FIXED POINT RESULTS IN FUZZY METRIC-LIKE SPACES

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Abstract. In this paper, the concept of fuzzy metric-like spaces is introduced which generalizes the notion of fuzzy metric spaces given by George and Veeramani [8]. Some fixed point results for fuzzy contractive mappings on fuzzy metric-like spaces are derived. These results generalize several comparable results from the current literature. We also provide illustrative examples in support of our new results where result from current literature are not applicable.

1. **Introduction and preliminaries**

The evolution of fuzzy mathematics commenced with an introduction of the notion of fuzzy sets by Zadeh [25], as a new way to represent the vagueness in every day life. There are many practical problems where the nature of uncertainty in the behavior of a given system possesses fuzzy rather than stochastic nature.

The concept of a fuzzy metric space was introduced and generalized in many ways ([7, 9, 14]) . George and Veeramani [8, 9] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [14]. They obtained a Hausdorff and first countable topology on the modified fuzzy metric spaces, which has very important applications in quantum particle physics, particularly in connection with both string and ϵ^{∞} theory (see, [17] and references therein). In fuzzy metric spaces given by Kramosil and Michalek [14], Grabiec [11] gave the fuzzy version of Banach contraction principle. Subsequently, Many authors proved fixed point and common fixed point theorems in fuzzy metric spaces, (see [1–6, 10, 13, 15, 16, 20–24]).

Recently, Harandi [12] introduced the concept of metric-like spaces as a generalization of partial metric spaces and metric spaces (see [12]) and proved some fixed point results in such spaces. For definitions and examples of metric like spaces, we refer to [12].

The aim of this paper is to introduce the notion of fuzzy metric-like spaces, as a generalization of fuzzy metric spaces. Fixed point theorems for contractive mappings in fuzzy metric-like spaces are also proved. Examples are provided which illustrate the results.

Definition 1.1. [25] A fuzzy set A in a nonempty set X is a function with domain X and values in $[0, 1]$.

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Definition 1.2. [19] A binary operation $\star\colon [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $\{[0,1],\star\}$ is an abelian topological monoid with unit 1 such that $a\star b\le c\star d$ whenever $a\le c$ and $b\le d$, $a,b,c,d\in[0,1]$. Three typical examples of $t-$ norms are $a\star b=\min\{a,b\}$ (minimum $t-$ norm), $a\star b=ab$ (product $t-$ norm), and $a\star b=\max\{a+b-1,0\}$ (Lukasiewicz $t-$ norm).

Definition 1.3. [8] The triplet $(X,M,\star)$ is a fuzzy metric space if X is an arbitrary set, $\star$ is a continuous t-norm, M is a fuzzy set in $X^2\times(0,\infty)$ satisfying the following conditions:

(FM1) $M(x,y,t)>0$;
(FM2) $M(x,y,t)=1$ if and only if $x=y$;
(FM3) $M(x,y,t)=M(y,x,t)$;
(FM4) $M(x,y,t)\star M(y,z,s)\le M(x,z,t+s)$;
(FM5) $M(x,y,\cdot)\colon (0,\infty)\to[0,1]$ is a continuous mapping;

for all $x,y,z\in X$ and $s,t>0$.

Here M with $\star$ is called a fuzzy metric on X. Note that, $M(x,y,t)$ can be thought of as the definition of nearness between x and y with respect to t. It is known that $M(x,y,.)$ is nondecreasing for all $x,y\in X$. For examples of fuzzy metric spaces we refer to [18].

2. **Fuzzy Metric-Like Spaces**

In this section, we define the fuzzy metric-like spaces and give some examples and properties of fuzzy metric-like spaces.

Definition 2.1. The triplet $(X,\mathcal{F},\star)$ is a fuzzy metric-like space if X is an arbitrary set, $\star$ is a continuous t-norm, $\mathcal{F}$ is a fuzzy set in $X^2\times(0,\infty)$ satisfying the following conditions:

(FML1) $\mathcal{F}(x,y,t)>0$;
(FML2) if $\mathcal{F}(x,y,t)=1$ then $x=y$;
(FML3) $\mathcal{F}(x,y,t)=\mathcal{F}(y,x,t)$;
(FML4) $\mathcal{F}(x,y,t)\star\mathcal{F}(y,z,s)\le\mathcal{F}(x,z,t+s)$;
(FML5) $\mathcal{F}(x,y,\cdot)\colon(0,\infty)\to[0,1]$ is a continuous mapping;

for all $x,y,z\in X$ and $s,t>0$.

Here M with $\star$ is called a fuzzy metric-like on X. A fuzzy metric-like space satisfies all of the conditions of a fuzzy metric space except that $\mathcal{F}(x,x,t)$ may be less than 1 for all $t>0$ and for some (or may be for all) $x\in X$. Also, every fuzzy metric space is fuzzy metric-like space with unit self fuzzy distance, that is, with $\mathcal{F}(x,x,t)=1$ for all $t>0$ and for all $x\in X$.

Note that, the axiom (FM2) in Definition 3 gives the idea that when $x=y$ the degree of nearness of x and y is perfect, or simply 1, and then $M(x,x,t)=1$ for each $x\in X$ and for each $t>0$. While in fuzzy metric-like space, $M(x,x,t)$ may be less than 1, that is, the concept of fuzzy metric-like is applicable when the degree of nearness of x and y is not perfect for the case $x=y$.

By using the following propositions several examples of fuzzy metric-like spaces can be obtained.

Proposition 2.2. *Let (X,σ) be any metric-like space. Then the triplet $(X,\mathcal{F},\star)$ is a fuzzy metric like space, where $\star$ is defined by $a\star b = ab$ for all $a,b\in[0,1]$ and the fuzzy set $\mathcal{F}$ is given by*

$$\mathcal{F}(x,y,t)=\frac{kt^n}{kt^n+m\sigma(x,y)} \text{ for all } x,y\in X, t>0,$$

where $k\in\mathbb{R}^+$, $m>0$ and $n\geq 1$.

Proof. The proof of properties (FML1)-(FML3) and (FML5) are obvious. For (FML4), let $x,y,z\in X$ and $a=\sigma(x,y), b=\sigma(y,z), c=\sigma(x,z)$, then for all $t,s>0$ we know that

$$\begin{aligned} c\leq a+b \quad &\Rightarrow \quad kt^ns^nc\leq k(t+s)^ns^na+kt^n(t+s)^nb \\ &\Rightarrow \quad kt^ns^nc\leq (t+s)^n\left[ks^na+kt^nb+mab\right] \\ &\Rightarrow \quad kt^ns^n\left[k(s+t)^n+mc\right]\leq (t+s)^n\left[kt^n+ma\right]\left[ks^n+mb\right] \\ &\Rightarrow \quad \frac{kt^n}{kt^n+ma}\cdot\frac{ks^n}{ks^n+mb}\leq\frac{k(t+s)^n}{k(t+s)^n+mc} \\ &\Rightarrow \quad \mathcal{F}(x,y,t)\star\mathcal{F}(y,z,s)\leq\mathcal{F}(x,z,t+s). \end{aligned}$$

Therefore (FML4) is also satisfied and $(X,\mathcal{F},\star)$ is a fuzzy metric-like space. □

Remark 2.3. Note that the above Proposition holds even with the t-norm $a\star b=\min\{a,b\}$.

Remark 2.4. The Proposition 2.2 shows that every metric-like space induces a fuzzy metric-like space. For $k=n=m=1$ the induced fuzzy metric-like space $(X,\mathcal{F},\star)$ is called the standard fuzzy metric-like space, where

$$\mathcal{F}(x,y,t)=\frac{t}{t+\sigma(x,y)} \text{ for all } x,y\in X, t>0.$$

Example 2.5. Let $X=\mathbb{R}^+$, $k\in\mathbb{R}^+$ and $m>0$. Define $\star$ by $a\star b=ab$ and the fuzzy set $\mathcal{F}$ in $X^2\times(0,\infty)$ by

$$\mathcal{F}(x,y,t)=\frac{kt}{kt+m(\max\{x,y\})} \text{ for all } x,y\in X, t>0.$$

Then, since $\sigma(x,y)=\max\{x,y\}$ for all $x,y\in X$, is a metric-like on X (see [12]) therefore by Proposition 2.2, $(X,\mathcal{F},\star)$ is a fuzzy metric-like space, but it is not a fuzzy metric space, as $\mathcal{F}(x,x,t)=\dfrac{kt}{kt+mx}\neq 1$ for all $x>0$ and $t>0$.

Proposition 2.6. *Let (X,σ) be any metric-like space. Then the triplet $(X,\mathcal{F},\star)$ is a fuzzy metric-like space, where $\star$ is given by $a\star b=ab$ for all $a,b\in[0,1]$ and the fuzzy set $\mathcal{F}$ is defined by*

$$\mathcal{F}(x,y,t)=e^{-\sigma(x,y)/t^n} \text{ for all } x,y\in X, t>0,$$

where $n\geq 1$.

Proof. The proof of properties (FML1)-(FML3) and (FML5) are obvious. For (FML4), let $x, y, z \in X$ and $a = \sigma(x, y), b = \sigma(y, z), c = \sigma(x, z)$, then for all $t, s > 0$ we know that

$$\begin{aligned} c \leq a + b \quad &\Rightarrow \quad \frac{c}{(t+s)^n} \leq \frac{a+b}{(t+s)^n} \Rightarrow \frac{c}{(t+s)^n} \leq \frac{a}{t^n} + \frac{b}{s^n} \\ &\Rightarrow \quad e^{\frac{c}{(t+s)^n}} \leq e^{\left[\frac{a}{t^n} + \frac{b}{s^n}\right]} \\ &\Rightarrow \quad e^{\left[-\frac{a}{t^n} - \frac{b}{s^n}\right]} \leq e^{-\frac{c}{(t+s)^n}} \\ &\Rightarrow \quad \mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s) \leq \mathcal{F}(x, z, t + s). \end{aligned}$$

Therefore (FML4) is also satisfied and $(X, \mathcal{F}, \star)$ is a fuzzy metric-like space. □

Remark 2.7. Note that the above Proposition holds even with the t-norm $a \star b = \min\{a, b\}$.

Example 2.8. Let $X = \mathbb{R}^+$. Define $\star$ by $a \star b = ab$ and the fuzzy set $\mathcal{F}$ in $X^2 \times (0, \infty)$ by

$$\mathcal{F}(x, y, t) = \frac{1}{e^{\max\{x,y\}/t}}, \text{ for all } x, y \in X, t > 0.$$

Then, since $\sigma(x, y) = \max\{x, y\}$ for all $x, y \in X$ is a metric-like on X (see [12]) therefore by Proposition 2.6, $(X, \mathcal{F}, \star)$ is a fuzzy metric-like space, but not a fuzzy metric space, as $\mathcal{F}(x, x, t) = \frac{1}{e^{x/t}} \neq 1$ for all $x > 0$ and $t > 0$.

Example 2.9. Let $X = \mathbb{N}$. Define $\star$ by $a \star b = ab$ and fuzzy set $\mathcal{F}$ in $X^2 \times (0, \infty)$ by

$$\mathcal{F}(x, y, t) = \begin{cases} \dfrac{x}{y^3}, & \text{if } x \leq y; \\ \dfrac{y}{x^3}, & \text{if } y \leq x. \end{cases}$$

Then $(X, \mathcal{F}, \star)$ is a fuzzy metric-like space.

Proof. The proofs of properties (FML1),(FML3) and (FML5) are obvious. For (FML2), let $x, y \in X$, $x \leq y$ and $\mathcal{F}(x, y, t) = 1$. Then $\mathcal{F}(x, y, t) = x/y^3 = 1$, that is, $x = y^3$. As $x \leq y$ and $x, y \in X = \mathbb{N}$ we must have $x = y = 1$. Similarly, if $y \leq x$ and $\mathcal{F}(x, y, t) = 1$, we obtain $x = y = 1$. Thus, (FML2) is satisfied.
For (FML4), let $x, y, z \in X$, $t, s > 0$ and we consider the following cases:

(a) Suppose, $x \leq y \leq z$, then $\mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s) = \frac{x}{y^3}\frac{y}{z^3} = \frac{x}{y^2 z^3} \leq \frac{x}{z^3} = \mathcal{F}(x, z, t + s)$.

(b) Suppose, $x \leq z \leq y$, then $\mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s) = \frac{x}{y^3}\frac{z}{y^3} = \frac{xz}{y^6} \leq \frac{x}{z^3} = \mathcal{F}(x, z, t + s)$.

(c) Suppose, $y \leq x \leq z$, then $\mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s) = \frac{y}{x^3}\frac{y}{z^3} = \frac{y^2}{x^3 z^3} \leq \frac{x}{z^3} = \mathcal{F}(x, z, t + s)$.

(d) Suppose, $y \leq z \leq x$, then $\mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s) = \frac{y}{x^3}\frac{y}{z^3} = \frac{y^2}{x^3 z^3} \leq \frac{z}{x^3} = \mathcal{F}(x, z, t + s)$.

Similarly, for the cases (e) $z \leq x \leq y$ and (f) $z \leq y \leq x$ one can see that $\mathcal{F}(x, y, t) \star \mathcal{F}(y, z, s) \leq \mathcal{F}(x, z, t+s)$. Therefore (FML4) is also satisfied and $(X, \mathcal{F}, \star)$ is a fuzzy metric-like space. But it is not a fuzzy metric space, as $F(x, x, t) = \frac{1}{x^2} \neq 1$ for all $x > 1$ and $t > 0$. $\square$

Now we define convergence, Cauchy sequence in fuzzy metric-like spaces and the completeness of fuzzy metric-like space.

Definition 2.10. A sequence $\{x_n\}$ in a fuzzy metric-like space $(X, F, \star)$ is said to be convergent to $x \in X$ if

$$\lim_{n\to\infty} F(x_n, x, t) = F(x, x, t) \text{ for all } t > 0.$$

Definition 2.11. A sequence $\{x_n\}$ in a fuzzy metric-like space $(X, F, \star)$ is said to be Cauchy if $\lim\limits_{n\to\infty} F(x_{n+p}, x_n, t)$ for all $t > 0, p \geq 1$ exists and is finite.

Definition 2.12. A fuzzy metric like spaces $(X, F, \star)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that

$$\lim_{n\to\infty} F(x_n, x, t) = F(x, x, t) = \lim_{n\to\infty} F(x_{n+p}, x_n, t) \text{ for all } t > 0, p \geq 1.$$

Remark 2.13. In a fuzzy metric-like space, the limit of a convergent sequence may not be unique. For instance, for a fuzzy metric-like space $(X, F, \star)$ given in Example 2.5 with $m = k = 1$. Define a sequence $\{x_n\}$ in X by $x_n = 1 - \dfrac{1}{n}$ for all $n \in \mathbb{N}$. If $x \geq 1$ then

$$\lim_{n\to\infty} F(x_n, x, t) = \lim_{n\to\infty} \frac{t}{t + \max\{x_n, x\}} = \lim_{n\to\infty} \frac{t}{t + x} = F(x, x, t)$$

for all $t > 0$. Therefore the sequence $\{x_n\}$ converges to all $x \in X$ with $x \geq 1$.

Remark 2.14. In a fuzzy metric-like space, a convergent sequence may not be a Cauchy sequence. Consider a fuzzy metric-like space $(X, F, \star)$ given in Remark 2.13. Define a sequence $\{x_n\}$ in X by $x_n = 1 + (-1)^n$ for all $n \in \mathbb{N}$. If $x \geq 2$ then

$$\lim_{n\to\infty} F(x_n, x, t) = \lim_{n\to\infty} \frac{t}{t + \max\{x_n, x\}} = \lim_{n\to\infty} \frac{t}{t + x} = F(x, x, t)$$

for all $t > 0$. Therefore a sequence $\{x_n\}$ converges to all $x \in X$ with $x \geq 2$ but it is not a Cauchy sequence as $\lim\limits_{n\to\infty} F(x_n, x_{n+p}, t)$ does not exist.

Analogous to [11] we give the following definition.

Definition 2.15. Let $(X, F, \star)$ be a fuzzy metric-like space. A mapping $T\colon X \to X$ is said to be a fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{F(Tx, Ty, t)} - 1 \leq k[\frac{1}{F(x, y, t)} - 1] \tag{1}$$

for all $x, y \in X$ and $t > 0$. Here k is called the fuzzy contractive constant of T.

3. **Fixed Point Results**

In this section, we prove some fixed point theorems in fuzzy metric-like spaces.

Theorem 3.1. *Let $(X, \digamma, \star)$ be a complete fuzzy metric-like space and $T\colon X \to X$ a fuzzy contractive mapping with fuzzy contractive constant k, then T has a unique fixed point $u \in X$ and $\digamma(u, u, t) = 1$ for all $t > 0$.*

Proof. For arbitrary $x_0 \in X$, define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$ then x_n is a fixed point of T. Therefore, we assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ and $t > 0$, we obtain from (1) that

$$\frac{1}{\digamma(x_n, x_{n+1}, t)} - 1 = \frac{1}{\digamma(Tx_{n-1}, Tx_n, t)} - 1 \leq k[\frac{1}{\digamma(x_{n-1}, x_n, t)} - 1].$$

Setting $\digamma(x_n, x_{n+1}, t) = \digamma_n(t)$ and $1 - k = \lambda$, it follows from the above inequality that

$$\frac{1}{\digamma_n(t)} \leq \frac{k}{\digamma_{n-1}(t)} + \lambda, \quad \text{for all} \quad t > 0.$$

From successive applications of the above inequality, we obtain

$$\begin{aligned}
\frac{1}{\digamma_n(t)} &\leq \frac{k^n}{\digamma_0(t)} + k^{n-1}\lambda + k^{n-2}\lambda + \cdots + \lambda \\
&\leq \frac{k^n}{\digamma_0(t)} + (k^{n-1} + k^{n-2} + \cdots + 1)\lambda \\
&\leq \frac{k^n}{\digamma_0(t)} + 1 - k^n,
\end{aligned}$$

that is,

$$\frac{1}{\frac{k^n}{\digamma_0(t)} + 1 - k^n} \leq \digamma_n(t) \quad \text{for all} \quad t > 0, n \in \mathbb{N}. \tag{2}$$

If $n \in \mathbb{N}, p \geq 1$, then we have

$$\begin{aligned}
\digamma(x_{n+p}, x_n, t) &\geq \digamma(x_n, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, x_{n+p}, \frac{t}{2}) \\
&\geq \digamma(x_n, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, x_{n+2}, \frac{t}{2^2}) \star \digamma(x_{n+2}, x_{n+p}, \frac{t}{2^2}) \\
&\geq \digamma(x_n, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, x_{n+2}, \frac{t}{2^2}) \\
&\quad \star \cdots \star \digamma(x_{n+p-2}, x_{n+p-1}, \frac{t}{2^{p-1}}) \star \digamma(x_{n+p-1}, x_{n+p}, \frac{t}{2^{p-1}}) \\
&= \digamma_n(\frac{t}{2}) \star \digamma_{n+1}(\frac{t}{2^2}) \star \cdots \star \digamma_{n+p-2}(\frac{t}{2^{p-1}}) \star \digamma_{n+p-1}(\frac{t}{2^{p-1}}).
\end{aligned}$$

Using (2) in the above inequality, we obtain

$$\begin{aligned}
\digamma(x_{n+p}, x_n, t) &\geq \frac{1}{\frac{k^n}{\digamma_0(\frac{t}{2})} + 1 - k^n} \star \frac{1}{\frac{k^{n+1}}{\digamma_0(\frac{t}{2^2})} + 1 - k^{n+1}} \\
&\quad \star \cdots \star \frac{1}{\frac{k^{n+p-1}}{\digamma_0(\frac{t}{2^{p-1}})} + 1 - k^{n+p-1}} \\
&\geq \frac{1}{\frac{k^n}{\digamma_0(\frac{t}{2})} + 1} \star \frac{1}{\frac{k^n}{\digamma_0(\frac{t}{2^2})} + 1} \star \cdots \star \frac{1}{\frac{k^n}{\digamma_0(\frac{t}{2^{p-1}})} + 1}.
\end{aligned}$$

As, $k \in (0,1)$, therefore using the properties of continuous t-norm we obtain from the above inequality that

$$\lim_{n\to\infty} F(x_{n+p}, x_n, t) = 1 \text{ for all } t > 0, p \geq 1.$$

Therefore $\{x_n\}$ is a Cauchy sequence in $(X, F, \star)$. By completeness of $(X, F, \star)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} F(x_n, u, t) = \lim_{n\to\infty} F(x_{n+p}, x_n, t) = F(u, u, t) = 1 \text{ for all } t > 0, p \geq 1. \tag{3}$$

We show that u is a fixed point of T.
Again, for all $t > 0, n \geq 0$, we obtain from (1) that

$$\frac{1}{F(Tx_n, Tu, t)} - 1 \leq k[\frac{1}{F(x_n, u, t)} - 1] = \frac{k}{F(x_n, u, t)} - k,$$

that is,

$$\frac{1}{\frac{k}{F(x_n,u,t)} + 1 - k} \leq F(Tx_n, Tu, t).$$

Using the above inequality we obtain

$$\begin{aligned} F(u, Tu, t) &\geq F(u, x_{n+1}, \frac{t}{2}) \star F(x_{n+1}, Tu, \frac{t}{2}) \\ &= F(u, x_{n+1}, \frac{t}{2}) \star F(Tx_n, Tu, \frac{t}{2}) \\ &\geq F(u, x_{n+1}, \frac{t}{2}) \star \frac{1}{\frac{k}{F(x_n,u,\frac{t}{2})} + 1 - k}. \end{aligned}$$

Taking limit as $n \to \infty$ and using (3) in the above inequality we obtain that $F(u, Tu, t) = 1$, that is, $Tu = u$. Therefore, u is a fixed point of T and $F(u, u, t) = 1$ for all $t > 0$.

For uniqueness, let v be another fixed point of T. Suppose, $F(u, v, t) < 1$ for some $t > 0$, then it follows from (1) that

$$\begin{aligned} \frac{1}{F(u, v, t)} - 1 &= \frac{1}{F(Tu, Tv, t)} - 1 \\ &\leq k[\frac{1}{F(u, v, t)} - 1] \\ &< \frac{1}{F(u, v, t)} - 1, \end{aligned}$$

a contradiction. Therefore, we must have $F(u, v, t) = 1$, for all $t > 0$, and therefore $u = v$. □

The following example illustrates the above theorem.

Example 3.2. Let $X = [0,1]$. Define $\star$ by $a \star b = ab$ and fuzzy set F in $X^2 \times (0, \infty)$ by

$$F(x, y, t) = \frac{t}{t + \max\{x, y\}}, \text{ for all } x, y \in X, t > 0.$$

Then $(X, F, \star)$ is a complete fuzzy metric-like space. If $T \colon X \to X$ is given by

$$Tx = \begin{cases} 0, & \text{if } x = 1; \\ \frac{x}{2}, & \text{otherwise.} \end{cases}$$

Then T is a fuzzy contractive mapping with fuzzy contractive constant $k \in [\frac{1}{2}, 1)$. Note that all conditions of Theorem 3.1 are satisfied. Moreover, T has a unique fixed point $0 \in X$ and $\digamma(0,0,t) = 1$ for all $t > 0$. Note that, T is not a fuzzy contractive mapping with respect to the standard fuzzy metric on X given by

$$M(x,y,t) = \frac{t}{t + |x-y|} \text{ for all } x,y \in X, t > 0.$$

Indeed, for $x = 1$, and $y = \dfrac{9}{10}$ there is no k in $(0,1)$ satisfying

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq k[\frac{1}{M(x,y,t)} - 1] \text{ for all } t > 0.$$

Therefore the result of Gregori and Sapena (see Theorem 4.4 of [11]) is not applicable to this case.

Corollary 3.3. *Let $(X, \digamma, \star)$ be a complete fuzzy metric-like space and $T\colon X \to X$ a mapping satisfying*

$$\frac{1}{\digamma(T^n x, T^n y, t)} - 1 \leq k[\frac{1}{\digamma(x,y,t)} - 1]$$

for some positive integer n, and for all $x, y \in X$, $t > 0$, where $k \in (0,1)$. Then T has a unique fixed point $u \in X$ and $\digamma(u,u,t) = 1$ for all $t > 0$.

Proof. From Theorem 3.1, T^n has a unique fixed point $u \in X$ and $\digamma(u,u,t) = 1$ for all $t > 0$. As $T^n(Tu) = T(T^n u) = Tu$, so Tu is also a fixed point of T^n and by uniqueness we have $Tu = u$. Since the fixed point of T is also a fixed point of T^n, therefore fixed point of T is unique. $\square$

In the next the theorem we replace the completeness of space by applying an additional condition on mapping T.

Theorem 3.4. *Let $(X, \digamma, \star)$ be a fuzzy metric-like space and $T\colon X \to X$ a fuzzy contractive mapping with fuzzy contractive constant k. Suppose that there exists $u \in X$ such that $\digamma(u,Tu,t) \geq \digamma(x,Tx,t)$ for all $x \in X$ and $t > 0$, then u becomes a unique fixed point of T and $\digamma(u,u,t) = 1$ for all $t > 0$.*

Proof. Let $\digamma_x(t) = \digamma(x,Tx,t)$ for all $x \in X$ and $t > 0$. Then by assumption $\digamma_u(t) \geq \digamma_x(t)$ for all $x \in X$ and $t > 0$. We claim that $\digamma(u,Tu,t) = 1$ for all $t > 0$. Indeed, if $\digamma_u(t) = \digamma(u,Tu,t) < 1$ for some $t > 0$, then it follows from (1) that

$$\begin{aligned} \frac{1}{\digamma_{Tu}(t)} - 1 &= \frac{1}{\digamma(Tu,TTu,t)} - 1 \\ &\leq k[\frac{1}{\digamma(u,Tu,t)} - 1] = k[\frac{1}{\digamma_u(t)} - 1] \\ &< \frac{1}{\digamma_u(t)} - 1, \end{aligned}$$

that is, $\digamma_u(t) < \digamma_{Tu}(t)$, $Tu \in X$, a contradiction. Therefore, we have $\digamma_x(t) = \digamma(u,Tu,t) = 1$ for all $t > 0$, and so $Tu = u$. Following the similar argument as in Theorem 3.1, uniqueness of fixed point of T follows. If $\digamma(u,u,t) < 1$ for some $t > 0$, then from (1), we have

$$\frac{1}{\digamma(u,u,t)} - 1 = \frac{1}{\digamma(Tu,Tu,t)} - 1 \leq k[\frac{1}{\digamma(u,u,t)} - 1] < \frac{1}{\digamma(u,u,t)} - 1,$$

a contradiction. Therefore, $\digamma(u,u,t) = 1$. □

Remark 3.5. In the above theorem it is shown that in a fuzzy metric-like space, the self fuzzy distance of the fixed point of a fuzzy contractive mapping is always 1, that is, the degree of self nearness of the fixed point of a fuzzy contractive mapping is perfect.

Example 3.6. Let $X = [0,1] \cap \mathbb{Q}$. Define $\star$ by $a \star b = \max\{a+b-1, 0\}$ (Lukasiewicz $t-$norm) and the fuzzy set $\digamma$ in $X^2 \times (0,\infty)$ by

$$\digamma(x,y,t) = 1 - \frac{\max\{x,y\}}{1+t} \text{ for all } x,y \in X, t > 0.$$

Then $(X,\digamma,\star)$ is a fuzzy metric-like space. It is obvious that, $(X,\digamma,\star)$ is not a complete fuzzy metric-like space. Let $T\colon X \to X$ be a mapping defined by

$$Tx = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,\frac{1}{2}] \cap \mathbb{Q}; \\ \frac{x}{2}, & \text{if } x \in (\frac{1}{2},1] \cap \mathbb{Q}. \end{cases}$$

Then T is a fuzzy contractive mapping with fuzzy contractive constant $k \in [\frac{1}{2},1)$. Note that, $\digamma(0,T0,t) = 1 \geq \digamma(x,Tx,t)$ for all $x \in X$ and $t > 0$. Thus, all the conditions of Theorem 3.4 are satisfied and 0 is the unique fixed point of T and $\digamma(0,0,t) = 1$ for all $t > 0$.

Let (X,ρ) be a metric-like space and $(X,\digamma,\star)$ be the induced standard fuzzy metric-like space. Suppose $(X,\digamma,\star)$ has an additional property that $\lim_{t\to\infty} \digamma(x,y,t) = 1$ for all $x,y \in X$. In the next theorem we investigate the existence and uniqueness of fixed point of mapping with a different contractive condition in such spaces.

Theorem 3.7. *Let $(X,\digamma,\star)$ be a complete fuzzy metric-like space such that*

$$\lim_{t\to\infty} \digamma(x,y,t) = 1$$

for all $x,y \in X$ and $T\colon X \to X$ a mapping satisfying the condition

$$\digamma(Tx,Ty,\alpha t) \geq \digamma(x,y,t) \tag{4}$$

for all $x,y \in X$, $t > 0$, where $\alpha \in (0,1)$. Then T has a unique fixed point $u \in X$ and $\digamma(u,u,t) = 1$ for all $t > 0$.

Proof. For arbitrary $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for any $n \in \mathbb{N}$, then x_n is a fixed point of T. We assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ and $t > 0$ we obtain from (4) that

$$\digamma(x_n,x_{n+1},t) = \digamma(Tx_{n-1},Tx_n,t) \geq \digamma(x_{n-1},x_n,\frac{t}{\alpha}).$$

Setting $\digamma(x_n,x_{n+1},t) = \digamma_n(t)$ and repeating the above process we obtain

$$\digamma_n(t) \geq \digamma_0(\frac{t}{\alpha^n}) \text{ for all } t > 0, n \in \mathbb{N}. \tag{5}$$

If $n \in \mathbb{N}$, $p \geq 1$, then we have

$$\begin{aligned}
\digamma(x_{n+p}, x_n, t) &\geq \digamma(x_n, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, x_{n+p}, \frac{t}{2}) \\
&\geq \digamma(x_n, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, x_{n+2}, \frac{t}{2^2}) \star \digamma(x_{n+2}, x_{n+p}, \frac{t}{2^2}) \\
&\geq \digamma(x_n, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, x_{n+2}, \frac{t}{2^2}) \\
&\quad \star \cdots \star \digamma(x_{n+p-2}, x_{n+p-1}, \frac{t}{2^{p-1}}) \star \digamma(x_{n+p-1}, x_{n+p}, \frac{t}{2^{p-1}}) \\
&= \digamma_n(\frac{t}{2}) \star \digamma_{n+1}(\frac{t}{2^2}) \star \cdots \star \digamma_{n+p-2}(\frac{t}{2^{p-1}}) \star \digamma_{n+p-1}(\frac{t}{2^{p-1}}).
\end{aligned}$$

Using (5) in the above inequality we obtain

$$\digamma(x_{n+p}, x_n, t) \geq \digamma_0(\frac{t}{2\alpha^n}) \star \digamma_0(\frac{t}{2^2\alpha^{n+1}}) \star \cdots \star \digamma_0(\frac{t}{2^{p-1}\alpha^{n+p-1}}).$$

As $\alpha \in (0,1)$ and $\lim_{t\to\infty} \digamma(x,y,t) = 1$ for all $x, y \in X$, so using the properties of continuous t-norm we obtain from the above inequality that

$$\lim_{n\to\infty} \digamma(x_{n+p}, x_n, t) = 1 \text{ for all } t > 0, p \geq 1.$$

Therefore $\{x_n\}$ is a Cauchy sequence in $(X, \digamma, \star)$. By completeness of $(X, \digamma, \star)$ there exists $u \in X$ such that

$$\lim_{n\to\infty} \digamma(x_n, u, t) = \lim_{n\to\infty} \digamma(x_{n+p}, x_n, t) = \digamma(u, u, t) = 1 \text{ for all } t > 0, p \geq 1. \quad (6)$$

We now show that u is a fixed point of T. To prove this, we proceed as follows:

For all $t > 0, n \geq 0$, we obtain from (4) that

$$\begin{aligned}
\digamma(u, Tu, t) &\geq \digamma(u, x_{n+1}, \frac{t}{2}) \star \digamma(x_{n+1}, Tu, \frac{t}{2}) \\
&= \digamma(u, x_{n+1}, \frac{t}{2}) \star \digamma(Tx_n, Tu, \frac{t}{2}) \\
&\geq \digamma(u, x_{n+1}, \frac{t}{2}) \star \digamma(x_n, u, \frac{t}{2\alpha}).
\end{aligned}$$

Taking limit as $n \to \infty$ and using (6) in the above inequality we obtain that $\digamma(u, Tu, t) = 1$. Therefore, u is a fixed point of T and $\digamma(u, u, t) = 1$, for all $t > 0$.

For uniqueness, let v be another fixed point of T. Using (4) we obtain

$$\digamma(u, v, t) = \digamma(Tu, Tv, t) \geq \digamma(u, v, \frac{t}{\alpha}),$$

that is,

$$\digamma(u, v, t) \geq \digamma(u, v, \frac{t}{\alpha}) \quad \text{for all} \quad t > 0.$$

As the above inequality holds for all $t > 0$, we obtain

$$\digamma(u, v, t) \geq \digamma(u, v, \frac{t}{\alpha^n}) \quad \text{for all} \quad n \in \mathbb{N}.$$

Taking limit as $n \to \infty$ and using the fact that $\lim_{t\to\infty} \digamma(x, y, t) = 1$ for all $x, y \in X$ it follows that $\digamma(u, v, t) = 1$, so $u = v$. $\square$

Next example shows that the condition $\lim_{t\to\infty} F(x,y,t) = 1$ for all $x, y \in X$ in Theorem 3.7 is not superfluous.

Example 3.8. Let $X = \{0,1\}$ and $c > 2$ be a fixed number. Define $\star$ by $a \star b = \max\{a+b-1, 0\}$ (Lukasiewicz $t-$norm) and the fuzzy set F in $X^2 \times (0,\infty)$ by

$$F(1,0,t) = F(0,1,t) = 1 - \frac{1}{c}, F(0,0,t) = F(1,1,t) = 1 - \frac{2}{c}.$$

Then $(X, F, \star)$ is a complete fuzzy metric-like space. Let $T\colon X \to X$ be a mapping defined by

$$T0 = 1, T1 = 0.$$

Then, all the conditions of Theorem 4, except $\lim_{t\to\infty} F(x,y,t) = 1$ for all $x, y \in X$ are satisfied with arbitrary $\alpha \in (0,1)$. Note that T has no fixed point in X.

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