Iranian Journal of Fuzzy Systems **Vol. 11, No. 2,** (2014) pp. 17-25

THE INCLUSION-EXCLUSION PRINCIPLE FOR IF-STATES

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Abstract. Applying two definitions of the union of IF-events, P. Grzegorzewski gave two generalizations of the inclusion-exclusion principle for IF-events. In this paper we prove an inclusion-exclusion principle for IF-states based on a method which can also be used to prove Grzegorzewski's inclusion-exclusion principle for probabilities on IF-events. Finally, we give some applications of this principle by extending some results regarding the classical probabilities to the case of the IF-states.

1. Introduction

In science as well as in decision making, the information available is always incomplete, and information processing deals with uncertainty information (fuzziness, randomness, vagueness, etc.). A generalized theory of uncertainty was developed in [17] and it was also studied in [18]. In recent years, many approaches of new probability models try to describe the uncertainty. L. Zadeh was the first to define the fuzzy events and to develop a probability theory for fuzzy events ([16]).
Intuitionistic Fuzzy Sets (IFS), introduced and studied by Atanassov ([1], [2]), are an extension of fuzzy set theory in which not only a membership degree is given, but also a non-membership degree, which is more or less independent. Recently, the intuitionistic fuzzy sets have been applied to develop theories modelling imprecision and pattern recognition ([8]). Various concepts of probability on IFS have been proved to be very successful for minimizing the uncertainty of initial information which involves human judgement.
In [5] and [4] a general form of probabilities on IFS is given and a new representation theorem is proved for the so-called φ-probabilities including a large variety of special cases.
Using the additivity based on the Łukasiewicz connectives, in [13], B. Riečan gave an axiomatic characterization of a probability on IFS-events and proved in [14] a representation theorem for it.
Another approach of the probability on IF-events was defined in [6] using *Gödel* connectives. Applying two definitions of the union of IF-events, P. Grzegorzewski generalized in [7] the classical inclusion-exclusion principle for the case of IF-events.
In this paper we prove an inclusion-exclusion principle for IF-states based on a

Received: November 2012; Revised: February 2013; Accepted: December 2013
Key words and phrases: IF-set, IF-event, IFS-probability, IF-state, Inclusion-exclusion principle, Boole inequality, Bonferroni inequality, Łukasiewicz connectives.

method which can also be used to prove Grzegorzewski's inclusion-exclusion principle for probabilities on IF-events. Finally, we give some applications of this principle by extending some results regarding the classical probabilities to the case of the IF-states.

2. Preliminaries

Let $(\Omega, \mathcal{S}, P)$ be a classical probability space, where Ω is a universe of discourse, $\mathcal{S}$ is a σ-field of subsets of Ω and P is a probability measure over Ω.
An *IF-set* A in Ω is given by an ordered triple $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in \Omega\}$, where $\mu_A, \nu_A : \Omega \to [0,1]$ satisfy the condition $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in \Omega$.
The functions μ_A and ν_A are respectively called the *membership* and the *non-membership* functions.
For every $x \in \Omega$, let $\theta_A(x) = 1 - \mu_A(x) - \nu_A(x)$.
Then θ determines the *degree of uncertainty* and $\theta_A(x)$ is called *IF-index* of x in A. An *IF-event* A is defined as an IF-subset of Ω such that μ_A, ν_A are Borel measurable.
We will denote by $IFS(\Omega)$ the family of all IF-sets in Ω and by $\mathcal{F}$ the family of all IF-events. We also define:

$$0_\Omega = \{(x, 0, 1) \mid x \in \Omega\}, 1_\Omega = \{(x, 1, 0) \mid x \in \Omega\}.$$

Since an IF-event A is well defined by the functions μ_A, ν_A, we will use the notation $A = (\mu_A, \nu_A)$. Therefore $0_\Omega = (0, 1)$ and $1_\Omega = (1, 0)$.

The basic operations on the IF-events A, B are defined by Atanassov in [1] and [2] as follows:

$$A \subseteq B \text{ iff } \mu_{\mathrm{A}} \leq \mu_{\mathrm{B}} \text{ and } \nu_{\mathrm{A}} \geq \nu_{\mathrm{B}}$$
$$A = B \text{ iff } \mu_{\mathrm{A}} = \mu_{\mathrm{B}} \text{ and } \nu_{\mathrm{A}} = \nu_{\mathrm{B}}$$
$$A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$$
$$A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$
$$A \oplus B = (\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B)$$
$$A \odot B = (\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B)$$
$$A^c = (\nu_A, \mu_A).$$

The following operations were used by Riečan in [12]-[13]:

$$(\mu_A, \nu_A) \oplus_{\mathrm{L}} (\mu_B, \nu_B) = (\mu_A \oplus_{\mathrm{L}} \mu_B, \nu_A \odot_{\mathrm{L}} \nu_B)$$
$$(\mu_A, \nu_A) \odot_{\mathrm{L}} (\mu_B, \nu_B) = (\mu_A \odot_{\mathrm{L}} \mu_B, \nu_A \oplus_{\mathrm{L}} \nu_B),$$

where $x \oplus_{\mathrm{L}} y = (x + y) \wedge 1$, $x \odot_{\mathrm{L}} y = (x + y - 1) \vee 0$ are the Łukasiewicz connectives.

The classical inclusion-exclusion principle for measures states that the equality

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + \sum_{i<j<k} \mu(A_i \cap A_j \cap A_k) - \cdots$$
$$+ (-1)^{n+1} \mu(A_1 \cap A_2 \cap \cdots \cap A_n)$$

holds for any sequence $(A_i)_{i=1}^n$ from the domain of a measure μ.

Grzegorzewski defined in [7] the probability of the IF-event A to be a number in the interval $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ with $\mathcal{P}^\flat(A) = \int_\Omega \mu_A dP$, $\mathcal{P}^\sharp(A) = 1 - \int_\Omega \nu_A dP$. If $\theta_A = 0$, then $\mathcal{P}^\flat(A) = \mathcal{P}^\sharp(A)$ and the probability of the IF-event A reduces to the probability of a fuzzy set in the sense of Zadeh's definition ([16]).
Based on this concept of probability on IF-events, Grzegorzewski proved two versions of the inclusion-exclusion principle ([7]).
In [10], Kuková and Navara investigated the forms of the inclusion-exclusion principle that hold for different types of fuzzy set operations. They proved that the only continuous fuzzy operations which satisfy the inclusion-exclusion principle are the Gödel ones, the product operations and some of their ordinal sums.

The notion of an *L-state* have been introduced in [15] for the case of the pair of operations $(\oplus_L, \odot_L)$ and it was proved in [14] that for an L-state $m : \mathcal{F} \longrightarrow [0,1]$, there exists $\alpha \in [0,1]$ such that $m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$.
In the same manner we define the notion of an IF-state for the case of the pair of operations $(\oplus, \odot)$.

If $A_n = (\mu_{A_n}, \nu_{A_n})$ and $A = (\mu_A, \nu_A)$ we will write $A_n \nearrow A$ if $\mu_{A_n}(\omega) \nearrow \mu_A(\omega)$ and $\nu_{A_n}(\omega) \searrow \nu_A(\omega)$ for all $\omega \in \Omega$.

Definition 2.1. A mapping $m : \mathcal{F} \longrightarrow [0,1]$ is an *IF-state*, if the following properties are satisfied:
(1) $m(1_\Omega) = 1$, $m(0_\Omega) = 0$;
(2) $m(A \oplus B) = m(A) + m(B) - m(A \odot B)$ for all $A, B \in \mathcal{F}$;
(3) $A_n \nearrow A$ implies $m(A_n) \nearrow m(A)$.

The next result can be proved in a similar way as in [14].

Proposition 2.2. *Let $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ be the Grzegorzewski's probability where $\mathcal{P}^\flat(A) = \int_\Omega \mu_A dP$ and $\mathcal{P}^\sharp(A) = 1 - \int_\Omega \nu_A dP$. If $\alpha \in [0,1]$, then the mapping $m : \mathcal{F} \longrightarrow [0,1]$ defined by $m(A) = (1-\alpha)\mathcal{P}^\flat(A) + \alpha\mathcal{P}^\sharp(A)$ is an IF-state.*

Theorem 2.3. [9] *Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 1, \ldots, n$ and let m be an IF-state such that $m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$. Then m satisfies the inclusion-exclusion principle:*

$$m(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n m(A_i) - \sum_{i<j} m(A_i \cap A_j) + \sum_{i<j<k} m(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} m(\bigcap_{i=1}^n A_i).$$

3. Inclusion-exclusion Principle for IF-states

In this section we prove an inclusion-exclusion principle for IF-states based on a method which can also be used to prove Grzegorzewski's inclusion-exclusion principle for probabilities on IF-events.
For the IF-events $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ we will use the notations:

$$A + B = (\mu_A + \mu_B, \nu_A + \nu_B), \;\; A - B = (\mu_A - \mu_B, \nu_A - \nu_B),$$

$$\alpha A = (\alpha\mu_A, \alpha\mu_A), \; \alpha \in \mathbb{R}.$$

Note that $\oplus$ is defined between two IF-events, but, in general, $\sum_{i=1}^n A_i$ is not an IF-event. The same mention for $A - B$ and αA.

Lemma 3.1. *Let A_i, B be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i}),\ i = 1, \ldots, n$, $B = (\mu_B, \nu_B)$. Then:*

$$(\sum_{i=1}^{n} A_i) \oplus B = \sum_{i=1}^{n} (A_i \oplus B) - (n-1)(\mu_B, 0).$$

Proof. For $n = 1$ it is obvious.
If $n = 2$, then we have:
$(A_1 + A_2) \oplus B = (\mu_{A_1} + \mu_{A_2}, \nu_{A_1} + \nu_{A_2}) \oplus (\mu_B, \nu_B) = (\mu_{A_1} + \mu_{A_2} + \mu_B - \mu_{A_1}\mu_B - \mu_{A_2}\mu_B, \nu_{A_1}\nu_B + \nu_{A_2}\nu_B)$.
$A_1 \oplus B + A_2 \oplus B = (\mu_{A_1} + \mu_B - \mu_{A_1}\mu_B, \nu_{A_1}\nu_B) + (\mu_{A_2} + \mu_B - \mu_{A_2}\mu_B, \nu_{A_2}\nu_B)$
$= (\mu_{A_1} + \mu_{A_2} + 2\mu_B - \mu_{A_1}\mu_B - \mu_{A_2}\mu_B, \nu_{A_1}\nu_B + \nu_{A_2}\nu_B) = (A_1 + A_2) \oplus B + (\mu_B, 0)$.
It follows that $(A_1 + A_2) \oplus B = A_1 \oplus B + A_2 \oplus B - (\mu_B, 0)$.
Assume $(\sum_{i=1}^{n} A_i) \oplus B = \sum_{i=1}^{n} (A_i \oplus B) - (n-1)(\mu_B, 0)$ and we have:
$(\sum_{i=1}^{n+1} A_i) \oplus B = (\sum_{i=1}^{n} A_i + A_{n+1}) \oplus B = (\sum_{i=1}^{n} A_i) \oplus B + A_{n+1} \oplus B - (\mu_B, 0)$
$= \sum_{i=1}^{n} (A_i \oplus B) - (n-1)(\mu_B, 0) + A_{n+1} \oplus B - (\mu_B, 0) = \sum_{i=1}^{n+1} (A_i \oplus B) - n(\mu_B, 0)$,
which proves the assertion, according to the induction principle. □

Lemma 3.2. *Let A_i, B be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i}),\ i = 1, \ldots, n$, $B = (\mu_B, \nu_B)$.*
(1) *If $n = 2k$, then*
$(A_1 - A_2 + A_3 - A_4 + \cdots - A_{2k}) \oplus B = A_1 \oplus B - A_2 \oplus B + \cdots - A_{2k} \oplus B + (\mu_B, 0)$;
(2) *If $n = 2k + 1$, then*
$(A_1 - A_2 + A_3 - A_4 + \cdots - A_{2k} + A_{2k+1}) \oplus B = A_1 \oplus B - A_2 \oplus B + \cdots - A_{2k} \oplus B + A_{2k+1} \oplus B$.

Proof. The proof can be done by induction similarly as in Lemma 3.1. □

Theorem 3.3. *Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i}),\ i = 1, \ldots, n$. Then:*

$$\bigoplus_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i - \sum_{i<j}^{n} (A_i \odot A_j) + \cdots + (-1)^{n+1} \bigodot_{i=1}^{n} A_i.$$

Proof. If $n = 1$, it is obvious.
For $n = 2$ we have:
$A_1 + A_2 - A_1 \odot A_2 = (\mu_{A_1}, \nu_{A_1}) + (\mu_{A_2}, \nu_{A_2}) - (\mu_{A_1}\mu_{A_2}, \nu_{A_1} + \nu_{A_2} - \nu_{A_1}\nu_{A_2})$
$= (\mu_{A_1} + \mu_{A_2} - \mu_{A_1}\mu_{A_2}, \nu_{A_1} + \nu_{A_2} - \nu_{A_1} - \nu_{A_2} + \nu_{A_1}\nu_{A_2})$
$= (\mu_{A_1} + \mu_{A_2} - \mu_{A_1}\mu_{A_2}, \nu_{A_1}\nu_{A_2}) = A_1 \oplus A_2$.
Thus $A_1 \oplus A_2 = A_1 + A_2 - A_1 \odot A_2$.

Suppose $n = 2m$. We have:
$\bigoplus_{i=1}^{2m} A_i = (\bigoplus_{i=1}^{2m-1} A_i) \oplus A_{2m} = (\sum_{i_1=1}^{2m-1} A_{i_1} - \sum_{i_1<i_2}^{2m-1} A_{i_1} \odot A_{i_2}$

$+ \sum_{i_1<i_2<i_3}^{2m-1} A_{i_1} \odot A_{i_2} \odot A_{i_3} - \cdots + A_1 \odot A_2 \odot \cdots \odot A_{2m-1}) \oplus A_{2m}$

$= (\sum_{i_1=1}^{2m-1} A_{i_1}) \oplus A_{2m} - (\sum_{i_1<i_2}^{2m-1} A_{i_1} \odot A_{i_2}) \oplus A_{2m}$

$+(\sum_{i_1<i_2<i_3}^{2m-1} A_{i_1} \odot A_{i_2} \odot A_{i_3}) \oplus A_{2m} - \cdots + (A_1 \odot A_2 \odot \cdots \odot A_{2m-1}) \oplus A_{2m}$

$= \sum_{i_1=1}^{2m-1} (A_{i_1} \oplus A_{2m}) - (\binom{2m-1}{1} - 1)(\mu_{A_{2m}}, 0)$

$$
\begin{aligned}
&-\sum_{i_1<i_2}^{2m-1}(A_{i_1}\odot A_{i_2})\oplus A_{2m}+(\binom{2m-1}{2}-1)(\mu_{A_{2m}},0)\\
&+\sum_{i_1<i_2<i_3}^{2m-1}(A_{i_1}\odot A_{i_2}\odot A_{i_3})\oplus A_{2m}-(\binom{2m-1}{3}-1)(\mu_{A_{2m}},0)\\
&\vdots\\
&-\sum_{i_1<i_2<\cdots<i_{2m-2}}^{2m-1}(A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-2}})\oplus A_{2m}+(\binom{2m-1}{2m-2}-1)(\mu_{A_{2m}},0)\\
&+(A_1\odot A_2\odot\cdots\odot A_{2m-1})\oplus A_{2m}\\
&=\sum_{i_1=1}^{2m-1}(A_{i_1}+A_{2m}-A_{i_1}\odot A_{2m})-(\binom{2m-1}{1}-1)(\mu_{A_{2m}},0)\\
&-\sum_{i_1<i_2}^{2m-1}(A_{i_1}\odot A_{i_2}+A_{2m}-A_{i_1}\odot A_{i_2}\odot A_{2m})+(\binom{2m-1}{2}-1)(\mu_{A_{2m}},0)\\
&+\sum_{i_1<i_2<i_3}^{2m-1}(A_{i_1}\odot A_{i_2}\odot A_{i_3}+A_{2m}-A_{i_1}\odot A_{i_2}\odot A_{i_3}\odot A_{2m})\\
&-(\binom{2m-1}{3}-1)(\mu_{A_{2m}},0)\\
&\vdots\\
&-\sum_{i_1<i_2<\cdots<i_{2m-2}}^{2m-1}(A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-2}}+A_{2m}\\
&-A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-2}}\odot A_{2m})+(\binom{2m-1}{2m-2}-1)(\mu_{A_{2m}},0)\\
&+A_1\odot A_2\odot\cdots\odot A_{2m-1}+A_{2m}-A_1\odot A_2\odot\cdots\odot A_{2m-1}\odot A_{2m}\\
&=\sum_{i_1=1}^{2m-1}A_{i_1}-\sum_{i_1<i_2}^{2m}A_{i_1}\odot A_{i_2}+\sum_{i_1<i_2<i_3}^{2m}A_{i_1}\odot A_{i_2}\odot A_{i_3}-\cdots\\
&+\sum_{i_1<i_2<\cdots<i_{2m-1}}^{2m}A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-1}}-A_1\odot A_2\odot\cdots\odot A_{2m-1}\odot A_{2m}\\
&+[\binom{2m-1}{1}-\binom{2m-1}{2}+\binom{2m-1}{3}-\cdots-\binom{2m-1}{2m-2}+1]A_{2m}\\
&-[\binom{2m-1}{1}-\binom{2m-1}{2}+\binom{2m-1}{3}-\cdots-\binom{2m-1}{2m-2}](\mu_{A_{2m}},0)\\
&=\sum_{i_1=1}^{2m}A_{i_1}-\sum_{i_1<i_2}^{2m}A_{i_1}\odot A_{i_2}+\sum_{i_1<i_2<i_3}^{2m}A_{i_1}\odot A_{i_2}\odot A_{i_3}-\cdots\\
&+\sum_{i_1<i_2<\cdots<i_{2m-1}}^{2m}A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-1}}-A_1\odot A_2\odot\cdots\odot A_{2m-1}\odot A_{2m}\\
&-[1-\binom{2m-1}{1}+\binom{2m-1}{2}-\binom{2m-1}{3}+\cdots+\binom{2m-1}{2m-2}\\
&-\binom{2m-1}{2m-1}]A_{2m}+[1-\binom{2m-1}{1}+\binom{2m-1}{2}-\binom{2m-1}{3}+\cdots+\binom{2m-1}{2m-2}\\
&-\binom{2m-1}{2m-1}](\mu_{A_{2m}},0)\\
&=\sum_{i_1=1}^{2m}A_{i_1}-\sum_{i_1<i_2}^{2m}A_{i_1}\odot A_{i_2}+\sum_{i_1<i_2<i_3}^{2m}A_{i_1}\odot A_{i_2}\odot A_{i_3}-\cdots\\
&+\sum_{i_1<i_2<\cdots<i_{2m-1}}^{2m}A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-1}}-A_1\odot A_2\odot\cdots\odot A_{2m-1}\odot A_{2m}\\
&-(1-1)^{2m-1}A_{2m}+(1-1)^{2m-1}(\mu_{A_{2m}},0)\\
&=\sum_{i_1=1}^{2m}A_{i_1}-\sum_{i_1<i_2}^{2m}A_{i_1}\odot A_{i_2}+\sum_{i_1<i_2<i_3}^{2m}A_{i_1}\odot A_{i_2}\odot A_{i_3}-\cdots\\
&+\sum_{i_1<i_2<\cdots<i_{2m-1}}^{2m}A_{i_1}\odot A_{i_2}\odot\cdots\odot A_{i_{2m-1}}-A_1\odot A_2\odot\cdots\odot A_{2m-1}\odot A_{2m}.
\end{aligned}
$$

Similarly, for $n=2m+1$ we get:

$\bigoplus_{i=1}^{2m+1} A_i = \sum_{i_1=1}^{2m+1} A_{i_1} - \sum_{i_1<i_2}^{2m+1} A_{i_1} \odot A_{i_2} + \sum_{i_1<i_2<i_3}^{2m+1} A_{i_1} \odot A_{i_2} \odot A_{i_3} - \cdots$
$- \sum_{i_1<i_2<\cdots<i_{2m}}^{2m+1} A_{i_1} \odot A_{i_2} \odot \cdots \odot A_{i_{2m}} + A_1 \odot A_2 \odot \cdots \odot A_{2m} \odot A_{2m+1}.$

We conclude that

$$\bigoplus_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i - \sum_{i<j}^{n} (A_i \odot A_j) + \cdots + (-1)^{n+1} \bigodot_{i=1}^{n} A_i. \qquad \square$$

Corollary 3.4. *Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 1, \ldots, n$.*
If $A = \bigoplus_{i=1}^{n} A_i$, then:
$\mu_A = \sum_{i=1}^{n} \mu_{A_i} - \sum_{i<j}^{n} \mu_{A_i} \odot \mu_{A_j} + \cdots + (-1)^{n+1} \mu_{A_1} \odot \mu_{A_2} \odot \cdots \odot \mu_{A_n}$
$\nu_A = \sum_{i=1}^{n} \nu_{A_i} - \sum_{i<j}^{n} \nu_{A_i} \oplus \nu_{A_j} + \cdots + (-1)^{n+1} \nu_{A_1} \oplus \nu_{A_2} \oplus \cdots \oplus \nu_{A_n},$
where $x \oplus y = x + y - xy$ and $x \odot y = xy$ for all $x, y \in [0, 1]$.

Theorem 3.5. *Let A_i be IF-events, $A_i = (\mu_{A_i}, \nu_{A_i})$, $i = 1, \ldots, n$ and let $m : \mathcal{F} \longrightarrow [0, 1]$ be an IF-state such that $m(A) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$. Then m satisfies the inclusion-exclusion principle:*

$$m(\bigoplus_{i=1}^{n} A_i) = \sum_{i=1}^{n} m(A_i) - \sum_{i<j}^{n} m(A_i \odot A_j) + \cdots + (-1)^{n+1} m(\bigodot_{i=1}^{n} A_i).$$

Proof. Since $m(A) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dP)$, we have:
$\sum_{i=1}^{n} m(A_i) = \sum_{i=1}^{n} \int_\Omega \mu_{A_i} dP + \sum_{i=1}^{n} \alpha(1 - \int_\Omega (\mu_{A_i} + \nu_{A_i}) dP)$
$= \int_\Omega (\sum_{i=1}^{n} \mu_{A_i}) dP + \alpha(\binom{n}{1} - \int_\Omega (\sum_{i=1}^{n} (\mu_{A_i} + \nu_{A_i})) dP).$
Similarly,
$\sum_{i<j} m(A_i \odot A_j) = \int_\Omega (\sum_{i<j} \mu_{A_i} \odot \mu_{A_j}) dP$
$+\alpha(\binom{n}{2} - \int_\Omega (\sum_{i<j} (\mu_{A_i} \odot \mu_{A_j} + \nu_{A_i} \oplus \nu_{A_j})) dP),$
$\sum_{i<j<k} m(A_i \odot A_j \odot A_k) = \int_\Omega (\sum_{i<j<k} \mu_{A_i} \odot \mu_{A_j} \odot \mu_{A_k}) dP$
$+\alpha(\binom{n}{3} - \int_\Omega (\sum_{i<j<k} (\mu_{A_i} \odot \mu_{A_j} \odot \mu_{A_k} + \nu_{A_i} \oplus \nu_{A_j} \oplus \nu_{A_k})) dP),$

$\vdots$

$m(A_1 \odot A_2 \odot \cdots \odot A_n) = \int_\Omega (\mu_{A_1} \odot \mu_{A_2} \odot \cdots \odot \mu_{A_n}) dP$
$+\alpha(\binom{n}{n} - \int_\Omega (\mu_{A_1} \odot \mu_{A_2} \odot \cdots \odot \mu_{A_n} + \nu_{A_1} \oplus \nu_{A_2} \oplus \cdots \oplus \nu_{A_n}) dP).$

Put $A = \bigoplus_{i=1}^{n} A_i$ and sum the previous equalities:

$\sum_i m(A_i) - \sum_{i<j} m(A_i \odot A_j) + \sum_{i<j<k} m(A_i \odot A_j \odot A_k) - \cdots$
$+(-1)^{n+1} m(A_1 \odot A_2 \odot \cdots \odot A_n) = \int_\Omega (\sum_i \mu_{A_i} - \sum_{i<k} \mu_{A_i} \odot \mu_{A_j} + \cdots$
$+(-1)^{n+1} \mu_{A_1} \odot \cdots \odot \mu_{A_n}) dP + \alpha(\binom{n}{1} - \binom{n}{2} + \cdots + (-1)^{n+1} \binom{n}{n})$
$-\alpha \int_\Omega (\sum_i \mu_{A_i} - \sum_{i<k} \mu_{A_i} \odot \mu_{A_j} + \cdots + (-1)^{n+1} \mu_{A_1} \odot \cdots \odot \mu_{A_n}) dP$
$-\alpha \int_\Omega (\sum_i \nu_{A_i} - \sum_{i<k} \nu_{A_i} \oplus \nu_{A_j} + \cdots + (-1)^{n+1} \nu_{A_1} \oplus \cdots \oplus \nu_{A_n}) dP$
$= \int_\Omega \mu_A dP + \alpha(1 - (1 - 1)^{n+1}) - \alpha \int_\Omega \mu_A dP - \alpha \int_\Omega \nu_A dP$
$= \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega (\mu_A + \nu_A) dP) = m(A). \qquad \square$

Remark 3.6. Similarly as in Theorem 3.5, we can prove the Grzegorzewski's inclusion-exclusion principle for the probability $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$, where $\mathcal{P}^\flat(A) = \int_\Omega \mu_A dP$ and $\mathcal{P}^\sharp(A) = 1 - \int_\Omega \nu_A dP$ (see [7]):
If we denote $T_k^{(n)} = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \mathcal{P}(A_{i_1} \odot A_{i_2} \odot \cdots \odot A_{i_k})$, then the probability $\mathcal{P}(\bigoplus_{k=1}^n A_k)$ is obtained as a solution of the equation

$$\mathcal{P}(\bigoplus_{k=1}^{n} A_k) + \sum_{k=1}^{n/2} T_{2k}^{(n)} = \sum_{k=1}^{n/2} T_{2k-1}^{(n)}$$

if n is even or a solution of the equation

$$\mathcal{P}(\bigoplus_{k=1}^{n} A_k) + \sum_{k=1}^{(n+1)/2-1} T_{2k}^{(n)} = \sum_{k=1}^{(n+1)/2} T_{2k-1}^{(n)}$$

if n is odd.

4. **Applications**

As applications of inclusion-exclusion principle, we extend the Boole and Bonferroni inequalities from classical probabilities (see for example [11]) to the case of IF-states.
These results can be easily proved by induction.

Let $A_1, A_2, \dots A_n$ be n IF-events ($n \ge 1$) and let $m : \mathcal{F} \longrightarrow [0,1]$ be an IF-state such that $m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$.

Proposition 4.1. *(Boole inequalities) The following hold:*
(1) $m(\bigoplus_{k=1}^n A_k) \le \sum_{k=1}^n m(A_k)$;
(2) $m(\bigcup_{k=1}^n A_k) \le \sum_{k=1}^n m(A_k)$.

Proposition 4.2. *(Bonferroni inequalities) The following hold:*
(1) $m(\bigodot_{k=1}^n A_k) \ge \sum_{k=1}^n m(A_k) - n + 1$;
(2) $m(\bigcap_{k=1}^n A_k) \ge \sum_{k=1}^n m(A_k) - n + 1$.

5. **Conclusions**

(1) P. Grzegorzewski proved in [7], [6] two versions of the inclusion-exclusion principle for probability on IF-events based on the pairs of operations $(\cup, \cap)$ and $(\oplus, \odot)$:
$A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$, $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$
$A \oplus B = (\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B)$, $A \odot B = (\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B)$,
assuming that:
$\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$, with $\mathcal{P}^\flat(A) = \int_\Omega \mu_A dP$, $\mathcal{P}^\sharp(A) = 1 - \int_\Omega \nu_A dP$.
In this paper we proved an inclusion-exclusion principle for an IF-state $m : \mathcal{F} \longrightarrow [0,1]$ such that $m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$, for the case of the pair of operations $(\oplus, \odot)$. We presented a new method which can also be used to prove Grzegorzewski's inclusion-exclusion principle for probabilities on IF-events.
(2) For the case of the pair $(\cup, \cap)$ of operations on IF-events, it was proved in [9] the inclusion-exclusion principle for an IF-state $m : \mathcal{F} \longrightarrow [0,1]$ such that

$m(A) = \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega(\mu_A + \nu_A)dQ)$.
Similarly as in Theorem 3.5, we can prove the inclusion-exclusion principle for the above IF-state and the pair $(\oplus, \odot)$ of operations on IF-events.
(3) K. Atanassov and B. Riečan introduced two new operations over intuitionistic fuzzy sets ([3]) which are analogue to the operations "subtraction" and "division". One can try to investigate properties of the probability on IF-events based on these operations.

Acknowledgements. For the second author this research has been supported by the grant VEGA 1/0621/11.
The authors are very grateful to the referees for their valuable suggestions which improved the presentation of the paper.

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