UNIFORMITIES IN FUZZY METRIC SPACES

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Abstract. The aim of this paper is to study induced (quasi-)uniformities in Kramosil and Michalek's fuzzy metric spaces. Firstly, I-uniformity in the sense of J. Gutiérrez García and I-neighborhood system in the sense of Höhle and Šostak are induced by the given fuzzy metric. It is shown that the fuzzy metric and the induced I-uniformity will generate the same I-neighborhood system. Secondly, the relationship between Hutton quasi-uniformities and I-quasi-uniformities is given and it is proved that the category of strongly stratified I-quasi-uniform spaces can be embedded in the category of Hutton quasi-uniform spaces as a bicoreflective subcategory. Also it is shown that two kinds of Hutton quasi-uniformities can generate the same I-uniformity in fuzzy metric spaces.

1. Introduction

It is well known that metric spaces play important roles in the research and applications of topology. Uniform spaces, as a bridge between metric spaces and topological spaces, are convenient tools for an investigation of topologies.

Probabilistic metric space, a generalization of the ordinary metric space, was first studied by Menger [15] and further developed by Schweizer and Sklar [16]. Inspired by the notion of probabilistic metric spaces, Kramosil and Michalek [14] in 1975 introduced the notion of fuzzy metric, a fuzzy set on the Cartesian product $X \times X \times \Re$ satisfying certain conditions. George and Veeramani [2, 3, 4] slightly modified the definition of Kramosil and Michalek's fuzzy metric space and associated each fuzzy metric space with a Hausdorff topology.

Up till now many topological structures and related theories have been defined and studied in probabilistic metric space and Kramosil and Michalek's (or George and Veeramani's) fuzzy metric space. For example, Höhle [10, 11] studied the associated topologies and the fuzzy uniformities in the probabilistic metric space; J. Gutiérrez García and M.A. de Prada Vicente [9] studied the Hutton [0,1]-quasi-uniformities generated by the George and Veeramani's fuzzy metric. Gregori et.al, in [5, 6] studied the convergence and completeness in George and Veeramani's fuzzy metric spaces. In [17], Yue endowed fuzzy metric with many-valued structures–fuzzifying topology and fuzzifying uniformity.

The motivation of this paper is to study some (quasi-)uniformities in Kramosil and Michalek's fuzzy metric spaces. Firstly, I-uniformity in the sense of J. Gutiérrez García and I-neighborhood system in the sense of Höhle and Šostak can be induced by the given fuzzy metric. It is shown that the I-neighborhood system induced by

Received: December 2013; Revised: September 2014; Accepted: December 2014
Key words and phrases: Fuzzy metric, I-uniformity, Hutton quasi-uniformity.

I-uniformity is in accordance with the I-neighborhood system induced by fuzzy metric. Secondly, another kind of Hutton quasi-uniformity a little different from that in [9] is induced by fuzzy metric, and it is shown that these two kinds of Hutton quasi-uniformities will generate the same I-uniformity. Also it is proved that the category of strongly stratified I-quasi-uniform spaces can be embedded in the category of Hutton quasi-uniform spaces as a bicoreflective subcategory.

2. **Preliminaries**

In this paper, I denotes the unit interval $[0, 1]$ and I^X denotes the set of all maps from X to I. Similarly, $I^{X\times X}$ denotes the set of all maps from $X \times X$ to I.

Definition 2.1. A binary operation $* : I \times I \to I$ is called a left-continuous t-norm if it satisfies the following conditions:
(1) $*$ is associative and commutative;
(2) $a * 1 = a$ for all $a \in I$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(4) $*$ is left-continuous.

For each left-continuous t-norm $*$, the implication $\to$ can be determined by $a \to b = \bigvee\{c \in [0, 1] | a * c \leq b\}$ and $a * b \leq c \Leftrightarrow a \leq b \to c$ holds for $a, b, c \in I$.

The three most commonly used left-continuous t-norm are the minimum, the usual product and the Lukasievicz t-norm, denoted by $\wedge$, $\cdot$ and $\diamondsuit$ respectively, where $a\diamondsuit b = max\{0, a + b - 1\}$.

For $D \in I^X$ and $\alpha \in I$, we will denote by $\alpha * D$ and $\alpha \to D$ the elements of I^X defined for each $x \in X$ as
$(\alpha * D)(x) = \alpha * (D(x))$, and $(\alpha \to D)(x) = \alpha \to (D(x))$.

Proposition 2.2. *Suppose that $*$ is a left-continuous t-norm and $\to$ is the implication operation corresponding to $*$. Then the following conditions hold:*
(I1) $1 \to a = a$;
(I2) $a \leq b$ *if and only if* $a \to b = 1$;
(I3) $(a \to b) * (c \to d) \leq a * c \to b * d$;
(I4) $a \to (\bigwedge_{j\in J} a_j) = \bigwedge_{j\in J}(a \to a_j)$;
(I5) $(\bigvee_{j\in J} a_j) \to b = \bigwedge_{j\in J}(a_j \to b)$;
(I7) $a * (a \to b) \leq b$;
(I8) $a \to (b \to c) = b \to (a \to c) = a * b \to c$;
(I9) $(a \to b) \to b \geq b$;
(I10) $b \leq a \to b$.

Define $S(-,-) : I^X \times I^X \to I$ by $S(C, D) = \bigwedge_{x\in X} C(x) \to D(x)$. Then $S(-,-)$ is an I-partial order on I^X which is called fuzzy inclusion order, and the value $S(C, D)$ can be interpreted as the degree to which C is a subset of D.

Lemma 2.3. [1] *Let $S(-,-)$ be the fuzzy inclusion order. Then the following statements hold:*
(1) $C \leq D$ *if and only if* $S(C, D) = 1$;
(2) $S(C, D) \leq S(E, C) \to S(E, D)$;
(3) $S(C_1, D_1) * S(C_2, D_2) \leq S(C_1 * C_2, D_1 * D_2)$.

Definition 2.4. [14] A map $M : X \times X \times [0, +\infty) \to [0, 1]$ is called a fuzzy metric if it satisfies the following conditions:

(KM1) $M(x, y, 0) = 0$ for all $x, y \in X$;

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(KM4) $M(x, z, s + t) \geq M(x, y, s) * M(y, z, t)$ for all $x, y, z \in X$ and $s, t > 0$;

(KM5) $M(x, y, -) : [0, +\infty) \to [0, 1]$ is left-continuous.

The triple $(X, M, *)$ is called a fuzzy metric space.

In Definition 2.4, if M is a fuzzy set on $X \times X \times (0, +\infty)$ and (KM1), (KM5) are replaced with the following (GV1), (GV5), respectively, then $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani [2, 3, 4]:

(GV1) $M(x, y, t) > 0$ for all $t > 0$ and all $x, y \in X$;

(GV5) $M(x, y, -) : (0, +\infty) \to [0, 1]$ is continuous.

The fuzzy metric used in this paper is in the sense of Kramosil and Michalek.

Definition 2.5. [2, 3, 4] Let d be an ordinary metric on X and define $M^d : X \times X \times [0, +\infty) \to [0, 1]$ by $M^d(x, y, t) = \frac{t}{t+d(x,y)}$ when $t \neq 0$ and $M^d(x, y, t) = 0$ when $t = 0$. Then M^d is a fuzzy metric on X and is called the standard fuzzy metric induced by d.

Definition 2.6. [7] A map $\mathtt{u} : I^{X\times X} \to I$ is called an I-quasi-uniformity if it satisfies the following conditions:

($\mathtt{u}$1) $\mathtt{u}(0_{X\times X}) = 0$; $\mathtt{u}(1_{X\times X}) = 1$.

($\mathtt{u}$2) $\mathtt{u}(A) \leq \bigwedge_{x\in X} A(x, x)$;

($\mathtt{u}$3) $A, B \in I^{X\times X}$, $A \leq B$ implies $\mathtt{u}(A) \leq \mathtt{u}(B)$;

($\mathtt{u}$4) $\mathtt{u}(A) \wedge \mathtt{u}(B) \leq \mathtt{u}(A \wedge B)$ for $A, B \in I^{X\times X}$;

($\mathtt{u}$5) $\mathtt{u}(A) \leq \bigvee\{\alpha * \mathtt{u}(B) * \mathtt{u}(C) |\ B, C \in I^{X\times X}, \alpha * (B \circ C) \leq A\}$ for all $A \in I^{X\times X}$, where $(B \circ C)(x, z) = \bigvee_{y\in X} B(x, y) * C(y, z)$;

If an I-quasi-uniformity $\mathtt{u}$ further satisfies the following condition:

($\mathtt{u}$6) $\mathtt{u}(A) = \mathtt{u}(A^{-1})$, where $A^{-1}(x, y) = A(y, x)$,

then $\mathtt{u}$ is called an I-uniformity. The pair $(X, \mathtt{u})$ is called an I(-quasi)-uniform space.

An I-(quasi-)uniform space $(X, \mathtt{u})$ is called strongly stratified if it satisfies the following condition (SS):

(SS) $\forall \alpha \in I$, $\forall A \in I^{X\times X}$, $\alpha \to \mathtt{u}(A) = \mathtt{u}(\alpha \to A)$.

Let $(X, \mathtt{u})$ and $(Y, \mathtt{u}')$ be two I-(quasi-)uniform spaces. A map $F : X \to Y$ is called uniformly continuous if $\mathtt{u}'(B) \leq \mathtt{u}((F \times F)^{\leftarrow}(B))$ for all $B \in I^{Y\times Y}$. The category of strongly stratified I-quasi-uniform spaces and uniformly continuous maps is denoted by **SS*I*-QUnif**.

Definition 2.7. A map $\mathtt{B} : I^{X\times X} \to I$ is called a base of I-(quasi-)uniformity $\mathtt{u}$ if and only if $\mathtt{u}(A) = \bigvee_{B\leq A} \mathtt{B}(B)$ for all $A \in I^{X\times X}$. If $\mathtt{B}$ is a base of $\mathtt{u}$ and $supp\mathtt{B} = \{A |\ \mathtt{B}(A) > 0\}$ be a countable set, then $\mathtt{B}$ is called a countable base of $\mathtt{u}$.

Definition 2.8. [7] An I-(quasi-)uniformity $\mathtt{u}$ is said to be separated if and only if, for each pair $(x,y) \in X \times X$ with $x \neq y$, there exists $A \in I^{X\times X}$ such that $\mathtt{u}(A) > A(x,y)$.

Definition 2.9. [12] A map $\Im : I^X \to I$ is called an I-filter if it satisfies the following conditions:

(F1) $\Im(0_X) = 0$; $\Im(1_X) = 1$.

(F2) $D, E \in I^X$, $D \leq E$ implies $\Im(D) \leq \Im(E)$;

(F3) $\Im(D) \wedge \Im(E) \leq \Im(D \wedge E)$ for $D, E \in I^X$.

An I-filter is called stratified if it still satisfies:

(S) $\forall \alpha \in I$, $\forall D \in I^X$, $\alpha * \Im(D) \leq \Im(\alpha * D)$.

Definition 2.10. [12] Let $\mathcal{N} : X \to I^{I^X}$ be a map and define $\mathcal{N}(x) = N_x$. $\mathcal{N}$ is said to be an I-neighborhood system on X if N_x satisfies the following axioms for each $x \in X$:

(N1) $N_x(0_X) = 0$; $N_x(1_X) = 1$.

(N2) $D, E \in I^X$, $D \leq E$ implies $N_x(D) \leq N_x(E)$;

(N3) $N_x(D) \wedge N_x(E) \leq N_x(D \wedge E)$ for $D, E \in I^X$;

(N4) $N_x(D) \leq D(x)$ for all $D \in I^X$;

(N5) $N_x(D) \leq \bigvee\{N_x(E) |\ \forall y \in X, E(y) \leq N_y(E)\}$ for all $D \in I^X$.

From the above definition of I-neighborhood system, we know that N_x is an I-filter for each $x \in X$, and (N5) can be reformulated in the following form:

(N5$'$) $N_x(D) \leq N_x(N_-(D))$ for all $D \in I^X$.

where $N_-(D) \in I^X$ is defined by $N_-(D)(y) = N_y(D)$.

Let $H(I^X)$ denote the family of all maps $f : I^X \to I^X$ such that:

(1) $a \leq f(D)$ for all $D \in I^X$; (2) $f(\bigvee_{j\in J} D_j) = \bigvee_{j\in J} f(D_j)$.

Definition 2.11. [13] A nonempty subset $\mathtt{D}$ of $H(I^X)$ is called a Hutton quasi-uniformity if it satisfies the following conditions:

($\mathtt{D1}$) $f \in \mathtt{D}$; $f \leq g \in H(I^X)$, then $g \in \mathtt{D}$.

($\mathtt{D2}$) $f, g \in \mathtt{D}$ implies $f \wedge g \in \mathtt{D}$;

($\mathtt{D3}$) $f \in \mathtt{D}$ implies that there exists $\aleph : \mathtt{D} \times \mathtt{D} \to I$ such that

(i) $\bigvee\{\aleph(g,h) |\ g, h \in \mathtt{D}\} = 1$;

(ii) $\aleph(g,h) * (g \circ h) \leq f$ for all $g, h \in \mathtt{D}$.

The pair $(X, \mathtt{D})$ is called Hutton quasi-uniform space. A map $F : (X, \mathtt{D}) \to (Y, \mathtt{D}')$ is called uniformly continuous if $F^{\Leftarrow}(g) \in \mathtt{D}$ for all $g \in \mathtt{D}'$, where $F^{\Leftarrow}(g) = F^{\leftarrow} \circ g \circ F^{\rightarrow}$. The category of Hutton quasi-uniform spaces and uniformly continuous maps is denoted by **H-QUnif**.

Remark 2.12. The axiom ($\mathtt{D3}$) is weaker than the original axiom ($\mathtt{D3}'$):

($\mathtt{D3}'$) If $f \in \mathtt{D}$, then there exists $g \in \mathtt{D}$ such that $g \circ g \leq f$.

3. I-uniformity Induced by Fuzzy Metric

Let $(X, M, *)$ be a fuzzy metric space and define $M(-,-,t) : X \times X \to I$ by $M(-,-,t)(x,y) = M(x,y,t)$. From (KM4), it is easy to check that $M(-,-,\frac{t}{2}) \circ$

$M(-,-,\frac{t}{2}) \leq M(-,-,t)$. In this section, we will induce an I-uniformity from a given fuzzy metric and study some properties of it.

Lemma 3.1. *Let $(X, M, *)$ be a fuzzy metric space and define $\mathtt{u}_M : I^{X\times X} \to I$ as follows:*

$$\forall A \in I^{X\times X},\ \mathtt{u}_M(A) = \bigvee_{t>0} S(M(-,-,t), A) = \bigvee_{t>0} \bigwedge_{x,y} M(x,y,t) \to A(x,y).$$

Then $\mathtt{u}_M$ is an I-uniformity.

Proof. From the definition of $\mathtt{u}_M$, (u1),(u3) and (u6) are obvious and we only prove (u2),(u4) and (u5).

(u2):

$$\mathtt{u}_M(A) = \bigvee_{t>0} \bigwedge_{x,y} M(x,y,t) \to A(x,y) \leq \bigvee_{t>0} \bigwedge_{x\in X} M(x,x,t) \to A(x,x) = \bigwedge_{x\in X} A(x,x).$$

(u4):

$$\begin{aligned}
\mathtt{u}_M(A) \wedge \mathtt{u}_M(B) &= \bigvee_{t>0} S(M(-,-,t), A) \wedge \bigvee_{r>0} S(M(-,-,r), B) \\
&= \bigvee_{t>0} \bigvee_{r>0} S(M(-,-,t), A) \wedge S(M(-,-,r), B) \\
&\leq \bigvee_{t>0} \bigvee_{r>0} S(M(-,-,t) \wedge M(-,-,r), A \wedge B) \\
&= \bigvee_{l>0} S(M(-,-,l), A \wedge B) \\
&= \mathtt{u}_M(A \wedge B).
\end{aligned}$$

(u5):

$$\begin{aligned}
\mathtt{u}_M(A) &= \bigvee_{t>0} S(M(-,-,t), A) \\
&\leq \bigvee_{t>0} S(M(-,-,\frac{t}{2}) \circ M(-,-,\frac{t}{2}), A) \\
&= \bigvee_{t>0} S(D \circ D, A) * S(D, D) * S(D, D) \qquad (D = M(-,-,\frac{t}{2})) \\
&\leq \bigvee_{t>0} \bigvee_{B,C\in I^{X\times X}} S(B \circ C, A) * S(D, B) * S(D, C) \\
&\leq \bigvee_{B,C\in I^{X\times X}} \mathtt{u}_M(B) * \mathtt{u}_M(C) * S(B \circ C, A) \\
&= \bigvee_{\alpha*(B\circ C)\leq A} \alpha * \mathtt{u}_M(B) * \mathtt{u}_M(C).
\end{aligned}$$

□

Lemma 3.2. *$\mathtt{u}_M$ defined in Lemma 3.1 also satisfies the following properties:*

(1) $\forall \alpha \in I$, $\mathtt{u}_M(\alpha) = \alpha$.

(2) $\alpha * \mathtt{u}_M(A) \leq \mathtt{u}_M(\alpha * A)$.

(3) $\alpha \to \mathtt{u}_M(A) \geq \mathtt{u}_M(\alpha \to A)$. In particular if $\to$ preserves joins, then $\alpha \to \mathtt{u}_M(A) = \mathtt{u}_M(\alpha \to A)$.

(4) $\forall \alpha \in I$, $\forall s \in (0,1]$, $\mathtt{u}_M(\alpha * M(-,-,s)) = \alpha$.

Proof. (1)

$$\mathtt{u}_M(\alpha)=\bigvee_{t>0}\bigwedge_{x,y}M(x,y,t)\to\alpha=\alpha.$$

(2)

$$\begin{aligned}\alpha*\mathtt{u}_M(A) &= \alpha*\bigvee_{t>0}S(M(-,-,t),A)\\ &= \bigvee_{t>0}\alpha*S(M(-,-,t),A)\\ &\le \bigvee_{t>0}S(M(-,-,t),\alpha*A)\\ &= \mathtt{u}_M(\alpha*A).\end{aligned}$$

(3)

$$\begin{aligned}\alpha\to\mathtt{u}_M(A) &= \alpha\to\bigvee_{t>0}S(M(-,-,t),A)\\ &\ge \bigvee_{t>0}\alpha\to S(M(-,-,t),A)\\ &= \bigvee_{t>0}S(M(-,-,t),\alpha\to A)\\ &= \mathtt{u}_M(\alpha\to A).\end{aligned}$$

If $\to$ preserves joins, it is obvious that $\alpha\to\mathtt{u}_M(A)=\mathtt{u}_M(\alpha\to A)$.

(4)

$$\begin{aligned}\mathtt{u}_M(\alpha*M(-,-,s)) &= \bigvee_{t>0}S(M(-,-,t),\alpha*M(-,-,s))\\ &\ge S(M(-,-,s),\alpha*M(-,-,s))\\ &\ge \alpha.\end{aligned}$$

From (u2), $\mathtt{u}_M(\alpha*M(-,-,s))\le\alpha$ is obvious. Hence, $\mathtt{u}_M(\alpha*M(-,-,s))=\alpha$. □

Remark 3.3. (1) From Lemma 3.2 (2), we know that

$$\bigvee_{\alpha*(B\circ C)\le A}\alpha*\mathtt{u}_M(B)*\mathtt{u}_M(C)=\bigvee_{B\circ C\le A}\mathtt{u}_M(B)*\mathtt{u}_M(C).$$

Hence

$$\mathtt{u}_M(A)\le\bigvee_{B\circ C\le A}\mathtt{u}_M(B)*\mathtt{u}_M(C).$$

(2) According to the proof of (u5) in Lemma 3.1, we can also get the following result:

$$\begin{aligned}\mathtt{u}_M(A) &= \bigvee_{t>0}S(M(-,-,t),A)\\ &\le \bigvee_{t>0}S(M(-,-,\tfrac{t}{2})\circ M(-,-,\tfrac{t}{2}),A)\\ &= \bigvee_{t>0}S(M(-,-,\tfrac{t}{2})\circ M(-,-,\tfrac{t}{2}),A)*S(M(-,-,\tfrac{t}{2}),M(-,-,\tfrac{t}{2}))\\ &\le \bigvee_{t>0}\bigvee_{B\in I^{X\times X}}S(B\circ B,A)*S(M(-,-,\tfrac{t}{2}),B)\\ &\le \bigvee_{B\in I^{X\times X}}\mathtt{u}_M(B)*S(B\circ B,A)\\ &= \bigvee_{\alpha*(B\circ B)\le A}\alpha*\mathtt{u}_M(B).\end{aligned}$$

Similar to (1), we have

$$\bigvee_{\alpha*(B\circ B)\leq A} \alpha * \mathtt{u}_M(B) = \bigvee_{B\circ B\leq A} \mathtt{u}_M(B).$$

Hence

$$\mathtt{u}_M(A) \leq \bigvee_{B\circ B\leq A} \mathtt{u}_M(B).$$

Example 3.4. Let $X = \{x, y\}$ and $M : X \times X \times [0, +\infty) \to [0, 1]$ be defined by

$$M(a,b,t) = \begin{cases} 1, & a = b = x, t > 0 \\ 1, & a = b = y, t > 0 \\ 1, & a \neq b, t > \frac{1}{2}, \\ \frac{1}{2} + t, & a \neq b, t \leq \frac{1}{2}, \\ 0, & t = 0, \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space. Let $A \in I^{X\times X}$ be

$$A(a,b) = \begin{cases} 1, & a = b = x, \\ 1, & a = b = y, \\ \frac{2}{3}, & a = x, b = y, \\ \frac{1}{3}, & a = y, b = x, \end{cases}$$

Then

$$\begin{aligned} \mathtt{u}_M(A) &= \bigvee_{t>0} \bigwedge_{x,y} M(x,y,t) \to A(x,y) \\ &= \bigvee_{t>0} [(M(x,y,t) \to \frac{2}{3}) \wedge (M(y,x,t) \to \frac{1}{3})] \\ &= \bigvee_{t>0} M(y,x,t) \to \frac{1}{3} \\ &= \bigvee_{0<t\leq\frac{1}{2}} (\frac{1}{2} + t) \to \frac{1}{3} \\ &= \frac{1}{2} \to \frac{1}{3} \\ &= \begin{cases} \frac{5}{6}, & * = \diamondsuit, \\ \frac{1}{3}, & * = \wedge, \\ \frac{2}{3}, & * = \cdot, \end{cases} \end{aligned}$$

Example 3.5. Let d be an ordinary metric on X and M^d be the induced standard fuzzy metric. Take $* = \diamondsuit$. For $U \in 2^{X\times X}$, we have the following computations (here we do not distinguish $U \in 2^{X\times X}$ and its characteristic function):

$$\begin{aligned} \mathtt{u}_{M^d}(U) &= \bigvee_{t>0} \bigwedge_{x,y} M^d(x,y,t) \to U(x,y) \\ &= \bigvee_{t>0} \bigwedge_{(x,y)\notin U} \frac{t}{t + d(x,y)} \to 0 \\ &= \bigvee_{t>0} \bigwedge_{(x,y)\notin U} 1 - \frac{t}{t + d(x,y)} \\ &= \bigvee_{t>0} \frac{\bigwedge_{(x,y)\notin U} d(x,y)}{t + \bigwedge_{(x,y)\notin U} d(x,y)} \\ &= \begin{cases} 0, & \bigwedge_{(x,y)\notin U} d(x,y) = 0, \\ 1, & \bigwedge_{(x,y)\notin U} d(x,y) \neq 0, \end{cases} \end{aligned}$$

Hence we can consider $U \in \mathtt{u}_{M^d}$ if and only if $\bigwedge_{(x,y)\notin U} d(x,y) \neq 0$ if and only if there exists $\varepsilon > 0$ such that $\bigwedge_{(x,y)\notin U} d(x,y) \geq \varepsilon$ if and only if there exists $\varepsilon > 0$ such that $d(x,y) \geq \varepsilon$ for all $(x,y) \notin U$ if and only if there exists $\varepsilon > 0$ such that $\{(x,y)|d(x,y) < \varepsilon\} \subseteq U$.

Lemma 3.6. *Let $(X, M, *)$ be a fuzzy metric space and $\mathtt{u}_M$ be the induced I-uniformity. Then $\mathtt{u}_M$ is separated.*

Proof. Let $(x,y) \in X \times X$ with $x \neq y$. By (KM2), there exists $s > 0$ such that $M(x,y,s) < 1$. Take $A = M(-,-,s)$. Then $\mathtt{u}_M(A) = 1 > M(x,y,s) = A(x,y)$, as desired. □

Lemma 3.7. *Let $(X, M, *)$ be a fuzzy metric space and $\mathtt{u}_M$ be the induced I-uniformity. Then $\mathtt{u}_M$ has a countable base.*

Proof. Let $A_{r,s} = r * M(-,-,s)$ and set

$\mathcal{A} = \{A_{r,s}|\ r$ is a rational number in$[0,1)$ and s is a rational number in$(0,+\infty)\}$.

Then $\mathcal{A}$ is a countable set. Define $\mathtt{B} : I^{X\times X} \to I$ by

$$\mathtt{B}(A) = \begin{cases} \mathtt{u}_M(A), & A \in \mathcal{A} \\ 0, & \text{others} \end{cases}$$

We can assert that $\mathtt{B}$ is a countable base of $\mathtt{u}_M$. We need to show that $\mathtt{u}_M(A) = \bigvee_{B\leq A} \mathtt{B}(B)$ for all $A \in I^{X\times X}$. Since $\mathtt{u}_M(A) \geq \bigvee_{B\leq A} \mathtt{B}(B)$ is obvious, it suffices to show that $\mathtt{u}_M(A) \leq \bigvee_{B\leq A} \mathtt{B}(B)$. Let l be any rational number in $[0,1)$ such that $l < \mathtt{u}_M(A)$. Then there exists a rational number $t \in (0,+\infty)$ such that $l \leq S(M(-,-,t),A)$. Thus $l * M(-,-,t) = A_{l,t} \leq A$ and $A_{l,t} \in \mathcal{A}$. By $\mathtt{B}(A_{l,t}) = l$, we have $\mathtt{u}_M(A) \leq \bigvee_{B\leq A} \mathtt{B}(B)$ from the arbitrariness of l, as desired. □

In classical topology, a uniformity is metrizable if and only if it is separated and has a countable base. From Lemma 3.6 and Lemma 3.7, it is obtained that $\mathtt{u}_M$ is separated and has a countable base. It,s an I-uniformity metrizable if and only if it is separated and has a countable base? We leave it as an question.

Lemma 3.8. *Let $(X, M, *)$ be a fuzzy metric space and define $N_x^M : I^X \to I$ as follows:*

$$N_x^M(D) = \bigvee_{t>0} \bigwedge_{y\in X} M(x,y,t) \to D(y).$$

Then $\aleph^M : X \to I^{I^X}$ defined by $\aleph^M(x) = N_x^M$ is an I-neighborhood system.

Proof. We only prove

$$\begin{aligned} N_x^M(N_-^M(A)) &= \bigvee_{t>0} \bigwedge_{y} M(x,y,t) \to N_y^M(A) \\ &= \bigvee_{t>0} \bigwedge_{y} \{M(x,y,t) \to \bigvee_{t_1>0} \bigwedge_{z} M(y,z,t_1) \to A(z)\} \\ &\geq \bigvee_{t>0} \bigwedge_{y} \bigvee_{t_1>0} \bigwedge_{z} [M(x,y,t) \to (M(y,z,t_1) \to A(z))] \\ &= \bigvee_{t>0} \bigwedge_{y} \bigvee_{t_1>0} \bigwedge_{z} M(x,y,t) * M(y,z,t_1) \to A(z) \\ &\geq \bigvee_{t>0} \bigwedge_{y} \bigvee_{t_1>0} \bigwedge_{z} M(x,y,t+t_1) \to A(z) \\ &= \bigvee_{r>0} \bigwedge_{z} M(x,y,r) \to A(z) \\ &= N_x^M(A). \end{aligned}$$

□

From [7], $\mathtt{u}_M$ can generate the I-neighborhood system $\aleph^{\mathtt{u}} : X \to I^{I^X}$ in the following way:

$\aleph^{\mathtt{u}_M}(x) = N_x^{\mathtt{u}_M} : I^X \to I$ is defined by

$$\forall D \in I^X,\ N_x^{\mathtt{u}_M}(D) = \bigvee\{\alpha * \mathtt{u}(A) |\ \alpha * (A[x]) \leq D\}$$

where $A[x] \in I^X$ is defined by $A[x](y) = A(x, y)$. Since $\mathtt{u}_M$ satisfies (2) in Lemma 3.2, $N_x^{\mathtt{u}_M}(D)$ can also be simplified by

$$N_x^{\mathtt{u}_M}(D) = \bigvee_{A[x] \leq D} \mathtt{u}(A).$$

Now we have the following result.

Theorem 3.9. *The I-neighborhood system induced by M is equivalent to the I-neighborhood system induced by $\mathtt{u}_M$.*

Proof. We need to check that $N_x^{\mathtt{u}_M}(D) = N_x^M(D)$ for all $x \in X$ and $D \in I^X$.

$$\begin{aligned}
N_x^{\mathtt{u}_M}(D) &= \bigvee_{A[x] \leq D} \mathtt{u}_M(A) \\
&= \bigvee_{A[x] \leq D} \bigvee_{t>0} \bigwedge_{y,z} M(y, z, t) \to A(y, z) \\
&= \bigvee_{A[x] \leq D} \bigvee_{t>0} \bigwedge_{y,z} M(y, z, t) \to A[y](z) \\
&\leq \bigvee_{A[x] \leq D} \bigvee_{t>0} \bigwedge_{z} M(x, z, t) \to A[x](z) \\
&\leq \bigvee_{A[x] \leq D} \bigvee_{t>0} \bigwedge_{z} M(x, z, t) \to D(z) \\
&= \bigvee_{t>0} \bigwedge_{z} M(x, z, t) \to D(z) \\
&= N_x^M(D).
\end{aligned}$$

On the other hand, define $A^* : X \times X \to I$ by

$$A^*(y, z) = \begin{cases} 1, & y \neq x, \\ D(z), & y = x, \end{cases}$$

Then

$$\begin{aligned}
N_x^{\mathtt{u}_M}(D) &= \bigvee_{A[x] \leq D} \bigvee_{t>0} \bigwedge_{y,z} M(y, z, t) \to A(y, z) \\
&\geq \bigvee_{t>0} \bigwedge_{y,z} M(y, z, t) \to A^*(y, z) \\
&= \bigvee_{t>0} \bigwedge_{z} M(x, z, t) \to D(z) \\
&= N_x^M(D),
\end{aligned}$$

as desired. □

Remark 3.10. In [18], Zhang gave an enriched approach to many valued topology. If $*$ is also distributive over arbitrary meets, then the I-topology induced by I-neighborhood system in this paper will be a strong I-topology in the sense of Zhang.

4. The Relationship Between I-quasi-uniformity and Hutton Quasi-uniformity

In order to see that two Hutton quasi-uniformities can induce the same I-uniformities in fuzzy metric spaces, we first study the relationship between I-quasi-uniformities and Hutton quasi-uniformities. Some ideas of this section can be found in [8].

In [7, 8], $\Upsilon : H(I^X) \to I^{X\times X}$ and $\Lambda : I^{X\times X} \to H(I^X)$ are used to study the relationship between I-uniformities and L-valued Hutton uniformities, where $\Upsilon(f)(x,y) = \bigwedge_{\alpha\in I} \alpha \to (f(x_\alpha))(y)$ and $\Lambda(A)(D) = \bigvee_{x\in X} D(x) * A(x,-)$. The following lemma shows that I-quasi-uniformity can be induced by a Hutton quasi-uniformity from Υ.

Lemma 4.1. *Let $\mathtt{D}$ be a Hutton quasi-uniformity on X and define $\mathtt{u}_\mathtt{D} : I^{X\times X} \to I$ as follows:*

$$\mathtt{u}_\mathtt{D}(A) = \bigvee_{f\in \mathtt{D}} S(\Upsilon(f), A) = \bigvee_{f\in \mathtt{D}} \bigwedge_{x,y} \Upsilon(f)(x,y) \to A(x,y).$$

Then $\mathtt{u}_\mathtt{D}$ is an I-quasi-uniformity.

Proof. From the definition of $\mathtt{u}_\mathtt{D}$, it is easy to check ($\mathtt{u}1$)-($\mathtt{u}4$). We only verify that it satisfies ($\mathtt{u}5$).

($\mathtt{u}5$): Let $f \in \mathtt{D}$. By ($\mathtt{D}3$), there exists $\aleph : \mathtt{D}\times\mathtt{D} \to I$ such that

(i) $\bigvee\{\aleph(g,h)|\ g,h\in\mathtt{D}\} = 1$;

(ii) $\aleph(g,h) * (g\circ h) \le f$ for all $g,h\in\mathtt{D}$.

Then $\Upsilon(f) \ge \Upsilon(\aleph(g,h) * (g\circ h))$ for all $g,h\in\mathtt{D}$. Hence,

$$S(\Upsilon(f),A) \le S(\Upsilon(\aleph(g,h)*(g\circ h),A) \le S(\aleph(g,h)*(\Upsilon(g)\circ\Upsilon(h)),A)$$

for all $g,h\in\mathtt{D}$. So

$$\begin{aligned}
S(\Upsilon(f),A) &\le \bigwedge_{g,h\in\mathtt{D}} S(\aleph(g,h)*(\Upsilon(g)\circ\Upsilon(h)),A)\\
&= \bigwedge_{g,h\in\mathtt{D}} \aleph(g,h) \to S(\Upsilon(g)\circ\Upsilon(h),A)\\
&\le \bigwedge_{g,h\in\mathtt{D}} \aleph(g,h) \to (\bigvee_{g,h\in\mathtt{D}} S(\Upsilon(g)\circ\Upsilon(h),A))\\
&= (\bigvee_{g,h\in\mathtt{D}} \aleph(g,h)) \to \bigvee_{g,h\in\mathtt{D}} S(\Upsilon(g)\circ\Upsilon(h),A)\\
&= \bigvee_{g,h\in\mathtt{D}} S(\Upsilon(g)\circ\Upsilon(h),A)\\
&\le \bigvee_{g,h\in\mathtt{D}} \bigvee_{B,C\in I^{X\times X}} S(B\circ C,A)*S(\Upsilon(g),B)*S(\Upsilon(h),C)\\
&\le \bigvee\{\alpha*\mathtt{u}_\mathtt{D}(B)*\mathtt{u}_\mathtt{D}(C)|\alpha*(B\circ C)\le A\},
\end{aligned}$$

as desired. □

Similar to the proof of Theorem 3.3.6 in [8], we can prove the following Lemma 4.2.

Lemma 4.2. *Let $(X,\mathtt{u})$ be a strongly stratified I-quasi-uniform space and set $\mathtt{D}_\mathtt{u} = \{f\in H(I^X)|\ \mathtt{u}(\Upsilon(f)) = 1\}$. Then $\mathtt{D}_\mathtt{u}$ is a Hutton quasi-uniformity.*

Proof. It is easy to check ($\mathtt{D1}$)and ($\mathtt{D2}$). We only prove ($\mathtt{D3}$). Let $f \in \mathtt{D_u}$, then $\mathtt{u}(\Upsilon(f)) = 1$. From ($\mathtt{D5}$), we know

$$\mathtt{u}(\Upsilon(f)) \leq \bigvee\{\alpha * \mathtt{u}(B) * \mathtt{u}(C)|\ \alpha * (B \circ C) \leq \Upsilon(f)\}.$$

Let $\alpha * (B \circ C) \leq \Upsilon(f)$. Then $\alpha * (\Lambda(B) \circ \Lambda(C)) \leq f$. Since $g = \mathtt{u}(B) \to \Lambda(B) \in \mathtt{D_u}$ and $h = \mathtt{u}(C) \to \Lambda(C) \in \mathtt{D_u}$, then

$$\alpha * \mathtt{u}(B) * \mathtt{u}(C) * (g \circ h) \leq \alpha * (\Lambda(B) \circ \Lambda(C)) \leq f.$$

So $g \circ h \leq \alpha * \mathtt{u}(B) * \mathtt{u}(C) \to f$. Consequently, for $B, C \in I^{X\times X}$ where $\alpha * (B \circ C) \leq \Upsilon(f)$, we have

$$\alpha * \mathtt{u}(B) * \mathtt{u}(C) \leq \bigvee\{\beta | \exists g, h \in \mathtt{D_u}, s.t., g \circ h \leq \beta \to f\}.$$

This is to say

$$\bigvee\{\beta|\ \exists g, h \in \mathtt{D_u},\ \text{s.t.},\ g \circ h \leq \beta \to f\} = 1.$$

Define $\aleph : \mathtt{D_u} \times \mathtt{D_u} \to L$ as follows:

$$\aleph(g, h) = \bigvee\{\beta :|\ g \circ h \leq \beta \to f\}$$

Then

(i)$\bigvee\{\aleph(g, h)|\ g, h \in \mathtt{D}\} = 1$;

(ii)$\aleph(g, h) * (g \circ h) \leq f$ for all $g, h \in \mathtt{D}$. □

Lemma 4.3. *(1) If $(X, \mathtt{u})$ is a strongly stratified I-quasi-uniform space, then $\mathtt{u} = \mathtt{u}_{\mathtt{D_u}}$.*

(2) Let $(X, \mathtt{D})$ be a Hutton quasi-uniform space. Then $\mathtt{D} \subseteq \mathtt{D}_{\mathtt{u_D}}$.

Proof. (1) From the definition of $\mathtt{u}_{\mathtt{D_u}}$, we have $\mathtt{u}_{\mathtt{D_u}}(A) = \bigvee_{f\in \mathtt{D_u}} S(\Upsilon(f), A)$ for $A \in I^{X\times X}$. We can assert that $S(\Upsilon(f), A) \leq \mathtt{u}(A)$ for each $f \in \mathtt{D_u}$. In fact,

$$S(\Upsilon(f), A) \to \mathtt{u}(A) = \mathtt{u}(S(\Upsilon(f), A) \to A) \geq \mathtt{u}(\Upsilon(f)) = 1.$$

So $\mathtt{u}(A) \geq \mathtt{u}_{\mathtt{D_u}}(A)$. This is to say that $\mathtt{u} \geq \mathtt{u}_{\mathtt{D_u}}$.

On the other hand, let $f = \Lambda(\mathtt{u}(A) \to A)$. Then $f \in \mathtt{D_u}$ and $S(\Upsilon(f), A) = S(\mathtt{u}(A) \to A, A) \geq \mathtt{u}(A)$. Hence $\mathtt{u}_{\mathtt{D_u}}(A) \geq \mathtt{u}(A)$, as desired.

(2) It is trivial and omitted. □

Lemma 4.4. *(1) If $F : (X, \mathtt{D}) \to (Y, \mathtt{D}')$ is uniformly continuous, then $F : (X, \mathtt{u_D}) \to (Y, \mathtt{u_{D'}})$ is also uniformly continuous.*

(2) If $F : (X, \mathtt{u}) \to (Y, \mathtt{u}')$ is uniformly continuous, then $F : (X, \mathtt{D_u}) \to (Y, \mathtt{D_{u'}})$ is also uniformly continuous.

Proof. We only prove (1) and we need to prove that $\mathtt{u_{D'}}(B) \leq \mathtt{u_D}((F \times F)^{\leftarrow}(B))$ for all $B \in I^{Y\times Y}$. Since $F : (X, \mathtt{D}) \to (Y, \mathtt{D}')$ is uniformly continuous, it suffices to prove that $S(\Upsilon(F^{\Leftarrow}(g)), (F \times F)^{\leftarrow}(B)) \geq S(\Upsilon(g), B)$ for $g \in \mathtt{D}'$. In fact,

$$\begin{aligned}
S(\Upsilon(F^{\Leftarrow}(g)), (F\times F)^{\leftarrow}(B)) &= \bigwedge_{x,y} \Upsilon(F^{\Leftarrow}(g))(x, y) \to (F \times F)^{\leftarrow}(B)(x, y)\\
&= \bigwedge_{x,y} (\bigwedge_{\alpha\in I} \alpha \to F^{\Leftarrow}(g)(x_\alpha)(y)) \to B(F(x), F(y))\\
&= \bigwedge_{x,y} \Upsilon(g)(F(x), F(y)) \to B(F(x), F(y))\\
&\geq \bigwedge_{z,w} \Upsilon(g)(z, w) \to B(z, w)\\
&= S(\Upsilon(g), B),
\end{aligned}$$

as desired. □

From Lemma 4.3 and Lemma 4.4, we have the following result.

Theorem 4.5. **SSI-QUnif** *can be embedded in* **H-QUnif** *as a coreflective subcategory.*

5. Hutton Quasi-uniformity Induced by Fuzzy Metric

In this section, another kind of Hutton quasi-uniformity a little different from that in [9] is induced by fuzzy metric, and it is shown that these two kinds of Hutton quasi-uniformities will generate the same I-uniformity.

In [9], J. Gutiérrez García gave the construction of Hutton quasi-uniformity in the following way:

For $\varepsilon \in (0,1], t > 0$, define $W_{\varepsilon,t} : I^X \to I^X$ by

$$W_{\varepsilon,t}(x_\alpha)(y) = \alpha * [(1-\varepsilon) \to M(x,y,t)], W_{\varepsilon,t}(a) = \bigvee_{x\in X} W_{\varepsilon,t}(x_{a(x)}).$$

Then $\mathcal{B}_M = \{W_{\varepsilon,t} | \, \varepsilon \in (0,1], t > 0\}$ will be a base of one Hutton quasi-uniformity, denoted by $\mathtt{D}_{\mathcal{B}_M}$. If $* = \diamondsuit$, then $\mathtt{D}_{\mathcal{B}_M}$ is a Hutton uniformity.

Now we give another similar form for generating Hutton quasi-uniformity.

For $t > 0$, define $W_t : I^X \to I^X$ as follows:

$$W_t(x_\alpha)(y) = \alpha * M(x,y,t), W_t(a) = \bigvee_{x\in X} W_t(x_{a(x)}).$$

It is routine to check that $\mathcal{B}^M = \{W_t | t > 0\}$ is also a base of one Hutton quasi-uniformity, denoted by $\mathtt{D}_{\mathcal{B}^M}$. If $* = \diamondsuit$, then $\mathtt{D}_{\mathcal{B}_M}$ is also a Hutton uniformity.

$\mathtt{D}_{\mathcal{B}_M} \subseteq \mathtt{D}_{\mathcal{B}^M}$ is obvious. Is $\mathtt{D}_{\mathcal{B}_M} = \mathtt{D}_{\mathcal{B}^M}$ valid? The following example shows that it is not the case.

Example 5.1. Let $X = \{x, y\}$ and $M : X \times X \times [0, +\infty) \to [0, 1]$ be defined by

$$M(a,b,t) = \begin{cases} 1, & a = b, t > 0 \\ \frac{1}{2}, & a \neq b, t > 0 \\ 0, & t = 0 \end{cases}$$

Then $(X, M, \diamondsuit)$ is a fuzzy metric space.

$$W_t(x_{\frac{1}{2}})(a) = \begin{cases} \frac{1}{2}, & a = x, \\ 0, & a = y, \end{cases}$$

$$W_{\varepsilon,t}(x_{\frac{1}{2}})(a) = \begin{cases} \frac{1}{2}, & a = x, \\ \frac{1}{2}, & \varepsilon \geq \frac{1}{2}, a = y, \\ \varepsilon, & \varepsilon < \frac{1}{2}, a = y, \end{cases}$$

It is easy to check that $W_t \in \mathtt{D}_{\mathcal{B}^M}$, but $W_t \notin \mathtt{D}_{\mathcal{B}_M}$.

Let $(X, M, *)$ be a fuzzy metric space. According to Lemma 4.1, we have

$$\begin{aligned} \mathtt{u}_{\mathtt{D}_{\mathcal{B}^M}}(A) &= \bigvee_{f\in\mathcal{B}^M} S(\Upsilon(f), A) \\ &= \bigvee_{t>0} S(\Upsilon(W_t), A) \\ &= \bigvee_{t>0} \bigwedge_{x,y} \Upsilon(W_t)(x,y) \to A(x,y) \\ &= \bigvee_{t>0} \bigwedge_{x,y} [\bigwedge_{\alpha\in I} \alpha \to (\alpha * M(x,y,t))] \to A(x,y) \\ &= \bigvee_{t>0} \bigwedge_{x,y} M(x,y,t) \to A(x,y) \\ &= \mathtt{u}_M(A) \end{aligned}$$

and

$$\begin{aligned}
\mathtt{u}_{\mathtt{D}_{\mathcal{B}_M}}(A) &= \bigvee_{f\in\mathcal{B}_M} S(\Upsilon(f), A) \\
&= \bigvee_{\varepsilon,t} S(\Upsilon(W_{\varepsilon t}, A) \\
&= \bigvee_{\varepsilon,t}\bigwedge_{x,y} \Upsilon(W_{\varepsilon,t}(x,y) \to A(x,y) \\
&= \bigvee_{\varepsilon,t}\bigwedge_{x,y}[\bigwedge_{\alpha\in I} \alpha \to (\alpha * ((1-\varepsilon) \to M(x,y,t))] \to A(x,y) \\
&= \bigvee_{\varepsilon,t}\bigwedge_{x,y}((1-\varepsilon) \to M(x,y,t)) \to A(x,y).
\end{aligned}$$

Theorem 5.2. *Let $(X, M, *)$ be a fuzzy metric space. Then $\mathtt{u}_{\mathtt{D}_{\mathcal{B}^M}} = \mathtt{u}_{\mathtt{D}_{\mathcal{B}_M}}$.*

Proof. It needs to show $\mathtt{u}_{\mathtt{D}_{\mathcal{B}^M}}(A) = \mathtt{u}_{\mathtt{D}_{\mathcal{B}_M}}(A)$ for all $A \in I^{X\times X}$, i.e.,

$$\bigvee_{t>0}\bigwedge_{x,y} M(x,y,t) \to A(x,y) = \bigvee_{\varepsilon,t}\bigwedge_{x,y}((1-\varepsilon) \to M(x,y,t)) \to A(x,y).$$

$$\bigvee_{t>0}\bigwedge_{x,y} M(x,y,t) \to A(x,y) \geq \bigvee_{\varepsilon,t}\bigwedge_{x,y}((1-\varepsilon) \to M(x,y,t)) \to A(x,y)$$

is obvious. Conversely,

$$\begin{aligned}
\mathtt{u}_{\mathtt{D}_{\mathcal{B}_M}}(A) &= \bigvee_{\varepsilon,t}\bigwedge_{x,y}((1-\varepsilon) \to M(x,y,t)) \to A(x,y) \\
&\geq \bigvee_{\varepsilon,t}\bigwedge_{x,y}(1-\varepsilon) * (M(x,y,t) \to A(x,y)) \\
&\geq \bigvee_{\varepsilon,t}(1-\varepsilon) * (\bigwedge_{x,y}(M(x,y,t) \to A(x,y)) \\
&= \bigvee_{t}[(\bigvee_{\varepsilon}(1-\varepsilon)) * \bigwedge_{x,y}(M(x,y,t) \to A(x,y)) \\
&= \bigvee_{t}\bigwedge_{x,y} M(x,y,t) \to A(x,y) \\
&= \mathtt{u}_{\mathtt{D}_{\mathcal{B}^M}}(A),
\end{aligned}$$

as desired. □

Theorem 5.2 shows that two Hutton quasi-uniformities generate the same I-quasi-uniformity. In fact, they also generate the same I-neighborhood system. Now see the following theorem.

Theorem 5.3. *Let $(X, M, *)$ be a fuzzy metric space. Then M, $\mathtt{u}_M$, $\mathtt{D}_{\mathcal{B}^M}$ and $\mathtt{D}_{\mathcal{B}_M}$ will generate the same I-neighborhood system.*

Proof. We only verify that $N_x^{\mathcal{B}^M}(D) = N_x^M(D)$ for all $x \in X$ and $D \in I^X$, where $N_x^{\mathcal{B}^M}$ is the I-neighborhood system of x induced by $\mathtt{D}_{\mathcal{B}^M}$ and is defined by

$$N_x^{\mathcal{B}^M}(D) = \bigvee_{t>0}\bigvee\{\alpha \in [0,1] |\ \exists W_t \in \mathcal{B}^M, \text{ s.t., } W_t(x_\alpha) \leq D\}.$$

Suppose $W_t(x_\alpha) \leq D$. Then $W_t(x_\alpha)(y) \leq D(y)$ for all $y \in X$, i.e., $\alpha * M(x,y,t) \leq D(y)$. This is to say $\alpha \leq M(x,y,t) \to D(y)$ for all $y \in X$. Hence $\alpha \leq \bigwedge_{y\in X} M(x,y,t) \to D(y) \leq N_x^M(D)$. So $N_x^{\mathcal{B}^M}(D) \leq N_x^M(D)$.

Conversely, $\forall t > 0$, let $\alpha = \bigwedge_{y\in X} M(x,y,t) \to D(y)$. Then it is easy to check that $W_t(x_\alpha) \leq D$. Hence $\bigwedge_{y\in X} M(x,y,t) \to D(y) \leq N_x^{\mathcal{B}^M}(D)$. So $N_x^M(D) \leq N_x^{\mathcal{B}^M}(D)$ from the arbitrariness of t. □

Acknowledgements. This work is supported by NSFC (11201437), NSF of Shandong Province (ZR2011AQ010), the specialized Research Fund for the Doctoral Program of Higher Education(No.20110132120014) and the Fundamental Research Funds for the Central Universities(No.201213010, 201362030).

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