

Robust adaptive fuzzy output feedback attitude control of a satellite

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Abstract

This paper develops a robust indirect adaptive fuzzy output-feedback control scheme for attitude tracking of a rigid satellite in the presence of unknown inertia matrix, completely unknown nonlinear dynamics, and external disturbances. Unlike state-feedback approaches, the proposed method uses only measurable attitude outputs and does not require direct angular-velocity measurements. First, the Modified Rodrigues Parameters (MRPs) model of satellite dynamic is reformulated into a new state-space representation suitable for the integrated design of the adaptive fuzzy observer and controller. A robust adaptive fuzzy observer is developed to estimate the unavailable state variables from the measured outputs and approximate the unknown nonlinear dynamics together with the unknown inertia matrix in the presence of uncertainties and bounded disturbances. Based on the estimated information, an adaptive fuzzy output-feedback controller is designed to track a desired attitude trajectory. Lyapunov-based design establishes the stability of the proposed observer and closed-loop control system. Numerical simulations demonstrate accurate attitude tracking, effective state and uncertainty estimation, and robustness against external disturbances. The proposed framework therefore reduces dependence on angular-velocity sensors and prior knowledge of the satellite dynamics while maintaining reliable tracking performance.

Keywords: Rigid satellite, modified Rodrigues parameters, state observer, adaptive fuzzy control, Lyapunov stability.

1 Introduction

Attitude control is a central subsystem of a satellite because many mission functions depend directly on the ability to maintain or track a commanded orientation. Imaging payloads, communication antennas, navigation instruments, and precise sensors all require accurate pointing. Even modest attitude errors can reduce measurement quality, degrade payload performance, increase energy use, and in demanding missions, compromise the intended objectives [3, 24]. In practice, satellite attitude dynamics are affected by several sources of uncertainty. The inertia matrix may vary because of fuel consumption or internal payload motion, while gravity-gradient torque, solar radiation pressure, and aerodynamic effects introduce external disturbances. Actuator limits and possible failures create additional difficulties [1, 36]. A useful controller must therefore provide accurate tracking without becoming overly sensitive to modeling errors and environmental perturbations.

A separate challenge is the availability of system state variables. Direct measurement of every state variable, particularly angular velocity, may be unreasonable or undesirable because additional sensors increase hardware complexity, mass, power consumption, cost, and the number of potential failure points. This has encouraged the development of output-feedback attitude controllers that use measured outputs rather than complete state information [32].

These practical concerns have led to a broad range of control methods, including robust, adaptive, fault-tolerant, and intelligent techniques. The reported methods have improved tracking and robustness under uncertain operating conditions [34, 35]. Nevertheless, it remains difficult to address unknown nonlinear dynamics, inertia uncertainty, not all state measurements, and external disturbances within a single design [7].

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The choice of attitude coordinates is equally important because it shapes the kinematic model, controller structure, and stability analysis [24]. Common representations include Euler angles, direction cosine matrices, quaternions, Rodrigues parameters, and Modified Rodrigues Parameters (MRPs) [23, 31]. Each representation offers a different balance between physical interpretation, dimensionality, singularity properties, and computational convenience.

Euler angles are intuitive, but their gimbal-lock singularity limits their use in large-angle maneuvers [24]. Quaternions avoid kinematic singularities, yet they represent a three-dimensional attitude with four parameters, require preservation of a unit-norm constraint, and have a double-covering property that must be handled in control design [28].

MRPs provide a minimal three-parameter representation and do not require a normalization constraint [4, 31]. Their compact kinematic form is convenient for nonlinear control and observer design [8, 20]. MRPs are singular at a complete rotation of 360° . In general, this limitation can be managed through switching between the principal set and its equivalent shadow set [8, 21, 31]. Shadow-set switching, however, does not happen in the present method. The initial conditions and reference trajectory considered here are far from a nonsingular MRP operating region.

The present study uses MRPs to describe satellite attitude kinematics. Rather than employing the MRP equations directly, the dynamic model is recast in a new state-space form suited for our observer and controller design. Within this structure, a robust adaptive fuzzy observer estimates the unavailable state variables from the measured attitude outputs, while an adaptive fuzzy system approximates the completely unknown nonlinear dynamics. Also, the inertia matrix of the system is approximated by an adaptive mechanism.

Adaptive control has been widely studied for satellite attitude tracking because it can compensate for parametric uncertainty through online adjustment. Such controllers can reduce sensitivity to inertia variation without requiring exact knowledge of the satellite model [1, 34]. Their effectiveness, however, depends on the adaptation mechanism and may deteriorate when rapid changes occur in the parameters.

Sliding-mode control is another widely used approach because of its robustness to matched uncertainties and disturbances. Conventional and higher-order sliding-mode controllers have been applied to improve transient behavior and preserve tracking accuracy during demanding maneuvers [22, 38]. Finite-time and predefined-time variants have also been proposed to accelerate convergence [18, 33]. Many of these designs, however, still assume that full-state information or a sufficiently accurate dynamic model is available.

Output-feedback methods address the sensing limitation more directly. By combining the controller with an observer, they reduce dependence on angular-velocity measurements and additional sensors in spacecraft systems [19, 37]. In some reported designs, however, state and uncertainty estimation are developed as separate modules [24].

Fuzzy logic systems and neural networks offer a different way to handle uncertain nonlinear dynamics because they can approximate unknown functions without a complete analytical model. Adaptive fuzzy control has therefore become a useful option for satellite attitude tracking under parametric uncertainty and external disturbances [8, 11]. Earlier intelligent controllers often assumed that all states were measurable, whereas more recent output-feedback designs seek to operate with attitude measurements alone.

Among the closest related studies, Huo et al. [11] considered an adaptive fuzzy observer-based attitude tracking for a rigid spacecraft. Their design used an adaptive fuzzy observer to estimate the unavailable angular velocity and approximate unknown nonlinear terms. The formulation also addressed actuator faults and input saturation. But the system inertia matrix is assumed known, and the controller is a back-stepping method. The present study differs in the robust adaptive fuzzy observer and robust adaptive controller. Also, the system inertia matrix is assumed unknown.

Subsequent studies have improved particular aspects of uncertain attitude control. Zhu et al. combined a fuzzy disturbance observer with adaptive sliding-mode control under time-varying output constraints [39]. Hatami et al. addressed actuator misalignment and environmental disturbances through adaptive fault-tolerant control [10], while Gao et al. developed a finite-time adaptive fuzzy method to improve transient behavior [9]. Su and Shen later considered predefined-time fault-tolerant tracking under multiple uncertainties [26]. Related recent works have likewise concentrated on disturbance rejection, output constraints, fault accommodation, and prescribed convergence properties [9, 12, 26, 39].

Recent developments in fuzzy modeling and adaptive control have provided effective tools for representing and controlling uncertain nonlinear systems. Khalifa et al. [14] introduced an interval type-2 Takagi-Sugeno-Kang fuzzy-based Hammerstein structure for nonlinear networked systems, demonstrating the capability of fuzzy models to represent nonlinear behavior in the presence of network-related effects. Subsequently, an interval type-2 fuzzy Wiener model was developed for nonlinear engineering systems, in which the model parameters were updated using a Lyapunov-based mechanism [13]. Building on this modeling framework, adaptive interval type-2 fuzzy control was further investigated for nonlinear networked Wiener systems subject to packet dropouts and time-varying delays, with online parameter adaptation and compensation mechanisms incorporated into the control structure [6]. More recently, general type-2 and interval type-3 fuzzy approaches have been employed to enhance the representation of uncertainties and improve adaptive control or nonlinear system modeling capabilities [15, 16, 17].

Table 1 contains some studies relevant to satellite attitude control and includes related rigid-spacecraft work when the various aspects of attitude control are technically compared. Taken together, these studies reveal two remaining

Table 1: Comparison of representative attitude control methods for rigid satellites and the proposed method.

Study	Application / Problem	Measurement Availability	Main Uncertainty or Constraint	Core Approach
Xiao et al. (2017) [32]	Rigid-satellite attitude tracking	Attitude measured; angular velocity unavailable	Known inertia and system dynamics without disturbances	Output-feedback tracking control in Euler coordinates
Zhang et al. (2017) [36]	Fault-tolerant attitude control of a rigid satellite	Attitude and angular velocity available	Disturbances and thruster faults	Sliding-mode observer with finite-time fault-tolerant control
Huo et al. (2017) [11]	Rigid-spacecraft attitude tracking with a new state-space formulation	Attitude measured; angular velocity unavailable	Unknown dynamics, disturbances, actuator faults, and saturation	Adaptive fuzzy observer with back-stepping fault-tolerant control
Adnane et al. (2020) [1]	Attitude control of a rigid satellite	Attitude and angular velocity available	Torque fault estimation	Adaptive feedback linearization and fault estimation
Su and Shen (2024) [26]	Rigid spacecraft attitude tracking	Attitude and angular velocity available	Nominal inertia with uncertainty, disturbances, actuator faults, and input saturation	Adaptive sliding-mode control with prescribed performance
Proposed method	Rigid-satellite attitude tracking with a new state-space formulation	Attitude measured; angular velocity unavailable	Unknown inertia matrix, completely unknown dynamics, and disturbances	Robust adaptive fuzzy observer and robust adaptive control

issues. First, fuzzy output-feedback methods commonly reconstruct unavailable states and approximate nonlinearities, but inertia uncertainty is often neglected. In addition, the role of a dedicated compact state-space reformulation in organizing the observer, approximator, robust controller and specifically in adaptive inertia estimation has received less attention.

In view of the above discussion, this paper develops a robust observer-based adaptive fuzzy controller for a rigid satellite with completely unknown nonlinear dynamics and inertia. Thus, firstly the conventional MRP kinematic model is reformulated in a new state-space form suited to observer and controller design. Second, a robust adaptive fuzzy observer estimates the unavailable system states using the measured outputs by approximating the unknown nonlinear dynamics and inertia matrix and also an estimate of the upper bound of the lumped uncertainties. Finally, based on the unknown nonlinear dynamics and inertia matrix approximations, a robust controller is designed for attitude tracking. The observer and closed-loop stability properties are established separately through Lyapunov analysis.

The paper is configured in the following form: Section 2 presents the rigid satellite dynamical equations in the MRPs and transforms the model into a new dynamical one. Section 3 discusses the design of a robust adaptive fuzzy observer to estimate the system's states. Section 4 introduces a robust controller for the rigid satellite based on the observer. Section 5 provides a simulation study and discussion of the suggested method. Finally, Section 6 concludes the article.

2 Rigid satellite dynamics

As discussed earlier, the modified Rodriguez parameters (MRP) model serves as a refined classical approach for representing the rotational dynamics of a satellite. Therefore, rigid satellite dynamics is described as:

$$\dot{\sigma}_b = G(\sigma_b) \omega_b, \quad (1a)$$

$$J\dot{\omega}_b = -S(\omega_b)J\omega_b + u + d. \quad (1b)$$

The modified Rodrigues parameters (MRPs), represented by $\sigma_b = [\sigma_{b1}, \sigma_{b2}, \sigma_{b3}]^T \in R^3$ defines the satellite's attitude in the body coordinates relative to the inertial coordinates. $\omega_b \in R^3$ indicates the angular velocity of the satellite, and its total inertia matrix is given as $J \in R^{3 \times 3}$, which is an unknown matrix. The symbols $u \in R^3$

and $d \in R^3$ signify the control input torque and the disturbance vector. The disturbance torque arises from model uncertainties and external influences. Finally, $s(a)$ characterizes the skew-symmetric matrix associated with $a \in R^3$, such that for every vector $b \in R^3$, $s(a)b = a \times b$. As a result, $s(a)$ is described as:

$$s(a) = \begin{bmatrix} 0 & -a_{31} & a_{21} \\ a_{31} & 0 & -a_{11} \\ -a_{21} & a_{11} & 0 \end{bmatrix}. \quad (2)$$

Also, $G(\sigma_b)$ in (1a) is defined as:

$$G(\sigma_b) = 0.25 \left[(1 - \sigma_b^T \sigma_b) I_3 + 2s(\sigma_b) + 2\sigma_b^T \sigma_b \right]. \quad (3)$$

Here, $I_3 \in R^{3 \times 3}$ is the identity matrix.

Assumption 1. $d(t)$ is presumed to be bounded, namely, $\|d(t)\| \leq D$, where $D > 0$ is an unknown parameter.

The assumption is reasonable for practical systems where the uncertainty can be constrained within physical or operational limits.

To get ready transform the system (1) to a new dynamic model, firstly, the dynamic equation (1b) is written as

$$\dot{\omega}_b = -J^{-1}s(\omega_b)J\omega_b + J^{-1}u + J^{-1}d. \quad (4)$$

Let define $J_{inv} = J^{-1}$ as the new unknown matrix, then (4) is simplified as

$$\dot{\omega}_b = -J_{inv}s(\omega_b)J\omega_b + J_{inv}u + J_{inv}d. \quad (5)$$

The latter can also be rewritten as follows

$$\dot{\omega}_b = F(\omega_b) + J_{inv}u + w_b, \quad (6)$$

in which $F(\omega_b) = -J_{inv}s(\omega_b)J\omega_b$ is considered an unknown nonlinear function vector and $w_b = J_{inv}d$, represents the new disturbance vector. Thus, system (1) can be rearranged as

$$\dot{\sigma}_b = G(\sigma_b)\omega_b, \quad (7a)$$

$$\dot{\omega}_b = F(\omega_b) + J_{inv}u + w_b. \quad (7b)$$

Now, the dynamic equations in (7) are first converted into a companion-form representation, and afterward a robust adaptive fuzzy estimator is introduced to concurrently approximate both the states and unknown dynamics of the system.

$G(\sigma_b)$ is a matrix function with the following characteristic [5]:

$$G(\sigma_b)G^T(\sigma_b) = \left(\frac{1 + \sigma_b^T \sigma_b}{4} \right)^2 I_3. \quad (8)$$

Noticing (8), it can be inferred

$$G^{-1}(\sigma_b) = \left(\frac{4}{1 + \sigma_b^T \sigma_b} \right)^2 G^T(\sigma_b). \quad (9)$$

Regarding (7a) and (9), one can get

$$\omega_b = G^{-1}(\sigma_b)\dot{\sigma}_b = \left(\frac{4}{1 + \sigma_b^T \sigma_b} \right)^2 G^T(\sigma_b)\dot{\sigma}_b = H(\sigma_b, \dot{\sigma}_b). \quad (10)$$

The following can be obtained by replacing (10) and (7b) in the time derivative of (7a), i.e.

$$\ddot{\sigma}_b = \dot{G}(\sigma_b)\omega_b + G(\sigma_b)\dot{\omega}_b = \dot{G}(\sigma_b)H(\sigma_b, \dot{\sigma}_b) + G(\sigma_b)(F(\omega_b) + J_{inv}u + w_b), \quad (11)$$

or

$$\ddot{\sigma}_b = H_F(\sigma_b, \dot{\sigma}_b) + G(\sigma_b)J_{inv}u + w, \quad (12)$$

in which, $\mathbf{H}_F(\sigma_b, \dot{\sigma}_b) = \dot{\mathbf{G}}(\sigma_b) \mathbf{H}(\sigma_b, \dot{\sigma}_b) + \mathbf{G}(\sigma_b) F(\omega_b)$ is an unknown function vector of σ_b and $\dot{\sigma}_b$ and $w = \mathbf{G}(\sigma_b) \mathbf{w}_b$ is the new uncertainty.

Defining

$$\mathbf{X} = \begin{pmatrix} \sigma_b \\ \dot{\sigma}_b \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 \\ \dot{\mathbf{X}}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \quad (13)$$

the dynamic equation (12) can be formed as

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}[\mathbf{H}_F(\mathbf{X}) + \mathbf{G}(\mathbf{X}_1) \mathbf{J}_{inv} \mathbf{u} + \mathbf{w}], \quad (14a)$$

$$\mathbf{y} = \sigma_b = \mathbf{C}\mathbf{X}, \quad (14b)$$

in which

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I}_3 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ \mathbf{I}_3 \end{bmatrix}, \mathbf{C} = [\mathbf{I}_3 \quad 0]. \quad (15)$$

For achieving satellite attitude tracking control under conditions of uncertain matrix $\mathbf{J}_{inv}$, completely unknown $\mathbf{H}_F(\mathbf{X})$ and non-measurable ω_b and also in the presence of disturbance vector $\mathbf{w}$, an adaptive fuzzy observer-based control will be proposed.

3 Adaptive fuzzy observer design

In this part, to reduce the construction cost and minimize the use of required sensors, the angular velocities of the system are assumed to be non-measurable. To this end, an adaptive fuzzy estimator is proposed to estimate $\mathbf{X}$ along with approximating unknown dynamics and parameters of the system. The following presents the suggested observer.

$$\dot{\hat{\mathbf{X}}} = \mathbf{A}\hat{\mathbf{X}} + \mathbf{B} \left[\hat{\mathbf{H}}_F(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \hat{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{u}_o \right] - \mathbf{L}(\hat{\mathbf{y}} - \mathbf{y}), \quad (16a)$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{X}}. \quad (16b)$$

In which $\hat{\mathbf{X}}$ is the estimate of $\mathbf{X}$, $\hat{\mathbf{y}}$ represents the estimate of the output y , matrix $\mathbf{L}$ is the estimator gain matrix, designated such that guarantees the matrix $\mathbf{A} - \mathbf{L}\mathbf{C}$ to be Hurwitz, $\hat{\mathbf{H}}_F(\cdot)$ and $\hat{\mathbf{J}}_{inv}$ are estimates of $\mathbf{H}_F(\cdot)$ and $\mathbf{J}_{inv}$ respectively and $\mathbf{u}_o$ is a robust term. Here, an adaptive fuzzy inference system is used to approximate $\mathbf{H}_F(\cdot)$ and an adaptive scheme to estimate $\mathbf{J}_{inv}$.

3.1 Fuzzy inference system

A useful universal approximator in engineering applications is fuzzy implication system which can be employed to approximate unknown dynamics of a system [30]. Some rules can write a zero-order TSK fuzzy system as below [27]

$$R^{(\ell)} : IF \ z_1 \text{ is } F_1^\ell \text{ and } \dots \ z_n \text{ is } F_n^\ell \text{ THEN } \ y \text{ is } \bar{y}^\ell \quad \ell = 1, 2, \dots, N, \quad (17)$$

$\mathbf{Z} = [z_1, \dots, z_n]^T$ is vector of the inputs and $y \in R$ denotes the scalar output of the fuzzy inference system. F_i^ℓ is the input fuzzy set z_i in the rule ℓ , $\bar{y}^\ell$ indicates the output constant value associated with that rule and N represents the number of fuzzy rules. The fuzzy system output can be computed as follows

$$y(\mathbf{Z}|\boldsymbol{\theta}) = \frac{\sum_{\ell=1}^N \left[\prod_{i=1}^n \mu_{F_i^\ell}(z_i) \right] \bar{y}^\ell}{\sum_{\ell=1}^N \left[\prod_{i=1}^n \mu_{F_i^\ell}(z_i) \right]} = \boldsymbol{\theta}^T \boldsymbol{\xi}(\mathbf{Z}), \quad (18)$$

$\mu_{F_i^\ell}(\cdot)$ is membership function of fuzzy set F_i^ℓ and $\boldsymbol{\theta} = [\bar{y}^1, \dots, \bar{y}^N]^T \in R^N$ is the vector of adjustable parameters.

Also $\boldsymbol{\xi}(\mathbf{Z}) = [\xi^1(\mathbf{Z}), \dots, \xi^N(\mathbf{Z})]^T \in R^N$ is the fuzzy function vector, where each component is defined as

$$\xi^\ell(\mathbf{Z}) = \frac{\prod_{i=1}^n \mu_{F_i^\ell}(z_i)}{\sum_{\ell=1}^N \left[\prod_{i=1}^n \mu_{F_i^\ell}(z_i) \right]}. \quad (19)$$

According to the (18), the fuzzy approximation of the function vector $\mathbf{H}_F(\mathbf{X})$ is described as

$$\hat{\mathbf{H}}_F(\sigma_b, \sigma_b | \theta) = \hat{\mathbf{H}}_F(\mathbf{X} | \theta) = \begin{bmatrix} \hat{h}_{f_1}(\mathbf{X}) \\ \hat{h}_{f_2}(\mathbf{X}) \\ \hat{h}_{f_3}(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} \theta_1^T \xi(\mathbf{X}) \\ \theta_2^T \xi(\mathbf{X}) \\ \theta_3^T \xi(\mathbf{X}) \end{bmatrix} = \theta \xi(\mathbf{X}), \quad (20)$$

where $\theta = [\theta_1 | \theta_2 | \theta_3]^T \in R^{3 \times N}$ is the matrix of tuning parameters.

Concerning the universal approximation capability of fuzzy inference systems[38], for $\hat{\mathbf{H}}_F(\mathbf{X} | \theta)$ introduced by fuzzy approximator (20), the optimal parameters matrix $\theta^* = [\theta_1^* | \theta_2^* | \theta_3^*]^T \in R^{3 \times N}$ is defined as

$$\theta^* = \arg \min_{\theta \in R^{3 \times N}} \left(\sup_{\mathbf{X} \in R^3} \|\theta^* \xi(\mathbf{X}) - \mathbf{H}_f(\mathbf{X})\| \right). \quad (21)$$

The minimum error of the fuzzy approximation system can be expressed as

$$\varepsilon = \theta^* \xi(\mathbf{X}) - \mathbf{H}_F(\mathbf{X}). \quad (22)$$

3.2 Convergence analysis of the proposed observer

Defining

$$\tilde{\mathbf{X}} = \hat{\mathbf{X}} - \mathbf{X}, \quad (23)$$

$$\tilde{\mathbf{y}} = \hat{\mathbf{y}} - \mathbf{y}, \quad (24)$$

$$\tilde{\mathbf{J}}_{inv} = \hat{\mathbf{J}}_{inv} - \mathbf{J}_{inv}, \quad (25)$$

$$\tilde{\theta} = \theta - \theta^*, \quad (26)$$

as the state, output observation, inertia, and parameter matrix estimation errors, respectively and using (14) and (16), the subsequent dynamical equations can be calculated.

$$\dot{\tilde{\mathbf{X}}} = \mathbf{A}\tilde{\mathbf{X}} + \mathbf{B} \left[\hat{\mathbf{H}}_F(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{u}_o - \mathbf{H}_F(\mathbf{X}) - \mathbf{w} \right] - \mathbf{L}\tilde{\mathbf{y}}, \quad (27a)$$

$$\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{X}}. \quad (27b)$$

Regarding (20), (22), (26), and (27b), (27a) can be manipulated and simplified as below

$$\begin{aligned} \dot{\tilde{\mathbf{X}}} &= \mathbf{A}\tilde{\mathbf{X}} + \mathbf{B} \left[\theta \xi(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{u}_o - \theta^* \xi(\mathbf{X}) + \varepsilon - \mathbf{w} \right] - \mathbf{L}\mathbf{C}\tilde{\mathbf{X}} \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{X}} + \mathbf{B} \left[(\theta - \theta^*) \xi(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{u}_o + \theta^* (\xi(\hat{\mathbf{X}}) - \xi(\mathbf{X})) + \varepsilon - \mathbf{w} \right] \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{X}} + \mathbf{B} \left[\tilde{\theta} \xi(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{u}_o + \mathbf{w}_F \right]. \end{aligned} \quad (28)$$

In (28), $\mathbf{w}_F$ is called the lumped uncertainty, defined as

$$\mathbf{w}_F = -\mathbf{w} + \varepsilon + \theta^* (\xi(\hat{\mathbf{X}}) - \xi(\mathbf{X})). \quad (29)$$

Therefore, the final dynamical system of the observation error can be written as

$$\dot{\tilde{\mathbf{X}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{X}} + \mathbf{B} \left[\tilde{\theta} \xi(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{u}_o + \mathbf{w}_F \right]. \quad (30a)$$

$$\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{X}}. \quad (30b)$$

If s is the Laplace variable, then the system (30) can be written in transfer function representation as

$$\tilde{\mathbf{y}} = \mathbf{Z}(s) \left[\tilde{\theta} \xi(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{w}_F + \mathbf{u}_o \right], \quad (31)$$

where $\mathbf{Z}(s)$ denotes the stable transfer function that characterizes the state-space representation $(\mathbf{A} - \mathbf{L}\mathbf{C}, \mathbf{B}, \mathbf{C})$, that is,

$$\mathbf{Z}(s) = \mathbf{C}(\mathbf{S}\mathbf{I} - \mathbf{A}_o)^{-1} \mathbf{B}, \quad (32)$$

in which, $\mathbf{A}_o = \mathbf{A} - \mathbf{L}\mathbf{C}$. The transfer function $\mathbf{Z}(s)$ can be converted to a strictly positive real (SPR) system by multiplication by an appropriate $\mathbf{T}(s)$ transfer function; in other words, (31) can be written as

$$\tilde{\mathbf{y}} = \mathbf{Z}(s)\mathbf{T}(s) \left[\tilde{\boldsymbol{\theta}}\boldsymbol{\xi}(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1)\tilde{\mathbf{J}}_{inv}\mathbf{u} + \mathbf{w}_o + \mathbf{u}_o \right], \quad (33)$$

$\mathbf{Z}(s)\mathbf{T}(s)$ is a transfer function matrix with SPR property and $\mathbf{w}_0$ is defined as

$$\mathbf{w}_0 = (\mathbf{T}^{-1}(s) - \mathbf{I}) \left(\tilde{\boldsymbol{\theta}}\boldsymbol{\xi}(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1)\tilde{\mathbf{J}}_{inv}\mathbf{u} + \mathbf{u}_o \right) + \mathbf{T}^{-1}(s)\mathbf{w}_F. \quad (34)$$

Assume that $\|\mathbf{w}_0(t)\| < \delta$, where $\delta > 0$ is an unknown parameter. Let $\hat{\delta}$ denote an estimate for δ , and $\tilde{\delta}$ denotes the estimation error as

$$\tilde{\delta} = \hat{\delta} - \delta, \quad (35)$$

$\mathbf{Z}(s)\mathbf{T}(s)$ can be shown in state-space realization; therefore (33) can be expressed as

$$\dot{\tilde{\mathbf{X}}} = \mathbf{A}_o\tilde{\mathbf{X}} + \mathbf{B}_s \left[\tilde{\boldsymbol{\theta}}\boldsymbol{\xi}(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1)\tilde{\mathbf{J}}_{inv}\mathbf{u} + \mathbf{w}_o + \mathbf{u}_o \right], \quad (36a)$$

$$\tilde{\mathbf{y}} = \mathbf{C}\tilde{\mathbf{X}}. \quad (36b)$$

Now, the subsequent theorem proves the convergence of the adaptive fuzzy observer (16).

Theorem 3.1. *For the system described by (14), the adaptive fuzzy observer given by (16) with the adaptive laws in (37), (38), (39) and the robust term in (40) guarantees the observer that error dynamics (27) asymptotically tends to zero.*

$$\dot{\boldsymbol{\theta}} = -\Gamma_{\boldsymbol{\theta}}\tilde{\mathbf{y}}\boldsymbol{\xi}^T(\hat{\mathbf{X}}), \quad (37)$$

$$\dot{\tilde{\mathbf{J}}}_{inv} = -\Gamma_{\mathbf{J}}\mathbf{G}^T(\mathbf{X}_1)\tilde{\mathbf{y}}\mathbf{u}^T, \quad (38)$$

$$\dot{\hat{\delta}} = \gamma_{\delta}\|\tilde{\mathbf{y}}\|, \quad (39)$$

$$\mathbf{u}_o = -\frac{\tilde{\mathbf{y}}}{\|\tilde{\mathbf{y}}\|}\hat{\delta}. \quad (40)$$

In the above equations, $\Gamma_{\boldsymbol{\theta}}$ and $\Gamma_{\mathbf{J}}$ are suitable diagonal matrices with positive elements and γ_{δ} is a positive scalar constant.

Proof. A positive definite function for (36) is designated as

$$V_o = \frac{1}{2}\tilde{\mathbf{X}}^T\mathbf{P}_o\tilde{\mathbf{X}} + \frac{1}{2}\text{tr}\left(\Gamma_{\boldsymbol{\theta}}^{-1}\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\theta}}^T\right) + \frac{1}{2}\text{tr}\left(\Gamma_{\mathbf{J}}^{-1}\tilde{\mathbf{J}}_{inv}\tilde{\mathbf{J}}_{inv}^T\right) + \frac{1}{2\gamma_{\delta}}\tilde{\delta}^2, \quad (41)$$

where tr represents the trace function and $\mathbf{P}_o$ indicates a symmetric positive matrix. Differentiation of (41) with respect to time and replacing (36a) into it, one can write

$$\begin{aligned} \dot{V}_o = & \frac{1}{2}\tilde{\mathbf{X}}^T \left(\mathbf{A}_o^T\mathbf{P}_o + \mathbf{P}_o\mathbf{A}_o \right) \tilde{\mathbf{X}} + \tilde{\mathbf{X}}^T\mathbf{P}_o\mathbf{B}_s \left[\tilde{\boldsymbol{\theta}}\boldsymbol{\xi}(\hat{\mathbf{X}}) + \mathbf{G}(\mathbf{X}_1)\tilde{\mathbf{J}}_{inv}\mathbf{u} + \mathbf{w}_o + \mathbf{u}_o \right] \\ & + \text{tr}\left(\Gamma_{\boldsymbol{\theta}}^{-1}\dot{\tilde{\boldsymbol{\theta}}}\tilde{\boldsymbol{\theta}}^T\right) + \text{tr}\left(\Gamma_{\mathbf{J}}^{-1}\dot{\tilde{\mathbf{J}}}_{inv}\tilde{\mathbf{J}}_{inv}^T\right) + \frac{1}{\gamma_{\delta}}\tilde{\delta}\dot{\tilde{\delta}}. \end{aligned} \quad (42)$$

Noticing (36b), since $(\mathbf{A}_o, \mathbf{B}_s, \mathbf{C})$ is a representation of the SPR state space, the following holds [2].

$$\mathbf{A}_o^T\mathbf{P}_o + \mathbf{P}_o\mathbf{A}_o = -\mathbf{Q}_o, \quad (43)$$

$$\mathbf{P}_o\mathbf{B}_s = \mathbf{C}^T. \quad (44)$$

In which, $\mathbf{Q}_o$ is also a symmetric positive matrix. By replacing (43) and (44) into (42), and noting (36b), (42) can be simplified as

$$\dot{V}_o = -\frac{1}{2}\tilde{\mathbf{X}}^T\mathbf{Q}_o\tilde{\mathbf{X}} + \tilde{\mathbf{y}}^T\tilde{\boldsymbol{\theta}}\boldsymbol{\xi}(\hat{\mathbf{X}}) + \tilde{\mathbf{y}}^T\mathbf{G}(\mathbf{X}_1)\tilde{\mathbf{J}}_{inv}\mathbf{u} + \tilde{\mathbf{y}}^T\mathbf{w}_0 + \tilde{\mathbf{y}}^T\mathbf{u}_o + \text{tr}\left(\Gamma_{\boldsymbol{\theta}}^{-1}\dot{\tilde{\boldsymbol{\theta}}}\tilde{\boldsymbol{\theta}}^T\right) + \text{tr}\left(\Gamma_{\mathbf{J}}^{-1}\dot{\tilde{\mathbf{J}}}_{inv}\tilde{\mathbf{J}}_{inv}^T\right) + \frac{1}{\gamma_{\delta}}\tilde{\delta}\dot{\tilde{\delta}}. \quad (45)$$

Considering Schwartz's inequality along with the properties of the trace function, the following relationship can be achieved.

$$\tilde{\mathbf{y}}^T \mathbf{w}_o \leq \|\tilde{\mathbf{y}}\| \|\mathbf{w}_o\|, \quad (46)$$

$$\tilde{\mathbf{y}}^T \tilde{\boldsymbol{\theta}} \xi(\hat{\mathbf{X}}) = \xi^T(\hat{\mathbf{X}}) \tilde{\boldsymbol{\theta}}^T \tilde{\mathbf{y}} = \text{tr}(\tilde{\mathbf{y}} \xi^T(\hat{\mathbf{X}}) \tilde{\boldsymbol{\theta}}^T), \quad (47)$$

$$\tilde{\mathbf{y}}^T \mathbf{G}(\mathbf{X}_1) \tilde{\mathbf{J}}_{inv} \mathbf{u} = \mathbf{u}^T \tilde{\mathbf{J}}_{inv}^T \mathbf{G}^T(\mathbf{X}_1) \tilde{\mathbf{y}} = \text{tr}(\mathbf{G}^T(\mathbf{X}_1) \tilde{\mathbf{y}} \mathbf{u}^T \tilde{\mathbf{J}}_{inv}^T). \quad (48)$$

Regarding the three latter properties, one can infer from (45) that

$$\begin{aligned} \dot{V}_o \leq & -\frac{1}{2} \tilde{\mathbf{X}}^T \mathbf{Q}_o \tilde{\mathbf{X}} + \text{tr}(\tilde{\mathbf{y}} \xi^T(\tilde{\mathbf{X}}) \tilde{\boldsymbol{\theta}}^T) + \text{tr}(\mathbf{G}^T(\mathbf{X}_1) \tilde{\mathbf{y}} \mathbf{u}^T \tilde{\mathbf{J}}_{inv}^T) + \|\tilde{\mathbf{y}}\| \|\mathbf{w}_o\| + \tilde{\mathbf{y}}^T \mathbf{u}_o \\ & + \text{tr}(\Gamma_{\boldsymbol{\theta}}^{-1} \dot{\boldsymbol{\theta}} \tilde{\boldsymbol{\theta}}^T) + \text{tr}(\Gamma_{\mathbf{J}}^{-1} \dot{\mathbf{J}}_{inv} \tilde{\mathbf{J}}_{inv}^T) + \frac{1}{\gamma_{\delta}} \dot{\delta} \hat{\delta}. \end{aligned} \quad (49)$$

Since $\|\mathbf{w}_o(t)\| < \delta$ and utilizing $\mathbf{u}_o$ in (40), (49) can be arranged to yield

$$\dot{V}_o \leq -\frac{1}{2} \tilde{\mathbf{X}}^T \mathbf{Q}_o \tilde{\mathbf{X}} + \text{tr}[(\Gamma_{\boldsymbol{\theta}}^{-1} \dot{\boldsymbol{\theta}} + \tilde{\mathbf{y}} \xi^T(\tilde{\mathbf{X}})) \tilde{\boldsymbol{\theta}}^T] + \text{tr}[(\Gamma_{\mathbf{J}}^{-1} \dot{\mathbf{J}}_{inv} + \mathbf{G}^T(\mathbf{X}_1) \tilde{\mathbf{y}} \mathbf{u}^T) \tilde{\mathbf{J}}_{inv}^T] + \|\tilde{\mathbf{y}}\| \delta - \|\tilde{\mathbf{y}}\| \hat{\delta} + \frac{1}{\gamma_{\delta}} \dot{\delta} \hat{\delta}. \quad (50)$$

Finally, using the adaptive laws (37), (38), (39) and the definition (35), the latter results

$$\dot{V}_o \leq -\frac{1}{2} \tilde{\mathbf{X}}^T \mathbf{Q}_o \tilde{\mathbf{X}}. \quad (51)$$

It indicates that $V_o(t)$. Utilizing Barbalat's lemma, we can show $\lim_{t \rightarrow \infty} \tilde{\mathbf{X}} \rightarrow 0$ and also boundedness of $\boldsymbol{\theta}$ and $\mathbf{J}_{inv}$. The proof ends. $\square$

4 Controller design

In this part, based on the estimated states and approximated dynamics obtained in the previous section, a robust controller is suggested for the system described in (14). To begin, a tracking error is introduced as

$$\mathbf{E} = \mathbf{X} - \mathbf{X}_d = \begin{bmatrix} \mathbf{X}_1 \\ \dot{\mathbf{X}}_1 \end{bmatrix} - \begin{bmatrix} \mathbf{X}_{1d} \\ \dot{\mathbf{X}}_{1d} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 \\ \dot{\mathbf{E}}_1 \end{bmatrix}. \quad (52)$$

Define a new variable $\boldsymbol{\psi}$ as below

$$\boldsymbol{\psi} = \boldsymbol{\Lambda} \mathbf{E} = [\boldsymbol{\Lambda}_1 \mid \mathbf{I}_3] \begin{bmatrix} \mathbf{E}_1 \\ \dot{\mathbf{E}}_1 \end{bmatrix} = \dot{\mathbf{E}}_1 + \boldsymbol{\Lambda}_1 \mathbf{E}_1, \quad (53)$$

where, $\boldsymbol{\Lambda}_1 \in R^{3 \times 3}$ is a proper designing positive matrix.

Taking the time derivative of (53) and replacing (52) and (14a) into it, one can write

$$\dot{\boldsymbol{\psi}} = \boldsymbol{\Lambda} \dot{\mathbf{E}} = \boldsymbol{\Lambda} (\dot{\mathbf{X}} - \dot{\mathbf{X}}_d) = \boldsymbol{\Lambda} \mathbf{A} \mathbf{X} + \boldsymbol{\Lambda} \mathbf{B} [\mathbf{H}_F(\mathbf{X}) + \mathbf{G}(\mathbf{X}_1) \mathbf{J}_{inv} \mathbf{u} + \mathbf{w}] - \boldsymbol{\Lambda} \dot{\mathbf{X}}_d. \quad (54)$$

Since $\boldsymbol{\Lambda} \mathbf{B} = \mathbf{I}_{3 \times 3}$, (54) can be simplified as

$$\dot{\boldsymbol{\psi}} = \boldsymbol{\Lambda} \mathbf{A} \mathbf{X} + \mathbf{H}_F(\mathbf{X}) + \mathbf{G}(\mathbf{X}_1) \mathbf{J}_{inv} \mathbf{u} + \mathbf{w} - \boldsymbol{\Lambda} \dot{\mathbf{X}}_d. \quad (55)$$

Let the following control signal

$$\mathbf{u}^* = \mathbf{J}_{inv}^{-1} \mathbf{G}^{-1}(\mathbf{X}_1) [-\boldsymbol{\Lambda} \mathbf{A} \mathbf{X} - \mathbf{H}_F(\mathbf{X}) - \mathbf{w} - \mathbf{K} \boldsymbol{\psi} + \boldsymbol{\Lambda} \dot{\mathbf{X}}_d], \quad (56)$$

in which, $\mathbf{K} \in R^{3 \times 3}$ is a design positive matrix. By applying the control effort (56) to the system (14), it can easily show that the vector $\boldsymbol{\psi}$ will converge to zero asymptotically i.e., $\lim_{t \rightarrow \infty} \boldsymbol{\psi} \rightarrow 0$ and noticing (53), it can conclude $\lim_{t \rightarrow \infty} \mathbf{E} \rightarrow 0$.

Since $\mathbf{J}_{inv}$, $\mathbf{H}_F(\mathbf{X})$ and w are unknown and $\mathbf{X}$ is not completely measurable, the given control signal (56) cannot be applied. However, the succeeding control input is offered.

$$\mathbf{u} = \hat{\mathbf{J}}_{inv}^{-1} \mathbf{G}^{-1}(\mathbf{X}_1) \left[-\mathbf{\Lambda} \mathbf{A} \widehat{\mathbf{X}} - \widehat{\mathbf{H}}_F(\widehat{\mathbf{X}}) - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{\Lambda} \dot{\mathbf{X}}_d + \mathbf{u}_c \right]. \quad (57)$$

In the previous section, $\hat{\mathbf{H}}_F(\hat{\mathbf{X}})$, $\hat{\mathbf{J}}_{inv}$ and $\hat{\mathbf{X}}$ have been obtained as the estimates of $\mathbf{H}_F(\mathbf{X})$, $\mathbf{J}_{inv}$ and $\mathbf{X}$, respectively. $\mathbf{u}_c$ is designated later as a compensation term.

By some manipulation in (55) and substituting (57) into it, one can write

$$\begin{aligned} \dot{\boldsymbol{\psi}} &= \mathbf{\Lambda} \mathbf{A} \mathbf{X} + \mathbf{H}_F(\mathbf{X}) + \mathbf{G}(\mathbf{X}_1) \left(\mathbf{J}_{inv} - \hat{\mathbf{J}}_{inv} \right) \mathbf{u} + \mathbf{G}(\mathbf{X}_1) \hat{\mathbf{J}}_{inv} \mathbf{u} + \mathbf{w} - \mathbf{\Lambda} \dot{\mathbf{X}}_d \\ &= \mathbf{\Lambda} \mathbf{A} \mathbf{X} + \mathbf{H}_F(\mathbf{X}) + \mathbf{G}(\mathbf{X}_1) \left(\mathbf{J}_{inv} - \hat{\mathbf{J}}_{inv} \right) \mathbf{u} + \left[-\mathbf{\Lambda} \mathbf{A} \widehat{\mathbf{X}} - \widehat{\mathbf{H}}_F(\widehat{\mathbf{X}}) - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{\Lambda} \dot{\mathbf{X}}_d + \mathbf{u}_c \right] \\ &= \mathbf{\Lambda} \mathbf{A} (\mathbf{X} - \widehat{\mathbf{X}}) - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{G}(\mathbf{X}_1) \left(\mathbf{J}_{inv} - \hat{\mathbf{J}}_{inv} \right) \mathbf{u} + \left(\mathbf{H}_F(\mathbf{X}) - \widehat{\mathbf{H}}_F(\widehat{\mathbf{X}}) \right) + \mathbf{w} + \mathbf{u}_c. \end{aligned} \quad (58)$$

Defining

$$\mathbf{w}_m = \mathbf{G}(\mathbf{X}_1) \left(\mathbf{J}_{inv} - \hat{\mathbf{J}}_{inv} \right) \mathbf{u} + \left(\mathbf{H}_F(\mathbf{X}) - \widehat{\mathbf{H}}_F(\widehat{\mathbf{X}}) \right). \quad (59)$$

As the system modeling error, and

$$\widetilde{\mathbf{E}} = \mathbf{E} - \widehat{\mathbf{E}} = \mathbf{X} - \widehat{\mathbf{X}} = -\widetilde{\mathbf{X}}. \quad (60)$$

As the observation error, (58) can be written as

$$\dot{\boldsymbol{\psi}} = \mathbf{\Lambda} \mathbf{A} \widetilde{\mathbf{E}} - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{w}_m + \mathbf{w} + \mathbf{u}_c. \quad (61)$$

Given that

$$\boldsymbol{\psi} = \mathbf{\Lambda} \mathbf{E} = \mathbf{\Lambda} (\widehat{\mathbf{E}} + \widetilde{\mathbf{E}}) = \mathbf{\Lambda} \widehat{\mathbf{E}} + \mathbf{\Lambda} \widetilde{\mathbf{E}} = \widehat{\boldsymbol{\psi}} + \mathbf{\Lambda} \widetilde{\mathbf{E}}, \quad (62)$$

it can result

$$\dot{\boldsymbol{\psi}} = \mathbf{\Lambda} \dot{\mathbf{E}} = \dot{\widehat{\boldsymbol{\psi}}} + \mathbf{\Lambda} \dot{\widetilde{\mathbf{E}}}. \quad (63)$$

Regarding (61) and (63), one can deduce

$$\dot{\widehat{\boldsymbol{\psi}}} + \mathbf{\Lambda} \dot{\widetilde{\mathbf{E}}} = \mathbf{\Lambda} \mathbf{A} \widetilde{\mathbf{E}} - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{w}_m + \mathbf{w} + \mathbf{u}_c, \quad (64)$$

or

$$\dot{\widehat{\boldsymbol{\psi}}} = \mathbf{\Lambda} (-\dot{\widetilde{\mathbf{E}}} + \mathbf{A} \widetilde{\mathbf{E}}) - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{w}_m + \mathbf{w} + \mathbf{u}_c. \quad (65)$$

Remarking $\mathbf{\Lambda} \mathbf{B} = \mathbf{I}_{3 \times 3}$ and the following fact

$$\dot{\widetilde{\mathbf{E}}} = \mathbf{A} \widetilde{\mathbf{E}} + \mathbf{B} \dot{\widetilde{\mathbf{E}}}_2. \quad (66)$$

Simplifying (65) that

$$\dot{\widehat{\boldsymbol{\psi}}} = -\dot{\widetilde{\mathbf{E}}}_2 - \mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{w}_m + \mathbf{w} + \mathbf{u}_c. \quad (67)$$

Define

$$\mathbf{w}_t = \mathbf{w}_m + \mathbf{w} - \dot{\widetilde{\mathbf{E}}}_2, \quad (68)$$

as the new lumped uncertainty, (67) is made simpler as

$$\dot{\widehat{\boldsymbol{\psi}}} = -\mathbf{K} \widehat{\boldsymbol{\psi}} + \mathbf{w}_t + \mathbf{u}_c. \quad (69)$$

Assume an unknown constant $\rho > 0$ such that

$$\|\mathbf{w}_t\| \leq \rho. \quad (70)$$

Theorem 4.1. *For the system in (14), the control input (57) together with (71) and (72), ensures that the tracking error $\mathbf{E}$ introduced by (52) tends asymptotically towards zero.*

$$\mathbf{u}_c = -\hat{\rho} \frac{\widehat{\boldsymbol{\psi}}}{\|\widehat{\boldsymbol{\psi}}\|}, \quad (71)$$

$$\dot{\hat{\rho}} = \gamma_\rho \|\widehat{\boldsymbol{\psi}}\|. \quad (72)$$

In which $\hat{\rho}$ is an estimate of ρ and γ_ρ is a designed positive constant.

Proof. As shown before, by applying control signal (57) to the system (14), dynamic equation (69) is obtainable. Now, the next positive function is defined for (69).

$$V_c = \frac{1}{2} \hat{\psi}^T \hat{\psi} + \frac{1}{2\gamma_\rho} \tilde{\rho}^2. \quad (73)$$

In the above equation, $\tilde{\rho}$ is defined as the estimation error of ρ by

$$\tilde{\rho} = \hat{\rho} - \rho. \quad (74)$$

By calculating the time derivative of (73) and substituting (69) into it, one can obtain

$$\dot{V} = \hat{\psi}^T \dot{\hat{\psi}} + \frac{1}{\gamma_\rho} \tilde{\rho} \dot{\hat{\rho}} = -\hat{\psi}^T \mathbf{K} \hat{\psi} + \hat{\psi}^T \mathbf{w}_t + \hat{\psi}^T \mathbf{u}_c + \frac{1}{\gamma_\rho} \tilde{\rho} \dot{\hat{\rho}}. \quad (75)$$

By replacing (71) and (72) into (75) and mentioning (74), one can obtain

$$\dot{V} = -\hat{\psi}^T \mathbf{K} \hat{\psi} + \hat{\psi}^T \mathbf{w}_t - \hat{\psi}^T \hat{\rho} \frac{\hat{\psi}}{\|\hat{\psi}\|} + \tilde{\rho} \|\hat{\psi}\| = -\hat{\psi}^T \mathbf{K} \hat{\psi} + \hat{\psi}^T \mathbf{w}_t - \hat{\rho} \|\hat{\psi}\| + \tilde{\rho} \|\hat{\psi}\| = -\hat{\psi}^T \mathbf{K} \hat{\psi} + \hat{\psi}^T \mathbf{w}_t - \rho \|\hat{\psi}\|. \quad (76)$$

Remarking (70), one can infer

$$\hat{\psi}^T \mathbf{w}_t \leq \|\hat{\psi}\| \|\mathbf{w}_t\| \leq \rho \|\hat{\psi}\|. \quad (77)$$

Noticing the latter, (76) results in

$$\dot{V} \leq -\hat{\psi}^T \mathbf{K} \hat{\psi}. \quad (78)$$

Noticing (78) and (73) and using Barbalat lemma, it can easily be shown that $\lim_{t \rightarrow \infty} \hat{\psi} \rightarrow 0$ and consequently $\lim_{t \rightarrow \infty} \hat{\mathbf{E}} \rightarrow 0$. Finally, noting (60) and the result of Theorem 1, it follows that $\lim_{t \rightarrow \infty} \mathbf{E} \rightarrow 0$. $\square$

5 Numerical simulation

As illustrated in Figure 1, the simulation is implemented in a closed-loop configuration. At each simulation step, the desired attitude σ_d , the measurable satellite attitude output σ_b , and the estimated quantities are supplied to the robust controller to generate the commanded control torque ($\mathbf{u}$). The control torque is then applied to the rigid-satellite model in the presence of the bounded external disturbance ($\mathbf{d}$). Using the measurable attitude output together with the control input, the robust adaptive fuzzy observer–estimator reconstructs the unavailable state information and updates the estimates of the unknown nonlinear dynamics and inertia-related parameters. These estimates are subsequently fed back to the controller, and the procedure is repeated throughout the simulation interval to evaluate the attitude-tracking and estimation performance of the proposed closed-loop system.

5.1 Simulation setup and parameter selection

Numerical simulations are conducted to evaluate the state estimation, attitude-tracking, and robustness performance of the proposed robust adaptive fuzzy output-feedback control scheme. The numerical results presented in this section correspond to a single deterministic simulation run over the interval $0 \leq t \leq 100$ s. The satellite parameters, initial conditions, controller and observer parameters, and fuzzy system parameters are reported in Tables 2-4. Table 2 presents the rigid-satellite inertia matrix, the initial plant conditions, and the external-disturbance torque. Table 3 summarizes the initial observer conditions, the controller gains and adaptation parameters. Table 4 provides the numerical parameters of the fuzzy approximation system.

The controller gain matrix is selected as $\mathbf{K} = \mathbf{I}_{3 \times 3}$. The robust-design coefficient is set to $\gamma_\rho = 0.1$, while the adaptation gains associated with the fuzzy parameters and inertia estimation are selected as $\Gamma_\theta = \mathbf{I}_{3 \times 3}$ and $\Gamma_{\mathbf{J}} = 10\mathbf{I}_{3 \times 3}$, respectively. The initial estimate of the inverse inertia matrix is chosen as $\hat{\mathbf{J}}_{inv}(0) = 0.1\mathbf{I}_{3 \times 3}$. The observer initial conditions are selected to differ from the corresponding plant initial conditions to assess the convergence of the proposed state observer.

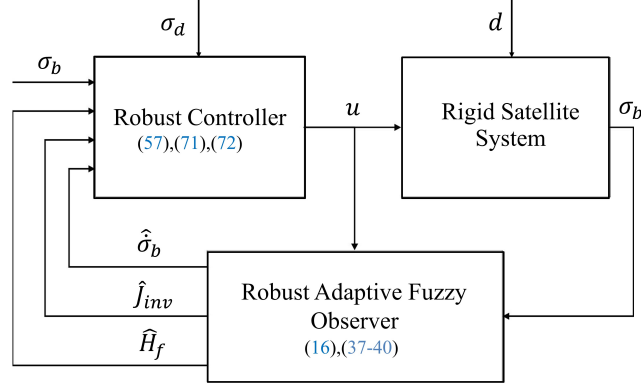


Figure 1: Block diagram of the proposed control method

Table 2: Satellite parameters, plant initial conditions, and external disturbance

Variable or parameter	Numerical value	Unit
J	$\begin{bmatrix} 40 & 1.2 & 0.9 \\ 1.2 & 42.5 & 1.4 \\ 0.9 & 1.4 & 50.2 \end{bmatrix}$	kgm^2
$\sigma_b(0)$	$[0.03, -0.02, -0.005]^T$	Dimensionless
$\dot{\sigma}_b(0)$	$[-0.01, 0.02, -0.01]^T$	s^{-1}
$d(t)$	$2 \times 10^{-4} \begin{bmatrix} \sin(0.8t) \\ \cos(0.5t) \\ \cos(0.3t) \end{bmatrix}$	N m

Table 3: Observer initial conditions and controller/adaptation parameters

Parameter	Numerical value
$\hat{\sigma}_b(0)$	$[0.02, -0.03, -0.02]^T$
$\hat{\dot{\sigma}}_b(0)$	$[0.01, -0.01, 0.02]^T$
K	$I_{3 \times 3}$
γ_ρ	0.1
Γ_Θ	$I_{3 \times 3}$
Γ_J	$10I_{3 \times 3}$
$\hat{J}_{\text{inv}}(0)$	$0.1I_{3 \times 3}$

5.2 Fuzzy approximation-system configuration

The fuzzy approximation system employs six inputs consisting of the three MRP components, σ_{b1} , σ_{b2} , and σ_{b3} , and their derivatives, $\dot{\sigma}_{b1}$, $\dot{\sigma}_{b2}$, and $\dot{\sigma}_{b3}$ and three Gaussian membership functions are assigned to each input, i.e. $3^6 = 729$ rules. The Gaussian membership function associated with the j th fuzzy set of the i th input is expressed as

$$\mu_{ij}(x_i) = \exp \left[-\frac{(x_i - c_{ij})^2}{2\alpha_{ij}^2} \right],$$

where c_{ij} and α_{ij} denote the center and standard deviation of the corresponding membership function, respectively. The parameters have been reported in Table 4. The centers have been chosen based on fuzzy input variations and the

Table 4: Configuration of the fuzzy approximation system

Parameter	Numerical value or description
Number of fuzzy inputs	6
Input vector	$x_f = [\sigma_{b1}, \sigma_{b2}, \sigma_{b3}, \hat{\sigma}_{b1}, \hat{\sigma}_{b2}, \hat{\sigma}_{b3}]^T$
Membership-function type	Gaussian
Membership functions per input	3
Centers for each σ_{bi}	$-0.2, 0, 0.2, \quad i = 1, 2, 3$
Standard deviation for each σ_{bi}	0.1
Centers for each $\hat{\sigma}_{bi}$	$-0.05, 0, 0.05, \quad i = 1, 2, 3$
Standard deviation for each $\hat{\sigma}_{bi}$	0.028

standard deviations have been designated to have a completeness about 0.5. The consequent fuzzy parameters have been adjusted online according to the proposed adaptive law.

5.3 Simulation results

All numerical results presented in the following figures correspond to a single deterministic simulation run. The initial conditions, reference trajectory, satellite parameters, controller gains, disturbance profile, and solver settings were fixed. No random sampling, stochastic measurement noise, or Monte Carlo trials have been used.

Figures 2-7 illustrate the state estimation, tracking, and control performance of the proposed observer and controller. Figure 2 compares the actual MRP attitude components with their estimated values. The close agreement between the actual and estimated trajectories after the initial transient demonstrates the convergence of the proposed nonlinear observer. Figure 3 compares the desired attitude trajectory with the corresponding satellite output and shows that the system output follows the reference signals with a small tracking error.

The attitude-tracking errors are presented in Figure 4. The errors decrease rapidly after the initial transient and remain close to zero during the subsequent tracking interval. Figure 5 compares the actual derivative-state variables with their observer estimates, whereas Figure 6 presents the corresponding estimation errors. These results demonstrate that the unavailable state variables are reconstructed with bounded and diminishing estimation errors under the considered disturbance.

Finally, Figure 7 presents the commanded control torque generated by the proposed controller. The control signals exhibit comparatively large values during the initial transient due to initial difference between the desired commands and outputs and subsequently remain bounded with smaller amplitudes during steady tracking. These results are consistent with reducing the initial tracking and estimation errors while maintaining the desired attitude trajectory.

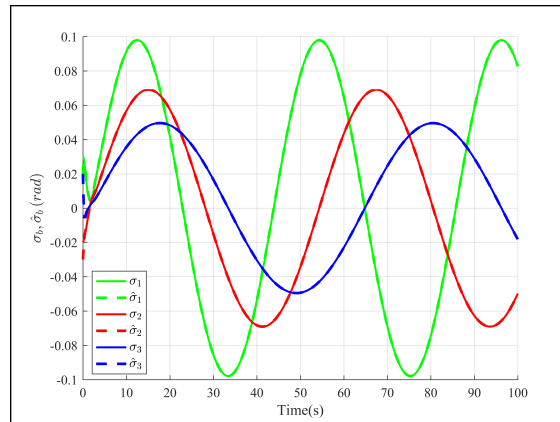


Figure 2: Comparison of the satellite attitude with its estimated value

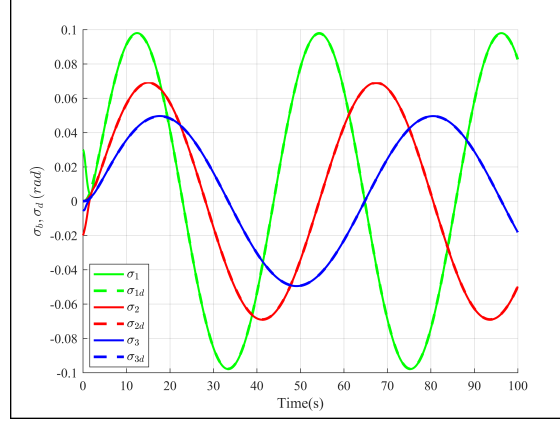


Figure 3: Comparison of the desired attitude trajectory with system output

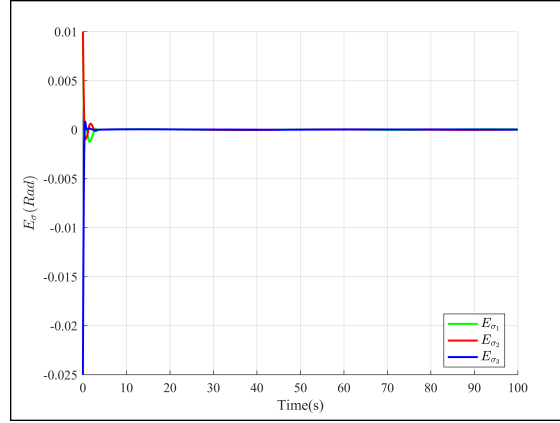
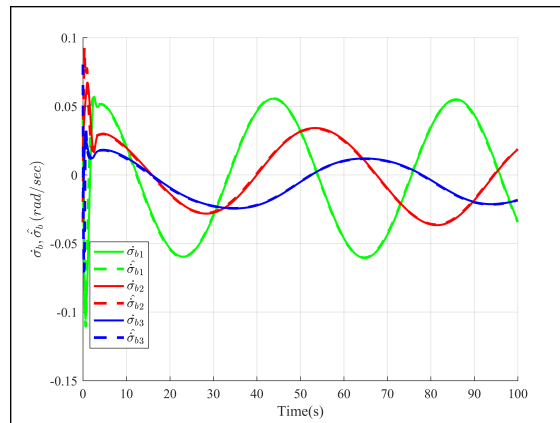


Figure 4: Satellite attitude tracking error

Figure 5: Comparison of $\dot{\sigma}_b$ with its estimated value

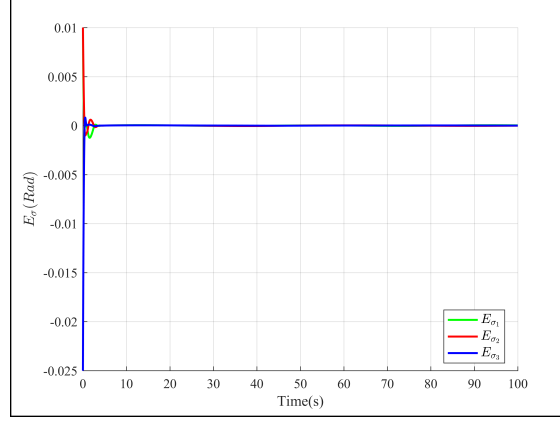
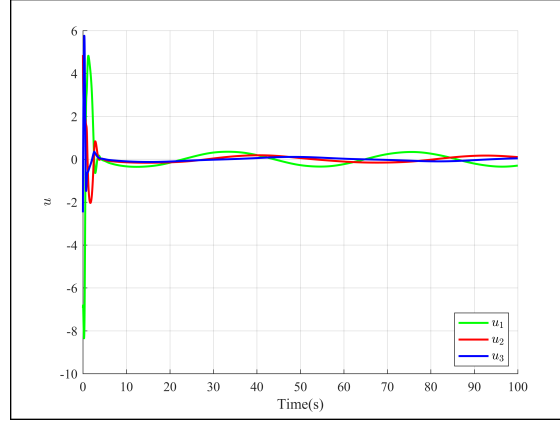
Figure 6: $\dot{\sigma}_b$ estimation error

Figure 7: control input torqued

5.4 Discussion

The numerical results indicate that the proposed output feedback framework achieves satisfactory satellite attitude tracking despite the absence of angular-velocity measurements, the unknown inertia matrix, the completely unknown nonlinear dynamics, and the presence of bounded disturbances. This behavior results from the coordinated operation of the MRP-based state-space reformulation, the robust adaptive fuzzy observer, the adaptive inertia-estimation mechanism, and the robust observer-based controller. The state-space reformulation of the MRP-based satellite model provides a convenient structure for the integrated design of the observer and controller.

Within this representation, the robust adaptive fuzzy observer estimates the unavailable state variables using the measurable attitude outputs. The reduction in the state estimation errors after the initial transient causes the estimated variables provide the controller with sufficiently accurate state information during the subsequent tracking interval. Measurement of the angular velocity is therefore not required in the proposed implementation.

The adaptive mechanisms play a complementary role. A controller designed in the presence of uncertainty and the unknown satellite inertia matrix and nonlinear dynamic model. In the proposed controller, the fuzzy system approximates the unknown nonlinear dynamics of the system, while the associated adaptive laws update the fuzzy approximator and inertia-related parameters using available measured and estimated signals. The robust observer and controller further compensate for the effects of bounded disturbances and residual fuzzy approximation errors.

Asymptotic stability of the proposed observer and controller is proven with the Lyapunov-based design. The theoretical development establishes the boundedness of the relevant closed-loop signals and the convergence properties of the estimation and tracking errors under the stated assumptions. The bounded trajectories and decreasing errors shown in the numerical results confirm these analytical conclusions for the reported deterministic simulation scenario.

From an implementation perspective, removing the need for angular-velocity measurements reduces dependence on additional sensors and also decreases cost of the control system implementation. This benefit is accompanied by a

higher computational demand arising from the observer design including fuzzy approximator, and online parameter updates in the adaptation laws. In particular, the computational load increases with the size of the fuzzy rule base. A larger rule base may improve approximation flexibility, but it also increases the number of calculations and adaptive parameters that must be handled in real time. The selection of the fuzzy system dimension is a tradeoff between the approximation accuracy and computational resources.

In a practical satellite attitude-control system, the output of the proposed controller represents the commanded torque, whereas the torque actually generated by the actuator is limited by its physical characteristics. The actuator constraints are not considered in the present theoretical formulation, which assumes that the commanded torque can be delivered to the satellite without actuator constraints.

6 Conclusion

This paper presents an observer-based adaptive fuzzy controller for attitude control of satellites, modeled with Modified Rodrigues Parameters (MRP). The designed control system effectively tracks the desired reference attitude while accounting for the system uncertainties and unknown dynamics. A state observer was designed to simultaneously estimate uncertain inertial matrix, nonlinear dynamics and states of the system, enhancing system robustness. By employing a fuzzy inference system, the nonlinearities of the satellite system were approximated, enabling precise control and compensation for model uncertainties. The proposed adaptive fuzzy controller ensured accurate output tracking to the desired reference values, achieving asymptotic stability of the overall system, namely, stability of the suggested observer and control methodology. Simulations demonstrated the usefulness of the suggested control strategy in accurately tracking the satellite's attitude, showcasing its potential for practical applications in attitude control of satellites.

References

- [1] A. Adnane, A. Bellar, M. A. S. Mohammed, J. Hong, et al., *Adaptive attitude controller design for a rigid satellite with feedback linearization approach*, The Journal of the Astronautical Sciences, **67** (2020), 1445-1469. <https://doi.org/10.1007/s40295-020-00227-y>
- [2] K. J. Astrom, B. Wittenmark, *A survey of adaptive control applications*, Proceedings of 1995 34th IEEE Conference on Decision and Control, (1995), 649-654. <https://doi.org/10.1109/CDC.1995.478986>
- [3] E. Canuto, C. Novara, D. Carlucci, C. P. Montenegro, et al., *Spacecraft dynamics and control: The embedded model control approach*, Butterworth-Heinemann, 2018. <https://doi.org/10.1016/c2016-0-00420-5>
- [4] J. L. Crassidis, F. L. Markley, *Sliding mode control using modified Rodrigues parameters*, Journal of Guidance, Control, and Dynamics, **19**(6) (1996), 1381-1383. <https://doi.org/10.2514/3.21798>
- [5] H. Du, S. Li, C. Qian, *Finite-time attitude tracking control of spacecraft with application to attitude synchronization*, IEEE Transactions on Automatic Control, **56**(11) (2011), 2711-2717. <https://doi.org/10.1109/TAC.2011.2159419>
- [6] A. M. El-Nagar, T. R. Khalifa, M. A. El-Brawany, M. El-Bardini, et al., *Adaptive interval type-2 fuzzy controller for nonlinear networked Wiener systems subject to packet dropout and time-varying delay*, ISA Transactions, **128**(Part A) (2022), 565-580. <https://doi.org/10.1016/j.isatra.2021.10.018>
- [7] L. Fan, H. Huang, K. Zhou, *Robust fault-tolerant attitude control for satellite with multiple uncertainties and actuator faults*, Chinese Journal of Aeronautics, **33**(12) (2020), 3380-3394. <https://doi.org/10.1016/j.cja.2020.06.026>
- [8] A. R. Fazlyab, F. F. Saberi, M. Kabgarian, *Adaptive attitude controller for a satellite based on neural network in the presence of unknown external disturbances and actuator faults*, Advances in Space Research, **57**(1) (2016), 367-377. <https://doi.org/10.1016/j.asr.2015.10.026>
- [9] S. Gao, Y. Jing, G. M. Dimirovski, Y. Zheng, *Adaptive fuzzy fault-tolerant control for the attitude tracking of spacecraft within finite time*, Acta Astronautica, **189** (2021), 166-180. <https://doi.org/10.1016/j.actaastro.2021.08.033>
- [10] M. Hatami, R. Nadafi, M. Kabgarian, *Fault-tolerant adaptive attitude tracking control of satellite with uncertainty and actuator misalignment in presence of environmental disturbances*, Scientific Reports, **15** (2025), 17947. <https://doi.org/10.1038/s41598-025-98468-0>

- [11] B. Huo, Y. Xia, L. Yin, M. Fu, *Fuzzy adaptive fault-tolerant output feedback attitude-tracking control of rigid spacecraft*, IEEE Transactions on Systems, Man, and Cybernetics: Systems, **47**(8) (2017), 1898-1908. <https://doi.org/10.1109/TSMC.2016.2564918>
- [12] U. Javaid, Z. Zhen, S. Shahid, D. S. Ibrahim, et al., *Output feedback attitude control of flexible spacecraft under actuator misalignment and input nonlinearities*, Journal of Vibration and Control, **30** (2024), 1783-1801. <https://doi.org/10.1177/10775463231171386>
- [13] T. R. Khalifa, A. M. El-Nagar, M. A. El-Brawany, E. A. El-Araby, et al., *A novel fuzzy Wiener-based nonlinear modelling for engineering applications*, ISA Transactions, **97** (2020), 130-142. <https://doi.org/10.1016/j.isatra.2019.07.017>
- [14] T. R. Khalifa, A. M. El-Nagar, M. A. El-Brawany, E. A. El-Araby, et al., *A novel Hammerstein model for nonlinear networked systems based on an interval type-2 fuzzy Takagi-Sugeno-Kang system*, IEEE Transactions on Fuzzy Systems, **29**(2) (2020), 275-285. <https://doi.org/10.1109/TFUZZ.2020.3007460>
- [15] T. R. Khalifa, X. Yu, A. Sharafian, X. Zhong, et al., *Interval Type-3 fuzzy Wiener model for nonlinear dynamic systems: Application to continuous stirred tank reactor*, Chaos, Solitons and Fractals, **199**(Part 1) (2025), 116584. <https://doi.org/10.1016/j.chaos.2025.116584>
- [16] T. R. Khalifa, X. Yu, X. Zhong, Z. Wu, *Adaptive general Type-2 fuzzy model-based control for nonlinear networked systems with packet dropouts*, ISA Transactions, **159** (2025), 257-277. <https://doi.org/10.1016/j.isatra.2025.02.009>
- [17] T. R. Khalifa, X. Yu, X. Zhong, Z. Wu, *Indirect adaptive interval Type-3 fuzzy tracking control for nonlinear discrete-time networked control systems with DoS attacks*, IEEE Transactions on Cybernetics, **55**(10) (2025), 4967-4980. <https://doi.org/10.1109/TCYB.2025.3591555>
- [18] J. Liu, Y. Cui, H. Wang, Y. Tian, et al., *Rigid spacecraft attitude control based on finite-time non-singular sliding mode and extended state disturbance observer with input hysteresis*, International Journal of Robust and Nonlinear Control, **35**(10) (2025), 4187-4200. <https://doi.org/10.1002/rnc.7896>
- [19] K. Liu, Y. Wang, Y. Li, Y. Zhang, et al., *Velocity-free adaptive neural-fuzzy predefined-time attitude control for spacecraft*, IEEE Transactions on Aerospace and Electronic Systems, **61**(3) (2025), 6354-6372. <https://doi.org/10.1109/TAES.2025.3526744>
- [20] N. Nazari, H. Moladavoudi, J. Beyramzad, *Finite time sliding mode control for agile rigid satellite with CMG actuators using fast high-order sliding mode observer*, Aerospace Systems, **7** (2024), 363-383. <https://doi.org/10.1007/s42401-024-00283-4>
- [21] S. A. O'Keefe, H. Schaub, *Shadow set considerations for modified Rodrigues parameter attitude filtering*, Journal of Guidance, Control, and Dynamics, **37**(6) (2014), 2030-2035. <https://doi.org/10.2514/1.G000405>
- [22] P. Pal, G. G. Jin, S. Bhakta, V. Mukherjee, *Adaptive chaos synchronization of an attitude control of satellite: A backstepping based sliding mode approach*, Heliyon, **8**(11) (2022), e11730. <https://doi.org/10.1016/j.heliyon.2022.e11730>
- [23] O. Rodrigues, *Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire*, Journal de Mathématiques Pures et Appliquées, **5** (1840), 380-440,
- [24] H. Schaub, J. L. Junkins, *Analytical mechanics of space systems*, Analytical Mechanics of Space Systems, 2018. <https://doi.org/10.2514/4.105210>
- [25] M. Shi, B. Wu, J. Tian, *Adaptive fault-tolerant attitude tracking control for spacecraft with input quantization*, Acta Astronautica, **218** (2024), 298-306. <https://doi.org/10.1016/j.actaastro.2024.02.004>
- [26] Y. Su, S. Shen, *Adaptive predefined-time fault-tolerant attitude tracking control for rigid spacecraft with guaranteed performance*, Acta Astronautica, **214** (2024), 677-688. <https://doi.org/10.1016/j.actaastro.2023.11.015>

- [27] T. Takagi, M. Sugeno, *Fuzzy identification of systems and its applications to modeling and control*, IEEE Transactions on Systems, Man, and Cybernetics, **SMC-15**(1) (1985), 116-132. <https://doi.org/10.1109/TSMC.1985.6313399>
- [28] P. Tsiotras, *Stabilization and optimality results for the attitude control problem*, Journal of Guidance, Control, and Dynamics, **19**(4) (1996), 772-779. <https://doi.org/10.2514/3.21698>
- [29] L. X. Wang, *A course in fuzzy systems and control*, Prentice-Hall, Inc., 1996.
- [30] C. H. Wang, H. L. Liu, T. C. Lin, *Direct adaptive fuzzy-neural control with state observer and supervisory controller for unknown nonlinear dynamical systems*, IEEE Transactions on Fuzzy Systems, **10**(1) (2002), 39-49. <https://doi.org/10.1109/91.983277>
- [31] T. F. Wiener, *Theoretical analysis of gimballess inertial reference equipment using delta-modulated instruments*, Massachusetts Institute of Technology, 1962. <http://hdl.handle.net/1721.1/14454>
- [32] B. Xiao, Z. Fu, Y. Guo, *Attitude output feedback tracking control of satellites without angular velocity observers*, in: IECON 2017-43rd Annual Conference of the IEEE Industrial Electronics Society, IEEE, (2017), 6544-6548. <https://doi.org/10.1109/IECON.2017.8217141>
- [33] B. Xiao, S. Yin, L. Wu, *A structure simple controller for satellite attitude tracking maneuver*, IEEE Transactions on Industrial Electronics, **64**(2) (2017), 1436-1446. <https://doi.org/10.1109/TIE.2016.2611576>
- [34] D. Ye, Y. Li, B. Xiao, W. Li, *Robust finite-time adaptive control algorithm for satellite fast attitude maneuver*, Journal of the Franklin Institute, **357**(16) (2020), 11558-11583. <https://doi.org/10.1016/j.jfranklin.2019.07.029>
- [35] L. You, Y. Dong, S. Zhaowei, *Time efficient sliding mode controller based on Bang–Bang logic for satellite attitude control*, Aerospace Science and Technology, **75** (2018), 342-352. <https://doi.org/10.1016/j.ast.2017.12.042>
- [36] A. A. Zhang, B. C. Lv, C. Z. Zhang, D. Z. She, *Finite time fault tolerant attitude control-based observer for a rigid satellite subject to thruster faults*, IEEE Access, **5** (2017), 16808-16817. <https://doi.org/10.1109/ACCESS.2017.2745701>
- [37] Y. Zheng, H. Zhang, Y. Wang, *Observer-based fault-tolerant fixed-time attitude tracking control for rigid spacecraft without angular velocity measurement*, Transactions of the Institute of Measurement and Control, **46**(15) (2024). <https://doi.org/10.1177/01423312241236135>
- [38] Z. Zhou, F. Zhu, D. Xu, Z. Gao, *An interval-estimation-based anti-disturbance sliding mode control strategy for rigid satellite with prescribed performance*, ISA Transactions, **105** (2020), 63-76. <https://doi.org/10.1016/j.isatra.2020.05.048>
- [39] Y. Zhu, X. Ning, S. Wang, Z. Wang, *Fixed-time fuzzy disturbance observer-based adaptive integral sliding-mode spacecraft attitude control under time-varying output constraints*, Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering, **238**(5) (2024), 779-790. <https://doi.org/10.1177/09596518231214284>