KRASNER $F^{(m,n)}$-HYPERRINGS

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ABSTRACT. In this paper, the notion of fuzzy Krasner (m, n)-hyperrings ($F^{(m,n)}$-hyperrings) by using the notion of F^m-hyperoperations and F^n-operations is introduced and some related properties are investigated. In this regards, relationships between Krasner $F^{(m,n)}$-hyperrings and Krasner (m, n)-hyperrings are considered. We shall prove that every Krasner $F^{(m,n)}$-hyperring is extended by a Krasner $F^{(2,n)}$-hyperring. The concepts of normal F-hyperideals and homomorphisms of Krasner $F^{(m,n)}$-hyperrings are adopted. Also, the quotient of Krasner $F^{(m,n)}$-hyperrings by defining regular relations are studied. Finally, the classical isomorphism theorems of groups are generalized to Krasner $F^{(m,n)}$-hyperrings provided the F-hyperideals considered in them are normal.

1. **Introduction**

In this section, we describe the motivation and a survey of related works. Following the introduction of hypergroups by Marty in 1934 which is a natural extension of classical groups, many papers and books concerning hyperstructure theory have appeared in literature (see [1, 2, 3, 14]). In a classical group, the composition of two elements is an element, while in an hypergroup, the composition of two elements is a set. Since then, this theory has had applications to several domains. On the other hand, the notion of n-group is another generalization of group. It seems that the first idea of investigations of n-ary algebras goes back to Krasner's lecture at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. But the first article concerning the theory of n-groups was written (under inspiration of Emmy Noether) by Dörnte in 1928. In [5], Davvaz and Vougiouklis introduced the concept of n-hypergroups as a generalization of both hypergroups in the sense of Marty and n-groups. Then this concept was studied by many authors, for example see Leoreanu-Fotea and Davvaz [11], Davvaz et al [8].

Hyperring theory is an important branch of hyperstructure and has studied by a variety of authors. Some review of hyperring theory can be found in [6]. Krasner hyperring is a well known type of hyperrings [9]. For the first time M. Krasner who gave his name of an important class of hyperrings studied Krasner hyperrings, which is a triple $(R, +, \cdot)$, where $(R, +)$ is a canonical hypergroup and $(R, \cdot)$ is a semigroup, such that $\cdot$ is distributive with respect to $+$ (see [10]). Recently, Krasner

Received: December 2012; Revised: June 2014; Accepted: October 2014

Key words and phrases: Hyperstructure, Hyperoperation, F^m-hyperoperation, Krasner (m, n)-hyperring, Krasner $F^{(m,n)}$-hyperring, F-hyperideal, Regular and strongly regular relation.

(m,n)-hyperrings which are a suitable generalization of Krasner hyperrings were introduced and analyzed by Mirvakili and Davvaz [12].

In algebraic structures or hyperstructures, there are three isomorphism theorems that describe the relationship between quotients, homomorphisms, and subobjects.

Zadeh in 1965 [15] introduced the notion of fuzzy set which has achieved a great success in various fields and then fuzzy set theory developed by many others in mathematics and other branches of science. The concept of a fuzzy subgroup has defined and studied by Rosenfeld in 1971 [13]. He formulated the concept of a fuzzy subgroup of a group. The relationships between the fuzzy sets and algebraic hyperstructures have been considered by Ameri, Borzooei, Corsini, Davvaz, Kazanci, Leoreanu-Fotea, Yamak, Zahedi, Zhan and many others (for example see [4, 7, 16]). In [16, 17], Zahedi and Hasankhani, introduced the notion of F-polygroups. Basic algebraic results on F-polygroups are obtained and a notion of homomorphism for F-polygroups is introduced. The product of F-polygroups and the F-subpolygroups are also studied. Let I be the unit interval $[0,1]$ and for an arbitrary set H let I^H (resp. I^H_*) be the set of all (non empty) fuzzy subsets of H. An F-hyperoperation (or fuzzy hyperoperation) on H is a function $* : H \times H \to I^H_*$. If $a \in H$ and μ, η are in I^H_* then

$$\mu * \eta = \bigcup_{x \in \mathrm{supp}(\mu), y \in \mathrm{supp}(\eta)} x * y, \qquad a * \mu = \bigcup_{x \in \mathrm{supp}(\mu)} a * x,$$

where $\mathrm{supp}(\mu) = \{x \in H \mid \mu(x) \neq 0\}$. A couple $(H, *)$, where $*$ is an F-hyperoperation on H, is called an F-polygroup if the following four conditions are satisfied: (i) $(x * y) * z = x * (y * z)$, for every x, y, z in H; (ii) there exists $e \in H$ such that $x \in \mathrm{supp}(x * e \cap e * x)$, for every $x \in H$; (iii) for each $x \in H$, there exists a unique element $x^{-1} \in H$ such that $e \in \mathrm{supp}(x * x^{-1} \cap x^{-1} * x)$; (iv) $z \in \mathrm{supp}(x * y) \Rightarrow x \in \mathrm{supp}(z * y^{-1}) \Rightarrow y \in \mathrm{supp}(x^{-1} * z)$, for every x, y, z in H. Indeed, a hyperoperation assigns to every pair of elements of H a non-empty subset of H, while a fuzzy hyperoperation assigns to every pair of elements of H a non-zero fuzzy subset of H. Several important applications of fuzzy algebra, such as in automata theory and coding theory can be found in [2].

In this paper, we introduce the notion of Krasner $F^{(m,n)}$-hyperrings, F-hyperideals and then we obtain some related basic results. Further more, we extend a Krasner F-hyperring to a Krasner $F^{(m,n)}$-hyperring. In particular, we show that every Krasner $F^{(m,n)}$-hyperring is extended by a Krasner $F^{(2,n)}$-hyperring. Moreover, we introduce the notions of normal F-hyperideals, regular and strongly regular relations, quotient Krasner $F^{(m,n)}$-hyperrings and finally we state isomorphism theorems for Krasner $F^{(m,n)}$-hyperrings.

2. **Basic Definitions**

In this section, we gather all definitions we require of hyperstructures and fuzzy subsets. We shall use the notation x_i^j to denote the sequence $x_i, x_{i+1}, \ldots, x_j$. Also, the sequence $\overbrace{a, \ldots, a}^{i}$ will be denoted by $\overset{(i)}{a}$. Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the family of all non-empty subsets of H. In general, for a positive integer n an *n-hyperoperation* on H is a mapping $f : H^n \longrightarrow \mathcal{P}^*(H)$ where H^n denotes the set of n-tuples over H. If for all $(x_1, \ldots, x_n) \in H^n$, the set $f(x_1^n)$ is singleton, then f is called an *n-operation*.

If $A_1, \ldots, A_n$ are non-empty subsets of H, then we denote

$$f(A_1, \ldots, A_n) = \bigcup\{f(x_1, \ldots, x_n) \mid x_i \in A_i, 1 \leq i \leq n\}.$$

An n-hyperoperation f on H is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $i, j \in \{1, \ldots, n\}$ and $x_1^{2n-1} \in H$. We use the notation $f_{(k)}(x_1^{k(n-1)+1})$ to denote $\underbrace{f(f(\ldots f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$, where $k \geq 1$ and $x_1^{k(n-1)+1} \in H$.

In the sequel, I is the unit interval $[0, 1] \subseteq \mathbb{R}$. A *fuzzy subset* of H is a mapping $\mu : H \longrightarrow I$. We denote the set of all fuzzy subsets of H by I^H, that is $I^H = \{\mu \mid \mu : H \longrightarrow [0, 1]$ is a function$\}$. Let $\mu, \eta \in I^H$ and let $\{\mu_\alpha : \alpha \in \Lambda\}$ be a collection of fuzzy subsets of H, where Λ is a non-empty index set. Then, we define the fuzzy subsets $\mu \cup \eta$ and $\bigcup\limits_{\alpha \in \Lambda} \mu_\alpha$ as follows: for all $x \in H$, $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\}$ and $(\bigcup\limits_{\alpha \in \Lambda} \mu_\alpha)(x) = \bigvee\limits_{\alpha \in \Lambda} \{\mu_\alpha(x)\}$. If $\mu \in I^H$, then the *support* of μ, is defined by $supp(\mu) = \{x \in H \mid \mu(x) \neq 0\}$. If $A \subseteq H$ and $t \in I$, then we define $A_t \in I^H$ as follows:

$$A_t(x) = \begin{cases} t & \text{if } x \in A, \\ 0 & \text{if } x \in H \backslash A. \end{cases}$$

In particular, if A is a singleton set, say $\{a\}$, then $\{a\}_t$ is called a *fuzzy point* and is sometimes denoted by a_t. We denote χ_X the characteristic function of set X. Let $I_*^H = I^H \backslash \{0\}$. Then, by an *$F^n$-hyperoperation* on H we mean a function $f : H^n \longrightarrow I_*^H$, in other words for any $x_1, \ldots, x_n \in H^n$, $f(x_1, \ldots, x_n)$ is a non-zero fuzzy subset of H. If for all $(x_1, \ldots, x_n) \in \mathcal{H}^n$, the set $supp(f(x_1, \ldots, x_n))$ is singleton, then f is called an *F^n-operation*. If $\mu_1, \ldots, \mu_n \in I_*^H$, then $f(\mu_1, \ldots, \mu_n)$ is defined by

$$f(\mu_1, \ldots, \mu_n) = \bigcup_{x_i \in supp(\mu_i)} f(x_1, \ldots, x_n).$$

Let $\mu_1, \ldots, \mu_n, \mu \in I_*^H$, $A \in \mathcal{P}^*(H)$ and $x_1^n \in H$. Then, for $i \in \{1, \ldots, n\}$

(1) $f(x_1^{i-1}, \mu, x_{i+1}^n)$ denotes $f(\chi_{\{x_1\}}, \ldots, \chi_{\{x_{i-1}\}}, \mu, \chi_{\{x_{i+1}\}}, \ldots, \chi_{\{x_n\}})$,
(2) $f(x_1^{i-1}, A, x_{i+1}^n)$ denotes $f(\chi_{\{x_1\}}, \ldots, \chi_{\{x_{i-1}\}}, \chi_A, \chi_{\{x_{i+1}\}}, \ldots, \chi_{\{x_n\}})$,
(3) $f(\mu_1^{i-1}, x, \mu_{i+1}^n)$ denotes $f(\mu_1^{i-1}, \chi_{\{x\}}, \mu_{i+1}^n)$,
(4) $f(\mu_1^{i-1}, A, \mu_{i+1}^n)$ denotes $f(\mu_1^{i-1}, \chi_A, \mu_{i+1}^n)$.

For an F^n-hyperoperation (F^n-operation) f on H, the couple (H, f) is called F^n-semihypergroup (F^n-semigroup) if f is associative, i.e.,

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $i, j \in \{1, \ldots, n\}$ and $x_1^{2n-1} \in H$.

3. Krasner $F^{(m,n)}$-hyperrings

In this section, we introduce the notion of an $F^{(m,n)}$-hyperring which is a generalization of ideas presented by S. Mirvakili and B. Davvaz [12] and Zahedi and Hasankhani [16, 17]. We recall the following background from [12]. A Krasner (m,n)-hyperring is an algebraic hyperstructure (R,f,g) which satisfies the following axioms:

(1) $(R,f)$is a canonical m-hypergroup, i.e., f is an associative m-hyperoperation and R is equipped with a unitary operation $^{-1}: R \longrightarrow R$ such that the following axioms hold:

  (i) there exists an element $e \in R$ such that $e^{-1} = e$ and $f(\overset{(i-1)}{e}, x, \overset{(m-i)}{e}) = x$,

  (ii) $x \in f(x_1^m)$ implies $x_i \in f(x_{i-1}^{-1}, \ldots, x_1^{-1}, x, x_m^{-1}, \ldots, x_{i+1}^{-1})$,

  (iii) $f(x_1^m) = f(x_{\sigma(1)}^{\sigma(m)})$, for all $x_1^m \in R$ and for all $\sigma \in \mathbb{S}_m$,

(2) g is an associative n-operation,

(3) g is distributive with respect to f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, $1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)),$$

(4) for every $x_2^n \in R$ we have

$$g(e, x_2^n) = g(x_2, e, x_3^n) = \ldots = g(x_2^n, e) = \{e\},$$

where e is the identity element of (R,f).

If e and e' are two identity elements of R, then

$$\{e\} = g(e, \overset{(n-1)}{e'}) = \{e'\},$$

and so identity element of a Krasner (m,n)-hyperring is unique.

Definition 3.1. R is a non-empty set, f is an F^m-hyperoperation and g is an F^n-operation. A Krasner $F^{(m,n)}$-hyperring is an algebraic hyperstructure $\mathcal{R} = (R,f,g)$ which satisfies the following axioms:

(1) (R,f) is a canonical F^m-hypergroup, i.e., f is associative and R is equipped with a unitary operation $^{-1}: R \longrightarrow R$ such that the following axioms hold:

  (i) there exists $e \in R$ such that $e^{-1} = e$ and $supp(f(\overset{(i-1)}{e}, x, \overset{(m-i)}{e})) = \{x\}$, for all $x \in R$ and for all $i \in \{1, \ldots, m\}$,
  (in this case we say e is an F-*identity* element of R),

  (ii) $z \in supp(f(x_1^m))$ implies $x_i \in supp(f(x_{i-1}^{-1}, \ldots, x_1^{-1}, z, x_m^{-1}, \ldots, x_{i+1}^{-1}))$, for all $i \in \{1, \ldots, m\}$ and for all $z, x_1^m \in R$,

  (iii) $f(x_1^m) = f(x_{\sigma(1)}^{\sigma(m)})$, for all $x_1^m \in R$ and for all $\sigma \in \mathbb{S}_m$,

(2) (R,g) is an F^n-semigroup,

(3) g is distributive with respect to f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, $1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)),$$

(4) for every $x_2^n \in R$ we have

$$supp(g(e, x_2^n)) = supp(g(x_2, e, x_3^n)) = \ldots = supp(g(x_2^n, e)) = \{e\},$$

where e is the F-identity element of (R, f).

$\mathcal{R}$ is called *commutative* if

$$g(x_1^n) = g(x_{\sigma(1)}^{\sigma(n)}), \text{ for all } x_1^n \in R \text{ and for every } \sigma \in \mathbb{S}_n.$$

If e and e' are two F-identity elements of R, then

$$\{e\} = supp(g(e, \overset{(n-1)}{e'})) = \{e'\},$$

and so F-identity element of a Krasner $F^{(m,n)}$-hyperring is unique.

It is clear that every Krasner $F^{(m,0)}$-hyperring is a canonical F^m-hypergroup and every Krasner $F^{(0,n)}$-hyperring is an F^n-semigroup. For simplicity, a Krasner (2, 2)-hyperring is called a Krasner hyperring and a Krasner $F^{(2,2)}$-hyperring is called a Krasner F-hyperring. The next proposition can be proved easily using previously defined notions and thus we omit its proof.

Proposition 3.2. *Let (R, f) be a canonical F^n-hypergroup, $\mu_1, \ldots, \mu_n, \mu \in I_*^R$ and $x_1, \ldots, x_n, x$ be arbitrary elements of R. Then,*

(1) $(x^{-1})^{-1} = x$,
(2) $(supp(f(x_1^n)))^{-1} = supp(f(x_n^{-1}, \ldots, x_1^{-1}))$, *where* $A^{-1} = \{a^{-1} \mid a \in A\}$,
(3) $supp(f(x_1^{i-1}, \mu, x_{i+1}^n)) = \bigcup_{t \in supp(\mu)} supp(f(x_1^{i-1}, t, x_{i+1}^n))$, *for all* $i \in \{1, \ldots, n\}$,
(4) $supp(f(\mu_1, \ldots, \mu_n)) = \bigcup_{x_i \in supp(\mu_i)} supp(f(x_1, \ldots, x_n))$.

Example 3.3. Let $t \in (0, 1]$ and let G be a group such that $x^2 = e$, for all $x \in G$. We define an F^m-operation f on G as follows:

$$f(x_1, \ldots, x_m)(z) = e_t(x_1 \ldots x_m z), \text{ for all } x_1, \ldots, x_m, z \in G.$$

Then, (G, f) is a canonical F^m-group. It is not difficult to see that (G, f, g) is a Krasner $F^{(m,n)}$-hyperring, where g is an F^n-operation defined by

$$g(x_1, \ldots, x_n) = e_t, \text{ for all } x_1, \ldots, x_n \in G.$$

Example 3.4. Let $t \in (0, 1]$ and let $G =< a >$ be the cyclic group of order 3. For every ordered m-tuple $\mathcal{A} = (a_1, \ldots, a_m)$ over G, we remove non-identity iterated entries in $\mathcal{A}$ in the following way: for each iterated entry we keep the first occurrence and replace the rest with e. We denote the new m-tuple by $C(\mathcal{A})$. Now we consider an F^m-hyperoperation f on G with the following properties:

(i) $f(a_1, \ldots, a_m) = f(a_{\sigma(1)}, \ldots, a_{\sigma(m)})$, for all $\sigma \in \mathbb{S}_m$,
(ii) $f(\mathcal{A}) = f(C(\mathcal{A}))$, for all $\mathcal{A} \in G^m$,
(iii) $f(x, \overset{(m-1)}{e}) = x_t$, for all $x \in G$,
(iv) $f(a^2, a, \overset{(m-2)}{e}) = G_t$.

Let $g : G \longrightarrow I_*^G$ be an F^2-operation with the following table:

g	e	a	a^2
e	e_t	e_t	e_t
a	e_t	a_t	$(a^2)_t$
a^2	e_t	$(a^2)_t$	a_t

It is easy to verify that (G, f, g) is a Krasner $F^{(m,2)}$-hyperring.

Example 3.5. Let $t_1, t_2 \in (0, 1]$. Consider $(F, +, .)$ a field, G a subgroup of $(F^*, .)$ and take $F/G = \{aG \mid a \in F\}$ with the F^m-hyperoperation and the F^n-operation given by

$$f(x_1G, \ldots, x_mG)(zG) = (x_1G + \ldots + x_mG)_{t_1}(z), \text{ for all } x_1^m, z \in F,$$

$$g(x_1G, \ldots, x_nG)(zG) = (x_1 \ldots x_n)_{t_2}(z), \text{ for all } x_1^n, z \in F.$$

If we define the unitary operation $^{-1}$ on F/G by $(xG)^{-1} = -xG$, then $(F/G, f, g)$ is a Krasner $F^{(m,n)}$-hyperring.

Theorem 3.6. *(1) Let (R, f', g') be a Krasner (m, n)-hyperring. Then, (R, f, g) is a Krasner $F^{(m,n)}$-hyperring, where*

$$f(x_1^m) = \chi_{f'(x_1^m)}, \text{ for all } x_1^m \in R,$$
$$g(x_1^n) = \chi_{g'(x_1^n)}, \text{ for all } x_1^n \in R.$$

((R, f, g) is called the Krasner $F^{(m,n)}$-hyperring induced by (R, f', g')).

(2) Let (R, f, g) be a Krasner $F^{(m,n)}$-hyperring. Then, (R, f', g') is a Krasner (m, n)-hyperring, where

$$f'(x_1^m) = supp(f(x_1^m)), \text{ for all } x_1^m \in R,$$
$$g'(x_1^n) = supp(g(x_1^n)), \text{ for all } x_1^n \in R.$$

((R, f', g') is called the Krasner (m, n)-hyperring extracted from (R, f, g)).

Proof. 1) Since $supp(f(x_1^m)) = f'(x_1^m)$ and (R, f') is an m-hypergroup, it follows that (R, f) is an F^m-hypergroup. Similarly, from $supp(g(y_1^n)) = g'(y_1^n)$ and that (R, g') is an n-semigroup we conclude that (R, g) is an F^n-semigroup. For every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, $1 \leq i \leq n$ from

$$\begin{aligned} g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) &= \bigcup_{t \in supp(f(x_1^m))} g(a_1^{i-1}, t, a_{i+1}^n) \\ &= \chi_{g'(a_1^{i-1}, f'(x_1^m), a_{i+1}^n)}. \end{aligned}$$

and that g' is distributive with respect to f', we obtain that g is distributive with respect to f. Finally, for every $x_2^n \in R$ we have

$$supp(g(e, x_2^n)) = g'(e, x_2^n) = \{e\}.$$

In a similar manner, we have $supp(g(x_2, e, x_3^n)) = \ldots = supp(g(x_2^n, e)) = \{e\}$.

2) An argument similar to that in (1) establishes (2). □

Let (R, f, g) be a Krasner $F^{(m,n)}$-hyperring and let S be a non-empty subset of R that is support closed under f and g, i.e., $supp(f(x_1^m)) \subseteq S$ and $supp(g(y_1^n)) \subseteq S$ for all $x_1^m, y_1^n \in S$. If (S, f, g) is itself a Krasner $F^{(m,n)}$-hyperring, then S is called a Krasner F-*subhyperring* of (R, f, g). A Krasner F-subhyperring $\mathcal{I}$ of a Krasner $F^{(m,n)}$-hyperring (R, f, g) is an F-*hyperideal* provided

$$x_1^n \in R \text{ and } i \in \{1, \ldots, n\} \Longrightarrow supp(g(x_1^{i-1}, \mathcal{I}, x_{i+1}^n)) \subseteq \mathcal{I}.$$

We conclude that if $\mathcal{I}$ and $\mathcal{J}$ are F-hyperideals of a Krasner $F^{(m,n)}$-hyperring (R, f, g) such that $\mathcal{I} \cap \mathcal{J} = \{e\}$, then $supp(g(a, b, x_3^n)) = \{e\}$ when $a \in \mathcal{I}$ and $b \in \mathcal{J}$ and $x_3^n \in R$. It can be shown that the union of a chain $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \ldots$ of F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $\mathcal{R}$ is an F-hyperideal of $\mathcal{R}$.

Example 3.7. Let a belong to a Krasner $F^{(m,2)}$-hyperring (R, f, g). Let

$$S = \{s \in R \mid supp(g(a, s)) = \{e\}\} \text{ and } T = \{supp(g(a, g(t, a))) \mid t \in R\}.$$

Then, S and T are Krasner F-subhyperrings.

Example 3.8. In the Krasner $F^{(m,n)}$-hyperring defined in Example 3.3, for each $x \in G$, the subset $\{e, x\}$ is an F-hyperideal and in the Krasner $F^{(m,2)}$-hyperring defined in Example 3.4 we have just two F-hyperideals $\{e\}$ and G.

It would be useful to have some criterions for deciding whether a given subset of a Krasner $F^{(m,n)}$-hyperring is an F-hyperideal or not. This is the purpose of the next lemma.

Lemma 3.9. *Let (R, f, g) be a Krasner $F^{(m,n)}$-hyperring and let $\mathcal{I}$ be a non-empty subset of R. Then, $\mathcal{I}$ is an F-hyperideal if and only if*

(1) $e \in \mathcal{I}$,
(2) $x^{-1} \in \mathcal{I}$, *for all* $x \in \mathcal{I}$,
(3) $supp(f(x_1^m)) \subseteq \mathcal{I}$, *for all* $x_1^m \in \mathcal{I}$,
(4) $supp(g(x_1^{i-1}, \mathcal{I}, x_{i+1}^n)) \subseteq \mathcal{I}$, *for all* $x_1^n \in R$ *and for all* $i \in \{1, \ldots, n\}$.

Proof. $\Rightarrow$) Since $(\mathcal{I}, f)$ is an F^m-hypergroup, conditions (1), (2) and (3) are satisfied. Condition (4) follows from definition of F-hyperideal.

$\Leftarrow$) Associativity and commutativity of f on $\mathcal{I}$ is inherited from associativity and commutativity of f on R. Using conditions (1), (2) and (3) it follows that $(\mathcal{I}, f)$ is a canonical F^m-hypergroup. Obviously, conditions (2), (3) and (4) of Definition 3.1 hold. □

Corollary 3.10. *Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring and let $\mathcal{I}$ be a non-empty subset of R. Then, $\mathcal{I}$ is an F-hyperideal of $\mathcal{R}$ if and only if $\mathcal{I}$ is a hyperideal of $\mathcal{R}' = (R, f', g')$ where $\mathcal{R}'$ is the Krasner (m, n)-hyperring extracted from $\mathcal{R}$.*

Corollary 3.11. *If $\{\mathcal{I}_\lambda \mid \lambda \in \Lambda\}$ is a set of F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $\mathcal{R} = (R, f, g)$, then $\mathcal{I} = \bigcap_{\lambda \in \Lambda} \mathcal{I}_\lambda$ is an F-hyperideal of $\mathcal{R}$.*

Let X be a non-empty subset of a Krasner $F^{(m,n)}$-hyperring $\mathcal{R} = (R, f, g)$. We define the F-hyperideal generated by X, to be the intersection of all F-hyperideals of $\mathcal{R}$ contain X. An F-hyperideal generated by X is denoted by $\langle X \rangle$. Notice that there exist at least one such F-hyperideal, R itself. That $\langle X \rangle$ is an F-hyperideal follows from Corollary 3.11. In a real sense $\langle X \rangle$ is the smallest F-hyperideal of $\mathcal{R}$ containing X. Let $A_1, \ldots, A_n$ be subsets of R. We denote the product of A_i's by $\prod\limits_{i=1}^{n} A_i$ and

$$\prod_{i=1}^{n} A_i = \bigcup_{k=1}^{\infty} \{x \in supp(f_{(k)}([g(a_{i1}^{in})]_{i=1}^{i=m_k})) \mid a_{ij} \in A_j,\ m_k = k(m-1)+1\}.$$

Lemma 3.12. *Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring and let $\mathcal{I}_1^m$ be F-hyperideals of $\mathcal{R}$. Then,*

(1) *$supp(f(\mathcal{I}_1^m))$ is an F-hyperideal of $\mathcal{R}$,*
(2) *$supp(f(\mathcal{I}_1^{i-1}, e, \mathcal{I}_{i+1}^m))$ is an F-hyperideal of $supp(f(\mathcal{I}_1^m))$,*

for all $i \in \{1, \ldots, m\}$.

Proof. (1) Let $\mathcal{R}' = (R, f', g')$ be the Krasner (m, n)-hyperring extracted from $\mathcal{R}$. By Corollary 3.10, $\mathcal{I}_1^m$ are hyperideals of $\mathcal{R}'$ and so by Lemma 3.4 of [12], $f'(\mathcal{I}_1^m)$ is a hyperideal of $\mathcal{R}'$. Since $f'(\mathcal{I}_1^m) = supp(f(\mathcal{I}_1^m))$, by Corollary 3.10, $supp(f(\mathcal{I}_1^m))$ is an F-hyperideal of $\mathcal{R}$.

(2) It is clear. □

Lemma 3.13. *Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring and let $\mathcal{I}_1^m$ be F-hyperideals of $\mathcal{R}$. Then,*

(1) *if $\mathcal{I}_1^n$ are subsets of R and $\mathcal{I}_j$ is an F-hyperideal of $\mathcal{R}$, for at least one $j \in \{1, \ldots, n\}$ and $\mathcal{R}$ is commutative, then $\prod\limits_{i=1}^{n} \mathcal{I}_i$ is an F-hyperideal of $\mathcal{R}$,*
(2) *$\langle \prod\limits_{i=1}^{n} \mathcal{I}_i \rangle \subseteq \bigcap\limits_{i=1}^{n} \mathcal{I}_i$, where $\mathcal{I}_1^n$ are F-hyperideals of $\mathcal{R}$.*

Proof. (1) Let $\mathcal{R}' = (R, f', g')$ be the Krasner (m, n)-hyperring extracted from $\mathcal{R}$. It is easy to see that

$$\prod_{i=1}^{n} \mathcal{I}_i = \bigcup_{k=1}^{\infty} \{x \in f'_{(k)}([g'(a_{i1}^{in})]_{i=1}^{i=m_k}) \mid a_{ij} \in \mathcal{I}_j,\ m_k = k(m-1)+1\}.$$

By Corollary 3.10, $\mathcal{I}_j$ is a hyperideal of $\mathcal{R}'$ and since $\mathcal{R}'$ is commutative, by Lemma 3.4 of [12], $\prod\limits_{i=1}^{n} \mathcal{I}_i$ is a hyperideal of $\mathcal{R}'$. Therefore by Corollary 3.10, $\prod\limits_{i=1}^{n} \mathcal{I}_i$ is an F-hyperideal of $\mathcal{R}$.

(2) By (1), $\prod\limits_{i=1}^{n} \mathcal{I}_i$ is an F-hyperideal and therefore we have $\langle \prod\limits_{i=1}^{n} \mathcal{I}_i \rangle = \prod\limits_{i=1}^{n} \mathcal{I}_i$. On the other hand, since for each $1 \leq i \leq n$, $\mathcal{I}_i$ is an F-hyperideal, by definition of $\prod\limits_{i=1}^{n} \mathcal{I}_i$ we have $\prod\limits_{i=1}^{n} \mathcal{I}_i \subseteq \bigcap\limits_{i=1}^{n} \mathcal{I}_i$. □

Lemma 3.14. *Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring and let $\mathcal{I}$ be an F-hyperideal of $\mathcal{R}$. Then,*

(1) *if $a_2^m \in \mathcal{I}$, then $supp(f(\mathcal{I}, a_2^m)) = \mathcal{I}$,*
(2) *if $x \in \mathcal{I}$ and $a_3^m \in R$, then $supp(f(\mathcal{I}, x, a_3^m)) = supp(f(\mathcal{I}, e, a_3^m))$,*
(3) *if $s \in supp(f(\mathcal{I}, r, \overset{(m-2)}{e}))$, then $supp(f(\mathcal{I}, r, \overset{(m-2)}{e})) = supp(f(\mathcal{I}, s, \overset{(m-2)}{e}))$,*
(4) *if $s_i \in supp(f(\mathcal{I}, r_i, \overset{(m-2)}{e}))$ for every $i \in \{2, \ldots, m\}$, then $supp(f(\mathcal{I}, s_2^m)) = supp(f(\mathcal{I}, r_2^m))$.*

Proof. (1) Let $\mathcal{R}' = (R, f', g')$ be the Krasner (m,n)-hyperring extracted from $\mathcal{R}$. Then we have $supp(f(\mathcal{I}, a_2^m)) = f'(\mathcal{I}, a_2^m)$. On the other hand by Lemma 3.4 of [12] we have $f'(\mathcal{I}, a_2^m) = \mathcal{I}$.

(2) Let $x \in \mathcal{I}$ and let $a_3^m \in R$. Then, by using (1) we have

$$\begin{aligned} supp(f(\mathcal{I}, x, a_3^m)) = supp(f(\mathcal{I}, f(x, \overset{(m-1)}{e}), a_3^m)) &= supp(f(f(\mathcal{I}, x, \overset{(m-2)}{e}), e, a_3^m)) \\ &= supp(f(\mathcal{I}, e, a_3^m)). \end{aligned}$$

(3) Let $s \in supp(f(\mathcal{I}, r, \overset{(m-2)}{e}))$, then

$$\begin{aligned} supp(f(\mathcal{I}, s, \overset{(m-2)}{e})) &\subseteq supp(f(\mathcal{I}, f(\mathcal{I}, \overset{(m-2)}{e}, r), \overset{(m-2)}{e})) \\ &= supp(f(f(\mathcal{I}, \mathcal{I}, \overset{(m-2)}{e}), r, \overset{(m-2)}{e})) \\ &= supp(f(\mathcal{I}, r, \overset{(m-2)}{e})). \end{aligned}$$

On the other hand $s \in supp(f(\mathcal{I}, r, \overset{(m-2)}{e}))$ implies that $r \in supp(f(\mathcal{I}, s, \overset{(m-2)}{e}))$ and so $supp(f(\mathcal{I}, r, \overset{(m-2)}{e})) \subseteq supp(f(\mathcal{I}, s, \overset{(m-2)}{e}))$.

(4) Using (3) we have

$$\begin{aligned} supp(f(\mathcal{I}, s_2^m)) &= supp(f(\mathcal{I}, f(\mathcal{I}, s_2, \overset{(m-2)}{e}), \ldots, f(\mathcal{I}, s_m, \overset{(m-2)}{e}))) \\ &= supp(f(\mathcal{I}, f(\mathcal{I}, r_2, \overset{(m-2)}{e}), \ldots, f(\mathcal{I}, r_m, \overset{(m-2)}{e}))) \\ &= supp(f(\mathcal{I}, r_2^m)). \end{aligned}$$

□

Lemma 3.15. *Let $(R, \circ, *)$ be a Krasner F-hyperring and $\mu_1, \mu_2 \in I_*^R$. Then, for every $x \in R$ we have $(\mu_1 \circ \mu_2) * x = (\mu_1 * x) \circ (\mu_2 * x)$.*

Proof. Let t be an arbitrary element of R. Then,

$$\begin{aligned} ((\mu_1 \circ \mu_2) * x)(t) &= \bigvee \{(y * x)(t) \mid y \in supp(\mu_1 \circ \mu_2)\} \\ &= \bigvee \{((r \circ s) * x)(t) \mid r \in supp(\mu_1), s \in supp(\mu_2)\} \\ &= \bigvee \{((r * x) \circ (s * x))(t) \mid r \in supp(\mu_1), s \in supp(\mu_2)\} \\ &= \bigvee \{(u \circ v)(t) \mid u \in supp(\mu_1 * x), v \in supp(\mu_2 * x)\} \\ &= ((\mu_1 * x) \circ (\mu_2 * x))(t). \end{aligned}$$

□

Corollary 3.16. *If $(R,\circ,*)$ is a Krasner F-hyperring and $\mu_1,\ldots,\mu_n \in I_*^R$, then*

$$(\mu_1 \circ \ldots \circ \mu_n) * x = (\mu_1 * x) \circ \ldots \circ (\mu_n * x).$$

Corollary 3.17. *If $(R,\circ,*)$ is a Krasner F-hyperring and $a_1^n, x_1^m \in R$, then for every $i \in \{1,\ldots,n\}$ we have*

$$a_1 * \ldots * a_{i-1} * (x_1 \circ \ldots \circ x_m) * a_{i+1} * \ldots * a_n =$$

$$(a_1 * \ldots * a_{i-1} * x_1 * a_{i+1} * \ldots * a_n) \circ \ldots \circ (a_1 * \ldots * a_{i-1} * x_m * a_{i+1} * \ldots * a_n).$$

Theorem 3.18. *Let $(R,\circ,*)$ be a Krasner F-hyperring. If f is an F^m-hyperoperation and g is an F^n-operation on R as follows:*

$$f(x_1^m) = x_1 \circ \ldots \circ x_m, \text{ for all } x_1^m \in R,$$
$$g(x_1^n) = x_1 * \ldots * x_n, \text{ for all } x_1^n \in R,$$

then (R,f,g) is a Krasner $F^{(m,n)}$-hyperring.

((R,f,g) is called the Krasner $F^{(m,n)}$-hyperring extended by $(R,\circ,*)$).

Proof. Since $\circ$ and $*$ are well-defined, f and g are well-defined. For every $i \in \{1,\ldots,n\}$ we have

$$\begin{aligned} f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) &= \bigcup_{t \in supp(f(x_i^{m+i-1}))} f(x_1^{i-1}, t, x_{m+i}^{2m-1}) \\ &= \bigcup_{t \in supp(x_i \circ \ldots \circ x_{m+i-1})} x_1 \circ \ldots x_{i-1} \circ t \circ x_{m+i} \circ \ldots \circ x_{2m-1} \\ &= x_1 \circ \ldots \circ x_{2m-1}. \end{aligned}$$

Therefore, f is associative. Now, let e be F-identity element of $(R,\circ)$. Then, for every $i \in \{1,\ldots,n\}$ we have

$$\begin{aligned} supp(f(\overset{(i-1)}{e}, x, \overset{(m-i)}{e})) = supp(f(\overset{(m-1)}{e}, x)) &= supp(\underbrace{e \circ \ldots \circ e}_{m-1} \circ x) \\ &= \bigcup_{y \in supp(\underbrace{e \circ \ldots \circ e}_{m-1})} supp(y \circ x) \\ &= supp(e \circ x) = \{x\}. \end{aligned}$$

If $z \in supp(f(x_1^m))$, then we have $z \in supp(x_1 \circ \ldots \circ x_m)$ and since $(R,\circ)$ is commutative, there exists $t \in supp(x_1 \circ \ldots x_{i-1} \circ x_{i+1} \circ \ldots \circ x_m)$ such that $z \in supp(x_i \circ t)$. By hypothesis we have $x_i \in supp(z \circ t^{-1})$. Using Proposition 3.2, we have $x_i \in supp(f(x_{i-1}^{-1}, \ldots, x_1^{-1}, z, x_m^{-1}, \ldots, x_{i+1}^{-1}))$. Therefore (R,f) is a canonical F^m-hypergroup. Using Corollary 3.17, we can prove that g is distributive with respect to f. One can easily check that other conditions of Definition 3.1 are valid and so (R,f,g) is a Krasner $F^{(m,n)}$-hyperring. □

Theorem 3.19. *Let (R,f,g) be a Krasner $F^{(m,n)}$-hyperring. If we define the F^2-hyperoperation $\circ$ as follows:*

$$x \circ y = f(x, y, \overset{(m-2)}{e}), \quad \forall x, y \in R,$$

then (R, f, g) *is extended by* $(R, \circ, g)$.

Proof. It is easy to verify that $(R, \circ)$ is a canonical F^2-hypergroup. For every $\mu_1, \mu_2 \in I_*^R$ we have

$$\mu_1 \circ \mu_2 = \bigcup_{\substack{x \in supp(\mu_1) \\ y \in supp(\mu_2)}} x \circ y = \bigcup_{\substack{x \in supp(\mu_1) \\ y \in supp(\mu_2)}} f(x, y, \overset{(m-2)}{e}) = f(\mu_1, \mu_2, \overset{(m-2)}{e}).$$

Now, we can show that g is distributive with respect to $\circ$. Let $a_1^n, x_1^2 \in R$ be arbitrary elements. Then for every $i \in \{1, \ldots, n\}$ we have

$$\begin{aligned} g(a_1^{i-1}, x_1 \circ x_2, a_{i+1}^n) &= g(a_1^{i-1}, f(x_1, x_2, \overset{(m-2)}{e}), a_{i+1}^n) \\ &= f(g(a_1^{i-1}, x_1, a_{i+1}^n), g(a_1^{i-1}, x_2, a_{i+1}^n), \overset{(m-2)}{g(a_1^{i-1}, e, a_{i+1}^n)}) \\ &= f(g(a_1^{i-1}, x_1, a_{i+1}^n), g(a_1^{i-1}, x_2, a_{i+1}^n), \overset{(m-2)}{e}) \\ &= g(a_1^{i-1}, x_1, a_{i+1}^n) \circ g(a_1^{i-1}, x_2, a_{i+1}^n). \end{aligned}$$

This completes the proof. □

Corollary 3.20. *Every Krasner* $F^{(m,n)}$*-hyperring is extended by a Krasner* $F^{(2,n)}$*-hyperring.*

4. **Quotient Krasner $F^{(m,n)}$-hyperrings**

The goal of this section is to introduce an equivalence relation on a Krasner $F^{(m,n)}$-hyperring and to construct a quotient Krasner $F^{(m,n)}$-hyperring.

Let θ be an equivalence relation on a non-empty set H. If A and B are non-empty subsets of H, then

(1) $A\,\overline{\theta}\,B$ means that for every $a \in A$, there exists $b \in B$ such that $a\,\theta\,b$ and for every $b \in B$ there exists $a \in A$ such that $a\,\theta\,b$,
(2) we write $A\,\overline{\overline{\theta}}\,B$ if for every $a \in A$ and for every $b \in B$, one has $a\,\theta\,b$.

An equivalence relation θ defined on a canonical F^m-hypergroup (R, f) is called *regular* if for every $x_1^m, y_1^m \in R$, $x_1 \theta y_1, \ldots, x_m \theta y_m$ we have $supp(f(x_1^m))\,\overline{\theta}\,supp(f(y_1^m))$ and θ is called *strongly regular* if $x_1 \theta y_1, \ \ldots \ , x_m \theta y_m$ implies that

$$supp(f(x_1^m))\,\overline{\overline{\theta}}\,supp(f(y_1^m)).$$

It is easy to verify that for a regular relation θ we have

$$\{\theta[x] \mid x \in supp(f(x_1, \ldots, x_m))\} = \{\theta[x] \mid x \in supp(f(\theta[x_1], \ldots, \theta[x_m]))\},$$

where $\theta[x]$ is the equivalence class of x. Also, whenever θ is a strongly regular relation, reflexivity of θ implies that for every $z_1, z_2 \in supp(f(x_1^m))$ we have $\theta[z_1] = \theta[z_2]$ and so $\{\theta[x] \mid x \in supp(f(x_1^m))\}$ is singleton.

In the sequel, we shall use the notation $\theta_{[x_i]}^{[x_j]}$ to denote the sequence$\theta[x_i], \ldots, \theta[x_j]$.

Theorem 4.1. *Let (R, f, g) be a Krasner $F^{(m,n)}$-hyperring and let θ be a regular relation on R. Then, $[R : \theta] = \{\theta[x] \mid x \in R\}$ is a Krasner (m, n)-hyperring with the m-hyperoperation $f|_\theta$ and n-operation $g|_\theta$ defined as follows:*

$$f|_\theta(\theta_{[x_1]}^{[x_m]}) = \{\theta[z] \mid z \in supp(f(x_1^m))\}, \text{ for all } x_1^m \in R,$$

$$g|_\theta(\theta_{[x_1]}^{[x_n]}) = \{\theta[supp(g(x_1^n))]\}, \text{ for all } x_1^n \in R.$$

Proof. Since θ is a regular relation, $f|_\theta$ is well-defined. Associativity of f implies that $f|_\theta$ is associative. If we define the unitary operation $^{-1} : [R : \theta] \longrightarrow [R : \theta]$ by $(\theta[x])^{-1} = \theta[x^{-1}]$, then $([R : \theta], f|_\theta)$ is a canonical m-hypergroup.

We can check that $g|_\theta$ is associative and that $g|_\theta$ is distributive with respect to $f|_\theta$. Obviously, for every $\theta_{[x_2]}^{[x_n]} \in [R : \theta]$ we have

$$g|_\theta(\theta[e], \theta_{[x_2]}^{[x_n]}) = g|_\theta(\theta[x_2], \theta[e], \theta_{[x_3]}^{[x_n]}) = \ldots = g|_\theta(\theta_{[x_2]}^{[x_n]}, \theta[e]) = \{\theta[e]\}.$$

This completes the proof. □

We let $[R : \theta]_i$ denote the *quotient* Krasner $F^{(m,n)}$-hyperring induced by the (m, n)-hyperring $([R : \theta], f|_\theta, g|_\theta)$.

Let $\mathcal{I}$ be an F-hyperideal of a Krasner $F^{(m,n)}$-hyperring (R, f, g). We define the relation $\mathcal{I}^*$ on R as follows:

$$x\mathcal{I}^*y \text{ if and only if } x \in supp(f(\mathcal{I}, y, \overset{(m-2)}{e})).$$

It is not difficult to see that the relation $\mathcal{I}^*$ is an equivalence relation.

Definition 4.2. Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring. An F-hyperideal $\mathcal{I}$ of $\mathcal{R}$ is called *normal* if for every $x \in R$,

$$supp(f(\mathcal{I}, f(x, x^{-1}, \overset{(m-2)}{e}), \overset{(m-2)}{e})) \subseteq \mathcal{I}.$$

For example, in the Krasner $F^{(m,n)}$-hyperring defined in Example 3.3, every F-hyperideal of (G, f, g) is normal.

If $\mathcal{I}$ is a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring $\mathcal{R} = (R, f, g)$ and $x \in R$ be an arbitrary element, then

$$\mathcal{I} = supp(f(\mathcal{I}, \overset{(m-1)}{e})) \subseteq supp(f(\mathcal{I}, f(x, x^{-1}, \overset{(m-2)}{e}), \overset{(m-2)}{e})).$$

Thus, for every $x \in R$, we have $supp(f(\mathcal{I}, f(x, x^{-1}, \overset{(m-2)}{e}), \overset{(m-2)}{e})) = \mathcal{I}$.

Remark 4.3. It can be easily shown that if $\mathcal{I}$ is a normal F-hyperideal and $a_1^m \in R$ then $x \in supp(f(\mathcal{I}, f(a_1^m), \overset{(m-2)}{e}))$ implies that $supp(f(a_1^m)) \subseteq supp(f(\mathcal{I}, x, \overset{(m-2)}{e}))$.

Lemma 4.4. *For F-hyperideals $\mathcal{I}_1^m$ of a Krasner $F^{(m,n)}$-hyperring, where $\mathcal{I}_j$ is normal, for some $1 \le j \le m$, we have*

(1) $\bigcap_{i=1}^{m} \mathcal{I}_i$ *is a normal F-hyperideal of $\mathcal{I}_k$, where $1 \le k \le m$,*
(2) $\mathcal{I}_j$ *is a normal F-hyperideal of $supp(f(\mathcal{I}_1^m))$.*

Proof. It is straightforward. □

Proposition 4.5. *Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring, and let $\mathcal{I}$ be a normal F-hyperideal of $\mathcal{R}$. Then, for $x, y \in R$, the following are equivalent:*

(1) $x\mathcal{I}^*y$,

(2) $supp(f(x, y^{-1}, \overset{(m-2)}{e})) \subseteq \mathcal{I}$,

(3) $supp(f(x, y^{-1}, \overset{(m-2)}{e})) \cap \mathcal{I} \neq \emptyset$.

Proof. $1 \Longrightarrow 2)$ Since $x \in supp(f(\mathcal{I}, y, \overset{(m-2)}{e}))$, we have

$$\begin{aligned} supp(f(x, y^{-1}, \overset{(m-2)}{e})) &\subseteq supp(f(f(\mathcal{I}, y, \overset{(m-2)}{e}), y^{-1}, \overset{(m-2)}{e})) \\ &= supp(f(\mathcal{I}, f(y, y^{-1}, \overset{(m-2)}{e}), \overset{(m-2)}{e})) \\ &= \mathcal{I}. \end{aligned}$$

$2 \Longrightarrow 3)$ It is obvious.

$3 \Longrightarrow 1)$ Let $z \in supp(f(x, y^{-1}, \overset{(m-2)}{e})) \cap \mathcal{I}$. Then we have $x \in supp(f(z, y, \overset{(m-2)}{e})) \subseteq supp(f(\mathcal{I}, y, \overset{(m-2)}{e}))$. That is, $x\mathcal{I}^*y$. □

Proposition 4.6. *Let $\mathcal{R} = (R, f, g)$ be a Krasner $F^{(m,n)}$-hyperring, and let $\mathcal{I}$ be a normal F-hyperideal of $\mathcal{R}$. Then, for every $z \in supp(f(x, y, \overset{(m-2)}{e}))$ we have*

$$supp(f(\mathcal{I}, f(x, y, \overset{(m-2)}{e}), \overset{(m-2)}{e})) = supp(f(\mathcal{I}, z, \overset{(m-2)}{e})).$$

Proof. Let $z \in supp(f(x, y, \overset{(m-2)}{e}))$ be an arbitrary element. Then

$$supp(f(\mathcal{I}, z, \overset{(m-2)}{e})) \subseteq supp(f(\mathcal{I}, f(x, y, \overset{(m-2)}{e}), \overset{(m-2)}{e})).$$

Now, we prove the converse inclusion. Let $a \in supp(f(\mathcal{I}, f(x, y, \overset{(m-2)}{e}), \overset{(m-2)}{e}))$ be an arbitrary element. Then we have $y \in supp(f(f(\mathcal{I}, x^{-1}, \overset{(m-2)}{e}), a, \overset{(m-2)}{e}))$. Therefore,

$$\begin{aligned} supp(f(x, y, \overset{(m-2)}{e})) &\subseteq supp(f(x, f(f(\mathcal{I}, x^{-1}, \overset{(m-2)}{e}), a, \overset{(m-2)}{e}), \overset{(m-2)}{e}) \\ &= supp(f(f(x, f(\mathcal{I}, x^{-1}, \overset{(m-2)}{e}), \overset{(m-2)}{e}), a, \overset{(m-2)}{e})) \\ &= supp(f(\mathcal{I}, a, \overset{(m-2)}{e})). \end{aligned}$$

This implies that $z \in supp(f(\mathcal{I}, a, \overset{(m-2)}{e}))$ and so we have $a \in supp(f(\mathcal{I}, z, \overset{(m-2)}{e}))$. □

Lemma 4.7. *If $\mathcal{I}$ is a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring (R, f, g), then*

(1) $\mathcal{I}^*[x] = supp(f(\mathcal{I}, x, \overset{(m-2)}{e}))$,

(2) *for every $a_2^m \in R$ we have $supp(f(\mathcal{I}, a_2^m)) = \mathcal{I}^*[x]$, where $x \in supp(f(\mathcal{I}, a_2^m))$.*

Proof. (1) We have

$$\begin{aligned}\mathcal{I}^*[x] &= \{y \in R \mid y\mathcal{I}^*x\} \\ &= \{y \in R \mid y \in supp(f(\mathcal{I}, x, \overset{(m-2)}{e}\} \\ &= supp(f(\mathcal{I}, x, \overset{(m-2)}{e})).\end{aligned}$$

(2) Let $x \in supp(f(\mathcal{I}, a_2^m))$ be an arbitrary element. We show that

$$supp(f(\mathcal{I}, a_2^m)) = supp(f(\mathcal{I}, x, \overset{(m-2)}{e})).$$

We have

$$\begin{aligned}supp(f(\mathcal{I}, x, \overset{(m-2)}{e})) \subseteq supp(f(\mathcal{I}, f(\mathcal{I}, a_2^m), \overset{(m-2)}{e})) &= supp(f(\mathcal{I}, \mathcal{I}, \overset{(m-2)}{e}), a_2^m) \\ &= supp(f(\mathcal{I}, a_2^m)).\end{aligned}$$

Also, by the first part and previous argument we have $x \in supp(f(\mathcal{I}, x, \overset{(m-2)}{e}) \subseteq supp(f(\mathcal{I}, f(\mathcal{I}, a_2^m), \overset{(m-2)}{e}))$. This implies that $supp(f(\mathcal{I}, a_2^m)) \subseteq supp(f(\mathcal{I}, x, \overset{(m-2)}{e}))$. Hence the desired result follows. □

For a set A, we define $\mathcal{I}^*[A] = \bigcup\limits_{a \in A} \mathcal{I}^*[a]$. It is obvious that $\mathcal{I}^*[\mathcal{I}^*[A]] = \mathcal{I}^*[A]$ and $\mathcal{I}^*[A] = supp(f(\mathcal{I}, A, \overset{(m-2)}{e}))$.

Corollary 4.8. *If $\mathcal{I}$ is a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring (R, f, g) and $supp(f(\mathcal{I}, a_2^m)) \cap supp(f(\mathcal{I}, b_2^m)) \neq \emptyset$, then*

$$supp(f(\mathcal{I}, a_2^m)) = supp(f(\mathcal{I}, b_2^m)).$$

Let $\mathcal{I}$ be a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring (R, f, g). We now define the relation $\mathcal{I}_*$ on R as follows:

$x\mathcal{I}_*y$ if and only if there exist $r_2^m \in R$ such that $x, y \in supp(f(\mathcal{I}, r_2^m))$, $\quad \forall x, y \in R$.

By Lemma 4.7, we have $\mathcal{I}^* = \mathcal{I}_*$.

Lemma 4.9. *Let $\mathcal{I}$ be a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring (R, f, g). Then, for all $x_1^m \in R$ we have*

(1) $\mathcal{I}^*[supp(f(x_1^m))] = \mathcal{I}^*[x]$, *where* $x \in supp(f(x_1^m))$,
(2) $\mathcal{I}^*[supp(f(x_1^m))] = supp(f(\mathcal{I}^*{}_{[x_1]}^{[x_m]}))$,
(3) $\mathcal{I}^*[supp(f(\mathcal{I}^*{}_{[x_1]}^{[x_m]}))] = supp(f(\mathcal{I}^*{}_{[x_1]}^{[x_m]}))$.

Proof. (1) Assume that $x \in supp(f(x_1^m))$. It is clear that $\mathcal{I}^*[x] \subseteq \mathcal{I}^*[supp(f(x_1^m))]$. Now, we prove the reverse inclusion. By Lemma 4.7 we have

$$\mathcal{I}^*[supp(f(x_1^m))] = supp(f(\mathcal{I}, f(x_1^m), \overset{(m-2)}{e})).$$

Since $x \in \mathcal{I}^*[supp(f(x_1^m))]$ by Remark 4.3 we have $supp(f(x_1^m)) \subseteq supp(f(\mathcal{I}, x, \overset{(m-2)}{e})) = I^*[x]$. Therefore we have $\mathcal{I}^*[supp(f(x_1^m))] \subseteq \mathcal{I}^*[x]$ which completes the proof.

(2) Using Lemma 4.7, we have

$$\begin{aligned} supp(f(\mathcal{I}^*{}^{[x_m]}_{[x_1]})) &= supp(f(f(\mathcal{I}, x_1, \overset{(m-2)}{e}), \ldots, f(\mathcal{I}, x_m, \overset{(m-2)}{e}))) \\ &= supp(f(\mathcal{I}, f(x_1^m), \overset{(m-2)}{e})) \\ &= \mathcal{I}^*[supp(f(x_1^m))]. \end{aligned}$$

(3) Using (2), we have

$$\mathcal{I}^*[supp(f(\mathcal{I}^*{}^{[x_m]}_{[x_1]}))] = \mathcal{I}^*[\mathcal{I}^*[supp(f(x_1^m))]] = \mathcal{I}^*[supp(f(x_1^m))].$$

□

Corollary 4.10. *The relation $\mathcal{I}^*$ is a strongly regular relation.*

Proof. Let $x_1\mathcal{I}^*y_1, \ldots, x_m\mathcal{I}^*y_m$, $x \in supp(f(x_1^m))$ and $y \in supp(f(y_1^m))$. Then,

$$\begin{aligned} x \in supp(f(x_1^m)) &\subseteq supp(f(f(\mathcal{I}, y_1, \overset{(m-2)}{e}), \ldots, f(\mathcal{I}, y_m, \overset{(m-2)}{e}))) \\ &= supp(f(\mathcal{I}, f(y_1^m), \overset{(m-2)}{e})) \\ &= \mathcal{I}^*[supp(f(y_1^m))] \\ &= \mathcal{I}^*[y]. \end{aligned}$$

Therefore, we have $x\mathcal{I}^*y$. □

Let $\mathcal{I}$ be a normal F-hyperideal of a Krasner $F^{(m,n)}$-hyperring (R, f, g). Since $\mathcal{I}^*$ is a strongly regular relation, by Theorem 4.1, $[R : \mathcal{I}^*]$ is a Krasner (m, n)-hyperring with m-operation $f|_{\mathcal{I}^*}$ and n-operation $g|_{\mathcal{I}^*}$ as follows:

$$f|_{\mathcal{I}^*}(\mathcal{I}^*{}^{[x_m]}_{[x_1]}) = \{\mathcal{I}^*[z]\}, \quad \forall z \in supp(f(x_1^m)),$$

$$g|_{\mathcal{I}^*}(\mathcal{I}^*{}^{[x_n]}_{[x_1]}) = \{\mathcal{I}^*[supp(g(x_1^n))]\},$$

and so $[R : \mathcal{I}^*]_i$ is a quotient Krasner $F^{(m,n)}$-hyperring.

5. Isomorphism Theorems of Krasner $F^{(m,n)}$-Hyperrings

In this section, with respect to the concepts of normal F-hyperideals and strongly regular relations and homomorphisms, isomorphism theorems for Krasner $F^{(m,n)}$-hyperrings are stated and proved.

Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two $F^{(m,n)}$-hyperrings. A *homomorphism* from R_1 to R_2 is a mapping $\varphi : R_1 \longrightarrow R_2$ such that

(1) $\varphi(e_{R_1}) = e_{R_2}$, where e_{R_1} and e_{R_2} are F-identity elements,
(2) $\varphi(supp(f_1(x_1^m))) = supp(f_2(\varphi_{x_1}^{x_m}))$,
(3) $\varphi(supp(g_1(y_1^n))) = supp(g_2(\varphi_{y_1}^{y_n}))$,

hold for all $x_1^m, y_1^n \in R_1$, where $\varphi_{a_i}^{a_j}$ denotes the sequence $\varphi(a_i), \ldots, \varphi(a_j)$.

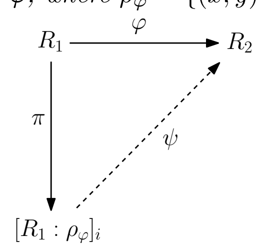
An injective homomorphism is called a *monomorphism* and an onto homomorphism is called an *epimorphism*. An injective and onto homomorphism is called an *isomorphism*. We say that R_1 is *isomorphic* to R_2, denoted by $R_1 \cong R_2$, if there exists an isomorphism from R_1 to R_2.

Lemma 5.1. *Let $\mathcal{R}_1 = (R_1, f_1, g_1)$ and $\mathcal{R}_2 = (R_2, f_2, g_2)$ be two $F^{(m,n)}$-hyperrings and let $\varphi : R_1 \longrightarrow R_2$ be a homomorphism. Then,*

(1) $\varphi(x^{-1}) = (\varphi(x))^{-1}, \quad \forall x \in R_1,$
(2) *φ is injective if and only if $ker\varphi = \{e_{R_1}\}$, where $ker\varphi = \{x \in R_1 \mid \varphi(x) = e_{R_2}\}$,*
(3) *$ker\varphi$ is an F-hyperideal of $\mathcal{R}_1$,*
(4) *$Im\varphi$ is a Krasner $F^{(m,n)}$-subhyperring of $\mathcal{R}_2$.*

Let θ be a regular relation on a Krasner $F^{(m,n)}$-hyperring (R, f, g). As is well known, the natural map $\pi : R \longrightarrow [R : \theta]_i$ by $\pi(x) = \theta(x)$ is an epimorphism. π is called *canonical* homomorphism.

Lemma 5.2. *Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two $F^{(m,n)}$-hyperrings and let $\varphi : R_1 \longrightarrow R_2$ be a homomorphism. Then, there exists a monomorphism $\psi : [R_1 : \rho_\varphi]_i \longrightarrow R_2$ such that $\psi \circ \pi = \varphi$, where $\rho_\varphi = \{(x, y) \in R_1 \times R_1 \mid \varphi(x) = \varphi(y)\}$.*



Proof. First, we show that ρ_φ is a regular relation on R_1 and then $[R_1 : \rho_\varphi]_i$ is defined. For $x_1^m, y_1^m \in R_1$ if $x_1\rho_\varphi y_1, \ldots, x_m\rho_\varphi y_m$ and $a \in supp(f_1(x_1^m))$, then we have

$$\begin{aligned} \varphi(a) \in \varphi(supp(f_1(x_1^m))) = supp(f_2(\varphi_{x_1}^{x_m})) &= supp(f_2(\varphi_{y_1}^{y_m})) \\ &= \varphi(supp(f_1(y_1^m))). \end{aligned}$$

Therefore, there exists $b \in supp(f_1(y_1^m))$ such that $\varphi(a) = \varphi(b)$. This implies that $supp(f_1(x_1^m))\overline{\overline{\rho_\varphi}}supp(f_1(y_1^m))$. Hence, ρ_φ is regular. Now, we define $\psi(\rho_\varphi[x]) = \varphi(x)$. It is easy to see that ψ is a monomorphism and $\psi \circ \pi = \varphi$. □

Theorem 5.3. *Let γ and θ be regular relations on a Krasner $F^{(m,n)}$-hyperring (R, f, g) such that $\gamma \subseteq \theta$. Then, there exists a regular relation μ on $[R : \gamma]_i$ such that $[[R : \gamma]_i : \mu]_i$ is isomorphic to $[R : \theta]_i$.*

Proof. We define the map $\varphi : [R : \gamma]_i \longrightarrow [R : \theta]_i$ by $\varphi(\gamma[x]) = \theta[x]$. Since $\gamma \subseteq \theta$, φ is well-defined. For $\gamma[x_1], \ldots, \gamma[x_m] \in [R : \gamma]_i$ we have

$$\begin{aligned} \varphi(f|_\gamma(\gamma_{[x_1]}^{[x_m]})) = \{\varphi(\gamma[z]) \mid z \in supp(f(x_1^m))\} &= \{\theta[z] \mid z \in supp(f(x_1^m))\} \\ &= f|_\theta(\theta_{[x_1]}^{[x_m]}) \\ &= f|_\theta(\varphi(\gamma[x_1]), \ldots, \varphi(\gamma[x_m])). \end{aligned}$$

Also, for $\gamma[y_1], \ldots, \gamma[y_n] \in [R : \gamma]_i$ we have

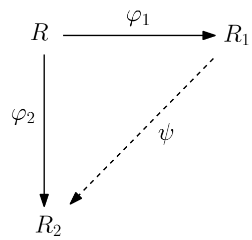
$$\varphi(g|_\gamma(\gamma_{[y_1]}^{[y_n]})) = \varphi(\gamma[supp(g(y_1^n))]) = \theta[supp(g(y_1^n))] = g|_\theta(\theta_{[y_1]}^{[y_n]}).$$

Therefore, φ is a homomorphism. Now, if

$$\mu = \{(\gamma[x], \gamma[y]) \in [R:\gamma]_i \times [R:\gamma]_i \mid \varphi(\gamma[x]) = \varphi(\gamma[y])\},$$

then by Lemma 5.2, there exists a monomorphism $\psi : [[R:\gamma]_i : \mu]_i \longrightarrow [R:\theta]_i$ such that $\psi \circ \pi = \varphi$, and so ψ is an isomorphism. □

Lemma 5.4. *Let (R, f, g), (R_1, f_1, g_1) and (R_2, f_2, g_2) be Krasner $F^{(m,n)}$-hyperrings and let $\varphi_1 : R \longrightarrow R_1$ and $\varphi_2 : R \longrightarrow R_2$ be epimorphisms such that $\varphi_1^{-1} \circ \varphi_1 \subseteq \varphi_2^{-1} \circ \varphi_2$. Then, there exists a unique epimorphism $\psi : R_1 \longrightarrow R_2$ such that $\psi \circ \varphi_1 = \varphi_2$.*



Proof. Since φ_1 is onto, for every $z_1 \in R_1$ there exists $x \in R$ such that $\varphi_1(x) = z_1$. We define $\psi : R_1 \longrightarrow R_2$ by $\psi(z_1) = \varphi_2(x)$. If $\varphi_1(y) = z_1$ $(y \in R)$, we have $(x, y) \in \varphi_1^{-1} \circ \varphi_1 \subseteq \varphi_2^{-1} \circ \varphi_2$, and so $\varphi_2(x) = \varphi_2(y)$. This proves that ψ is well-defined. We prove that ψ is an epimorphism. Clearly, we have $\psi(e_{R_1}) = e_{R_2}$. Now, if $x_1^m \in R_1$ are arbitrary elements, then there exist $y_1^m \in R$ such that $\varphi_1(y_i) = x_i$, $(1 \leq i \leq m)$ and we have

$$\begin{aligned} supp(f_2(\psi_{x_1}^{x_m})) = supp(f_2(\varphi_2{}_{y_1}^{y_m})) &= \varphi_2(supp(f(y_1^m))) \\ &= \{\varphi_2(t) \mid t \in supp(f(y_1^m))\} \\ &= \{\varphi_2(t) \mid \varphi_1(t) \in supp(f_1(x_1^m))\} \\ &= \{\psi(\varphi_1(t)) \mid \varphi_1(t) \in supp(f_1(x_1^m))\} \\ &= \psi(supp(f_1(x_1^m))). \end{aligned}$$

Also, for every $x_1^n \in R_1$, there exist $y_1^n \in R$ such that $\varphi_1(y_i) = x_i$, $(1 \leq i \leq n)$ and we have

$$\begin{aligned} supp(g_2(\psi_{x_1}^{x_n})) = supp(g_2(\varphi_2{}_{y_1}^{y_n})) = \varphi_2(supp(g(y_1^n))) &= \psi(\varphi_1(supp(g(y_1^n))) \\ &= \psi(supp(g_1(x_1^n))). \end{aligned}$$

It is routine to check that ψ is surjective and $\psi \circ \varphi_1 = \varphi_2$. The uniqueness is evident. □

Theorem 5.5. *If γ and θ are regular relations on a Krasner $F^{(m,n)}$-hyperring (R, f, g) such that $\gamma \subseteq \theta$, then there exists an epimorphism $[R:\gamma]_i \longrightarrow [R:\theta]_i$.*

Proof. Let $\pi_1 : R \longrightarrow [R:\gamma]_i$ and $\pi_2 : R \longrightarrow [R:\theta]_i$ be canonical homomorphisms. Since $\gamma = {\pi_1}^{-1} \circ \pi_1$ and $\theta = {\pi_2}^{-1} \circ \pi_2$, by Lemma 5.4 the proof is completed. □

Proposition 5.6. *Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be two Krasner $F^{(m,n)}$-hyperrings and $R = R_1 \times R_2 = \{(x, y) \mid x \in R_1, y \in R_2\}$. We define $f_\otimes : R^m \longrightarrow I_*^R$ and*

$g_\otimes : R^n \longrightarrow I_*^R$ *as follows:*

$$f_\otimes\big((x_1,y_1),\ldots,(x_m,y_m)\big)(a,b) = \min\{f_1(x_1,\ldots,x_m)(a), f_2(y_1,\ldots,y_m)(b)\},$$

$$g_\otimes\big((x_1,y_1),\ldots,(x_n,y_n)\big)(a,b) = \min\{g_1(x_1,\ldots,x_n)(a), g_2(y_1,\ldots,y_n)(b)\}.$$

for all $(a,b) \in R$. *Then,* $(R, f_\otimes, g_\otimes)$ *is a Krasner* $F^{(m,n)}$*-hyperring.*

Recall that for relations ρ and σ on R the relation product is

$$\rho \circ \sigma = \{(x,y) \in R^2 \mid (x,u) \in \rho, (u,y) \in \sigma \text{ for some } u \in R\}.$$

The diagonal relation $\triangle$ on R is the set $\{(a,a) \mid a \in R\}$ and the full relation R^2 is denoted by ∇.

Theorem 5.7. *Let* (R,f,g) *be a Krasner* $F^{(m,n)}$*-hyperring and* θ, θ^* *be regular relations on* R *such that* $\theta \cap \theta^* = \Delta$ *and* $\theta \circ \theta^* = \nabla$. *Then,*

$$R \cong [R:\theta]_i \times [R:\theta^*]_i$$

under the map $\psi(x) = (\theta[x], \theta^*[x])$.

Proof. If $x, y \in R$ and $\psi(x) = \psi(y)$ then $\theta[x] = \theta[y]$ and $\theta^*[x] = \theta^*[y]$, so $(x,y) \in \theta \cap \theta^*$; hence $x = y$. This means that ψ is injective. Now, let $x, y \in R$ are given. Since $\theta \circ \theta^* = \nabla$, there exists z in R such that $x\theta z$ and $z\theta^* y$, hence $\psi(z) = (\theta[z], \theta^*[z]) = (\theta[x], \theta^*[y])$, so ψ is onto. Now, for every $x_1, \ldots, x_m \in R$, we show that

$$\psi(supp(f(x_1^m))) = supp(f_\otimes(\psi_{x_1}^{x_m})).$$

We have

$$\begin{aligned}
\psi(supp(f(x_1^m))) &= \{\psi(x) \mid x \in supp(f(x_1^m))\} \\
&= \{(\theta[x], \theta^*[x]) \mid x \in supp(f(x_1^m))\} \\
&\subseteq \{(\theta[x], \theta^*[y]) \mid x, y \in supp(f(x_1^m))\} \\
&= f|_\theta(\theta_{[x_1]}^{[x_m]}) \times f|_{\theta^*}(\theta^*{}_{[x_1]}^{[x_m]}) \\
&= supp(f_\otimes(\psi_{x_1}^{x_m})),
\end{aligned}$$

and so $\psi(supp(f(x_1^m))) \subseteq supp(f_\otimes(\psi_{x_1}^{x_m}))$.

Conversely, suppose that $(\theta[a], \theta^*[b]) \in supp(f_\otimes(\psi_{x_1}^{x_m}))$. Then,

$$(\theta[a], \theta^*[b]) \in \{(\theta[x], \theta^*[y]) \mid x, y \in supp(f(x_1^m))\}.$$

Since $\theta \circ \theta^* = \nabla$, there exists c in R such that $a\,\theta\, c$ and $c\,\theta^* b$, and so $(\theta[a], \theta^*[b]) = (\theta[c], \theta^*[c])$, where $c \in supp(f(x_1^m))$. Therefore, $(\theta[a], \theta^*[b]) \in \psi(supp(f(x_1^m)))$. Finally, let $y_1^n \in R$ be arbitrary elements. Putting $supp(g(y_1^n)) = \{t\}$ we have

$$\begin{aligned}
\psi(supp(g(y_1^n))) = \{\psi(t)\} &= \{(\theta[t], \theta^*[t])\} \\
&= g|_\theta(\theta_{[y_1]}^{[y_n]}) \times g|_{\theta^*}(\theta^*{}_{[y_1]}^{[y_n]}) \\
&= supp(g_\otimes((\theta_{[y_1]}^{[y_n]}, \theta^*{}_{[y_1]}^{[y_n]})) \\
&= supp(g_\otimes(\psi_{y_1}^{y_n})),
\end{aligned}$$

where $(\theta^{[y_n]}_{[y_1]}, \theta^{*[y_n]}_{[y_1]})$ denotes the sequence $(\theta[y_1], \theta^*[y_1]), \ldots, (\theta[y_n], \theta^*[y_n])$. This completes the proof. □

We now present the first isomorphism theorem.

Theorem 5.8. *(First Isomorphism Theorem). Let $\mathcal{R}_1 = (R_1, f_1, g_1)$ and $\mathcal{R}_2 = (R_2, f_2, g_2)$ be two $F^{(m,n)}$-hyperrings and let $\varphi : R_1 \longrightarrow R_2$ be a homomorphism such that $K = ker\varphi$ is a normal F-hyperideal of $\mathcal{R}_1$. Then, $[R_1 : K^*]_i \cong Im\varphi$.*

Proof. We consider the map $\psi : [R_1 : K^*]_i \longrightarrow Im\varphi$ by $\psi(K^*[x]) = \varphi(x)$. By the following argument ψ is well-defined.

$$\begin{aligned}
K^*[x] = K^*[y] \quad &\Leftrightarrow \quad supp(f_1(K, x, \overset{(m-2)}{e_{R_1}})) = supp(f_1(K, y, \overset{(m-2)}{e_{R_1}}))\\
&\Rightarrow \quad \varphi(supp(f_1(K, x, \overset{(m-2)}{e_{R_1}}))) = \varphi(supp(f_1(K, y, \overset{(m-2)}{e_{R_1}})))\\
&\Leftrightarrow \quad supp(f_2(\varphi(K), \varphi(x), \overset{(m-2)}{e_{R_2}})) = supp(f_2(\varphi(K), \varphi(y), \overset{(m-2)}{e_{R_2}}))\\
&\Leftrightarrow \quad supp(f_2(e_{R_2}, \varphi(x), \overset{(m-2)}{e_{R_2}})) = supp(f_2(e_{R_2}, \varphi(y), \overset{(m-2)}{e_{R_2}}))\\
&\Leftrightarrow \quad \varphi(x) = \varphi(y).
\end{aligned}$$

Obviously, $\psi(K^*[e_{R_1}]) = \varphi(e_{R_1}) = e_{R_2}$ and for every $K^{*[x_m]}_{[x_1]} \in [R_1 : K^*]_i$, we have

$$\begin{aligned}
\psi(f_1|_{K^*}(K^{*[x_m]}_{[x_1]})) = \psi(K^*[z]) \quad &= \quad \varphi(supp(f_1(x_1^m)))\\
&= \quad supp(f_2(\varphi^{x_m}_{x_1}))\\
&= \quad supp(f_2(\psi(K^*[x_1]), \ldots, \psi(K^*[x_m]))),
\end{aligned}$$

where z is an arbitrary element of $supp(f_1(x_1^m))$. Also, for every $K^{*[y_n]}_{[y_1]} \in [R_1 : K^*]_i$, we have

$$\begin{aligned}
\psi(g_1|_{K^*}(K^{*[y_n]}_{[y_1]})) = \psi(K^*[supp(g_1(y_1^n))]) \; &= \; \varphi(supp(g_1(y_1^n)))\\
&= \; supp(g_2(\varphi^{y_n}_{y_1}))\\
&= \; supp(g_2(\psi(K^*[y_1]), \ldots, \psi(K^*[y_n]))).
\end{aligned}$$

Therefore, ψ is a homomorphism.

If $y \in Im\varphi$ is an arbitrary element, then there exist $x \in R_1$ such that $y = \varphi(x) = \psi(K^*[x])$ which implies that ψ is onto. We have

$$\begin{aligned}
ker\varphi \quad &= \quad \{K^*[x] \in [R_1 : K^*]_i \mid \psi(K^*[x]) = e_{R_2}\}\\
&= \quad \{K^*[x] \in [R_1 : K^*]_i \mid \varphi(x) = e_{R_2}\}\\
&= \quad \{K^*[x] \in [R_1 : K^*]_i \mid x \in K\}\\
&= \quad \{K^*[e_{R_1}]\}.
\end{aligned}$$

Therefore, ψ is an isomorphism and so $[R_1 : K^*]_i \cong Im\varphi$. □

We are now in a position to state and prove the second and third isomorphism theorems in Krasner $F^{(m,n)}$-hyperrings.

Theorem 5.9. *(Second Isomorphism Theorem). If $\mathcal{I}_1^m$ are F-hyperideals of a Krasner $F^{(m,n)}$-hyperring $(\mathcal{R}, f, g)$ such that $\mathcal{I}_j$ is normal for some $j \in \{1, \ldots, m\}$, then*

$$[supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) : (supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \cap \mathcal{I}_j)^*]_i \cong [supp(f(\mathcal{I}_1^m)) : \mathcal{I}_j^*]_i.$$

Proof. By Lemma 3.12, $supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m))$ is an F-hyperideal of $supp(f(\mathcal{I}_1^m))$. By Lemma 4.4, $\mathcal{I}_j$ is a normal F-hyperideal of $supp(f(\mathcal{I}_1^m))$ and so

$$supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \cap \mathcal{I}_j$$

is a normal F-hyperideal of $supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m))$. Therefore, $[supp(f(\mathcal{I}_1^m)) : \mathcal{I}_j^*]$ and $[supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) : (supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \cap \mathcal{I}_j)^*]$ are defined. We consider the map $\psi : supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \longrightarrow [supp(f(\mathcal{I}_1^m)) : \mathcal{I}_j^*]_i$ by $\psi(x) = \mathcal{I}_j^*[x]$. Clearly, $\psi(e) = \mathcal{I}_j^*[e]$ and for every $x_1^m \in supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m))$ we have

$$\begin{aligned} \psi(supp(f(x_1^m))) = \{\psi(x) \mid x \in supp(f(x_1^m))\} &= \{\mathcal{I}_j^*[x] \mid x \in supp(f(x_1^m))\} \\ &= \{\mathcal{I}_j^*[z]\} \\ &= f|_{\mathcal{I}_j^*}(\mathcal{I}_j^*{}_{[x_1]}^{[x_m]}) \\ &= f|_{\mathcal{I}_j^*}(\psi_{x_1}^{x_m}), \end{aligned}$$

where $z \in supp(f(x_1^m))$. Also, for every $y_1^n \in supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m))$ we have

$$\psi(supp(g(y_1^n))) = \mathcal{I}_j^*[supp(g(y_1^n))] = g|_{\mathcal{I}_j^*}(\mathcal{I}_j^*{}_{[y_1]}^{[y_n]}) = g|_{\mathcal{I}_j^*}(\psi_{y_1}^{y_n}).$$

Therefore, ψ is a homomorphism. Let $\mathcal{I}_j^*[a] \in [supp(f(\mathcal{I}_1^m)) : \mathcal{I}_j^*]_i$ be an arbitrary element. Then, there exist $a_i \in \mathcal{I}_i$; $1 \leq i \leq m$ such that $a \in supp(f(a_1^m))$. Now, for $x \in supp(f(a_1^{j-1}, e, a_{j+1}^m)) \subseteq supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m))$ we have

$$\begin{aligned} \psi(x) = \mathcal{I}_j^*[x] = \mathcal{I}_j^*[supp(f(a_1^{j-1}, e, a_{j+1}^m))] &= supp(f(\mathcal{I}_j, f(a_1^{j-1}, e, a_{j+1}^m), \overset{(m-2)}{e})) \\ &= supp(f(a_1^{j-1}, f(\mathcal{I}_j, \overset{(m-1)}{e}), a_{j+1}^m)) \\ &= supp(f(a_1^{j-1}, f(\mathcal{I}_j, a_j, \overset{(m-2)}{e}), a_{j+1}^m)) \\ &= supp(f(\mathcal{I}_j, f(a_1^m), \overset{(m-2)}{e})) \\ &= \mathcal{I}_j^*[supp(f(a_1^m))] \\ &= \mathcal{I}_j^*[a]. \end{aligned}$$

Therefore, ψ is onto. We have

$$\begin{aligned} ker\psi &= \{x \in supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \mid \mathcal{I}_j^*[x] = \mathcal{I}_j^*[e]\} \\ &= \{x \in supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \mid x \in \mathcal{I}_j\} \\ &= supp(f(\mathcal{I}_1^{j-1}, e, \mathcal{I}_{j+1}^m)) \cap \mathcal{I}_j. \end{aligned}$$

Hence, by the First Isomorphism Theorem the desired result holds. □

Theorem 5.10. *(Third Isomorphism Theorem). If $\mathcal{I}$ and $\mathcal{J}$ are normal F-hyperideals of a Krasner $F^{(m,n)}$-hyperring (R, f, g) such that $\mathcal{I} \subseteq \mathcal{J}$, then $[\mathcal{J} : \mathcal{I}^*]_i$ is a normal F-hyperideal of $[R : \mathcal{I}^*]_i$ and $[[R : \mathcal{I}^*]_i : [\mathcal{J} : \mathcal{I}^*]_i^*]_i \cong [R : \mathcal{J}^*]_i$.*

Proof. First, using Lemma 3.9 we show that $[\mathcal{J} : \mathcal{I}^*]_i$ is an F-hyperideal of $[R : \mathcal{I}^*]_i$. Since $e \in \mathcal{J}$, we have $\mathcal{I}^*[e] \in [\mathcal{J} : \mathcal{I}^*]_i$. If $\mathcal{I}^*[x] \in [\mathcal{J} : \mathcal{I}^*]_i$ then $x \in \mathcal{J}$ and so

$x^{-1} \in \mathcal{J}$ which implies that $(\mathcal{I}^*[x])^{-1} = \mathcal{I}^*[x^{-1}] \in [\mathcal{J} : \mathcal{I}^*]_i$. For every $\mathcal{I}^{*[x_m]}_{[x_1]} \in [\mathcal{J} : \mathcal{I}^*]_i$ and for $z \in supp(f(x_1^m)) \subseteq \mathcal{J}$ we have $f|_{\mathcal{I}^*}(\mathcal{I}^{*[x_m]}_{[x_1]}) = \{\mathcal{I}^*[z]\} \subseteq [\mathcal{J} : \mathcal{I}^*]_i$. For every $\mathcal{I}^{*[x_n]}_{[x_1]} \in [R : \mathcal{I}^*]_i$ and for all $i \in \{1, \dots, n\}$ we have

$$\begin{aligned} g|_{\mathcal{I}^*}(\mathcal{I}^{*[x_{i-1}]}_{[x_1]}, [\mathcal{J} : \mathcal{I}^*]_i, \mathcal{I}^{*[x_n]}_{[x_{i+1}]}) &= \bigcup_{t \in \mathcal{J}} g|_{\mathcal{I}^*}(\mathcal{I}^{*[x_{i-1}]}_{[x_1]}, \mathcal{I}^*[t], \mathcal{I}^{*[x_n]}_{[x_{i+1}]}) \\ &= \bigcup_{t \in \mathcal{J}} \{\mathcal{I}^*[supp(g(x_1^{i-1}, t, x_{i+1}^n))]\} \\ &\subseteq [\mathcal{J} : \mathcal{I}^*]_i. \end{aligned}$$

Thus, $[\mathcal{J} : \mathcal{I}^*]_i$ is an F-hyperideal of $[R : \mathcal{I}^*]_i$. For all $\mathcal{I}^*[x] \in [R : \mathcal{I}^*]$ and for every $t \in \mathcal{J}$ we have

$$\begin{aligned} f|_{\mathcal{I}^*}(\mathcal{I}^*[t], f|_{\mathcal{I}^*}(\mathcal{I}^*[x], \mathcal{I}^*[x^{-1}], \overset{(m-2)}{I^*[e]}), \overset{(m-2)}{\mathcal{I}^*[e]}) &= f|_{\mathcal{I}^*}(\mathcal{I}^*[t], \mathcal{I}^*[z], \overset{(m-2)}{\mathcal{I}^*[e]}) \\ &= \{\mathcal{I}^*[w]\} \\ &\subseteq [\mathcal{J} : \mathcal{I}^*]_i, \end{aligned}$$

where $z \in supp(f(x, x^{-1}, \overset{(m-2)}{e}))$ and

$$w \in supp(f(t, z, \overset{(m-2)}{e})) \subseteq supp(f(\mathcal{J}, f(x, x^{-1}, \overset{(m-2)}{e}), \overset{(m-2)}{e})) = \mathcal{J}.$$

So, $[\mathcal{J} : \mathcal{I}^*]_i$ is normal.

It is not difficult to see that $\psi : [R : \mathcal{I}^*]_i \longrightarrow [R : \mathcal{J}^*]_i$ is an onto homomorphism and $ker\varphi = [\mathcal{J} : \mathcal{I}^*]_i$. Therefore, by the First Isomorphism Theorem the desired result follows easily. □

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