SOME NEW FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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Abstract. Motivated by Samet et al. [Nonlinear Anal., 75(4) (2012), 2154-2165], we introduce the notions of α-ϕ-fuzzy contractive mapping and β-ψ-fuzzy contractive mapping and prove two theorems which ensure the existence and uniqueness of a fixed point for these two types of mappings. The presented theorems extend, generalize and improve the corresponding results given in the literature.

1. Introduction

The interest in fuzzy sets has been constantly growing from the starting paper of Zadeh [24] in 1965. Consequently, a large amount of theoretical and applied results is achieved in the directions of logic, mathematical analysis and general topology with many applications in economy and engineering. Now, an important theoretical development is the way of defining the concept of contractive mapping in fuzzy metric spaces. Indeed, in 1988 Grabiec [7] introduced the Banach contraction in a fuzzy metric space (in the sense of Kramosil and Michalek [9]) and extended the fixed point theorems of Banach and Edelstein to fuzzy metric spaces. In 2002, Gregori and Sapena [8] introduced the notion of fuzzy contractive mapping and proved some fixed point theorems in various classes of complete fuzzy metric spaces in the sense of George and Veeramani [5], Kramosil and Michalek [9] and Grabiec [7]. Soon after, Mihet [10] proposed a fuzzy fixed point theorem for a (weak) Banach contraction in M-complete fuzzy metric spaces. In this direction, Mihet [11, 12, 13] further extended the fixed point theory for contractive mappings in fuzzy metric spaces besides introducing some new types such as: Edelstein fuzzy contractive mappings, fuzzy ψ-contractive mappings, fuzzy contractive mappings of $(\epsilon - \lambda)$ type, etc. For more references on the development of fixed point theory in fuzzy metric spaces and its applications see [3, 4, 6, 14, 15, 16, 20, 21]. Very recently, Samet et al. [17] introduced the concept of α-ψ-contractive type mapping and utilized the same concept to prove several interesting fixed point theorems in the setting of metric spaces. Based on the same idea, we give some generalizations of the previous concepts of fuzzy contractive mappings [8, 12] in the setting of fuzzy metric spaces. Indeed, we introduce the notions of α-ϕ-fuzzy contractive mapping

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and β-ψ-fuzzy contractive mapping, that are useful to prove two corresponding fixed point theorems for such kind of contractive mappings. Moreover, we show that many existing results in the literature can be easily deduced from our theorems. In particular, the presented theorems extend, generalize and improve the results given in [8, 12]. Finally, we show that our results allow us to derive fixed point results in ordered fuzzy metric spaces.

2. **Preliminaries**

Consistent with Gregori and Sapena [8], and Mihet [12], the following definitions and results will be needed in the sequel.

Definition 2.1. (Schweizer and Sklar [18]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (t-norm) if the following conditions hold:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Four basic examples of continuous t-norms are: $a *_1 b = \min\{a, b\}, a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $\lambda \in (0,1)$, $a *_3 b = ab$, $a *_4 b = \max\{a + b - 1, 0\}$.

Definition 2.2. (Kramosil and Michalek [9]) A fuzzy metric space is a triple $(X, M, *)$, where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, +\infty)$, satisfying, for all $x, y \in X$, the following properties:

(KM-1) $M(x, y, 0) = 0$;
(KM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
(KM-3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;
(KM-4) $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left continuous;
(KM-5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $z \in X$ and for all $t, s > 0$.

If, in the above definition, the triangular inequality (KM-5) is replaced by the following condition:

(NA) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and $t, s > 0$,

then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space. It is easy to check that (NA) implies (KM-5), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

In order to introduce a Hausdorff topology on the fuzzy metric space, George and Veeramani [5] modified the above definition as follows:

Definition 2.3. (George and Veeramani [5]) A fuzzy metric space is a triple $(X, M, *)$, where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$, satisfying, for all $x, y \in X$, the following properties:

(GV-1) $M(x, y, t) > 0$ for all $t > 0$;

(GV-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
(GV-3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;
(GV-4) $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous;
(GV-5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $z \in X$ and for all $t, s > 0$.

From now on, we will work in fuzzy metric spaces on the sense of George and Veeramani.

Definition 2.4. [5, 7, 19] Let $(X, M, *)$ be a fuzzy metric space. Then:

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$, that is $\lim_{n\to+\infty} x_n = x$, if $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called M-Cauchy, if for each $\epsilon \in (0, 1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$.
(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called G-Cauchy if $\lim_{n\to+\infty} M(x_n, x_{n+m}, t) = 1$ for each $m \in \mathbb{N}$ and $t > 0$.

Now, a fuzzy metric space $(X, M, *)$ is called M-complete (G-complete) if every M-Cauchy (G-Cauchy) sequence is convergent.

Definition 2.5. (Di Bari and Vetro [4]) Let $(X.M, *)$ be a fuzzy metric space. The fuzzy metric M is said to be triangular if the following condition holds:

$$\left(\frac{1}{M(x,y,t)} - 1\right) \leq \left(\frac{1}{M(x,z,t)} - 1\right) + \left(\frac{1}{M(y,z,t)} - 1\right) \tag{1}$$

for all $x, y, z \in X$ and for all $t > 0$.

3. Fixed Point Theorems for α-ϕ-fuzzy Contractive Mappings

We start this section by introducing the new notions of α-ϕ-fuzzy contractive and α-admissible mappings in fuzzy metric spaces.
Denote by Φ the family of all right continuous functions $\phi : [0, +\infty) \to [0, +\infty)$, with $\phi(r) < r$ for all $r > 0$.

Remark 3.1. Note that for every function $\phi \in \Phi$, $\lim_{n\to+\infty} \phi^n(r) = 0$ for each $r > 0$, where $\phi^n(r)$ denotes the n-th iterate of ϕ.

Definition 3.2. Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is an α-ϕ-fuzzy contractive mapping if there exist two functions $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ and $\phi \in \Phi$ such that

$$\alpha(x, y, t)\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right) \tag{2}$$

for all $x, y \in X$ and for all $t > 0$.

Remark 3.3. If $\alpha(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$, and $\phi(r) = kr$ for all $r > 0$ and for some $k \in (0, 1)$, then Definition 3.2 reduces to the definition of fuzzy contractive mapping given by Gregori and Sapena [8]. It follows that a fuzzy contractive mapping is an α-ϕ-fuzzy contractive mapping; but the converse is not necessarily true (see Example 3.7 given below).

Definition 3.4. Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is α-admissible if there exists a function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ such that, for all $t > 0$,

$$x, y \in X, \ \alpha(x, y, t) \geq 1 \Longrightarrow \alpha(Tx, Ty, t) \geq 1.$$

Now, we are ready to state and prove our first result.

Theorem 3.5. *Let $(X, M, *)$ be a G-complete fuzzy metric space. Let $T : X \to X$ be an α-ϕ-fuzzy contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$, for all $t > 0$;*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$, for all $t > 0$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1, t) = \alpha(x_1, x_2, t) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1 \text{ for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{3}$$

By (3), we have

$$\begin{aligned}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) &= \left(\frac{1}{M(Tx_{n-1}, Tx_n, t)} - 1\right) \\ &\leq \alpha(x_{n-1}, x_n, t)\left(\frac{1}{M(Tx_{n-1}, Tx_n, t)} - 1\right).\end{aligned}$$

Using (2) with $x = x_{n-1}$ and $y = x_n$ from the above inequality, by the property of ϕ ($\phi(r) < r$ for all $r > 0$), we obtain

$$\begin{aligned}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) &\leq \phi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right) \\ &< \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right).\end{aligned}$$

Therefore, $M(x_n, x_{n+1}, t) > M(x_{n-1}, x_n, t)$ for all $n \in \mathbb{N}$ and thus $\{M(x_{n-1}, x_n, t)\}$ is an increasing sequence of positive real numbers in [0,1].

Let $S(t) = \lim_{n \to +\infty} M(x_{n-1}, x_n, t)$; we show that $S(t) = 1$ for all $t > 0$. We suppose that there is $t_0 > 0$ such that $S(t_0) < 1$, then from

$$\left(\frac{1}{M(x_n, x_{n+1}, t_0)} - 1\right) \leq \phi\left(\frac{1}{M(x_{n-1}, x_n, t_0)} - 1\right)$$

as $n \to +\infty$, using the right continuity of the function ϕ, we deduce that

$$\frac{1}{S(t_0)} - 1 \leq \phi\left(\frac{1}{S(t_0)} - 1\right) < \frac{1}{S(t_0)} - 1$$

that is a contradiction and so, we get $\lim_{n\to+\infty} M(x_{n-1}, x_n, t) = 1$ for all $t > 0$. Then, for a fixed $p \in \mathbb{N}$, we have

$$M(x_n, x_{n+p}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{p}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right)$$
$$* \cdots\cdots * M\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right) \to \overbrace{1 * \cdots\cdots * 1}^{p} = 1,$$

as $n \to +\infty$ and thus $\{x_n\}$ is a G-Cauchy sequence. Therefore, $\{x_n\}$ converges to x^* for some $x^* \in X$. Now, the continuity of T implies that $Tx_n \to Tx^*$ and so $\lim_{n\to+\infty} M(Tx_n, Tx^*, t) = 1$ for all $t > 0$. It follows that

$$\lim_{n\to+\infty} M(x_{n+1}, Tx^*, t) = \lim_{n\to+\infty} M(Tx_n, Tx^*, t) = 1$$

for all $t > 0$, that is, $x_n \to Tx^*$. By the uniqueness of the limit, we get $x^* = Tx^*$, that is, x^* is a fixed point of T. □

In the next theorem, we omit the continuity hypothesis of T.

Theorem 3.6. *Let $(X, M, *)$ be a G-complete fuzzy metric space. Let M be triangular and $T : X \to X$ be an α-ϕ-fuzzy contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point.

Proof. Following the proof of Theorem 3.5, we get that $\{x_n\}$ is a G-Cauchy sequence in the G-complete fuzzy metric space $(X, M, *)$. Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. On the other hand, from (3) and the hypothesis (iii), we have

$$\alpha(x_n, x^*, t) \geq 1, \text{ for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{4}$$

Now using successively, (1), (4) and (2), also in view of (GV-3), we obtain

$$\begin{aligned}\left(\frac{1}{M(Tx^*, x^*, t)} - 1\right) &\leq \left(\frac{1}{M(Tx^*, Tx_n, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x^*, t)} - 1\right)\\ &\leq \alpha(x_n, x^*, t)\left(\frac{1}{M(Tx_n, Tx^*, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x^*, t)} - 1\right)\\ &\leq \phi\left(\frac{1}{M(x_n, x^*, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x^*, t)} - 1\right).\end{aligned}$$

Letting $n \to +\infty$, since ϕ is continuous at $r = 0$, we obtain

$$\left(\frac{1}{M(Tx^*, x^*, t)} - 1\right) = 0, \text{ that is, } Tx^* = x^*.$$

□

The following example shows that the generalization given by Definition 3.2 offers many possibilities to study the existence of a fixed point for a mapping.

Example 3.7. Let $X = \left\{\frac{1}{n},\, n \in \mathbb{N}\right\} \cup \{0, 2\}$, $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and for all $t > 0$. Clearly, $(X, M, *)$ is a G-complete fuzzy metric space.

Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \dfrac{x^2}{4} & \text{if } x \in X \setminus \{2\}, \\ \\ 2 & \text{if } x = 2, \end{cases}$$

and the function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1 & \text{if } x, y \in X \setminus \{2\}, \\ \\ 0 & \text{otherwise}, \end{cases}$$

for all $t > 0$. Clearly, T is an α-ϕ-contractive mapping with $\phi(r) = r/2$ for all $r \geq 0$. In fact, if at least one between x and y is equal to 2, then $\alpha(x, y, t) = 0$ and so (2) holds trivially. Otherwise, if both x and y are in $X \setminus \{2\}$, then $\alpha(x, y, t) = 1$ and so (2) becomes

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \frac{1}{2}\left(\frac{1}{M(x, y, t)} - 1\right)$$

that is always true since $x + y \leq 2$.

Now, let $x, y \in X$ such that $\alpha(x, y, t) \geq 1$ for all $t > 0$, this implies that $x, y \in X \setminus \{2\}$ and by the definitions of T and α, we have

$$Tx = \frac{x^2}{4} \in X \setminus \{2\},\ Ty = \frac{y^2}{4} \in X \setminus \{2\} \text{ and } \alpha(Tx, Ty, t) = 1 \text{ for all } t > 0,$$

that is, T is α-admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$, indeed for $x_0 = 1$ we have $\alpha(1, T(1), t) = 1$.

Finally, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of the function α, it follows that $x_n \in X \setminus \{2\}$ for all $n \in \mathbb{N}$ and hence $x \in X \setminus \{2\}$. Therefore $\alpha(x_n, x, t) = 1$ for all $n \in \mathbb{N}$. Thus, all the hypotheses of Theorem 3.5 are satisfied. Here 0 and 2 are two fixed points of T.

However, T is not a fuzzy contractive mapping [8]. To see this consider $x = 2$ and $y = 1$, then we have

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) = \frac{7}{4t} \nleq \frac{k}{t} = k\left(\frac{1}{M(x, y, t)} - 1\right) \text{ (since } k \in (0, 1)).$$

Remark 3.8. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\left(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1\right) \le k\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)$$

for all $n \in \mathbb{N}$ and for all $t > 0$. In the conclusions of their paper, Yun et al. [23] observed that every fuzzy contractive sequence is Cauchy in both George and Veeramani sense and Grabiec sense. Here, in proving Theorems 3.5 and 3.6, we used the G-completeness of fuzzy metric space $(X, M, *)$. Thus it will be interesting to see whether these results will remain true in a M-complete fuzzy metric space.

Now, we give a sufficient condition to obtain the uniqueness of the fixed point in the previous theorems. Precisely, we consider the following hypothesis:

(H) for all $x, y \in X$ and for all $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \ge 1$ and $\alpha(y, z, t) \ge 1$.

Theorem 3.9. *Adding the condition (H) to the hypotheses of Theorem 3.5 (resp. Theorem 3.6), we obtain the uniqueness of the fixed point of T.*

Proof. Suppose that x^* and y^* are two fixed points of T. If $\alpha(x^*, y^*, t) \ge 1$, then by (2) we conclude easily that $x^* = y^*$. Assume that $\alpha(x^*, y^*, t) < 1$, then from (H) there exists $z \in X$ such that

$$\alpha(x^*, z, t) \ge 1 \text{ and } \alpha(y^*, z, t) \ge 1. \tag{5}$$

Since T is α-admissible, from (5) we get

$$\alpha(x^*, T^n z, t) \ge 1 \text{ and } \alpha(y^*, T^n z, t) \ge 1 \text{ for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{6}$$

Using (2) and (6), we have

$$\begin{aligned}\left(\frac{1}{M(x^*, T^n z, t)} - 1\right) &= \left(\frac{1}{M(Tx^*, T(T^{n-1}z), t)} - 1\right)\\ &\le \alpha(x^*, T^{n-1}z, t)\left(\frac{1}{M(Tx^*, T(T^{n-1}z), t)} - 1\right)\\ &\le \phi\left(\frac{1}{M(x^*, T^{n-1}z, t)} - 1\right).\end{aligned}$$

This implies that

$$\left(\frac{1}{M(x^*, T^n z, t)} - 1\right) \le \phi^n\left(\frac{1}{M(x^*, z, t)} - 1\right), \text{ for all } n \in \mathbb{N}.$$

Then, letting $n \to +\infty$, we have

$$T^n z \to x^*. \tag{7}$$

Similarly, for $n \to +\infty$ we get also

$$T^n z \to y^*. \tag{8}$$

Using (7) and (8), the uniqueness of the limit gives us $x^* = y^*$. □

In view of Remark 3.3 and to show the usefulness of our theorems, we prove the following classical theorem of Gregori and Sapena [8].

Theorem 3.10. *Let $(X, M, *)$ be a G-complete fuzzy metric space. Let $T : X \to X$ be a fuzzy contractive mapping. Then T has a unique fixed point.*

Proof. Let $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ be the function defined by $\alpha(x, y, t) = 1$, for all $x, y \in X$ and for all $t > 0$. Define also $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(r) = kr$ for all $r > 0$. Then T is an α-ϕ-contractive mapping. It is easy to show that all the hypotheses of Theorems 3.5 and 3.9 are satisfied. Consequently, T has a unique fixed point. □

Inspired by [1, 17], we show that our obtained theorems are also useful to deduce easily some fixed point results in ordered fuzzy metric spaces. We begin by giving the following two definitions.

Definition 3.11. Let $\preceq$ be an order relation on X. We say that $T : X \to X$ is a non-decreasing mapping with respect to $\preceq$, if $x \preceq y$ implies $Tx \preceq Ty$.

Definition 3.12. Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is a fuzzy order ϕ-contractive mapping if there exists $\phi \in \Phi$ such that the following implication holds:

$$x, y \in X,\ x \preceq y \Longrightarrow \left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right) \text{ for all } t > 0.$$

Theorem 3.13. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a G-complete fuzzy metric space. Let $\phi \in \Phi$ be such that $T : X \to X$ is a fuzzy order ϕ-contractive mapping and suppose that the following conditions hold:*

(i) *T is a non-decreasing mapping with respect to $\preceq$;*
(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, $M(x_0, Tx_0, t) > 0$ for all $t > 0$;*
(iii) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. Define the function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise,} \end{cases}$$

for all $t > 0$. The reader can show easily that T is α-ϕ-contractive and α-admissible. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. From (iii), this implies that $x_n \preceq x$ for all $n \in \mathbb{N}$, which gives us that $\alpha(x_n, x, t) = 1$ for all $n \in \mathbb{N}$ and $t > 0$. Thus all the hypotheses of Theorem 3.6 are satisfied and T has a fixed point. □

4. Fixed Point Theorems for β-ψ-fuzzy Contractive Mappings

In this section we present the new notions of β-ψ-fuzzy contractive and β-admissible mappings in fuzzy metric spaces.

Let Ψ be the class of all functions $\psi : [0, 1] \to [0, 1]$ such that

(i) ψ is non-decreasing and left continuous,

(ii) $\psi(r) > r$ for all $r \in (0, 1)$.

It can easily be shown (see, e.g, [22]) that if $\psi \in \Psi$, then $\psi(1) = 1$ and $\lim_{n\to+\infty} \psi^n(r) = 1$ for all $r \in (0, 1)$.

Definition 4.1. Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is a β-ψ-fuzzy contractive mapping if there exist two functions $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ and $\psi \in \Psi$ such that

$$M(x, y, t) > 0 \implies \beta(x, y, t)M(Tx, Ty, t) \geq \psi(M(x, y, t)) \tag{9}$$

for all $t > 0$ and for all $x, y \in X$ with $x \neq y$.

Remark 4.2. If $\beta(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$, then Definition 4.1 reduces to the definition of fuzzy ψ-contractive mapping given by Mihet [12]. It follows that a fuzzy ψ-contractive mapping is a β-ψ-fuzzy contractive mapping; but the converse is not true always (see Example 4.5 given below).

Definition 4.3. Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is β-admissible if there exists a function $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ such that, for all $t > 0$,

$$x, y \in X,\ \beta(x, y, t) \leq 1 \implies \beta(Tx, Ty, t) \leq 1.$$

Theorem 4.4. *Let $(X, M, *)$ be a M-complete non-Archimedean fuzzy metric space and $T : X \to X$ be a β-ψ-fuzzy contractive mapping satisfying the following conditions:*

(i) *T is β-admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;*
(iii) *for each sequence $\{x_n\}$ in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\beta(x_{m+1}, x_{n+1}, t) \leq 1$ for all $m, n \in \mathbb{N}$ with $m > n \geq k_0$ and for all $t > 0$;*
(iv) *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $t > 0$ and $x_n \to x$ as $n \to +\infty$, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.*

Then, T has a fixed point.

Proof. Let $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of T. Assume $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since T is β-admissible, we have

$$\beta(x_0, Tx_0, t) = \beta(x_0, x_1, t) \leq 1 \implies \beta(Tx_0, Tx_1, t) = \beta(x_1, x_2, t) \leq 1.$$

By induction, we get

$$\beta(x_n, x_{n+1}, t) \leq 1 \text{ for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{10}$$

Now, applying (9) with $x = x_{n-1}$ and $y = x_n$ and using (10), we obtain

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, Tx_n, t) \\ &\geq \beta(x_{n-1}, x_n, t)M(Tx_{n-1}, Tx_n, t) \\ &\geq \psi(M(x_{n-1}, x_n, t)). \end{aligned}$$

By induction, we get

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, x_1, t)), \text{ for all } n \in \mathbb{N}.$$

Since $\lim_{n\to+\infty} \psi^n(r) = 1$ for all $r \in (0,1)$, then we deduce that

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0.$$

Now, if the sequence $\{x_n\}$ is not M-Cauchy, then there are $\epsilon \in (0,1)$, $t > 0$ and $k_0 \in \mathbb{N}$ (by (iii)) such that, for each $k \in \mathbb{N}$ with $k \geq k_0$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon \text{ and } \beta(x_{m(k)}, x_{n(k)}, t) \leq 1.$$

Let, for each k, $m(k)$ be the least positive integer exceeding $n(k)$ satisfying the above property, that is

$$M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \epsilon \text{ and } M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \epsilon.$$

Then, for each positive integer $k \geq k_0$, we have

$$\begin{aligned} 1 - \epsilon &\geq M(x_{m(k)}, x_{n(k)}, t) \\ &\geq M(x_{m(k)-1}, x_{n(k)}, t) * M(x_{m(k)-1}, x_{m(k)}, t) \quad \text{(by (NA))} \\ &\geq (1-\epsilon) * M(x_{m(k)-1}, x_{m(k)}, t). \end{aligned}$$

Since $\lim_{n\to+\infty} (1-\epsilon) * M(x_{m(k)-1}, x_{m(k)}, t) = (1-\epsilon) * 1 = 1 - \epsilon$, it follows that

$$\lim_{n\to+\infty} M(x_{m(k)}, x_{n(k)}, t) = 1 - \epsilon.$$

Now, we get

$$\begin{aligned} &M(x_{m(k)}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)}, t) \text{ (by (NA))} \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)+1}, t) * M(x_{n(k)+1}, x_{n(k)}, t) \\ &= M(x_{m(k)}, x_{m(k)+1}, t) * M(Tx_{m(k)}, Tx_{n(k)}, t) * M(x_{n(k)+1}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * \beta(Tx_{m(k)}, Tx_{n(k)}, t) M(Tx_{m(k)}, Tx_{n(k)}, t) \text{ (by (iii))} \\ &\quad * M(x_{n(k)+1}, x_{n(k)}, t) \\ &\geq M(x_{m(k)}, x_{m(k)+1}, t) * \psi(M(x_{m(k)}, x_{n(k)}, t)) * M(x_{n(k)}, x_{n(k)+1}, t). \end{aligned}$$

Letting $k \to +\infty$, we obtain

$$1 - \epsilon \geq 1 * \psi(1-\epsilon) * 1 = \psi(1-\epsilon) > 1 - \epsilon$$

which is a contradiction and so $\{x_n\}$ is a Cauchy sequence. Since X is M-complete, there exists $x^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$.

On the other hand, from (10) and the hypothesis (iv) we have

$$\beta(x_n, x^*, t) \leq 1 \text{ for all } t > 0.$$

Now, by (NA) and (9), we get

$$\begin{aligned} M(Tx^*, x^*, t) &\geq M(Tx^*, Tx_n, t) * M(x_{n+1}, x^*, t) \\ &\geq \beta(x_n, x^*, t) M(Tx_n, Tx^*, t) * M(x_{n+1}, x^*, t) \\ &\geq \psi(M(x_n, x^*, t)) * M(x_{n+1}, x^*, t). \end{aligned}$$

Letting $n \to +\infty$ and since $\psi(1) = 1$, we conclude that $Tx^* = x^*$. □

The following example shows the usefulness of Definition 4.1.

Example 4.5. Let $X = (0, +\infty)$, $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $x, y \in X$ and for all $t > 0$. Clearly, $(X, M, *)$ is a M-complete non-Archimedean fuzzy metric space.

Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \in (0, 1], \\ 2 & \text{otherwise,} \end{cases}$$

and the function $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ by

$$\beta(x, y, t) = \begin{cases} 1 & \text{if } x, y \in (0, 1], \\ 2 & \text{otherwise,} \end{cases}$$

for all $t > 0$. It is easy to show that T is a β-ψ-contractive mapping with $\psi(r) = \sqrt{r}$, for all $r \in [0, 1]$. Clearly, T is β-admissible. Further, there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$, indeed for $x_0 = 1$ we have $\beta(1, T(1), t) = 1$.

Finally, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$, $x_n \to x \in X$ as $n \to +\infty$ and let $k_0 = 1$ such that for all $m, n \in \mathbb{N}$ we have $m > n \geq k_0$. By the definition of the function β, it follows that $x_n \in (0, 1]$ for all $n \in \mathbb{N}$. Now, if $x > 1$, we get $M(x_n, x, t) = \frac{\min\{x_n,x\}}{\max\{x_n,x\}} = \frac{x_n}{x} \leq \frac{1}{x} < 1$, that contradicts (i) of Definition 2.4, since $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$. Consequently, we obtain that $x \in (0, 1]$. Therefore $\beta(x_n, x, t) = 1$ and $\beta(x_{m+1}, x_{n+1}, t) = 1$ for all $m, n \in \mathbb{N}$. Thus, all the hypotheses of Theorem 4.4 are satisfied. Here 1 and 2 are two fixed points of T.

However, T is not a fuzzy ψ-contractive mapping [12]. To see this consider $x = \frac{1}{2}$ and $y = 3$, then we have

$$M(Tx, Ty, T) = \frac{\sqrt{1/2}}{2} \ngeq \sqrt{\frac{1/2}{3}} = \sqrt{M(x, y, t)} = \psi(M(x, y, t)).$$

To ensure the uniqueness of the fixed point, we will consider the following hypothesis:

(J) for all $x, y \in X$ and for all $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.

Theorem 4.6. *Adding the condition (J) to the hypotheses of Theorem 4.4, we obtain the uniqueness of the fixed point of T.*

Proof. The proof can be completed using a similar technique as given in the proof of Theorem 3.9. Therefore, to avoid repetitions, we omit the details. □

5. Conclusion

Motivated by Samet et al. [17], we proposed the concept of α-ϕ-fuzzy contractive mapping, which is weaker than the corresponding concept of fuzzy contractive mapping [8] and the concept of β-ψ-fuzzy contractive mapping, which is weaker than the corresponding concept of fuzzy-ψ-contractive mapping [12]. Moreover, we proved two theorems which ensure the existence and uniqueness of fixed points of these new types of contractive mappings. The new concepts lead to further investigations and applications. For example, using the recent ideas in the literature [2], it is possible to extend our results to the case of coupled fixed points in fuzzy metric spaces.

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