

## ON FIXED POINT THEOREMS FOR CONTRACTIVE-TYPE MAPPINGS IN FUZZY METRIC SPACES

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ABSTRACT. In this paper, we provide two different kinds of fixed point theorems in fuzzy metric spaces. The first kind is for the fuzzy  $\varepsilon$ -contractive type mappings and the second kind is for the fuzzy order  $\psi$ -contractive type mappings. They improve the corresponding conclusions in the literature.

### 1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [23] in 1965. After that, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. In the theory of fuzzy topological spaces, one of the main problems is to obtain an appropriate and consistent notion of a fuzzy metric space. This problem was investigated by many authors from different points of view [7, 8, 9, 12, 13, 19, 22]. Kramosil and Michalek [15] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [16]. Later, George and Veeramani [5] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek. For these two types of fuzzy metric spaces, we will call the former KM fuzzy metric space and the latter GV fuzzy metric space to avoid confusions in this paper.

These fuzzy metric spaces have been widely accepted as an appropriate notion of metric fuzziness in the sense that it provides rich topological structures which can be obtained, in many cases, from classical theorems. Further, these fuzzy metric spaces have very important applications in studying fixed point theorems for contraction type mappings [1, 3, 4, 18, 20, 24]. In this paper, we provide two different kinds of fixed point theorems in fuzzy metric spaces. The first kind is for the fuzzy  $\varepsilon$ -contractive type mappings and the second kind is for the fuzzy order  $\psi$ -contractive type mappings. They improve the corresponding conclusions in the literature.

For the sake of completeness, we briefly recall some notions from the theory of fuzzy metric spaces used in this paper. According to [14] a binary operation  $*$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a continuous  $t$ -norm if  $([0, 1], *)$  is an Abelian topological monoid with unit 1 such that  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$  for all  $a, b, c, d \in [0, 1]$ . Examples of  $t$ -norm are  $a * b = ab$  and  $a * b = \min\{a, b\}$ .

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**Definition 1.1.** (Kramosil and Michalek [15]) The 3-tuple  $(X, M, T)$  is said to be a KM fuzzy metric space if  $X$  is an arbitrary nonempty set,  $T$  is a  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times [0, \infty)$  satisfying the following conditions, for all  $x, y, z \in X, t, s > 0$ :

- (KM-1)  $M(x, y, 0) = 0$ ,
- (KM-2)  $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$ ,
- (KM-3)  $M(x, y, t) = M(y, x, t)$ ,
- (KM-4)  $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$ ,
- (KM-5)  $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous.

**Definition 1.2.** (George and Veeramani [5]) The 3-tuple  $(X, M, T)$  is said to be a GV fuzzy metric space if  $X$  is an arbitrary nonempty set,  $T$  is a continuous  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions, for all  $x, y, z \in X, t, s > 0$ :

- (GV-1)  $M(x, y, t) > 0$ ,
- (GV-2)  $M(x, y, t) = 1$  if and only if  $x = y$ ,
- (GV-3)  $M(x, y, t) = M(y, x, t)$ ,
- (GV-4)  $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$ ,
- (GV-5)  $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$  is continuous.

**Definition 1.3.** [11] Let  $(X, M, *)$  be a KM (or GV) fuzzy metric space. The fuzzy metric  $M$  (or the fuzzy metric space  $(X, M, *)$ ) is said to be non-Archimedean if it satisfies for each  $x, y, z \in X$  and each  $t > 0$

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t).$$

**Definition 1.4.** [6] A sequence  $(x_i)_{i \in \mathbb{N}}$  in a KM (or GV) fuzzy metric space  $(X, M)$  is said to be Cauchy if  $\lim_{i, j \rightarrow \infty} M(x_i, x_j, t) = 1$  for all  $t > 0$ ; a sequence  $(x_i)_{i \in \mathbb{N}}$  in  $X$  converges to  $x$  iff  $\lim_{i \rightarrow \infty} M(x_i, x, t) = 1$  for all  $t > 0$ ; a mapping  $f : X \rightarrow X$  is said to be continuous if  $fx_n \rightarrow fx$  whenever  $(x_n)$  is a sequence in  $X$  convergent to  $x$ .

**Definition 1.5.** [17] Let  $(X, M, *)$  be a KM (or GV) fuzzy metric space and  $0 < \varepsilon < 1$ . A mapping  $f : X \rightarrow X$  is called fuzzy  $\varepsilon$ -contractive if  $M(fx, fy, t) > M(x, y, t)$  whenever  $1 - \varepsilon < M(x, y, t) < 1$ .

**Theorem 1.6.** (Theorem 2.7 of [17]) Let  $(X, M, *)$  be a GV fuzzy metric space and let  $f : X \rightarrow X$  be a fuzzy  $\varepsilon$ -contractive mapping. Suppose that for some  $x \in X$ , the sequence  $(f^n x)_{n \in \mathbb{N}}$  contains a convergent subsequence and let  $\zeta \in X$  be its limit. If there exists  $t_0 > 0$  such that  $M(x, fx, t_0) > 1 - \varepsilon$  and  $M(\zeta, f\zeta, t_0) > 1 - \varepsilon$ , then  $\zeta$  is a fixed point of  $f$ .

**Definition 1.7.** [2] Let  $\preceq$  be an order relation on  $X$ . A mapping  $f : X \rightarrow X$  is called nondecreasing w.r.t  $\preceq$  if  $x \preceq y$  implies  $fx \preceq fy$ .

**Definition 1.8.** [2] Let  $(X, \preceq)$  be a partially ordered set, let  $(X, M, *)$  be a KM (or GV) fuzzy metric space, and let  $\psi$  be a function from  $[0, 1]$  to  $[0, 1]$ . A mapping  $f : X \rightarrow X$  is called a fuzzy order  $\psi$ -contractive mapping if the following implication holds:

$$x, y \in X, x \preceq y \Rightarrow [M(fx, fy, t) \geq \psi(M(x, y, t)) \quad \forall t > 0].$$

## 2. Main Results

In the first part of this section we show that actually only one of the conditions:  $M(x, f(x), t_0) > 1 - \varepsilon$  and  $M(\zeta, f\zeta, t_0) > 1 - \varepsilon$  is sufficient to guarantee the existence of a fixed point for a fuzzy  $\varepsilon$ -contractive mapping.

**Theorem 2.1.** *Let  $(X, M, *)$  be a GV fuzzy metric space and let  $f : X \rightarrow X$  be a fuzzy  $\varepsilon$ -contractive mapping. Suppose that for some  $x \in X$ , the sequence  $(f^n x)_{n \in \mathbb{N}}$  contains a convergent subsequence and let  $\zeta \in X$  be its limit. If there exists  $t_0 > 0$  such that  $M(\zeta, f\zeta, t_0) > 1 - \varepsilon$ , then  $\zeta$  is a fixed point of  $f$ .*

*Proof.* Due to Theorem 2.3 in [17], there exists a positive integer  $k$  such that  $f^k \zeta = \zeta$ . The desired result follows from Proposition 2.6 in [17]. □

**Theorem 2.2.** *Let  $(X, M, *)$  be a GV fuzzy metric space and let  $f : X \rightarrow X$  be a fuzzy  $\varepsilon$ -contractive mapping. Suppose that for some  $x \in X$ , the sequence  $(f^n x)_{n \in \mathbb{N}}$  contains a convergent subsequence and let  $\zeta \in X$  be its limit. If there exists  $t_0 > 0$  such that  $M(x, fx, t_0) > 1 - \varepsilon$ , then  $\zeta$  is a fixed point of  $f$ .*

*Proof.* Since  $f$  is a fuzzy  $\varepsilon$ -contractive mapping and  $M(x, fx, t_0) > 1 - \varepsilon$ , one can get that

$$1 - \varepsilon < M(x, fx, t_0) \leq M(f(x), f^2 x, t_0) \leq M(f^2 x, f^3 x, t_0) \leq \dots \tag{1}$$

From the above inequality, it follows that the sequence  $(M(f^n x, f^{n+1} x, t_0))_{n \in \mathbb{N}}$  is nondecreasing and bounded in  $[0, 1]$ , therefore it is convergent. On the other hand, suppose  $(f^{n_i} x)_{i \in \mathbb{N}}$  is a convergent subsequence of the iterative sequence  $(f^n x)_{n \in \mathbb{N}}$  and  $\zeta = \lim_{i \rightarrow \infty} f^{n_i} x$ . Now, by the continuities of  $f$  and fuzzy metric  $M$  (see Lemma 1.4 and 2.2 in [17]), we have that

$$M(f^{n_i}(x), f^{n_i+1} x, t) \rightarrow M(\zeta, f\zeta, t), \forall t > 0, \text{ as } i \rightarrow \infty. \tag{2}$$

Consequently, we get that

$$M(f^n x, f^{n+1} x, t_0) \rightarrow M(\zeta, f\zeta, t_0), \text{ as } n \rightarrow \infty. \tag{3}$$

By the monotonicity of the sequence  $(M(f^n x, f^{n+1} x, t_0))_{n \in \mathbb{N}}$  and the inequality (1), we obtain that  $M(\zeta, f\zeta, t_0) > 1 - \varepsilon$ . Thus the desired result follows from Theorem 2.1. □

Now we present an improved result which shows a sufficient and necessary condition for a periodic point to be a fixed point. For the sake of simplicity, denote zero and one time iteration as  $f^0 x = x$  and  $f^1 x = fx$ , respectively.

**Theorem 2.3.** *Let  $(X, M, *)$  be a GV fuzzy metric space and let  $f : X \rightarrow X$  be a fuzzy  $\varepsilon$ -contractive mapping. Suppose that for some  $x \in X$ , the sequence  $(f^n x)_{n \in \mathbb{N}}$  contains a convergent subsequence and let  $\zeta \in X$  be its limit. The point  $\zeta$  is a fixed point of  $f$  if and only if there exists some  $t_0 > 0$  such that  $M(f^{k_0} x, f^{k_0+1} x, t_0) > 1 - \varepsilon$  for some non-negative integer  $k_0$ .*

*Proof.* By Theorem 2.3 in [17], the point  $\zeta$  is a periodic point of  $f$ . Suppose  $\zeta$  is a fixed point of  $f$ , that is,  $f\zeta = \zeta$ . Thus we have  $M(\zeta, f\zeta, t) = 1$  for all  $t > 0$ . By the continuities of  $f$  and fuzzy metric  $M$  (see Lemma 1.4 and 2.2 in [17]), we have that

$$M(f^n x, f^{n+1} x, t) \rightarrow M(\zeta, f\zeta, t) = 1, \forall t > 0, \text{ as } n \rightarrow \infty. \tag{4}$$

Consequently, there exists a non-negative integer  $N$  such that  $M(f^n x, f^{n+1} x, t) > 1 - \varepsilon$  for all  $t > 0$  and  $n \geq N$ . Thus we can choose any  $t_0 > 0$  and  $k_0 \geq N$  with the required property.

Conversely, suppose there exists  $t_0 > 0$  such that  $M(f^{k_0} x, f^{k_0+1} x, t_0) > 1 - \varepsilon$  for some non-negative integer  $k_0$ . Then by a similar proof of Theorem 2.2, we can get the desired result.  $\square$

**Remark 2.4.** Theorem 2.3 is an improved version of Proposition 2.6 of [17].

**Example 2.5.** Let  $X = (0, \infty)$ ,  $a * b = ab$  and  $M(x, y, t) = \min\{x, y\}/\max\{x, y\}$  for all  $t > 0$ . Then  $(X, M, *)$  is a complete GV fuzzy metric space. Since  $\sqrt{t} > t$  for all  $t \in (0, 1)$ , the mapping  $f : X \rightarrow X$ ,  $fx = \sqrt{x}$ , is fuzzy  $\varepsilon$ -contractive for every  $\varepsilon \in (0, 1)$ . The iterative sequence  $(f^n(1/2))_{n \in \mathbb{N}}$  is convergent to 1. If let  $\varepsilon_0 = 1/2$  and  $t_0 = 1/2$ , then we have  $M(1/2, f(1/2), 1/2) = \sqrt{1/2} > 1 - \varepsilon_0$ . By Theorem 2, we get that 1 is a fixed point of  $f$ . However, if let  $\varepsilon_0 = 1/10$  and  $t_0 = 1/2$ , then by a simple calculation we have that  $M(1/2, f(1/2), 1/2) = \sqrt{1/2} < 1 - \varepsilon_0$  but  $M(f^2(1/2), f^3(1/2), 1/2) = \sqrt[8]{1/2} > 1 - \varepsilon_0$ . Now Theorem 2.2 cannot be applied, but it follows from Theorem 2.3 that 1 is a fixed point of  $f$ .

In [2], the authors proved the following fixed point theorem in complete ordered non-Archimedean fuzzy metric spaces with the t-norm of Hadžić-type.

**Theorem 2.6.** [2] *Let  $(X, \preceq)$  be a partially ordered set and  $(X, M, *)$  be a complete non-Archimedean KM fuzzy metric space with  $*$  of Hadžić-type. Let  $\psi : [0, 1] \rightarrow [0, 1]$  be a continuous, nondecreasing function and let  $f : X \rightarrow X$  be a fuzzy order  $\psi$ -contractive and nondecreasing mapping w.r.t  $\preceq$ . Suppose that either*

$$f \text{ is continuous,} \tag{5}$$

or

$x_n \preceq x \forall n$ , whenever  $\{x_n\} \subset X$  is nondecreasing sequence with  $x_n \rightarrow x \in X$  holds. If there exists  $x_0 \in X$  such that

$$x_0 \preceq fx_0, \quad \lim_{n \rightarrow \infty} \psi^n(M(x_0, fx_0, t)) = 1 \tag{6}$$

for each  $t > 0$ , then  $f$  has a fixed point.

Following the idea from [18], we prove that this theorem holds even if the t-norm of Hadžić-type is replaced by an arbitrary continuous t-norm. In other words, we can show that it is true in any complete non-Archimedean fuzzy metric space.

**Theorem 2.7.** *Let  $(X, \preceq)$  be a partially ordered set and  $(X, M, *)$  be a complete non-Archimedean KM fuzzy metric space. Let  $\psi : [0, 1] \rightarrow [0, 1]$  be a continuous, nondecreasing function and let  $f : X \rightarrow X$  be a fuzzy order  $\psi$ -contractive and nondecreasing mapping w.r.t  $\preceq$ . Suppose that either*

$$f \text{ is continuous,} \tag{7}$$

or

$x_n \preceq x \forall n$ , whenever  $\{x_n\} \subset X$  is nondecreasing sequence with  $x_n \rightarrow x \in X$  holds. If there exists  $x_0 \in X$  such that

$$x_0 \preceq fx_0, \quad \lim_{n \rightarrow \infty} \psi^n(M(x_0, fx_0, t)) = 1 \tag{8}$$

for each  $t > 0$ , then  $f$  has a fixed point.

*Proof.* Let  $x_n = fx_{n-1}$  for  $n \in \{1, 2, \dots\}$ . Since  $x_0 \preceq fx_0$  and  $f$  is nondecreasing w.r.t  $\preceq$ , we get

$$x_0 \preceq x_1 \preceq x_2 \preceq \dots \preceq x_n \preceq \dots \tag{9}$$

In order to prove  $\{x_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence, for any given  $t > 0$ , there are two cases to consider.

**Case 1:**  $M(x_0, fx_0, t) = 1$ .

In this case, we have  $\psi(1) = \lim_{n \rightarrow \infty} \psi(\psi^n(1)) = \lim_{n \rightarrow \infty} \psi^n(1) = 1$ , that is,  $\psi(1) = 1$ . Then since  $f$  is fuzzy order  $\psi$ -contractive, by induction we obtain

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, fx_0, t)) = \psi^n(1) = 1, \forall n \in \mathbb{N} \tag{10}$$

which implies  $M(x_n, x_{n+1}, t) = 1$  for all  $n \in \mathbb{N}$ . For any  $m, n \in \mathbb{N}$  satisfying  $m \geq n$ , we have

$$M(x_n, x_m, t) \geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * \dots * M(x_{m-1}, x_m, t) = 1.$$

Hence, it is obvious that for each  $\varepsilon \in (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1 - \varepsilon$  for all  $m, n \geq n_0$ .

**Case 2:**  $M(x_0, fx_0, t) < 1$ .

Suppose  $M(x_0, fx_0, t) = k_0 < 1$ . Since  $\psi$  is nondecreasing and  $\lim_{n \rightarrow \infty} \psi^n(k_0) = 1$ , we have  $\lim_{n \rightarrow \infty} \psi^n(k) = 1$  for  $k \in [k_0, 1]$ . In addition, we can get that  $\psi(1) = 1$  and  $\psi(k) > k$  for  $k \in [k_0, 1)$ . In fact, assume there exists  $k \in [k_0, 1)$  such that  $\psi(k) \leq k$ , again since  $\psi$  is nondecreasing we get that  $\psi^n(k) \leq k$  for all  $n \in \{1, 2, \dots\}$ , a contradiction. Since  $1 \geq \psi(1) = \lim_{k \rightarrow 1} \psi(k) \geq \lim_{k \rightarrow 1} k = 1$ , we have  $\psi(1) = 1$ .

As in Case 1 we can get

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, fx_0, t)) = \psi^n(k_0), \forall n \in \mathbb{N}. \tag{11}$$

By taking the limit as  $n \rightarrow \infty$  we have

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+1}, t) = 1. \tag{12}$$

Suppose there exists  $\varepsilon_0 \in (0, 1)$  such that for every  $i \in \mathbb{N}$  there are  $m(i), n(i) \in \mathbb{N}$  with  $m(i) > n(i) \geq i$  and

$$M(x_{n(i)}, x_{m(i)}, t) \leq 1 - \varepsilon_0. \tag{13}$$

Notice that if there exists such  $\varepsilon_0$ , then the above inequality also holds for any  $\varepsilon \in (0, \varepsilon_0]$ . Therefore, without loss of generality, we assume  $\varepsilon_0 \in (0, 1 - k_0)$ . For each  $i$ , let  $m(i)$  be the least integer exceeding  $n(i)$  satisfying the inequality (13), that is,

$$M(x_{n(i)}, x_{m(i)-1}, t) > 1 - \varepsilon_0. \tag{14}$$

Then, for each  $i \in \mathbb{N}$ , we have

$$\begin{aligned} 1 - \varepsilon_0 &\geq M(x_{n(i)}, x_{m(i)}, t) \\ &\geq M(x_{n(i)}, x_{m(i)-1}, t) * M(x_{m(i)-1}, x_{m(i)}, t) \\ &\geq (1 - \varepsilon_0) * M(x_{m(i)-1}, x_{m(i)}, t). \end{aligned} \tag{15}$$

Since  $\{M(x_n, x_{n+1}, t)\}_{n \in \mathbb{N}}$  is a nondecreasing sequence and  $\lim_{n \rightarrow \infty} M(x_n, x_{n+1}, t) = 1$ , from the above inequality we obtain

$$\lim_{i \rightarrow \infty} M(x_{n(i)}, x_{m(i)}, t) = 1 - \varepsilon_0. \quad (16)$$

Then, since  $x_{n(i)} \preceq x_{m(i)}$  and  $f$  is fuzzy order  $\psi$ -contractive, we get

$$\begin{aligned} 1 - \varepsilon_0 &\geq M(x_{n(i)}, x_{m(i)}, t) \\ &\geq M(x_{n(i)}, x_{n(i)+1}, t) * M(x_{n(i)+1}, x_{m(i)+1}, t) * M(x_{m(i)+1}, x_{m(i)}, t) \\ &\geq M(x_{n(i)}, x_{n(i)+1}, t) * \psi(M(x_{n(i)}, x_{m(i)}, t)) * M(x_{m(i)+1}, x_{m(i)}, t). \end{aligned} \quad (17)$$

Taking the limit as  $i \rightarrow \infty$  and using (17), we have

$$1 - \varepsilon_0 \geq 1 * \psi(1 - \varepsilon_0) * 1 = \psi(1 - \varepsilon_0). \quad (18)$$

Since we have assumed  $\varepsilon_0 \in (0, 1 - k_0)$ , that is,  $1 - \varepsilon_0 > k_0$ , from (18) we get

$$1 - \varepsilon_0 \geq \psi(1 - \varepsilon_0) > 1 - \varepsilon_0, \quad (19)$$

which is a contradiction. Therefore, for each  $\varepsilon \in (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1 - \varepsilon$  for all  $m, n \geq n_0$ .

From what has been discussed above, for the arbitrariness of  $t > 0$ , we get that  $\{x_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence. Since  $X$  is complete, then there exists  $x \in X$  satisfying  $\lim_{n \rightarrow \infty} x_n = x$ .

Now, if  $f$  is continuous then from  $x_{n+1} = fx_n$ , we have  $x = fx$ . While if the condition (8) holds, then we have

$$M(x_{n+1}, fx, t) = M(fx_n, fx, t) \geq \psi(M(x_n, x, t)), \forall n \in \mathbb{N}, t > 0. \quad (20)$$

Letting  $n \rightarrow \infty$  it follows

$$M(x, fx, t) \geq \psi(1) = 1, \forall t > 0, \quad (21)$$

which implies  $fx = x$ . We complete the whole proof here.  $\square$

**Example 2.8.** (This example is partly quoted from [2].) Let  $X = (0, \infty)$ ,  $a * b = ab$  and  $M(x, y, t) = \min\{x, y\} / \max\{x, y\}$  for all  $t > 0$ . Then  $(X, M, *)$  is a complete non-Archimedean KM fuzzy metric space. Consider the following relation on  $X$ :

$$x \preceq y \Leftrightarrow (x = y \text{ or } x, y \in [1, 4], x \leq y). \quad (22)$$

It is easy to see that  $\preceq$  is a partial order on  $X$ . Define a self map  $f$  of  $X$  as follows:

$$fx = \begin{cases} 2x, & 0 < x < 1, \\ \frac{x+5}{3}, & 1 \leq x \leq 4, \\ 2x-5, & x > 4. \end{cases} \quad (23)$$

Now, it is easy to see that  $f$  is continuous and nondecreasing w.r.t  $\preceq$ . In addition, define a continuous nondecreasing mapping  $\psi : [0, 1] \rightarrow [0, 1]$  as follows:

$$\psi(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < \frac{1}{4}, \\ \left(\frac{4\sqrt{2}-1}{2}\right)t + \frac{1-2\sqrt{2}}{4}, & \frac{1}{4} \leq t < \frac{1}{2}, \\ \sqrt{t}, & \frac{1}{2} \leq t \leq 1. \end{cases} \quad (24)$$

Now we prove that  $f$  is fuzzy order  $\psi$ -contractive. Indeed, by the definition of  $\psi$ , one can see  $\psi(t) \leq \sqrt{t}$  for all  $t \in [0, 1]$ . Thus from the argument of Example 2.5 in [2], it follows that  $M(fx, fy, t) \geq \psi(M(x, y, t))$  for all  $x, y \in X$  with  $x \preceq y$ . However, since  $*$  is not a t-norm of Hadžić-type, Theorem 2.4 (i.e., Theorem 2.3 of [2]) can not be applied. Furthermore, since  $\psi(t)$  is not always greater than  $t$  for  $t \in (0, 1)$ , Theorem 2.4 of [2] can not be applied, either. On the other hand, for  $x_0 = 2$ , we have  $2 = x_0 \preceq fx_0 = 7/3$  and for each  $t > 0$ ,

$$\lim_{n \rightarrow \infty} \psi^n(M(x_0, fx_0, t)) = \lim_{n \rightarrow \infty} \psi^n\left(\frac{6}{7}\right) = \lim_{n \rightarrow \infty} \left(\frac{6}{7}\right)^{\frac{1}{2^n}} = 1. \tag{25}$$

Hence all conditions of Theorem 2.5 are satisfied and so  $f$  has a fixed point on  $X$ .

### 3. Conclusions

In [17], by an elegant proof, D. Mihet answered into affirmative an open question raised by A. Razani in [21]. In this paper, we have improved the corresponding conclusions in [17] and given two examples to illustrate the validity of our results. In addition, we have generalized the results in [2] for the fuzzy order  $\psi$ -contractive type mappings.

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