\psi}_{b^-}\varkappa(z)\right) = [\mathfrak{a},\mathfrak{b}]^{\vartheta,\psi}_{b^-}\mathscr{H}\big(\varkappa(z)\big), \tag{87}$$

respectively.

Definition 3.2. [4, 26] *Let $\psi(z)$ is IF and $\psi^{'}(z) \neq 0$ for all $z \in [\mathfrak{a}, \mathfrak{b}]$, then the RS&LS granular ψ-CFD of order $\vartheta > 0$ for $\varkappa \in \Xi^{n}_{\mathfrak{R}}$ are defined by*

$$ {}^{gr}_{a^+}\mathfrak{D}^{\vartheta;\psi}\varkappa(z) = {}^{gr}\mathfrak{I}^{\mathfrak{n}-\vartheta,\psi}_{a^+}\left(\frac{1}{\psi^{'}(z)}\frac{d}{dz}\right)^{n}\varkappa(z) = \frac{1}{\Gamma(\mathfrak{n}-\vartheta)}\int_{\mathfrak{a}}^{z}\psi^{'}(p)(\psi(z)-\psi(p))^{\mathfrak{n}-\vartheta-1}\varkappa^{[\mathfrak{n}]}_{\psi}(p)dp, \tag{88}$$

and

$$ {}^{gr}_{b^-}\mathfrak{D}^{\vartheta;\psi}\varkappa(z) = {}^{gr}\mathfrak{I}^{n-\vartheta,\psi}_{b^-}\left(-\frac{1}{\psi^{'}(z)}\frac{d}{dz}\right)^{\mathfrak{n}}\varkappa(z) = \frac{1}{\Gamma(\mathfrak{n}-\vartheta)}\int_{z}^{b}\psi^{'}(p)(\psi(p)-\psi(z))^{\mathfrak{n}-\vartheta-1}(-1)^{\mathfrak{n}}\varkappa^{[\mathfrak{n}]}_{\psi}(p)dp, \tag{89}$$

where, $\varkappa^{[\mathfrak{n}]}_{\psi}(z) = \left(\frac{1}{\psi^{'}(z)}\frac{d}{dz}\right)^{\mathfrak{n}}\varkappa(z)$, respectively. The HMFs are defined by

$$\mathscr{H}\left({}^{gr}_{a^+}\mathfrak{D}^{\vartheta;\psi}\varkappa(z)\right) = {}^{C}_{a^+}\mathcal{D}^{\vartheta;\psi}\mathscr{H}\left(\varkappa(z)\right) \quad \text{and} \quad \mathscr{H}\left({}^{gr}_{b^-}\mathfrak{D}^{\vartheta;\psi}\varkappa(z)\right) = -{}^{C}_{b^-}\mathcal{D}^{\vartheta;\psi}\mathscr{H}\left(\varkappa(z)\right). \tag{90}$$

Remark 3.3. *The analytical derivations in Theorems 3.4–4.4 are carried out under the following functional-analytic setting. We assume that $\psi \in C^1([0,T],\mathfrak{R})$ is strictly increasing and satisfies $\psi'(z) > 0$ for all $z \in [0,T]$. The FVFs involved in the analysis are assumed to be granular measurable, piecewise continuous in the HMF sense, and of granular ψ-exponential order whenever the granular fuzzy LTWR-AF is applied. Hence, by Lemma 2.31, the corresponding LTWR-AFs are well defined. The granular transform and convolution operations are understood componentwise through the HMF representation for every membership level $\alpha \in [0,1]$ and every RDM coordinate vector $\alpha_{\varkappa_{\mathcal{S}}} \in [0,1]^{|\mathcal{S}|}$. The boundedness of the M–LF is ensured by assumption $(\mathbb{HA})$, while the measurability, growth, and Lipschitz conditions on the nonlinear term are governed by $(\mathbb{HFI})$–$(\mathbb{HFIII})$. The fixed-point argument is then performed on the complete metric space $\mathcal{C}([0,z],\Xi^{n}_{\mathfrak{R}})$ equipped with the Bielecki-type granular metric $\mathcal{D}^{gr}_{\chi,z}$, as established in Proposition 4.7.*

Theorem 3.4. *Let $\mathfrak{n}-1 < \vartheta < \mathfrak{n}$ and $z \in \mathcal{T}$. Assume that $\varkappa$ and $f(\cdot,\varkappa(\cdot))$ are granular measurable and of granular ψ-exponential order, so that their corresponding granular fuzzy LTWR-AFs exist in the sense of Lemma 2.31. Then the following fuzzy Volterra integral equation*

$$\varkappa(z) = \sum_{k=1}^{\mathfrak{n}}(\psi(z))^{k-1}\mathbb{E}_{\vartheta,k}\left(\lambda\big(\psi(z)\big)^{\vartheta}\right)\varkappa^{[k-1]}_{\psi}(0) + \int_0^z(\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}\left(\lambda\big(\psi(z)-\psi(q)\big)^{\vartheta}\right)f(q,\varkappa(q))\psi^{'}(q)dq, \tag{91}$$

is the mild solution to the Cauchy problem (84).

Proof. By the granular measurability and granular ψ-exponential order assumptions, Lemma 2.31 guarantees that the granular fuzzy LTWR-AFs of $\varkappa$ and $f(\cdot,\varkappa(\cdot))$ exist. Therefore, applying the granular fuzzy LTWR-AF to both sides of the GCFFDEs involving the granular ψ-CFD in (84), together with the linearity property, is well defined and yields

$$\mathfrak{L}_{\psi}\left[\left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(z)\right] - \lambda\mathfrak{L}_{\psi}\left[\varkappa(z)\right] = \mathfrak{L}_{\psi}\left[f(z,\ \varkappa(z))\right]. \tag{92}$$

Using the HMF on Equation (92) and applying Remark 2.32, we get

$$\mathcal{L}_{\psi}\left[\mathscr{H}\left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(z)\right] - \lambda\mathcal{L}_{\psi}\left[\mathscr{H}\left(\varkappa(z)\right)\right] = \mathcal{L}_{\psi}\left[\mathscr{H}\left(f(z,\varkappa(z))\right)\right]. \tag{93}$$

Using Definition 3.2, Equation (93) can be written as

$$\mathcal{L}_{\psi}\left[{}^{C}_{0^+}\mathcal{D}^{\vartheta;\psi}\mathscr{H}\left(\varkappa(z)\right)\right] - \lambda\mathcal{L}_{\psi}\left[\mathscr{H}\left(\varkappa(z)\right)\right] = \mathcal{L}_{\psi}\left[\mathscr{H}\left(f(z,\varkappa(z))\right)\right], \tag{94}$$

or Equation (94) can be written more precisely as

$$\mathcal{L}_{\psi}\left[{}^{C}_{0^+}\mathcal{D}^{\vartheta;\psi}\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right] - \lambda\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right] = \mathcal{L}_{\psi}\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\right]. \tag{95}$$

For all $z \in (0,T]$, by employing Remark 2.28 on Equation (95), we obtain

$$p^{\vartheta}\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right] - \sum_{k=0}^{\mathfrak{n}-1}p^{\vartheta-k-1}\varkappa^{gr[k]}_{\psi}(0,\alpha,\alpha_{\varkappa_{\mathcal{S}}}) - \lambda\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right] = \mathcal{L}_{\psi}\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\right]. \tag{96}$$

Equation (96) can be explicitly written as

$$[p^{\vartheta}-\lambda]\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right]=\sum_{k=0}^{\mathfrak{n}-1}p^{\vartheta-k-1}\varkappa_{\psi}^{gr[k]}(0,\alpha,\alpha_{\varkappa_{\mathcal{S}}})+\mathcal{L}_{\psi}\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\right]. \tag{97}$$

Solving Equation (97) for $\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right]$, we obtain:

$$\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right]=\sum_{k=0}^{\mathfrak{n}-1}\frac{p^{\vartheta-k-1}}{p^{\vartheta}-\lambda}\varkappa_{\psi}^{gr[k]}(0,\alpha,\alpha_{\varkappa_{\mathcal{S}}})+\frac{1}{p^{\vartheta}-\lambda}\mathcal{L}_{\psi}\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\right]. \tag{98}$$

Using [23, Lemma 4.2], the following equation is obtained

$$\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right]=\mathcal{L}_{\psi}\left[\sum_{k=1}^{\mathfrak{n}}(\psi(z))^{k-1}\mathbb{E}_{\vartheta,k}(\lambda\big(\psi(z)\big)^{\vartheta})\right]\varkappa_{\psi}^{gr[k-1]}(0,\alpha,\alpha_{\varkappa_{\mathcal{S}}})+\mathcal{L}_{\psi}\left[(\psi(z))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(\lambda\big(\psi(z)\big)^{\vartheta})\right]\times\mathcal{L}_{\psi}\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\right]. \tag{99}$$

Applying the convolution Theorem 2.27 to Equation (99), we obtain the following:

$$\mathcal{L}_{\psi}\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})\right]=\mathcal{L}_{\psi}\left[\sum_{k=1}^{\mathfrak{n}}(\psi(z))^{k-1}\mathbb{E}_{\vartheta,k}(\lambda\big(\psi(z)\big)^{\vartheta})\right]\varkappa_{\psi}^{gr[k-1]}(0,\alpha,\alpha_{\varkappa_{\mathcal{S}}})+\mathcal{L}_{\psi}\left[(\psi(z))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(\lambda\big(\psi(z)\big)^{\vartheta})\circ_{\psi}f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\right]. \tag{100}$$

Applying the inverse Laplace transform yields the following solution

$$\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})=\sum_{k=1}^{\mathfrak{n}}(\psi(z))^{k-1}\mathbb{E}_{\vartheta,k}(\lambda\big(\psi(z)\big)^{\vartheta})\varkappa_{\psi}^{gr[k-1]}(0,\alpha,\alpha_{\varkappa_{\mathcal{S}}})+\int_{0}^{z}(\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(\lambda\big(\psi(z)-\psi(q)\big)^{\vartheta})f^{gr}(q,\varkappa^{gr}(q,\alpha,\alpha_{\varkappa_{\mathcal{S}}}))\psi^{'}(q)dq. \tag{101}$$

Finally, by applying Proposition 2.4, we obtain:

$$\varkappa(z)=\sum_{k=1}^{\mathfrak{n}}(\psi(z))^{k-1}\mathbb{E}_{\vartheta,k}(\lambda\big(\psi(z)\big)^{\vartheta})\varkappa_{\psi}^{[k-1]}(0)+\int_{0}^{z}(\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(\lambda\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi^{'}(q)dq. \tag{102}$$

Thus, the proof is complete. □

Example 3.5. *Consider the following granular Caputo fuzzy fractional cerebrospinal fluid (CSF) model*

$$\big({}^{gr}_{0^+}\mathfrak{D}^{\vartheta}\varkappa\big)(z)-K(I_f(z)+\frac{P_d}{R})\varkappa(z)=-\frac{K}{R}\varkappa^2(z). \tag{103}$$

Najariyan and Zhao [34] *obtained the approximate solution of the system (103). Suppose* $0<\vartheta<1$, $\lambda=K(I_f(z)+\frac{P_d}{R})$, $\psi(z)=z$, $f(z,\varkappa(z))=-\frac{K}{R}\varkappa^2(z)$, $\varkappa(0)=(7,11,15)$, *and* $z\in[0,2]$. *By Theorem 3.4, the AFS of the system (103) is*

$$\varkappa(z)=(7,11,15)\mathbb{E}_{\vartheta,1}(\lambda z^{\vartheta})+\int_{0}^{z}(z-q)^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(\lambda\big(z-q\big)^{\vartheta})f(q,\varkappa(q))dq. \tag{104}$$

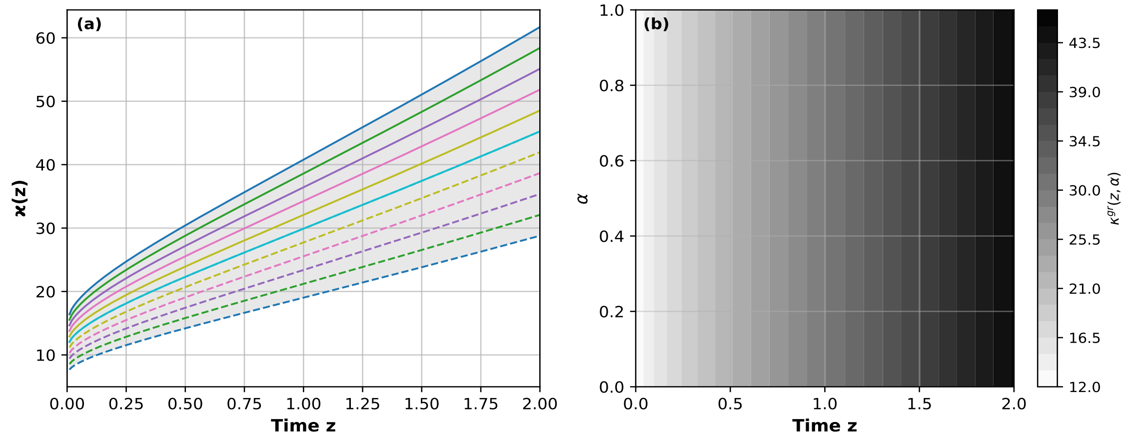

Figure 1: Graphical representation of the fuzzy solution described in Equation (104) and its contour analysis. (a) Time evolution of the fuzzy fractional solution $\varkappa(z)$ showing upper and lower bounds at different α-cut levels ($\alpha = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$). The solid curves represent upper solutions while dashed curves represent lower solutions, with the shaded gray region indicating the fuzzy uncertainty band at $\alpha = 0$. (b) Contour map of the HMF $\varkappa^{gr}(z, \alpha)$ illustrating how the fuzzy uncertainty evolves, where darker colors indicate higher solution values. The contour structure demonstrates the characteristic narrowing of the uncertainty band as α increases toward 1.

Physiological interpretation and model superiority:

1. *The CSF model describes the pressure dynamics in the brain's ventricular system, which is critical for diagnosing hydrocephalus and other neurological disorders. The variable $\varkappa(z)$ represents the CSF pressure (mmH_2O), K is the absorption coefficient* ($K = 2.30$ ($\mathrm{mL}^{-1}\ \mathrm{min}^{-1}$)), R is the resistance to CSF outflow ($R = 609$ ($\mathrm{mmH_2O}\ \mathrm{mL}^{-1}\,\mathrm{min}$)), P_d is the diastolic blood pressure ($P_d = 100$ ($\mathrm{mmH_2O}$)), and $I_f(z)$ is the CSF formation rate ($I_f(z) = 0.02$ (mLmin^{-1})). The initial condition $\varkappa(0) = (7, 11, 15)$ represents the uncertain initial CSF pressure in mmHg.

2. The term $-\frac{K}{R}\varkappa^2(z)$ in the system represents the nonlinearity that has a strong influence on the uncertainty propagation. The proposed fuzzy solution shows the non-symmetric uncertainty bands because the upper bound diverges faster than the lower bound converges. This non-symmetric uncertainty evolution is a direct consequence of the nonlinear nature of the fuzzy arithmetic and cannot be represented with linearized or deterministic models.

3. The fractional-order ϑ shows the memory effects in CSF absorption and circulation dynamics. At lower fractional orders ($\vartheta = 0.5$)–the uncertainty bandwidth expand more gradually indicating that stronger memory effects suppress uncertainty amplification. This observation has a medical implication. Patients with compromised CSF circulation (smaller ϑ) may have stable pressure trajectories, but exhibit the deficient adaptation to perturbations.

4. Najariyan and Zhao [34] solved this model using the approximate iterative scheme. Our technique has the following key advantages: (i) the AFS is expressed via M-LF, which accurately adopts the exact long-time asymptotic behavior, which is important for prediction of chronic disorders in CSF; (ii) granular representation using HMF avoids the diameter explosion phenomenon, which can be observed using Hukuhara-type fuzzy solutions and thus provides physically realistic uncertainty bands; and (iii) our approach is straightforward and it can be extended to the multi-compartment CSF models without requiring any structural change.

5. Classical deterministic CSF models (by setting $\alpha = 1$) yield the single pressure trajectory. However, the clinical recording of CSF pressure is highly variable in terms of inter-individual differences, measurement noise as well as due to physiological changes. Our proposed solution of the fuzzy CSF model offer a confidence interval, which can be compared with clinical data and used for validation of model predictions and estimation of patient-specific risk. This quantification of uncertainty is necessary for applications in personalized medicine.

6. If $\vartheta = 1$, then the proposed model will be transformed into the classical fuzzy ordinary differential equation. Comparison between the integer-order solution and fractional-order solution ($\vartheta < 1$ vs. $\vartheta = 1$) shows that the tails are heavier due to the fractional orders and consistent with the memory-dependent nature of CSF flow.

Experimental results on relaxation dynamics of CSF have demonstrated power-law form of its decay, which can be better modeled by a fractional-order rather than integer-order kernels, thus justifying the practical advantage of the proposed approach.

4 The E&U of mild solution for fuzzy initial value problem (FIVP) under granular ψ-CFD

We now investigate the E&U of a mild solution of the following FIVP to GCFFDS under granular ψ-CFD

$$\begin{cases} \left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(z) = \mathscr{F}(z,\varkappa(z)), \\ \varkappa(0) = \varkappa_0. \end{cases} \tag{105}$$

where $\mathscr{F} : [0,T] \times \Xi^n_{\mathfrak{R}} \longrightarrow \Xi^n_{\mathfrak{R}}$, ${}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}$ is the granular ψ-CFD. Moreover, $\varkappa(0) \in C([0,T], \Xi^n_{\mathfrak{R}})$ and $\psi(z)$ is the IF such that $\psi'(z) \neq 0$ over the domain. The function $\mathscr{F}(z,\varkappa(z))$ can be expressed as the sum of a linear and non-linear term $A\varkappa(z)$, and $f(z,\varkappa(z))$, respectively. Therefore, we can reformulate system (105) as follows

$$\begin{cases} \left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\varkappa\right)(z) & = A\varkappa(z) + f(z,\varkappa(z)), \\ \varkappa(0) & = \varkappa_0, \end{cases} \tag{106}$$

where A is an $\mathfrak{n} \times \mathfrak{n}$ fuzzy matrix and $f : \mathcal{T} \times \Xi^n_{\mathfrak{R}} \longrightarrow \Xi^n_{\mathfrak{R}}$ which will be specified later.

Definition 4.1. *Let $\mathcal{S} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$ be the finite set of uncertainty sources involved in the system, and let $\alpha_{\varkappa_{\mathcal{S}}} = (\alpha_{\varkappa_s})_{\mathfrak{s}\in\mathcal{S}} \in [0,1]^m$ be the corresponding RDM coordinate vector. A fuzzy-valued function $\varkappa : [0,T] \to \Xi^n_{\mathfrak{R}}$ is said to be granular measurable if, for every $\alpha \in [0,1]$ and every RDM coordinate vector $\alpha_{\varkappa_{\mathcal{S}}} \in [0,1]^m$, its HMF representation $z \longmapsto \varkappa^{gr}(z,\alpha,\alpha_{\varkappa_{\mathcal{S}}})$ is Lebesgue measurable on $[0,T]$. Similarly, for a mapping $f : \mathcal{T} \times \Xi^n_{\mathfrak{R}} \to \Xi^n_{\mathfrak{R}}$, we say that $f(\cdot,\mathfrak{w})$ is granular measurable if, for every fixed $\mathfrak{w} \in \Xi^n_{\mathfrak{R}}$, the mapping $z \longmapsto f^{gr}(z,\mathfrak{w};\alpha,\alpha_{\varkappa_{\mathcal{S}}})$ is Lebesgue measurable on $[0,T]$ for every $\alpha \in [0,1]$ and every $\alpha_{\varkappa_{\mathcal{S}}} \in [0,1]^m$.*

The following assumptions are used to establish the system's solvability (106):

($\mathbb{HA}$). There exists $M > 0$ such that

$$\left\|\mathbb{E}_{\vartheta,1}\left(A(\psi(z))^{\vartheta}\right)\right\| \leq M, \quad \text{and} \quad \left\|\mathbb{E}_{\vartheta,\vartheta}\left(A(\psi(z)-\psi(q))^{\vartheta}\right)\right\| \leq M, \tag{107}$$

for all $0 \leq q \leq z \leq T$ and $\vartheta \in (0,1]$.

($\mathbb{HF}$). The mapping $f : \mathcal{T} \times \Xi^n_{\mathfrak{R}} \longrightarrow \Xi^n_{\mathfrak{R}}$ satisfies:

($\mathbb{HFI}$). For every fixed $\mathfrak{w} \in \Xi^n_{\mathfrak{R}}$, the FVF $f(\cdot,\mathfrak{w}) : [0,T] \longrightarrow \Xi^n_{\mathfrak{R}}$ is granular measurable in the sense of Definition 4.1. Moreover, for every fixed $z \in [0,T]$, the mapping $f(z,\cdot) : \Xi^n_{\mathfrak{R}} \longrightarrow \Xi^n_{\mathfrak{R}}$ is continuous with respect to the metric $\mathcal{D}^{gr}$.

($\mathbb{HFII}$). There exists a constant C_0 such that

$\mathcal{D}^{gr}(f(z,\overline{0}),\overline{0}) \leq C_0, \forall z \in \mathcal{T}$.

($\mathbb{HFIII}$). There exists $C_1 > 0$ such that

$$\mathcal{D}^{gr}(f(z,\mathfrak{w}_1), f(z,\mathfrak{w}_2)) \leq C_1\mathcal{D}^{gr}(\mathfrak{w}_1,\mathfrak{w}_2),$$

for all $z \in \mathcal{T}$ and $\mathfrak{w}_1, \mathfrak{w}_2 \in \Xi^n_{\mathfrak{R}}$.

The supremum metric

$$\mathcal{D}^{gr}_{0,z}(\varkappa,\mathfrak{w}) = \sup_{p\in[0,\ z]} \mathcal{D}^{gr}(\varkappa(p),\mathfrak{w}(p)), \tag{108}$$

where, $\varkappa, \mathfrak{w} \in \mathcal{C}([0,z], \Xi^n_{\mathfrak{R}})$.

Remark 4.2. *Assumptions* $(\mathbb{HA})$–$(\mathbb{HFIII})$ *are sufficient conditions for ensuring the well-posedness of the proposed granular fuzzy fractional system. Assumption* $(\mathbb{HA})$ *provides a uniform bound for the M-LF generated by the linear part of the system. Assumption* $(\mathbb{HFI})$ *ensures the granular measurability of* $f(\cdot, \mathfrak{w})$ *and the continuity of* $f(z,\cdot)$ *with respect to the granular metric. Assumption* $(\mathbb{HFII})$ *provides growth control at the zero fuzzy element, whereas* $(\mathbb{HFIII})$ *imposes a Lipschitz condition that makes the solution operator contractive under the Bielecki-type granular metric. These assumptions are not claimed to be necessary or minimal; rather, they constitute a standard contraction framework adapted to the granular fuzzy metric setting.*

Lemma 4.3. *Suppose* $\psi \in C^1([\mathfrak{a},\mathfrak{b}],\mathfrak{R})$ *is an IF such that* $\psi'(z) \neq 0$ *on* $[\mathfrak{a},\mathfrak{b}]$*, then the following integral*

$$I(z) = \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi'(q)dq, \tag{109}$$

is bounded for all $z \in [\mathfrak{a},\mathfrak{b}]$*, i.e.,* $\exists\ C^{**} > 0$ *such that*

$$I(z) \le C^{**}, \quad \forall z \in [\mathfrak{a},\mathfrak{b}]. \tag{110}$$

Proof. Since $\psi(z)$ is increasing and differentiable over $[\mathfrak{a},\mathfrak{b}]$, we have $\psi(z)-\psi(q) \ge 0$ for every $q \in [0,z]$. By applying the Mean Value Theorem, there exists some $\xi \in (q,z)$ such that $\psi(z)-\psi(q) = \psi'(\xi)(z-q)$. Thus, plugging this value into the above integral, we have

$$I(z) = \int_0^z (\psi'(\xi)(z-q))^{\vartheta-1}\psi'(q)dq. \tag{111}$$

Since $\psi'(q)$ is continuous over $[\mathfrak{a},\mathfrak{b}]$, so, there exists $C^* > 0$ such that $0 < \psi'(q) \le C^*, \quad \forall q \in [\mathfrak{a},\mathfrak{b}]$. Therefore, by replacing the function term with its upper bound and applying the integral, we have $I(z) \le C^* C^{*^{\vartheta-1}} \frac{z^\vartheta}{\vartheta}$. Since $z \le b$ on $[\mathfrak{a},\mathfrak{b}]$, we have $I(z) \le \frac{C^{*^\vartheta} b^\vartheta}{\vartheta} = C^{**}$. This completes the proof. □

Theorem 4.4. *Let* $\vartheta \in (0,1]$ *and* $z \in \mathcal{T}$*. Assume that* $\varkappa$ *and* $f(\cdot,\varkappa(\cdot))$ *are granular measurable and of granular* ψ*-exponential order, so that their corresponding granular fuzzy LTWR-AFs exist in the sense of Lemma 2.31. Then the following fuzzy Volterra integral equation*

$$\varkappa(z) = \mathbb{E}_{\vartheta,1}\left(A\big(\psi(z)\big)^{\vartheta}\right)\varkappa(0) + \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}\left(A\big(\psi(z)-\psi(q)\big)^{\vartheta}\right) f(q,\varkappa(q))\psi'(q)dq, \tag{112}$$

is the mild solution to the Cauchy problem (106).

Proof. By the granular measurability and granular ψ-exponential order assumptions, Lemma 2.31 guarantees that the granular fuzzy LTWR-AFs of $\varkappa$ and $f(\cdot,\varkappa(\cdot))$ exist. Therefore, applying the granular fuzzy LTWR-AF to system (106) is well defined. For each $z \in \mathcal{T}$, the HMF of the GCFDSs (106) can be written as:

$${}^{C}_{0^+}\mathcal{D}^{\vartheta;\psi}\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})) = A_{gr}(\alpha,\alpha_{\varkappa_S})\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})) + f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S}))), \tag{113}$$

where $\alpha, \in [0,1]$ and $\alpha_{\varkappa_S})$ is the admissible RDM coordinate vector. Applying granular fuzzy LTWR-AF to the problem (113) and using Remark 2.28, we get

$$p^\vartheta\mathcal{L}_\psi\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S}))\right] - p^{\vartheta-1}\varkappa^{gr}(0,\alpha,\alpha_{\varkappa_S})) = A_{gr}(\alpha,\alpha_{\varkappa_S})\mathcal{L}_\psi\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S}))\right] + \mathcal{L}_\psi\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})))\right]. \tag{114}$$

After some simplification, the above Equation (114) turned into the following form

$$\mathcal{L}_\psi\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S}))\right] = \frac{p^{\vartheta-1}\varkappa^{gr}(0,\alpha,\alpha_{\varkappa_S}))}{p^\vartheta I - A_{gr}(\alpha,\alpha_{\varkappa_S})} + \frac{\mathcal{L}_\psi\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})))\right]}{p^\vartheta I - A_{gr}(\alpha,\alpha_{\varkappa_S})}. \tag{115}$$

Using Theorem 2.33, Equation (115) can be represented as:

$$\mathcal{L}_\psi\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S}))\right] = \mathcal{L}_\psi\left[\mathbb{E}_{\vartheta,1}(A_{gr}(\alpha,\alpha_{\varkappa_S})\big(\psi(z)\big)^{\vartheta})\right]\varkappa^{gr}(0,\alpha,\alpha_{\varkappa_S})) + \mathcal{L}_\psi\left[(\psi(z))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A_{gr}(\alpha,\alpha_{\varkappa_S})\big(\psi(z)\big)^{\vartheta})\right] \times \mathcal{L}_\psi\left[f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})))\right]. \tag{116}$$

Applying the convolution Theorem 2.27 to Equation (116) yields

$$\mathcal{L}_\psi\left[\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S}))\right]=\mathcal{L}_\psi\left[\mathbb{E}_{\vartheta,1}(A_{gr}(\alpha,\alpha_{\varkappa_S})\big(\psi(z)\big)^{\vartheta})\right]\varkappa^{gr}(0,\alpha,\alpha_{\varkappa_S}))+\mathcal{L}_\psi\left[(\psi(z))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A_{gr}(\alpha,\alpha_{\varkappa_S})\big(\psi(z)\big)^{\vartheta})\circ_\psi f^{gr}(z,\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})))\right]. \tag{117}$$

Applying the inverse LTWR-AF on both sides of Equation (117), we get:

$$\begin{aligned}\varkappa^{gr}(z,\alpha,\alpha_{\varkappa_S})) &= \mathbb{E}_{\vartheta,1}\left(A_{gr}(\alpha,\alpha_A)(\psi(z))^{\vartheta}\right)\varkappa^{gr}(0,\alpha,\alpha_{\varkappa_S}))\\ &\quad+\int_0^z(\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}\left(A_{gr}(\alpha,\alpha_A)(\psi(z)-\psi(q))^{\vartheta}\right)f^{gr}(q,\varkappa^{gr}(q,\alpha,\alpha_{\varkappa_S})))\psi'(q)dq.\end{aligned} \tag{118}$$

Finally, using Proposition 2.4, we derive the following solution

$$\varkappa(z)=\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\varkappa(0)+\int_0^z(\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi'(q)dq. \tag{119}$$

This completes the proof. □

Remark 4.5. *Now we consider the following granular fuzzy fractional differential system (GFFDS) of order $\vartheta\in(0,1)$:*

$$\begin{cases}\left({}^{gr}_{0^+}\mathfrak{D}^{\vartheta}\varkappa\right)(z) &= A\varkappa(z)+f(z,\varkappa(z)),\\ \quad\varkappa(0) &= \varkappa_0.\end{cases} \tag{120}$$

Dong et al. [12] *have obtained the AFS of the system (120). This system is a special case of our proposed system (106). If we take $\psi(z)=z$, then system (120) and the proposed system (106) are coincide. Moreover, the AFS of both systems in this case are identical.*

We are now going to define the MS-CP (106) as:

Definition 4.6. *A FVF $\varkappa\in\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}})$ is the MS-CP (106) if it satisfied Equation (112).*

Proposition 4.7. *By Corollary 2.15, $(\Xi^n_{\mathfrak{R}},\mathcal{D}^{gr})$, $(\Xi^n_{\mathfrak{R}},\mathcal{D}^{gr})$ is a complete granular metric space. For $\chi>0$, define*

$$\mathcal{D}^{gr}_{\chi,z}(\varkappa,\mathfrak{w})=\sup_{p\in[0,z]}e^{-\chi\psi(p)}\mathcal{D}^{gr}(\varkappa(p),\mathfrak{w}(p)), \tag{121}$$

for $\varkappa,\mathfrak{w}\in\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}})$. Then $\big(\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}}),\mathcal{D}^{gr}_{\chi,z}\big)$ is a complete metric space.

Proof. Since ψ is continuous on the compact interval $[0,z]$, the weight $e^{-\chi\psi(p)}$ is positive and bounded. Hence, there exist positive constants m_χ and M_χ such that $0<m_\chi\le e^{-\chi\psi(p)}\le M_\chi<\infty$, $p\in[0,z]$. Let $\mathcal{D}^{gr}_{0,z}(\varkappa,\mathfrak{w})=\sup_{p\in[0,z]}\mathcal{D}^{gr}(\varkappa(p),\mathfrak{w}(p))$. Then $m_\chi\mathcal{D}^{gr}_{0,z}(\varkappa,\mathfrak{w})\le\mathcal{D}^{gr}_{\chi,z}(\varkappa,\mathfrak{w})\le M_\chi\mathcal{D}^{gr}_{0,z}(\varkappa,\mathfrak{w})$. Thus, $\mathcal{D}^{gr}_{\chi,z}$ and $\mathcal{D}^{gr}_{0,z}$ are equivalent metrics. Now, let $\{\varkappa_m\}$ be a Cauchy sequence in $\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}})$ with respect to $\mathcal{D}^{gr}_{\chi,z}$. By the equivalence of the two metrics, $\{\varkappa_m\}$ is also Cauchy with respect to $\mathcal{D}^{gr}_{0,z}$. Hence, for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

$$\sup_{p\in[0,z]}\mathcal{D}^{gr}(\varkappa_m(p),\varkappa_l(p))<\varepsilon,\qquad m,l\ge N.$$

For every fixed $p\in[0,z]$, the sequence $\{\varkappa_m(p)\}$ is a Cauchy sequence in $(\Xi^n_{\mathfrak{R}},\mathcal{D}^{gr})$. Since $(\Xi^n_{\mathfrak{R}},\mathcal{D}^{gr})$ is complete, there exists $\varkappa(p)\in\Xi^n_{\mathfrak{R}}$ such that

$$\mathcal{D}^{gr}(\varkappa_m(p),\varkappa(p))\to 0\quad\text{as } m\to\infty.$$

Letting $l\to\infty$ in the supremum estimate gives

$$\sup_{p\in[0,z]}\mathcal{D}^{gr}(\varkappa_m(p),\varkappa(p))\le\varepsilon,\qquad m\ge N.$$

Thus, $\varkappa_m\to\varkappa$ uniformly on $[0,z]$ with respect to $\mathcal{D}^{gr}$. Since the uniform limit of continuous functions with values in a metric space is continuous, we obtain $\varkappa\in\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}})$. Consequently,

$$\mathcal{D}^{gr}_{0,z}(\varkappa_m,\varkappa)\to 0\quad\text{as } m\to\infty.$$

Using the equivalence of $\mathcal{D}^{gr}_{0,z}$ and $\mathcal{D}^{gr}_{\chi,z}$, we obtain

$$\mathcal{D}^{gr}_{\chi,z}(\varkappa_m,\varkappa)\to 0\quad\text{as } m\to\infty.$$

Hence, $\big(\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}}),\mathcal{D}^{gr}_{\chi,z}\big)$ is a complete metric space. □

Now we present the E&U of the MS-CP (106).

Theorem 4.8. *Under the assumptions* ($\mathbb{HA}$), ($\mathbb{HFI}$), ($\mathbb{HFII}$) *and* ($\mathbb{HFIII}$), *the system (106) has a unique mild solution on* $\mathcal{T}$.

Proof. We begin by introducing an operator $\mathfrak{F} : \mathcal{C}([0,z],\Xi^n_{\mathfrak{R}}) \longrightarrow \mathcal{C}([0,z],\Xi^n_{\mathfrak{R}})$ such that

$$\mathfrak{F}[\varkappa](z) = \mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\varkappa_0 + \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi^{'}(q)dq. \tag{122}$$

Let $\Omega_r = \big\{\varkappa \in \mathcal{C}([0,z],\Xi^n_{\mathfrak{R}}) : \mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{0}) \le r\big\}$ such that $\frac{MC_1\Gamma(\vartheta)}{\chi^{\vartheta}} < 1$, where $\chi > (MC_1\Gamma(\vartheta))^{1/\vartheta}$ and $r = \dfrac{M\|\varkappa_0\| + MC_0C^{**}}{1 - MC_1CC^{**}}$, where $MC_1CC^{**} < 1$. We proceed with the proof as follows:

Step 1: The operator $\mathfrak{F}$ maps Ω_r into itself. Specifically, $\forall z \in [0,T]$ and $\varkappa \in \Omega_r$, the following relation holds:

$$\begin{aligned}
\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\overline{0}) &= \mathcal{D}^{gr}\Bigg(\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\varkappa_0 + \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi^{'}(q)dq,\overline{0}\Bigg),\\
&\le \mathcal{D}^{gr}\Bigg(\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\varkappa_0,\overline{0}\Bigg) + \mathcal{D}^{gr}\Bigg(\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi^{'}(q)dq,\overline{0}\Bigg),\\
&\le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \int_0^z \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\|(\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)\mathcal{D}^{gr}\Bigg(f(q,\varkappa(q)),\overline{0}\Bigg)dq,\\
&\le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)\mathcal{D}^{gr}\Bigg(f(q,\varkappa(q)),\overline{0}\Bigg)dq,\\
&\le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)\Bigg(\mathcal{D}^{gr}(f(q,\varkappa(q)),f(q,\overline{0})) + \mathcal{D}^{gr}(f(q,\overline{0}),\overline{0})\Bigg)dq,\\
&\le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)\Bigg(C_1\mathcal{D}^{gr}(\varkappa(q),\overline{0}) + C_0\Bigg)dq,\\
&\le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \Bigg[C_0\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)dq\\
&\quad + C_1\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)e^{\chi\psi(q)}\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{0})dq\Bigg],\\
&\le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \Bigg[C_0\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)dq + C_1\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{0})\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)e^{\chi\psi(q)}dq\Bigg].
\end{aligned}$$

The last integral of the above inequality involves the term $e^{\chi\psi(q)}$. Since $\chi > 0$ and ψ is increasing, we have $e^{\chi\psi(q)} \le e^{\chi\psi(z)},\ 0 \le q \le z$. Hence, there exists a positive constant $C > 0$ such that $e^{\chi\psi(q)} \le C,\ 0 \le q \le z \le T$. One may take $C = e^{\chi\psi(T)}$. So, the aforementioned inequality can be written in the following form

$$\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\overline{0}) \le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \Bigg[C_0\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)dq + C_1C\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{0})\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi^{'}(q)dq\Bigg]. \tag{123}$$

Using Lemma 4.3, we can write Equation (123) as

$$\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\overline{0}) \le \|\mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\|\ \|\varkappa_0\| + \|\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})\| \Bigg[C_0C^{**} + C^{**}C_1Cr\Bigg]. \tag{124}$$

Using the hypothesis given in Equation (107), we get

$$\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\overline{0}) \le M\Bigg[\|\varkappa_0\| + C_0C^{**} + C^{**}C_1Cr\Bigg] \le M\|\varkappa_0\| + MC_0C^{**} + MC^{**}C_1Cr. \tag{125}$$

Multiplying both sides by $e^{-\chi\psi(z)}$ and taking the supremum over $z \in [0,T]$, we obtain

$$\mathcal{D}^{gr}_{\chi,z}(\mathfrak{F}[\varkappa],\overline{0}) \le r.$$

Hence, $\mathfrak{F}(\Omega_r) \subseteq \Omega_r$.

Step 2: To show that $\mathfrak{F}$ is continuous, consider a sequence $(\varkappa_n) \subset \Omega_r$ such that $(\varkappa_n) \longrightarrow \varkappa$ in Ω_r. Then for every $z \in [0,T]$, we have

$$\mathfrak{F}[\varkappa_n](z) = \mathbb{E}_{\vartheta,1}(A\big(\psi(z)\big)^{\vartheta})\varkappa_0 + \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa_n(q))\psi^{'}(q)dq. \tag{126}$$

By applying the hypothesis ($\mathbb{HFI}$), we have

$$(\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa_n(q))\psi^{'}(q) \longrightarrow (\psi(z)-\psi(q))^{\vartheta-1}\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi^{'}(q),$$

as n is very large. The hypothesis ($\mathbb{HFI}$) ensures the measurability and continuity of the FVF $f(z,\varkappa(z))$. Moreover, Lebesgue's dominated theorem guarantees that we can interchange the limit and the integral. This also allows us to show that $\mathcal{D}^{gr}(\mathfrak{F}[\varkappa_n](z), \mathfrak{F}[\varkappa](z)) \longrightarrow 0$ as $n \longrightarrow \infty$, confirming that $\mathfrak{F}$ is continuous. Therefore, by using hypotheses ($\mathbb{HFI}$) and Lebesgue's dominated theorem, we have

$$\mathcal{D}^{gr}(\mathfrak{F}[\varkappa_n](z),\mathfrak{F}[\varkappa](z)) \leq \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\mathcal{D}^{gr}\Big(\mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa_n(q))\psi^{'}(q), \mathbb{E}_{\vartheta,\vartheta}(A\big(\psi(z)-\psi(q)\big)^{\vartheta})f(q,\varkappa(q))\psi^{'}(q)\Big) \longrightarrow 0,$$

as $n \longrightarrow \infty$. Thus, $\mathfrak{F}$ is continuous on Ω_r.

Step 3: $\mathfrak{F}$ is a contraction mapping. For $\chi > 0$, consider the Bielecki-type granular metric

$$\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{\varkappa}) = \sup_{p\in[0,z]} e^{-\chi\psi(p)}\mathcal{D}^{gr}(\varkappa(p),\overline{\varkappa}(p)).$$

Let $\varkappa, \overline{\varkappa} \in \Omega_r$. Then, for each $z \in [0,T]$, using ($\mathbb{HA}$) and ($\mathbb{HFIII}$), we obtain

$$\begin{aligned}\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\mathfrak{F}[\overline{\varkappa}](z)) &\leq \int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi'(q)\left\|\mathbb{E}_{\vartheta,\vartheta}\left(A(\psi(z)-\psi(q))^{\vartheta}\right)\right\|\mathcal{D}^{gr}\big(f(q,\varkappa(q)),f(q,\overline{\varkappa}(q))\big)dq\\ &\leq MC_1\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}\psi'(q)\mathcal{D}^{gr}(\varkappa(q),\overline{\varkappa}(q))dq.\end{aligned} \tag{127}$$

By the definition of $\mathcal{D}^{gr}_{\chi,z}$, we have $\mathcal{D}^{gr}(\varkappa(q),\overline{\varkappa}(q)) \leq e^{\chi\psi(q)}\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{\varkappa})$. Therefore,

$$e^{-\chi\psi(z)}\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\mathfrak{F}[\overline{\varkappa}](z)) \leq MC_1\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{\varkappa})\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}e^{-\chi(\psi(z)-\psi(q))}\psi'(q)dq. \tag{128}$$

Using the substitution $s = \psi(z)-\psi(q)$, we get

$$\int_0^z (\psi(z)-\psi(q))^{\vartheta-1}e^{-\chi(\psi(z)-\psi(q))}\psi'(q)dq \leq \int_0^\infty s^{\vartheta-1}e^{-\chi s}ds = \frac{\Gamma(\vartheta)}{\chi^{\vartheta}}.$$

Hence,

$$e^{-\chi\psi(z)}\mathcal{D}^{gr}(\mathfrak{F}[\varkappa](z),\mathfrak{F}[\overline{\varkappa}](z)) \leq \frac{MC_1\Gamma(\vartheta)}{\chi^{\vartheta}}\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{\varkappa}).$$

Taking the supremum over $z \in [0,T]$, we obtain

$$\mathcal{D}^{gr}_{\chi,z}(\mathfrak{F}[\varkappa],\mathfrak{F}[\overline{\varkappa}]) \leq K_\chi\mathcal{D}^{gr}_{\chi,z}(\varkappa,\overline{\varkappa}), \qquad K_\chi = \frac{MC_1\Gamma(\vartheta)}{\chi^{\vartheta}}.$$

Choosing $\chi > (MC_1\Gamma(\vartheta))^{1/\vartheta}$ gives $0 < K_\chi < 1$. Thus, $\mathfrak{F} : \Omega_r \to \Omega_r$ is a contraction with respect to $\mathcal{D}^{gr}_{\chi,z}$. By Proposition 4.7, $\big(\mathcal{C}([0,z],\Xi^n_{\mathfrak{R}}),\mathcal{D}^{gr}_{\chi,z}\big)$ is a complete metric space. Moreover, Ω_r is a closed subset of this space and is therefore complete. Hence, by Banach's contraction principle, $\mathfrak{F}$ has a unique fixed point $\varkappa^* \in \Omega_r$. By the definition of $\mathfrak{F}$, this fixed point satisfies the mild integral equation (112). Therefore, $\varkappa^*$ is the unique mild solution of system (106). □

5 Supply chain SIQR model framework

Inspired by the epidemic model [11], a novel framework is introduced to analyze the dynamic interaction between the supply chain and the environment of during the life cycle of the products. The supply chain ecosystem divides its product population into four compartments: The compartment $\mathbb{S}(z)$ represents the stock sustainable and active products, which are those currently in use or circulation, typically during the introduction, growth, and maturity phases of their life cycle. These products are still functional and valuable but they are eventually entering the end-of-life phase. The compartment $\mathbb{I}(z)$ denotes the impact intensive-products, referring to those products whose production and operational phases impose significant environmental burdens, including high emissions and excessive use of natural resources. This compartment aligns with the growth and maturity phase of the product life cycle where the large-scale manufacturing and increased usage lead to the highest environmental impact in the supply chain. The compartment $\mathbb{Q}(z)$ represents quarantined products, refereing to those items that have been identified potentially defective and excluded from the active product cycle for further assessment. This compartment is similar to the quarantine phase and may involve the products undergoing the evaluation of carbon footprint and quality-control as investigations before reconsideration or disposal decisions. The compartment $\mathbb{R}(z)$ representing the recycled (or remanufactured) products depicts the products that have reached the end-of-life phase and have entered the reverse logistics process. In this phase, the products are recycled for reuse, allowing them to re-enter the supply chain process with a minimum environmental footprint.

The proposed model uses the granular ψ-CFD $({}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi})$ to capture the memory effect and non-local dynamics in supply chain systems, where past decisions continuously influence the present and future environmental and operational performance. The FVFs handle the uncertainty in parameters, including customer demand, environmental costs and recycling efficiency. The proposed system is defined by the following four equations: The dynamic model for the sustainable supply chain using the SIQR model can be formulated as follows: The compartment $\mathbb{S}(z)$ develops through various inflows and outflows. The new products from the extraction of the raw material and manufacturing are entered in state $\mathbb{S}(z)$ at a constant rate $\tilde{a}$. The state $\mathbb{S}(z)$ decreases as some of the products move towards the state $\mathbb{I}(z)$ at a rate $-\tilde{\ell}\mathbb{S}(z)\mathbb{I}(z)$, where $\tilde{\ell}$ is the environmental impact propagation rate. The state of $\mathbb{Q}(z)$ is allowed to return to the sustainable state $\mathbb{S}(z)$ at a rate of $\tilde{\omega}\mathbb{Q}(z)$. Similarly, $\mathbb{R}(z)$ re-enter to $\mathbb{S}(z)$, at a rate $\tilde{\sigma}\mathbb{R}(z)$, highlighting the circularity in the system. Additionally, sustainable products can leave the system permanently at a rate $-\tilde{\mu}\mathbb{S}(z)$ accounting for attrition. Thus, the equation for the rate of change of $\mathbb{S}(z)$ can be written in the following form:

$$\frac{d\mathbb{S}(z)}{dz} := \tilde{a} - \tilde{\ell}\mathbb{S}(z)\mathbb{I}(z) + \tilde{\omega}\mathbb{Q}(z) + \tilde{\sigma}\mathbb{R}(z) - \tilde{\mu}\mathbb{S}(z). \tag{129}$$

The interaction of the supply chain with the environment is a complex physical phenomenon with memory effects. We employ the GCFD to capture the dynamic behavior of the proposed model more accurately and efficiently. In this research, the GCFD with respect to the AF ψ is used to represent the rate of change across various compartments of the dynamic system. Furthermore, the fuzzy dynamic systems approach is typically used to study the complex interaction between the supply chain and the environment, which contains uncertain factors and lacks information. This technique allows for a more realistic representation of uncertainty inside the dynamic model. Accordingly, we introduce the concept of GCFD with respect to AF ψ to represent the rate of change of the compartment $\mathbb{S}$ as follows.

$${}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{S}(z) := \tilde{a} - \tilde{\ell}\mathbb{S}(z)\mathbb{I}(z) + \tilde{\omega}\mathbb{Q}(z) + \tilde{\sigma}\mathbb{R}(z) - \tilde{\mu}\mathbb{S}(z). \tag{130}$$

Here, ${}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}$ is the GCFD of order ϑ with respect to AF ψ and $\mathbb{S}, \mathbb{I}, \mathbb{Q}, \mathbb{R}$: $\mathfrak{R} \longrightarrow \Xi_{\mathfrak{R}}$ are the FVFs. Using the same reasoning, we construct GCFFDEs involving granular ψ-CFD that express the rate of change of three states $\mathbb{I}$, $\mathbb{Q}$, and $\mathbb{R}$ as follows:

The impact-intensive compartment $\mathbb{I}$ describes the overall dynamics of products whose lifespan produces considerable environmental impact. The influx into the compartment $\mathbb{I}(z)$ is represented as $\tilde{\ell}\mathbb{S}(z)\mathbb{I}(z)$, representing the growth of impact-intensive goods that result from the expansion of production under unsustainable practices. The depletion from this compartment is expressed by $-(\tilde{\nu} + \tilde{\gamma} + \tilde{\mu})\mathbb{I}(z)$, indicating the three different pathways. First, $-\tilde{\nu}\mathbb{I}(z)$ shows the direct remanufacturing where products are assigned to the recycling stream $\mathbb{R}(z)$, at a rate $\tilde{\nu}$. Secondly, $-\tilde{\gamma}\mathbb{I}(z)$ represents the quarantine where non-compliant products are confirmed and sent to the quarantined compartment $\mathbb{Q}(z)$, at a rate of $\tilde{\gamma}$. Third, $-\tilde{\mu}\mathbb{I}(z)$ signifies the disposal showing that products excluded from the system at a rate of $\tilde{\mu}$ permanently. Thus, the GCFDE that describes the rate of the compartment $\mathbb{I}(z)$ is given by

$${}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{I}(z) := \tilde{\ell}\mathbb{S}(z)\mathbb{I}(z) - (\tilde{\nu} + \tilde{\gamma} + \tilde{\mu})\mathbb{I}(z). \tag{131}$$

The compartment $\mathbb{Q}(z)$ denotes the products excluded temporarily from the active supply chain for quality assessment. Inflow to compartment $\mathbb{Q}(z)$, results from $+\tilde{\gamma}\mathbb{I}(z)$ (products transiting from $\mathbb{I}(z)$ by environmental load), and outflow, represented as $-(\tilde{\eta} + \tilde{\mu} + \tilde{\omega})\mathbb{Q}(z)$, depicts the three distinct pathways. First, the expression $-\tilde{\eta}\mathbb{Q}(z)$ accounts for the clearing-to-recycling process that quantifies the flow of quarantined products sent to the recycling stream of compartment $\mathbb{R}(z)$ occurring at rate $\tilde{\eta}$. Second, the term $-\tilde{\omega}\mathbb{Q}(z)$ denotes the re-entry into $\mathbb{S}(z)$, showing that the environmentally approved products return to $\mathbb{S}(z)$ at rate $\tilde{\omega}$. Third, $-\tilde{\mu}\mathbb{Q}(z)$ represents for disposal, indicating the elimination of products that are not appropriate for recovery at rate $\tilde{\mu}$. So the GCFDE that represents the rate of change of $\mathbb{Q}$ is given by

$$ {}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{Q}(z) := \tilde{\gamma}\mathbb{I}(z) - (\tilde{\eta} + \tilde{\mu} + \tilde{\omega})\mathbb{Q}(z). \tag{132} $$

The compartment $\mathbb{R}(z)$ denotes the reverse-logistics stream and describes the mechanisms through which end-of-use products are re-entering $\mathbb{S}(z)$ through the remanufacturing process. The inflow into $\mathbb{R}(z)$ from $\tilde{\nu}\mathbb{I}(z)$, describes the products leaving directly from the compartment $\mathbb{I}(z)$to recycling. Furthermore, the compartment $\mathbb{Q}(z)$ participates in recycling at rate $\tilde{\eta}$, denoted by the term $\tilde{\eta}\mathbb{Q}(z)$. The outflow from $\mathbb{R}(z)$ is represented as $-(\tilde{\sigma} + \tilde{\mu})\mathbb{R}(z)$, accounting for the two major segments. First, the term $-\tilde{\sigma}\mathbb{R}(z)$ represents the remanufacturing, whereas the recycled products are reprocessed and reintegrated into $\mathbb{S}(z)$ at rate $\tilde{\sigma}$. Second, the term $-\tilde{\mu}\mathbb{R}(z)$ shows the losses during the recycling process including the wastage that permanently excluded from the system at rate $\tilde{\mu}$. Therefore, the GCFDE that characterizes the evolution $\mathbb{R}(z)$ takes the form is

$$ {}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{R}(z) := \tilde{\nu}\mathbb{I}(z) + \tilde{\eta}\mathbb{Q}(z) - (\tilde{\sigma} + \tilde{\mu})\mathbb{R}(z). \tag{133} $$

Thus, the uncertain of dynamic behavior of the closed-loop supply chain is modeled by the following granular Caputo-type fuzzy fractional SIQR model

$$ \begin{cases} {}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{S}(z) := \tilde{a} - \tilde{\ell}\mathbb{S}(z)\mathbb{I}(z) + \tilde{\omega}\mathbb{Q}(z) + \tilde{\sigma}\mathbb{R}(z) - \tilde{\mu}\mathbb{S}(z), \\ {}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{I}(z) := \tilde{\ell}\mathbb{S}(z)\mathbb{I}(z) - (\tilde{\nu} + \tilde{\gamma} + \tilde{\mu})\mathbb{I}(z), \\ {}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{Q}(z) := \tilde{\gamma}\mathbb{I}(z) - (\tilde{\eta} + \tilde{\mu} + \tilde{\omega})\mathbb{Q}(z), \\ {}^{gr}_{0^+}\mathfrak{D}^{\vartheta;\psi}\mathbb{R}(z) := \tilde{\nu}\mathbb{I}(z) + \tilde{\eta}\mathbb{Q}(z) - (\tilde{\sigma} + \tilde{\mu})\mathbb{R}(z), \end{cases} \tag{134} $$

with UICs $\mathbb{S}(0)=\mathbb{S}_0, \mathbb{I}(0)=\mathbb{I}_0, \mathbb{Q}(0)=\mathbb{Q}_0$ and $\mathbb{R}(0)=\mathbb{R}_0$. Table 1 represents the parameter description of the SIQR model (134).

Table 1: Parameter descriptions for the SIQR model

Parameter	Descriptions
$\tilde{a}$	The rate of logging in the network of new product
$\tilde{\ell}$	The rate of the environmental impact propagation
$\tilde{\mu}$	The rate of logging out the network of the product
$\tilde{\gamma}$	The probability that a high-impact product is identified and quarantined for assessment
$\tilde{\nu}$	The rate at which impact-intensive products moves into the recycling
$\tilde{\eta}$	The rate at which quarantined products are release to a recycled stream
$\tilde{\omega}$	The rate at which quarantined products are re-enter to the sustainable state
$\tilde{\sigma}$	The rate at which recycled products are re-enter the market as new sustainable products

5.1 Simulation environment and numerical implementation

Using the fuzzy parameter values of system (134) and the numerical scheme described in [32], we obtain the graphical results for the granular Caputo-type fuzzy fractional SIQR model. All parameters in system (134) are taken as triangular fuzzy numbers (TFNs) to represent uncertainty in their estimated values. These TFNs describe the variations in product flow, environmental impact, recycling, and disposal rates. In the graphical analysis, we use the following fuzzy parameter values:

- $\tilde{a} = (2, 3, 4)$: New product introduction rate based on manufacturing supply-chain throughput rates [11].
- $\tilde{\ell} = (0.5, 0.55, 0.6)$: Environmental impact propagation rate calibrated from carbon-footprint studies in apparel and electronics industries.
- $\tilde{\mu} = (0.10, 0.15, 0.20)$: Product disposal rate representing the removal of end-of-life items from supply chains.
- $\tilde{\gamma} = (0, 0.001, 0.002)$: Environmental audit rate representing occasional checks of product compliance.
- $\tilde{\nu} = (0, 0.25, 0.5)$: Recycling rate representing the movement of product from use of recovery stream.
- $\tilde{\eta} = (0, 0.005, 0.01)$: Quarantine clearing rate for products approved after quality checks.
- $\tilde{\omega} = (0.01, 0.02, 0.03)$: Re-entry rate representing the return of approved products from quarantine to the sustainable product class.

- $\tilde{\sigma} = (0.01, 0.02, 0.03)$: Remanufacturing rate representing the movement of recovered products from $\mathbb{R}(z)$ to $\mathbb{S}(z)$.

TFNs are also used to model uncertainty in the initial product distribution among the compartments. The initial conditions for sustainable stock are specified as $\mathbb{S}(0) = (1.3, 1.5, 1.7)$, indicating the initial stock of sustainable products from a moderate to high baseline. The impact-intensive stock is specified as $\mathbb{I}(0) = (2.5, 2.75, 3)$, representing the uncertain initial stock of products associated with high environmental impact. The initial conditions of the quarantined stock are taken as $\mathbb{Q}(0) = (1, 1.25, 1.5)$, which represent that the product is initially under assessment. The remanufactured stock is initialized as $\mathbb{R}(0) = (0.01, 0.02, 0.03)$, indicating the initial availability of remanufactured products. All numerical simulations are performed in MATLAB. The time interval is discretized uniformly as $z_n = z_0 + nh, \qquad n = 0, 1, \ldots, N$, where $z_0 = 0$, $T = 5$, $h = 0.1$, and $N = 50$. The fractional order is fixed as $\vartheta = 0.9$, and the gamma function is evaluated using built-in `gamma()` function in MATLAB. We imply the fuzzy initial conditions using level-wise lower and upper representations to obtain the interval-valued initial states. The proposed numerical technique produces the family of the granular fuzzy solution trajectories. The detailed parameter descriptions of the proposed model are given in Table 1. Moreover, the computational cost of the memory implementations is evaluated in terms of the number of time points N and the number of fuzzy levels N_β and the number of state variables d. The cost increases linearly with the number of chosen kernels ψ. In the present implementation, we take $N_\beta = 6$ and $d = 4$ for the SIQR system. To investigate sensitivity with respect to the memory scale, different choices of ψ can be used, such as $\psi_1(z) = z$, $\psi_2(z) = \log(1 + z)$, and $\psi_3(z) = z^\rho$, $\rho > 0$. The kernel $\psi(z) = z$ recovers the classical Caputo memory structure, while nonlinear choices of ψ modify the accumulation of historical effects. The proposed model reduces to a crisp solution when all fuzzy initial values and parameters are replaced by their midpoints, and to the classical fuzzy Caputo model when $\psi(z) = z$. The proposed granular ψ-Caputo framework generalizes both the crisp fractional model and the classical fuzzy Caputo model by simultaneously preserving memory flexibility and granular uncertainty propagation.

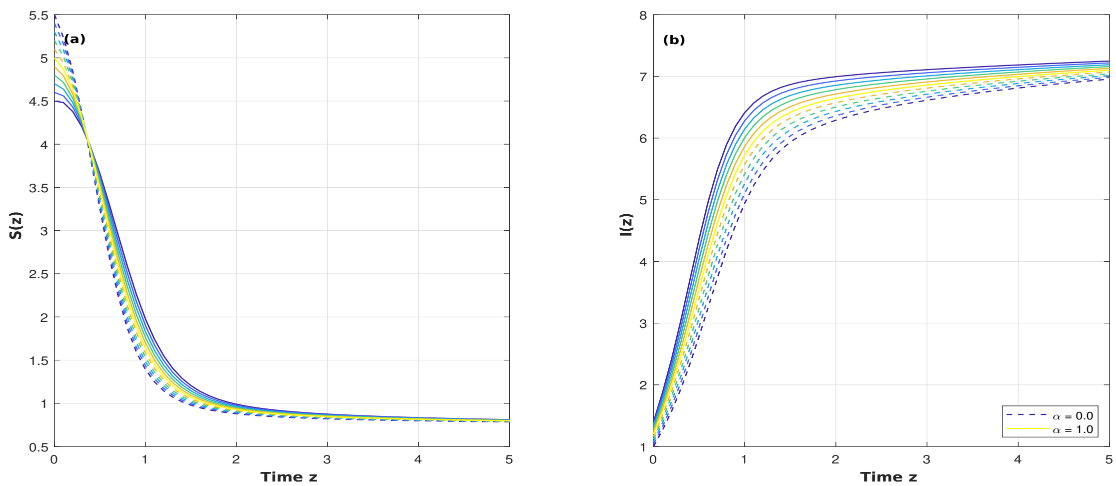

Figure 2: Time evolution analysis of the granular fuzzy fractional SIQR model (134) with fractional order $\vartheta = 0.9$. Dotted and solid curves denote the lower and upper α-cut trajectories, respectively. (a) Evolution of $\mathbb{S}(z)$ showing faster adjustment process compared with lower fractional order due to reduced memory effects on system dynamics. (b) The evaluation of $\mathbb{I}(z)$ exhibits a the shaper and more pronounced peak, suggesting that the weaker memory effects lead to a more immediate environmental impact response.

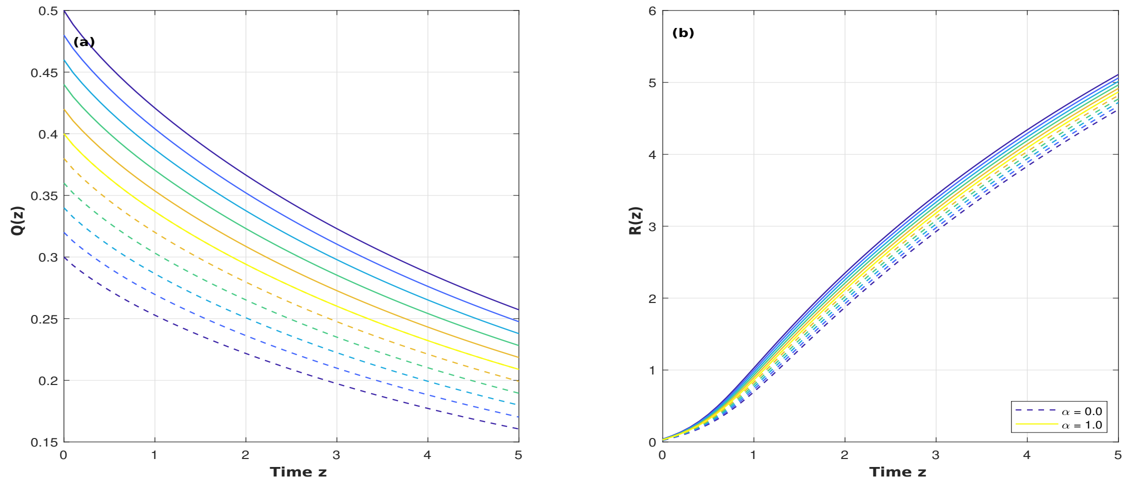

Figure 3: Time evolution analysis of the granular fuzzy fractional SIQR model (134) with fractional order $\vartheta = 0.9$. Dotted and solid curves denote the lower and upper α-cut trajectories, respectively. (a) Evolution of $\mathbb{Q}(z)$ exhibits a transient behavior similar to that of $\vartheta = 0.7$. (b) The trajectories of $\mathbb{R}(z)$ indicate a faster accumulation process towards steady-state level. The narrow trajectories at the fractional order $\vartheta = 0.9$ indicate that the system exhibits low memory effects and adopts more rapidly compared to the stronger memory effects.

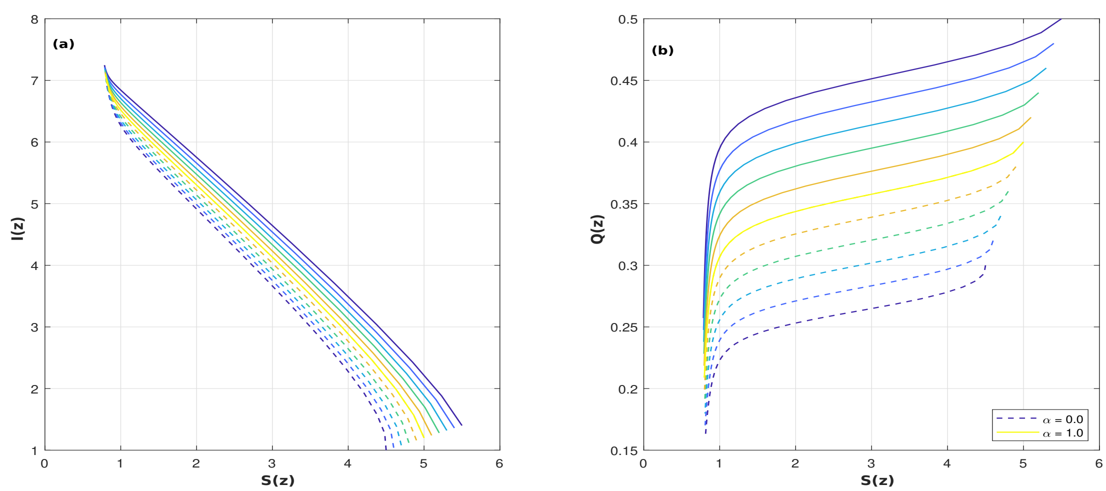

Figure 4: Phase portrait analysis of the granular fuzzy fractional SIQR model (134) with fractional order $\vartheta = 0.9$. (a) The $\mathbb{I}(z)$–$\mathbb{S}(z)$ trajectory indicate the trade-off between environmental impact and sustainable stock recovery. (b) The $\mathbb{Q}(z)$–$\mathbb{S}(z)$ trajectory shows the interaction between quarantine dynamics and sustainability performance. In both panels, solid and dashed curves represent the upper and lower bounds at different α-cut levels.

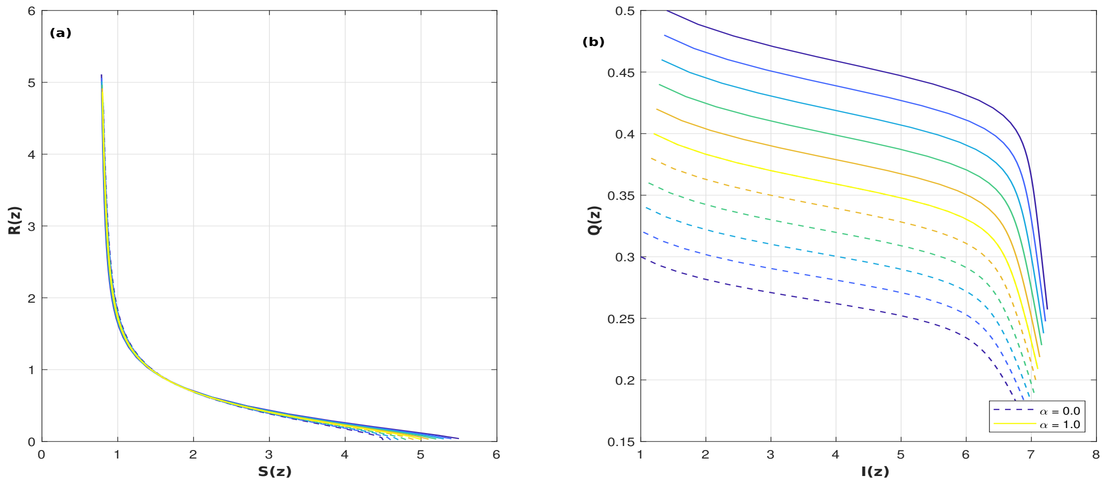

Figure 5: Phase portrait analysis of granular fuzzy fractional SIQR model (134) with a fractional-order $\vartheta = 0.9$. Dotted and solid curves denote the lower and upper α-cut trajectories, respectively. (a) The $\mathbb{R}(z)$– $\mathbb{S}(z)$ trajectory indicates the interaction between remanufactured and sustainable stock which exhibits the recovery activities in sustainable supply chain. (b) The $\mathbb{Q}(z)$–$\mathbb{I}(z)$ trajectory depicting the relationship between quarantine activity and the impact-intensive products. The narrow fuzzy bands in panel (a) indicate lower uncertainty, while the wider bands in panel (b) show the greater uncertainty in $\mathbb{Q}$–$\mathbb{I}$.

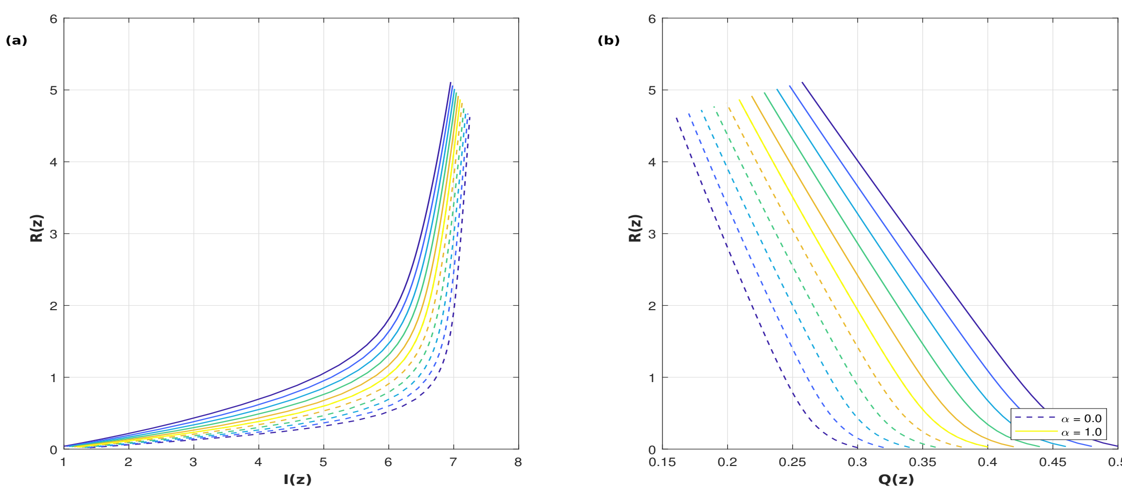

Figure 6: Phase portrait analysis of the granular fuzzy fractional SIQR model (134) with $\vartheta = 0.9$. Dotted and solid curves denote the lower and upper α-cut trajectories, respectively. (a) The $\mathbb{R}(z)$–$\mathbb{I}(z)$ trajectory illustrate the relationship between recycling and impact-intensive products. These trajectories indicate that increasing recycled stock directly impacts the recovery pathway. (b) The $\mathbb{R}(z)$–$\mathbb{Q}(z)$ trajectory represents the flow from quarantine to recycling activities, which indicates the recovered products' contribution to the growth of recycled stock. Both solution curves converge towards stable equilibrium points showing that the proposed model achieves long-term sustainability under appropriate interventions strategies.

The simulation results presented in Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6 provide managerial insights into inventory and safety-stock planing, recycling investment decisions, environmental compliance, memory effect assessment and uncertainty reduction strategies, in supply chains. Figure 7 indicates the comparison analysis of the four different compartments with various fractional-order $\vartheta = 0.7, 0.8, 0.9, 1.0$ for further represents the implications of the memory effects on the granular fuzzy fractional SIQR model.

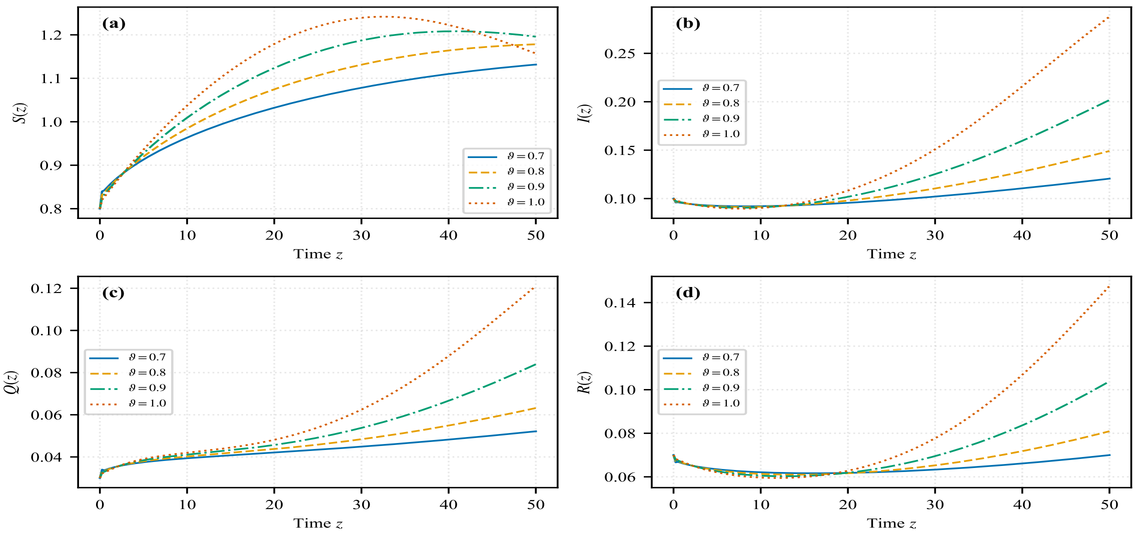

Figure 7: Comparison analysis of fuzzy fractional SIQR model (134) for different fractional-orders $\vartheta = 0.7, 0.8, 0.9, 1.0$. (a) Sustainable stock $\mathbb{S}(z)$. (b) Impact-intensive stock $\mathbb{I}(z)$. (c) Quarantined stock $\mathbb{Q}(z)$. (d) Recycled stock $\mathbb{R}(z)$. As the fractional-order approaches the classical integer-order $\vartheta = 1.0$, the memory-effect decreases and the system moves faster with sharper transitions. In contrast, low fractional-order at $\vartheta = 0.7$ exhibit strong memory effects.

6 Conclusions

In this research, we have presented a mathematical framework for granular fuzzy fractional calculus under the concept of granular ψ-CFD, unifying three fundamental concepts: (i) generalized memory kernels via arbitrary function ψ, (ii) multidimensional uncertainty representation through HMFs, and (iii) analytical solvability via granular fuzzy LTWR-AF. The fundamental properties, results, and relationships concerning the granular ψ-FFI, a granular ψ-CFD, and granular fuzzy LTWR-AF have been explored along with the framework of HMFs. Based on this theoretical foundation, the M-LF-based AFS of GCFFDEs and GCFFDS with respect to another function has been determined using the granular fuzzy LTWR-AF technique. An illustrative example was presented to demonstrate the applicability of the proposed technique. Moreover, the E&U of the fuzzy solutions for the abstract GCFFDS with fuzzy parameters and initial data has been investigated. To enhance the applicability of the proposed framework, the proposed methodology has been applied to a granular fuzzy SIQR supply-chain model under environmental disruptions. Numerical simulations and graphical analysis have demonstrated that the model can effectively describe the uncertainty propagation and memory-dependent behavior in supply chain dynamics.

6.1 Limitations and future research directions

Although the proposed framework provides significant advantages over existing approaches, several limitations remain. First, the evaluation of M-LF in the analytical fuzzy solution relies on infinite-series expansions, which may become computationally intensive for large-scale systems. In particular, increasing truncation order and dimensionality of the system can significantly increase the computational cost of evaluating the analytical fuzzy solution. Second, the analytical fuzzy solution depends on the choice of the arbitrary kernel function ψ to allow the framework to capture non-standard memory mechanisms. However, an inappropriate choice of kernel function ψ may effect the analytical closed-form fuzzy solutions. Third, the proposed framework is developed under gr-differentiability. Although this differentiability facilitates determining the analytical fuzzy solutions, it may exclude certain classes of highly non-convex fuzzy behaviors arising in complex uncertain environments. Future research may focus on developing efficient numerical schemes for solving the proposed framework to overcome the limitations associated with infinite series expansion of M-LF. Moreover, an adaptive and data-driven approach can be developed to identify the suitable choice of the kernel function ψ based on the system dynamics. Alternative mathematical technique including generalized fixed-point approaches can be introduced to minimize the shortcomings in granular differentiability. Furthermore, integrating the proposed framework with machine learning techniques may offer the hybrid intelligent models for real-time prediction, parameter estimation and decision support under uncertainty.

CRediT authorship contribution statement

Ghulam Muhammad, Qingyu Zhang, Muhammad Akram & Tofigh Allahviranloo: Concept, Design, Analysis, Writing, or revision of the manuscript.

Declaration of competing interest

The authors declare no conflicts of interests.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Declaration on the use of AI tools

AI and grammar-checking tools were used solely for language refinement and grammatical correction. The authors reviewed and approved all content and take full responsibility for the manuscript.

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