A FUZZY VERSION OF HAHN-BANACH EXTENSION THEOREM

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Abstract. In this paper, a fuzzy version of the analytic form of Hahn-Banach extension theorem is given. As application, the Hahn-Banach theorem for $r$-fuzzy bounded linear functionals on $r$-fuzzy normed linear spaces is obtained.

1. Introduction

Hahn-Banach theorem is one of the most famous and useful result in functional analysis. Ramakrishnan [15] established the norm-preserving fuzzy completion of a fuzzy normed algebra and gave a fuzzy extension of Hahn-Banach theorem. In the same year Rhie and Hwang [16] investigated the relation between fuzzy seminorms and crisp seminorms on a linear space $X$ and extended the analytic form of the Hahn-Banach theorem with the notion of fuzzy seminorm. In recent years, a fuzzy version of Hahn-Banach theorem on a vector space over the set of fuzzy real numbers and some related applications were proved by Binimol and Sunny Kuriakose [6, 7]. There are also many other fuzzy versions of Hahn-Banach theorem for fuzzy bounded linear operators on fuzzy normed spaces (see e.g. [2, 9, 12, 19] etc...).

In this paper, using the definition of fuzzy order due to L. A. Zadeh (see [21]), we assume that the set of real numbers $\mathbb{R}$ endowed with a fuzzy order $r$ instead of the natural order $\leq$ and prove a new fuzzy version of the analytic form of Hahn-Banach theorem. As application, the Hahn-Banach theorem for $r$-fuzzy bounded linear functionals on $r$-fuzzy normed linear spaces is obtained.

2. Preliminaries

We begin with a number of definitions related to fuzzy orders. We follow the notation and vocabulary of Zadeh [21] closely, and refer the reader to Amroune and Davvaz [1], Beg [3], Bernadette [4], Billot [5], Bodenhofer and et.al. [8], Kundu [10], Li and Yen [11], Ovchinnikov [13, 14], Stuti and Zedam [17], Venugopalan [18], Zadeh [21] and Zimmermann [22] for elementary definitions and facts about fuzzy order relations.

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [20] in 1965.

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Let $X$ be a nonempty set, a fuzzy subset $A$ of $X$ is characterized by its membership function $A : X \to [0, 1]$ and $A(x)$ is interpreted as the degree of membership of the element $x$ in the fuzzy subset $A$ for each $x \in X$.

In [21], Zadeh gave the following definition of fuzzy order.

**Definition 2.1.** [21] Let $X$ be a nonempty set. A Zadeh’s binary fuzzy partial order (briefly, fuzzy order) on $X$ is a fuzzy subset $r$ on $X \times X$ in which the following conditions are satisfied:

(i) for all $x \in X$, $r(x, x) = 1$, (fuzzy reflexivity);

(ii) for all $x, y \in X$, $(r(x, y) > 0$ and $x \neq y)$ implies $(r(y, x) = 0)$, (fuzzy antisymmetry);

(iii) for all $x, y, z \in X$, $r(x, z) \geq \max_{y \in X} \min \{r(x, y), r(y, z)\}$, (fuzzy transitivity).

Note that each crisp order $\leq$ on $X$ can be considered a fuzzy order defined by $r(x, y) = 1$ if $x \leq y$ and $r(x, y) = 0$ if $x$ and $y$ are incomparable elements.

A nonempty set $X$ with a fuzzy order $r$ defined on it is called fuzzy ordered set (for short, foset) and we denote it by $(X, r)$.

If $Y$ is a subset of a foset $(X, r)$, then the restriction of $r$ to $Y$ is a fuzzy order in $Y$ and is called induced fuzzy order.

A fuzzy order $r$ is linear (or total) on $X$ if for every $x, y \in X$, we have $r(x, y) > 0$ or $r(y, x) > 0$. If $x \neq y$, by the fuzzy antisymmetry of $r$, clearly only one of these conditions can be satisfied. A fuzzy ordered set $(X, r)$ in which $r$ is total is called a $r$-fuzzy chain. Conversely, if for any $x, y \in X$, $r(x, y) > 0$ if and only if $x = y$, then $(X, r)$ is called $r$-fuzzy antichain.

Next, we give some examples of fuzzy order.

**Example 2.2.** Let $X = \{a, b, c, d, e, f, g\}$. Then the fuzzy subset $r$ defined on $X \times X$ by the following table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.55</td>
<td>0.40</td>
<td>0.45</td>
<td>0.60</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.60</td>
<td>0.50</td>
<td>0.35</td>
<td>0.75</td>
</tr>
<tr>
<td>c</td>
<td>0.15</td>
<td>0</td>
<td>1</td>
<td>0.30</td>
<td>0.70</td>
<td>0.80</td>
<td>0.90</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.15</td>
<td>0</td>
</tr>
<tr>
<td>e</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.30</td>
<td>0.25</td>
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<td>f</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>g</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.20</td>
<td>1</td>
</tr>
</tbody>
</table>

is a fuzzy order on $X$.

**Example 2.3.** Let $x, y \in \mathbb{R}$. Then the fuzzy subset $r_\lambda$ defined for all $x, y \in \mathbb{R}$ by:

$$r_\lambda(x, y) = \begin{cases} 
1, & \text{if } x = y; \\
\min(1, \frac{y - x}{\lambda}), & \text{if } x < y; \\
0, & \text{if } x > y;
\end{cases}$$

is a total fuzzy order on $\mathbb{R}$.

Clearly, $0 \leq r_\lambda(x, y) \leq 1$ for all $x, y \in \mathbb{R}$. Thus $r_\lambda$ is well defined. Now let us show that $r_\lambda$ is a fuzzy order on $\mathbb{R}$.
1) For all \( x \in \mathbb{R} \), \( r_\lambda(x, x) = 1 \). Thus \( r_\lambda \) is fuzzy reflexive.

2) Let \( x, y \in \mathbb{R} \) with \( x \neq y \). Then, \( r_\lambda(x, y) > 0 \) is true only in the case \( x < y \). So, \( r_\lambda \) is fuzzy antisymmetric.

3) Let \( x, y, z \in \mathbb{R} \). Then, we have three cases to study.

3.i) If \( r_\lambda(x, z) = 1 \), then \( r_\lambda(x, z) \geq \min\{r_\lambda(x, y), r_\lambda(y, z)\} \), for all \( y \in \mathbb{R} \).

3.ii) If \( r_\lambda(x, z) = \frac{x-y}{x} > 0 \), then \( x < z \). Hence, for \( y \in \mathbb{R} \) we have three cases to consider:

(a) if \( x < z < y \), then \( r_\lambda(y, z) = 0 \).
(b) If \( x \leq y \leq z \), so \( \frac{x-y}{z} \geq \frac{y-z}{x} \). Hence, we get \( r_\lambda(x, z) \geq r_\lambda(y, z) \).
(c) If \( y < x < z \), then \( r_\lambda(x, y) = 0 \). Thus \( r_\lambda(x, z) \geq \min\{r_\lambda(x, y), r_\lambda(y, z)\} \), for all \( y \in \mathbb{R} \).

3.iii) If \( r_\lambda(x, z) = 0 \), then \( x > z \). So, for every \( y \in \mathbb{R} \) we have three cases:

(a) if \( x > z \geq y \), then \( r_\lambda(x, y) = 0 \).
(b) If \( x \geq y > z \), so \( r_\lambda(y, z) = 0 \).
(c) If \( y > x > z \), then \( r_\lambda(y, z) = 0 \).

Hence, \( r(x, z) \geq \min\{r_\lambda(x, y), r_\lambda(y, z)\} \), for all \( y \in \mathbb{R} \). Thus, \( r_\lambda \) is fuzzy transitive. Therefore, \( r_\lambda \) is a fuzzy order on \( \mathbb{R} \).

Since for all \( x, y \in \mathbb{R} \), such that \( x \neq y \) we have either \( x < y \) or \( y < x \). Then, we get either \( \min(1, \frac{y-x}{x}) > 0 \) or \( \min(1, \frac{x-y}{y}) > 0 \). Thus, \( r_\lambda \) is a total fuzzy order.

**Example 2.4.** Let \( X = \mathbb{R} \). Then, the fuzzy relation \( r \) defined for all \( x, y \in \mathbb{R} \) by:

\[
r(x, y) = \begin{cases} 
1, & \text{if } x = y; \\
0, & \text{if } x > y; \\
1 - \frac{x}{y}, & \text{if } 0 \leq x < y; \\
1 - \frac{y}{x}, & \text{if } x < y \leq 0; \\
1, & \text{if } x < 0 \text{ and } y > 0;
\end{cases}
\]

is a total fuzzy order on \( \mathbb{R} \).

Clearly, \( 0 \leq r(x, y) \leq 1 \) for all \( x, y \in \mathbb{R} \). Thus \( r \) is well defined. Now let us show that \( r \) is a fuzzy order on \( \mathbb{R} \).

1) For all \( x \in \mathbb{R} \), \( r(x, x) = 1 \). Thus \( r \) is fuzzy reflexive.

2) Let \( x, y \in \mathbb{R} \) such that \( x \neq y \). Then, we have \( r(x, y)r(y, x) = 0 \). So, \( r \) is fuzzy antisymmetric.

3) Let \( x, y, z \in \mathbb{R} \). Then, we have four cases to study.

3.i) If \( r(x, z) = 1 \), then \( r(x, z) \geq \min\{r(x, y), r(y, z)\} \), for all \( y \in \mathbb{R} \).

3.ii) If \( r(x, z) = 0 \), then \( x > z \). Hence, for every \( y \in \mathbb{R} \) we distinguish the following subcases:

(a) If \( x > z \geq y \), then it holds that \( r(x, y) = 0 \).
(b) If \( x \geq y > z \), then it holds that \( r(y, z) = 0 \).
(c) If \( y > x > z \), then it holds that \( r(y, z) = 0 \).

Thus, \( r(x, z) \geq \min\{r(x, y), r(y, z)\} \), for all \( y \in \mathbb{R} \).
3.iii) If \( r(x, z) = 1 - \frac{x}{z} \), then \( 0 \leq x < z \). Hence, for \( y \in \mathbb{R} \) we have four cases to consider:

(a) If \( 0 \leq x < z < y \), then \( r(y, z) = 0 \).

(b) If \( 0 \leq x < y < z \), then \( 1 - \frac{x}{y} \geq 1 - \frac{y}{z} \). Hence, we get \( r(x, z) \geq r(y, z) \).

(c) If \( 0 \leq y < x < z \), then \( r(x, y) = 0 \).

(d) If \( y < 0 \leq x < z \), then \( r(x, y) = 0 \).

Thus \( r(x, z) \geq \min \{ r(x, y), r(y, z) \} \), for all \( y \in \mathbb{R} \).

3.iv) If \( r(x, z) = 1 - \frac{x}{z} \), then by using a similar argument as in the case (3.iii) we can see that \( r(x, z) \geq \min \{ r(x, y), r(y, z) \} \), for all \( y \in \mathbb{R} \).

Hence, \( r \) is fuzzy transitive. Thus, \( r \) is a fuzzy order on \( \mathbb{R} \).

As for all \( x, y \in \mathbb{R} \), such that \( x \neq y \) we have either \( x < y \) or \( y < x \), then we get either \( r(x, y) = 1 - \frac{x}{y} > 0 \) or \( r(y, x) = 1 - \frac{y}{x} > 0 \). Thus, \( r \) is a total fuzzy order.

**Definition 2.5.** Let \((X, r)\) be a fuzzy ordered set and \(A\) be a subset of \(X\).

(a) An element \( u \in X \) is an \( r \)-upper bound of \( A \) if \( r(x, u) > 0 \) for all \( x \in A \). The set of all \( r \)-upper bounds of \( A \) is denoted by \( A^\uparrow \). If \( u \) is the \( r \)-upper bound of \( A \) and \( u \in A \), then \( u \) is called a greatest element of \( A \). The \( r \)-lower bound and least element are defined analogously and the set of all \( r \)-lower bounds of \( A \) is denoted by \( A^\downarrow \).

(b) An element \( m \in A \) is called a maximal element of \( A \) if there is no \( x \neq m \) in \( A \) for which \( r(m, x) > 0 \). \( x = m \). Minimal elements are defined similarly.

(c) As usual, the \( r \)-supremum of \( A \) is defined by \( \sup_r(A) \) = the least element of \( r \)-upper bounds of \( A \) (if it exists). Similarly, the \( r \)-infimum of \( A \) defined by \( \inf_r(A) \) = the greatest element of \( r \)-lower bounds of \( A \) (if it exists).

We write \( x \lor_r y \) the \( r \)-supremum and \( x \land_r y \) the \( r \)-infimum of the set \( \{x, y\}\). For linear fuzzy order, \( x \lor_r y = \max_r \{x, y\} \) and \( x \land_r y = \min_r \{x, y\} \).

**Definition 2.6.** Let \( r \) be a fuzzy order on \( \mathbb{R} \) and \( x \in \mathbb{R} \). If \( r(0, x) > 0 \), then \( x \) is called an \( r \)-positive real number. The set of them all is denoted by \( \mathbb{R}^+_r \). Similarly, if \( r(x, 0) > 0 \) then \( x \) is called an \( r \)-negative real number, and the set of them all is denoted by \( \mathbb{R}^-_r \).

**Definition 2.7.** 1) Let \( r \) be a fuzzy order on \( \mathbb{R} \). We say that \( r \) is compatible with the addition if for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\), we have

\[
(r(x_1, y_1) > 0 \text{ and } r(x_2, y_2) > 0) \implies (r(x_1 + x_2, y_1 + y_2) > 0).
\]

2) The fuzzy order \( r \) is said to be compatible with the multiplication by scalars if for all \((x, y) \in \mathbb{R}^2\) and \( \lambda > 0 \), we have

\[
(r(x, y) > 0) \implies (r(\lambda x, \lambda y) > 0).
\]

**Example 2.8.** The fuzzy order relation given in Example 2.4 is compatible with the addition and multiplication by scalars on \( \mathbb{R} \).

(i) \( r \) is compatible with the addition. Indeed, let \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) such that \( r(x_1, y_1) > 0 \) and \( r(x_2, y_2) > 0 \). By the definition of \( r \) we get that \( x_1 \leq y_1 \) and
$x_2 \leq y_2$. Then, $x_1 + x_2 \leq y_1 + y_2$. Hence, $r(x_1 + x_2, y_1 + y_2) > 0$. Thus, $r$ is compatible with the addition.

(ii) $r$ is compatible with the multiplication by scalars. Indeed, let $(x, y) \in \mathbb{R}^2$ such that $r(x, y) > 0$ and $\lambda > 0$. By the definition of $r$ we get that $x \leq y$. Then, $\lambda x \leq \lambda y$. Hence, $r(\lambda x, \lambda y) > 0$. Thus, $r$ is compatible with the multiplication by scalars.

Therefore, $r$ is compatible with the addition and multiplication by scalar on $\mathbb{R}$.

Next, we show the following two propositions which we shall need for proving a fuzzy version of Hahn-Banach theorem.

**Proposition 2.9.** Let $\mathbb{R}_r = (\mathbb{R}, r)$ be the set of all real numbers endowed with a fuzzy order $r$ compatible with the addition and the multiplication by scalar, and $x, y \in \mathbb{R}$. Then we have the following:

(i) If $r(0, x) > 0$ then $r(-x, 0) > 0$.

(ii) If $r(0, x) > 0$ then $r(-x, x) > 0$.

**Proof.** Let $x, y \in \mathbb{R}_r$. i) Since $r(0, x) > 0$ and by the fuzzy reflexivity $r(-x, -x) = 1 > 0$, then from the compatibility of $r$ with the addition we have that $r(0 + (-x), x + (-x)) > 0$. Hence, $r(-x, 0) > 0$.

ii) We assume that $r(0, x) > 0$. It is clear from (i) that $r(-x, 0) > 0$. Then, from the compatibility of $r$ with the addition we have that $r(-x, x) > 0$. \qed

**Proposition 2.10.** Let $\mathbb{R}_r = (\mathbb{R}, r)$ be the set of all real numbers endowed with a fuzzy order $r$ compatible with the addition and multiplication by scalar, and let $x, y \in \mathbb{R}$ such that $x \neq y$. Then the following are equivalent.

(i) $r(x, y) > 0$;

(ii) There exists $\tau \in \mathbb{R}$ such that $r(x, \tau) > 0$ and $r(\tau, y) > 0$, (r-fuzzy density).

**Proof.** Let $x, y \in \mathbb{R}_r$ such that $x \neq y$ and $r(x, y) > 0$. For the one direction, let $\tau = \frac{x + y}{2}$. Since $r(x, x) = 1 > 0$ and $r(x, y) > 0$, from the compatibility of $r$ with the addition we get that $r(x + x, x + y) > 0$.

Now, by the compatibility of $r$ with the multiplication we obtain that

$$r(x, \frac{x + y}{2}) > 0.$$  

Thus, $r(x, \tau) > 0$.

In the same way we get that $r(\tau, y) > 0$.

The other direction follows directly from the fuzzy transitivity. \qed

3. Results

In this section we assume that $\mathbb{R}_r$ is the set of real numbers $\mathbb{R}$ endowed with a fuzzy order $r$ compatible with the addition and multiplication by scalar instead of the natural order $\leq$ and we shall prove a fuzzy version of Hahn-Banach extension theorem. The prove of this fuzzy version will follow the same steps as the crisp case. As application, we define the notion of $r$-fuzzy normed space with the help of $r$-fuzzy norm as a generalization of crisp normed space, we introduce the notion
of $r$-fuzzy bounded linear functional and we prove the Hahn-Banach theorem for $r$-fuzzy bounded linear functionals on $r$-fuzzy normed linear spaces.

**Definition 3.1.** Let $X$ be an real linear space, and $T$ a mapping of $X$ into $\mathbb{R}_r$. We say that $T$ is a $r$-fuzzy sublinear functional on $X$ if

i) $r(T(x + y), T(x) + T(y)) > 0$ for all $x, y \in X$, ($r$-subadditivity);

ii) $T(\lambda x) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}_r^+$, ($r$-positively homogeneous).

**Example 3.2.** The mapping $T : \mathbb{R}_r \to \mathbb{R}_r$ defined by $T(x) = |x|_r = \max_r\{x, -x\}$ is an $r$-fuzzy sublinear functional on $\mathbb{R}_r$.

The following is a useful fact for $r$-fuzzy sublinear functionals.

**Proposition 3.3.** If $T$ is an $r$-fuzzy sublinear functional on a real linear space $X$ then $r(\lambda T(x), T(\lambda x)) > 0$, for all $x \in X$ and $\lambda \in \mathbb{R}_r$.

**Proof.** Let $x \in X$ and $\lambda \in \mathbb{R}_r$. If $\lambda \in \mathbb{R}_r^+$ we have $T(\lambda x) = \lambda T(x)$. Hence,

$$r(\lambda T(x), T(\lambda x)) = 1 > 0. \quad (1)$$

If $\lambda \in \mathbb{R}_r^-$, then from Proposition 2.9(i) we get that $-\lambda \in \mathbb{R}_r^+$. As $\lambda T(x) = -(\lambda T(x))$ so by the $r$-positively homogeneous of $T$ we have $\lambda T(x) = -(\lambda T(x)) = -T(\lambda x)$. On the other hand, since $T(\lambda x - \lambda x) = T(0) = 0$, by the $r$-subadditivity of $T$ we have $r(T(\lambda x + (-\lambda x)), T(\lambda x) + T(-\lambda x)) > 0$. Hence, $r(0, T(\lambda x) + T(-\lambda x)) > 0$. Now, from the compatibility of $r$ with the addition we have $r(-T(\lambda x), T(\lambda x)) > 0$. Thus,

$$r(\lambda T(x), T(\lambda x)) > 0. \quad (2)$$

Therefore, (1) and (2) implies that $r(\lambda T(x), T(\lambda x)) > 0$, for all $x \in X$ and $\lambda \in \mathbb{R}_r$. □

**Theorem 3.4** (Fuzzy version of Hahn-Banach theorem). Let $X_0$ be a subspace of a real linear space $X$, $T$ a $r$-fuzzy sublinear functional on $X$, and $u_0$ be an linear functional on $X_0$ such that $r(u_0(x), T(x)) > 0$ for all $x \in X_0$. Then there exists a linear functional $u$ on $X$ extends $u_0$ to $X$ and satisfies $r(u(x), T(x)) > 0$, for all $x \in X$.

**Proof.** Let $y \in X$ such that $y \notin X_0$ and denote by $Y$ the vector subspace generated by $X_0 \cup \{y\}$, so

$$Y = \{x_0 + \lambda y \mid x_0 \in X_0 \text{ and } \lambda \in \mathbb{R}_r - \{0\}\}$$

Let $\tau \in \mathbb{R}_r$, and provisionally define

$$u(x_0 + \lambda y) = u_0(x_0) + \lambda \tau.$$

It is easy to show that $u$ is a linear extension of $u_0$ to $Y$; hence it remains to choose $\tau \in \mathbb{R}_r$ such that for all $x_0 \in X_0$, and $\lambda \in \mathbb{R}_r - \{0\}$,

$$r(u_0(x_0) + \lambda \tau, T(x_0 + \lambda y)) > 0. \quad (3)$$

For all $\lambda \in \mathbb{R}_r^+ - \{0\}$, replacing $x_0$ by $\lambda x_0$, using the $r$-positive homogeneity of $T$, and from the compatibility of $r$ with the multiplication, it suffices to see that

$$r(u_0(x_0) + \tau, T(x_0 + y)) > 0. \quad (4)$$
Therefore, from the \( r \)-fuzzy compatibility of \( r \) with the addition, it suffices to see that
\[
 r(u_0(x_0) + \tau, T(x_0 + y)) > 0. 
\]

For all \( \lambda \in \mathbb{R}^+_r - \{0\} \), replacing \( x_0 \) by \( \lambda x_0 \), using the \( r \)-positive homogeneity of \( T \), and from the compatibility of \( r \) with the multiplication, we observe that it suffices to see that
\[
 r(-u_0(x_0) - \tau, T(-x_0 - y)) > 0 
\]
\[
(5) 
\]
Therefore, from the \( r \)-fuzzy compatibility of \( r \) with the addition, it suffices to see that
\[
 r(-u_0(x_0) - T(-x_0 - y), \tau) > 0. 
\]
\[
(7) 
\]
To see the existence of \( \tau \in \mathbb{R}_r \) satisfying (5) and (6), start by observing that
\[
 r(-u_0(x_0) - T(-x_0 - y), T(x_0 + y) - u_0(x_0)) \geq \min\{r(-u_0(x_0), -u_0(x_0))\}, 
\]
\[
r(-T(-x_0 - y), T(x_0 + y)) \geq \min\{1, r(-T(-x_0 - y), T(x_0 + y))\}. 
\]
In addition, from Proposition 3.3 we have \( r(-T(-x_0 - y), T(x_0 + y)) > 0 \).
Then \( r(-u_0(x_0) - T(-x_0 - y), T(x_0 + y) - u_0(x_0)) > 0 \),
and therefore by Proposition 2.10 there exists \( \tau \in \mathbb{R}_r \) satisfies (5) and (6). Hence, there exists \( \tau \in \mathbb{R}_r \) satisfies (3).
Now, an application of Zorn’s Lemma complete the proof. \( \square \)

Next, we shall give an application of \( r \)-fuzzy Hahn-Banach theorem, but in this subsection, we assume that \( r \) is linear order on \( \mathbb{R} \) compatible with the addition and multiplication.

**Definition 3.5.** Let \( X \) be a real linear space. An \( r \)-fuzzy norm on \( X \) is a mapping \( x \mapsto \|x\|_r \) from \( X \) into \( \mathbb{R}^+_r \) such that for all \( x, y \in X \) and \( \lambda \in \mathbb{R}^+_r \), the following properties hold:

i) \( \|x\|_r = 0 \) if and only if \( x = 0 \).
ii) \( \|\lambda x\|_r = |\lambda|_r \|x\|_r \).
iii) \( r(\|x + y\|_r, \|x\|_r + \|y\|_r) > 0 \).

A linear space \( X \) equipped with an \( r \)-fuzzy norm \( \|\|_r \) is called an \( r \)-fuzzy normed linear space. We denote it by \( (X, \|\|_r) \).

**Example 3.6.** The \( r \)-fuzzy absolute value \( |x|_r = x \lor_r (-x) \) is an \( r \)-fuzzy norm on \( \mathbb{R}_r \).

i) Let \( x \in \mathbb{R}_r \), since \( r \) is a total order we have either \( r(0, x) > 0 \) or \( r(0, -x) > 0 \).
Then by Proposition 2.9(ii) we have either \( r(-x, x) > 0 \) or \( r(x, -x) > 0 \).
Hence, \( r(0, |x|_r) > 0 \).

ii) Obvious.

iii) \( \|\lambda x\|_r = \lambda x \lor_r (-\lambda x) = |\lambda|_r x \lor_r (-|\lambda|_r) = |\lambda|_r x \lor_r (-x) = |\lambda|_r \|x\|_r \).

iv) Let \( x, y \in \mathbb{R}_r \). To prove that \( r(\|x + y\|_r, \|x\|_r + \|y\|_r) > 0 \) six cases are considered.

a) If \( r(0, x) > 0 \) and \( r(0, y) > 0 \) then
\[
r(\|x + y\|_r, \|x\|_r + \|y\|_r) = r(x + y, x + y) = r > 0. 
\]
b) If \( r(x, 0) > 0 \) and \( r(y, 0) > 0 \) then
\[
r(|x + y|_r, |x|_r + |y|_r) = r(-x - y, -x - y) = r > 0.
\]
c) If \( r(0, x) > 0 \), \( r(y, 0) > 0 \) and \( r(x, -y) > 0 \) then
\[
r(|x + y|_r, |x|_r + |y|_r) = r(-x - y, x - y)
\]
\[
\geq \min\{r(-x, x), r(-y, -y)\} > 0.
\]
d) If \( r(0, x) > 0 \), \( r(y, 0) > 0 \) and \( r(-y, x) > 0 \) then
\[
r(|x + y|_r, |x|_r + |y|_r) = r(x + y, x - y)
\]
\[
\geq \min\{r(x, x), r(y, -y)\} > 0.
\]
e) If \( r(x, 0) > 0 \), \( r(0, y) > 0 \) and \( r(y, -x) > 0 \) then
\[
r(|x + y|_r, |x|_r + |y|_r) = r(-x - y, -x + y)
\]
\[
\geq \min\{r(-x, -x), r(-y, y)\} > 0.
\]
f) If \( r(x, 0) > 0 \), \( r(0, y) > 0 \) and \( r(-x, y) > 0 \) then
\[
r(|x + y|_r, |x|_r + |y|_r) = r(x + y, -x + y)
\]
\[
\geq \min\{r(x, -x), r(y, y)\} > 0.
\]

**Definition 3.7.** Let \((X, \|\cdot\|_r)\) and \((Y, \|\cdot\|_r)\) be \(r\)-fuzzy normed linear spaces. A linear operator \(u\) from \(X\) into \(Y\) is called an \(r\)-fuzzy bounded linear operator if there exists \(K \in \mathbb{R}_r^+\) such that
\[
r(\|u(x)\|_r, K\|x\|_r) > 0, \quad \text{for all } x \in X.
\]

**Remark 3.8.** The \(r\)-fuzzy norms in \(X\) and \(Y\) are different. But we use same notation \(\|\cdot\|_r\), because there is no confusion.

**Example 3.9.** Let \((X, \|\cdot\|_r)\) be an \(r\)-fuzzy normed linear space, we define an operator \(u : (X, \|\cdot\|_r) \rightarrow (Y, \|\cdot\|_r)\) by \(u(x) = \lambda x\) where \(\lambda \neq 0 \in \mathbb{R}\) is fixed. Clearly \(u\) is an \(r\)-fuzzy bounded linear operator.

In the following Lemma we describe the \(r\)-fuzzy boundedness of a linear operator between \(r\)-fuzzy normed linear spaces by means of an \(r\)-fuzzy norm of it.

**Lemma 3.10.** Let \(u\) be an \(r\)-fuzzy bounded linear operator from \((X, \|\cdot\|_r)\) into \((Y, \|\cdot\|_r)\). Then there exists an \(r\)-fuzzy norm of \(u\), denoted by \(\|u\|_r\), such that:
\[
r(\|u(x)\|_r, K\|x\|_r) > 0, \quad \text{for all } x \in X.
\]

**Proof.** Since \(u\) is an \(r\)-fuzzy bounded linear operator, there exists \(K \in \mathbb{R}_r^+\) such that
\[
r(\|u(x)\|_r, K\|x\|_r) > 0, \quad \text{for all } x \in X.
\]
From the compatibility of \(r\) with the multiplication we obtain
\[
r(\|u(x)\|_r, K\|x\|_r) > 0, \quad \text{for all } x \in X.
\]
Hence,
\[
r(\sup_r\{\|u(x)\|_r : x \in X\}, K) > 0, \quad \text{for all } x \in X.
\]
This means that \(\sup_r(\|u(x)\|_r)\) is finite.
Now we put \( \|u\|_r = \sup_r \{ \frac{\|u(x)\|_r}{\|x\|_r} : x \in X \} \). It is clear that \( \|u\|_r = 0 \) if and only if \( u = 0 \), and that \( \|\lambda u\|_r = |\lambda|_r \|u\|_r \). Since
\[
r(\|u + v(x)\|_r, \|u(x)\|_r + \|v(x)\|_r) > 0, \quad \text{for all } x \in X,
\]
it follows from the compatibility of \( r \) with the multiplication that
\[
r(\|u + v(x)\|_r, \|u(x)\|_r + \|v(x)\|_r) > 0, \quad \text{for all } x \in X.
\]
Then we obtain
\[
r(\|u + v\|_r, \|u\|_r + \|v\|_r) > 0.
\]
Hence, \( \|u\|_r \) is a \( r \)-fuzzy norm of \( u \).

In addition, as \( \|u\|_r = \sup_r \{ \frac{\|u(x)\|_r}{\|x\|_r} : x \in X \} \), we get
\[
r\left(\frac{\|u(x)\|_r}{\|x\|_r}, \|u\|_r\right) > 0, \quad \text{for all } x \in X,
\]
which implies
\[
r\left(\|u(x)\|_r, \|u\|_r \|x\|_r\right) > 0, \quad \text{for all } x \in X.
\]

Theorem 3.11. Let \( X_0 \) be a subspace of an \( r \)-fuzzy normed linear space \( X \), and \( u_0 \) be an \( r \)-fuzzy bounded linear functional on \( X_0 \). Then there exists an \( r \)-fuzzy bounded linear functional \( u \) on \( X \) such that \( u(x) = u_0(x) \) for all \( x \in X_0 \) and \( \|u\|_r = \|u_0\|_r \).

Proof. \( T(x) = \|u_0\|_r \|x\|_r \). It is easy to see that \( T(x) \) is an \( r \)-fuzzy sublinear functional on \( X \). Since \( u_0 \) is an \( r \)-fuzzy bounded linear functional on \( X_0 \), we obtain for all \( x \in X_0 \), that
\[
r(u_0(x), T(x)) = r(u_0(x), \|u_0\|_r \|x\|_r) > 0.
\]
Then from the \( r \)-fuzzy Hahn-Banach theorem there exists a linear functional \( u \) on \( X \) extends \( u_0 \) to \( X \) and satisfies \( r(u(x), \|u_0\|_r \|x\|_r) > 0 \), for all \( x \in X \). Moreover, for all \( x \in X \) we have
\[
r(u(-x), \|u_0\|_r \|x\|_r) > 0.
\]
This shows that
\[
r(-u(x), \|u_0\|_r \|x\|_r) > 0.
\]
Hence,
\[
r(|u(x)|_r, \|u_0\|_r \|x\|_r) > 0.
\]
Therefore, \( u \) is an \( r \)-fuzzy bounded linear functional on \( X \) and satisfies
\[
r(\|u\|_r, \|u_0\|_r) > 0.
\]
But \( u \) extends \( u_0 \), so \( r(\|u_0\|_r, \|u\|_r) > 0 \) and therefore \( \|u\|_r = \|u_0\|_r \). \( \square \)
REFERENCES


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