

APPLICATION OF PARAMETRIC FORM FOR RANKING OF FUZZY NUMBERS

R. EZZATI, S. KHEZERLOO AND S. ZIARI

ABSTRACT. In this paper, we propose a new approach for ranking all fuzzy numbers based on revising the ranking method proposed by Ezzati et al. [10]. To this end, we present and investigate some properties of the proposed approach in details. Finally, to illustrate the advantage of the proposed method, it is applied to several groups of fuzzy numbers and the results are compared with other related and familiar ones.

1. Introduction

In many applications, ranking of fuzzy numbers is an important and prerequisite procedure for decision makers. Firstly, in 1976, Jain [13, 14] proposed a method for ranking fuzzy numbers, then a large variety of methods were developed to rank fuzzy numbers. Wang and Kerre [21, 22] classified the ordering method into three categories and proposed seven reasonable properties to evaluate the ordering method. In 2007, Asady and Zendehnam [3] proposed a new method based on "Distance Minimizing", and then in 2009, Abbasbandy and Hajjari [1] proposed a new method for the ranking of trapezoidal fuzzy numbers and showed that their new method overcomes some drawbacks of distance minimizing. But, by their new method, all trapezoidal fuzzy numbers $(x_0, y_0, \sigma, \sigma)$ with different σ and also all triangular fuzzy numbers $(\frac{x_0+y_0}{2}, \sigma, \sigma)$ with different σ are in the same order. For example, consider the two fuzzy numbers, $A = (3, 2, 2)$ and $B = (3, 1, 1)$, see Figure 1, from [6], $Mag(A) = Mag(B) = 3$, i.e. $A \sim B$ and also consider $A = (-1, 1, 5, 5)$ and $B = (0, 2, 2)$ then $Mag(A) = Mag(B) = 0$ so $A \sim B$. However, it is clear that the result of ordering is not reasonable and fuzzy numbers A and B don't belong to an equivalence class.

Then Ezzati et al. in [10] proposed a new method that solved such problems, but this method also has drawbacks. By this method, all fuzzy numbers $(x_0, y_0, \sigma, \sigma)$ with the same $\frac{x_0+y_0}{2}$ and σ , are in the same order. For example, consider the two fuzzy numbers, $A = (2, 4, 1, 1)$ and $B = (3, 2, 2)$, see Figure 3, that $R(A, \delta = 1) = R(B, \delta = 1) = 5$ so $A \sim B$. Chen and Tang [5] presented the ranking method for the non-normal p-norm trapezoidal fuzzy numbers based on integral value approach, presented by Liou and Wang [17]. Amit Kumar et al. in [15] modified Chen and Tang's [5] approach for ranking of non-normal p-norm trapezoidal fuzzy numbers.

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and also in [16] they pointed out the shortcomings of Liou and Wang's [17] approach for the ranking of L-R type generalize fuzzy numbers. In [4] the authors compared different methods by means of numerical simulations. Abbasbandy et al. pointed out the shortcomings of "sign distance method" and in order to solve the problems presented a revised method. They stated that these problems were not effective in the ranking image of fuzzy numbers and were inconsistent with human intuition. This paper is organized as follows: In Section 2, we review some basic definitions and results on fuzzy numbers. In Section 3, we propose a new method for ranking fuzzy numbers. Comparing the presented ranking method with some other approaches, some numerical examples are given in Section 4. Finally, Section 5 gives our concluding remarks.

2. Preliminaries

There are various definitions for the concept of fuzzy numbers ([8, 11, 12])

Definition 2.1. [12] A fuzzy number is a fuzzy set $u : R \rightarrow [0, 1]$ satisfying the following properties:

- (i) u is upper semi-continuous,
- (ii) $u(x) = 0$ outside of interval $[0, 1]$,
- (iii) There are real numbers a, b, c and d such that $a \leq b \leq c \leq d$ and
 - a. $u(x)$ is monotonic increasing on $[a, b]$,
 - b. $u(x)$ is monotonic decreasing on $[c, d]$,
 - c. $u(x) = 1, \quad b \leq x \leq c$,

and the membership function u can be expressed as

$$u(x) = \begin{cases} u_L(x), & a \leq x \leq b \\ 1, & b \leq x \leq c \\ u_R(x), & c \leq x \leq d \\ 0, & \text{otherwise} \end{cases}$$

where $u_L : [a, b] \rightarrow [0, 1]$ and $u_R : [c, d] \rightarrow [0, 1]$ are left and right membership functions of fuzzy number u , respectively.

Definition 2.2. [19] An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$.
2. $\bar{u}(r)$ is a bounded left-continuous non-increasing function over $[0, 1]$.
3. $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \bar{u}(r) = \alpha$, $0 \leq r \leq 1$.

The trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, with $[u]_1 = [x_0, y_0]$ and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ is a fuzzy sets where the membership function is as

$$u(x) = \begin{cases} \frac{x-x_0+\sigma}{\sigma}, & x_0 - \sigma \leq x \leq x_0 \\ 1, & x_0 \leq x \leq y_0 \\ \frac{y_0+\beta-x}{\beta}, & y_0 \leq x \leq y_0 + \beta \\ 0, & \text{otherwise} \end{cases}$$

and if $x_0 = y_0$ then u is a triangular fuzzy numbers, denoted by $u = (x_0, \sigma, \beta)$. For $0 < r \leq 1$, denote $[u]_r = \{x \in R; u(x) \geq r\}$ and $[u]_0 = \{x \in R; u(x) > 0\}$. Then it is well-known that for any $r \in [0, 1]$, $[u]_r$ is a bounded closed interval. For arbitrary $u = (\underline{u}(r), \bar{u}(r))$, $v = (\underline{v}(r), \bar{v}(r))$ addition and scalar multiplication are defined by extension principle and equivalently represented as

$$\begin{aligned} (\underline{u+v})(r) &= \underline{u}(r) + \underline{v}(r), & (\overline{u+v})(r) &= \bar{u}(r) + \bar{v}(r), \\ (\underline{k\underline{u}})(r) &= k\underline{u}(r), & (\overline{k\underline{u}})(r) &= k\bar{u}(r), \quad k \geq 0, \\ (\underline{k\underline{u}})(r) &= k\underline{u}(r), & (\overline{k\underline{u}})(r) &= k\underline{u}(r), \quad k < 0. \end{aligned} \quad (1)$$

The collection of all fuzzy numbers with addition and multiplication as defined by equations (1) is denoted by E , which is a convex cone.

3. Proposed Method

For an arbitrary fuzzy number $u \in E$ with parametric form $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$ and $[u]_1 = [\underline{u}(1), \bar{u}(1)]$, we define:

$$\begin{aligned} Mag(u) &= \frac{1}{2} \int_0^1 (\underline{u}(r) + \bar{u}(r) + \underline{u}(1) + \bar{u}(1)) f(r) dr \\ Momag(u) &= \frac{1}{2} \int_0^1 (\underline{u}(r) - \bar{u}(r) + \underline{u}(1) - \bar{u}(1)) dr \end{aligned} \quad (2)$$

where the weight function $f(\alpha)$ is nonnegative and increasing on $[0, 1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha) d\alpha = \frac{1}{2}$. In this paper we use $f(r) = r$.

Now we introduce

$$R(u, \gamma) = Mag(u) + \gamma Momag(u). \quad (3)$$

Therefore for any two fuzzy numbers $u, v \in E$, we define the ranking of u, v as follows:

- $R(u, \gamma) > R(v, \gamma)$ if and only if $u \succ v$
- $R(u, \gamma) < R(v, \gamma)$ if and only if $u \prec v$
- $R(u, \gamma) = R(v, \gamma)$ if and only if $u \sim v$

where

$$\gamma = \begin{cases} 0, & Mag(u) \neq Mag(v) \\ 1, & Mag(u) = Mag(v) \text{ and } z_0 \geq 0 \\ -1, & Mag(u) = Mag(v) \text{ and } z_0 < 0. \end{cases} \quad (4)$$

where $z_0 = \frac{\underline{u}(1) + \bar{u}(1)}{2}$ or $z_0 = \frac{\underline{v}(1) + \bar{v}(1)}{2}$.

Then, this article formulates the order \succeq and \preceq as $u \succeq v$ if and only if $u \succ v$ or $u \sim v$, $u \preceq v$ if and only if $u \prec v$ or $u \sim v$.

Proposition 3.1. For two arbitrary fuzzy numbers u, v , we have

$$R(u + v, \gamma) = R(u, \gamma) + R(v, \gamma). \quad (5)$$

Proof. Suppose $u = (\underline{u}(r), \bar{u}(r))$, $[u]_1 = [\underline{u}(1), \bar{u}(1)]$ and $v = (\underline{v}(r), \bar{v}(r))$, $[v]_1 = [\underline{v}(1), \bar{v}(1)]$ then

$$\begin{aligned} \text{Mag}(u+v) &= \frac{1}{2} \int_0^1 ((\underline{u+v})(r) + (\overline{u+v})(r) + (\underline{u+v})(1) + (\overline{u+v})(1)) f(r) dr \\ &= \frac{1}{2} \int_0^1 (\underline{u}(r) + \bar{u}(r) + \underline{u}(1) + \bar{u}(1)) f(r) dr \\ &\quad + \frac{1}{2} \int_0^1 (\underline{v}(r) + \bar{v}(r) + \underline{v}(1) + \bar{v}(1)) f(r) dr \\ &= \text{Mag}(u) + \text{Mag}(v) \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} \text{Momag}(u+v) &= \frac{1}{2} \int_0^1 \left((\underline{u+v})(r) - (\overline{u+v})(r) + (\underline{u+v})(1) - (\overline{u+v})(1) \right) dr \\ &= \frac{1}{2} \int_0^1 (\underline{u}(r) - \bar{u}(r) + \underline{u}(1) - \bar{u}(1)) dr \\ &\quad + \frac{1}{2} \int_0^1 (\underline{v}(r) - \bar{v}(r) + \underline{v}(1) - \bar{v}(1)) dr \\ &= \text{Momag}(u) + \text{Momag}(v) \end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned} R(u+v, \gamma) &= \text{Mag}(u+v) + \gamma \text{Momag}(u+v) \\ &= \text{Mag}(u) + \text{Mag}(v) + \gamma (\text{Momag}(u) + \text{Momag}(v)) \\ &= R(u, \gamma) + R(v, \gamma). \end{aligned} \tag{8}$$

□

Reasonable axioms that Wang and Kerre [21] had proposed for fuzzy numbers ranking are studied in the following theorem.

Let R be an ordering method, S the set of fuzzy numbers for which the method R can be applied, and \mathcal{A} a finite subset of S . The statement "two elements u and v in \mathcal{A} satisfy that u has a higher ranking than v when R is applied to the fuzzy numbers in \mathcal{A} " will be written as " $u \succ v$ by R on \mathcal{A} ". " $u \sim v$ by R on \mathcal{A} ", and " $u \preceq v$ by R on \mathcal{A} " are similarly interpreted. The following proposition shows the reasonable properties (see [21]) of the ordering approach, R .

Theorem 3.2. *Let E be the set of fuzzy numbers for which the proposed method can be applied, \mathcal{A} and \mathcal{A}' be two arbitrary finite subsets of E . Then the following statements hold.*

- A_1 . For $u \in \mathcal{A}$, $u \preceq u$.
- A_2 . For $(u, v) \in \mathcal{A}^2$, $u \preceq v$ and $v \preceq u$, we should have $u \sim v$.
- A_3 . For $(u, v, w) \in \mathcal{A}^3$, $u \preceq v$ and $v \preceq w$, we should have $u \preceq w$.
- A_4 . For $(u, v) \in \mathcal{A}^2$, $\inf \text{supp}(u) > \sup \text{supp}(v)$, we should have $v \preceq u$.
- A'_4 . For $(u, v) \in \mathcal{A}^2$, $\inf \text{supp}(v) > \sup \text{supp}(u)$, we should have $u \prec v$.
- A_5 . Let $(u, v) \in (\mathcal{A} \cap \mathcal{A}')^2$. We obtain the ranking order $u \preceq v$ by R on \mathcal{A}' if and only if $u \preceq v$ by R on \mathcal{A} .
- A_6 . Let $u, v, u+w$ and $v+w$ be elements of E . If $u \preceq v$, then $u+w \preceq v+w$.
- A'_6 . Let $u, v, u+w$ and $v+w$ be elements of E . If $u \prec v$, then $u+w \prec v+w$ when $w \neq 0$.
- A_7 . Let u, v, uw and vw be elements of E and $w \geq 0$. If $u \preceq v$, then $uw \preceq vw$.

Proof. $A_1, A_2, A_3, A_5,$ and A_7 are clear. We show A_4, A'_4 and A_6 .

A_4 : Let $\inf \text{supp}(u) > \sup \text{supp}(v)$, then

$$\bar{v}(r) < \underline{u}(r), \quad r \in [0, 1]$$

therefore

$$\text{Mag}(v) < \text{Mag}(u)$$

so

$$\gamma = 0$$

then

$$v \preceq u$$

Similarly A'_4 is hold.

A_6 : Let $u \preceq v$ then

$$R(u, \gamma) \leq R(v, \gamma)$$

by adding $R(w, \gamma)$ and using Proposition (3.1) we have

$$u + w \preceq v + w$$

Similarly A'_6 is hold. □

Corollary 3.3. (1) If $u \preceq v$ and $\text{Mag}(u) \neq \text{Mag}(v)$ i.e. ($\gamma = 0$), then $-v \preceq -u$.

(2) If $u \preceq v$ and $\text{Mag}(u) = \text{Mag}(v)$ and $z_0 > 0$, i.e. ($\gamma = 1$), then $-v \preceq -u$.

(3) If $u \preceq v$ and $\text{Mag}(u) = \text{Mag}(v)$ and $z_0 = 0$, i.e. ($\gamma = 1$), then $-u \preceq -v$.

(4) If $u \preceq v$ and $\text{Mag}(u) = \text{Mag}(v)$ and $z_0 < 0$, i.e. ($\gamma = -1$), then $-v \preceq -u$.

Proof. Let

$$u = (\underline{u}(r), \bar{u}(r)), \quad [u]_1 = [\underline{u}(1), \bar{u}(1)],$$

then

$$-u = (-\bar{u}(r), -\underline{u}(r)), \quad [-u]_1 = [-\bar{u}(1), -\underline{u}(1)]$$

therefore

$$\text{Mag}(-u) = \frac{1}{2} \int_0^1 (-\bar{u}(r) - \underline{u}(r) - \bar{u}(1) - \underline{u}(1))f(r)dr = -\text{Mag}(u) \quad (9)$$

$$\text{Momag}(-u) = \frac{1}{2} \int_0^1 (-\bar{u}(r) + \underline{u}(r) - \bar{u}(1) + \underline{u}(1))dr = \text{Momag}(u) \quad (10)$$

Case 1: Since $u \preceq v$ then $\text{Mag}(u) < \text{Mag}(v)$. It is clear that $-\text{Mag}(u) > -\text{Mag}(v)$ so by using equation (9),

$$\text{Mag}(-u) > \text{Mag}(-v)$$

therefore $-v \preceq -u$.

Case 2: Since $u \preceq v$, $\text{Mag}(u) = \text{Mag}(v)$ and $z_0 > 0$ then $\text{Momag}(v) \geq \text{Momag}(u)$. By using equations (9) and (10),

$$\text{Mag}(-u) = \text{Mag}(-v)$$

and

$$\text{Momag}(-v) \geq \text{Momag}(-u)$$

since $z_0 > 0$ then $-z_0 < 0$ therefor ($\gamma = -1$) and hence

$$\text{Mag}(-v) - \text{Momag}(-v) \leq \text{Mag}(-u) - \text{Momag}(-u)$$

thus $-v \preceq -u$.

Case 3: Since $u \preceq v$, $Mag(u) = Mag(v)$ and $z_0 = 0$ then $Momag(v) \geq Momag(u)$. By using equations (9) and (10),

$$Mag(-u) = Mag(-v)$$

and

$$Momag(-v) \geq Momag(-u)$$

since $z_0 = 0$ then $-z_0 = 0$ therefor ($\gamma = 1$) and hence

$$Mag(-v) + Momag(-v) \geq Mag(-u) + Momag(-u)$$

thus $-u \preceq -v$.

Case 4: It is similar to Case(2). □

Corollary 3.4. For any arbitrary trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, we have

$$Mag(u) = \frac{(\beta - \sigma) + 6(x_0 + y_0)}{12}, \quad Momag(u) = \frac{-(\beta + \sigma) + 4(x_0 - y_0)}{4}$$

and also for any arbitrary triangular fuzzy number $u = (x_0, \sigma, \beta)$, we have

$$Mag(u) = x_0 + \frac{(\beta - \sigma)}{12}, \quad Momag(u) = -\frac{(\beta + \sigma)}{4}$$

Proof. For any arbitrary trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ and triangular fuzzy number $u = (x_0, \sigma, \beta)$, it is easy to see that

$$\underline{u}(r) = x_0 + (r - 1)\sigma, \quad \bar{u}(r) = y_0 + (1 - r)\beta$$

and

$$\underline{u}(r) = x_0 + (r - 1)\sigma, \quad \bar{u}(r) = x_0 + (1 - r)\beta$$

so, the proof is clear. □

Remark 3.5. For two arbitrary crisp numbers a and b , $Mag(a) \neq Mag(b)$.

4. Numerical Examples

In this section, we apply many examples to illustrate the proposed approach to rank trapezoidal fuzzy numbers.

Example 4.1. [6], Consider two fuzzy numbers, $A = (3, 2, 2)$ and $B = (3, 1, 1)$, (see Figure 1). By using our proposed method

$$R(A, \gamma = 1) = 2$$

and

$$R(B, \gamma = 1) = 2.5.$$

Thus, the ranking order of fuzzy numbers is $B \succ A$.

As fuzzy numbers A and B have the same and symmetric spread, most of the existing approaches fail to rank them appropriately. For instance, according to the obtained results by the proposed method [1], $Mag(A) = Mag(B) = 3$, i.e. $A \sim B$. Also by using the approaches in [6, 23, 24], the ranking order is the same, i.e.,

$A \sim B$. In [2], different ranking orders are obtained when different index values (p) are taken. When $p = 1$ and $p = 2$, the ranking order of fuzzy numbers is $A \sim B$ and $A \succ B$, respectively and also by the approach proposed in [10], the ranking order is $A \succ B$. Nevertheless, inconsistent results are produced when the distance index and the CV index of Cheng's approach [7] are respectively used. Moreover, the ranking order obtained by Wang's approach [23] is $A \succ B$. Additionally, by the approaches provided in [18, 20], different ranking orders are obtained when different indices of optimism are taken. However, decision makers prefer the result $A \succ B$ intuitively.

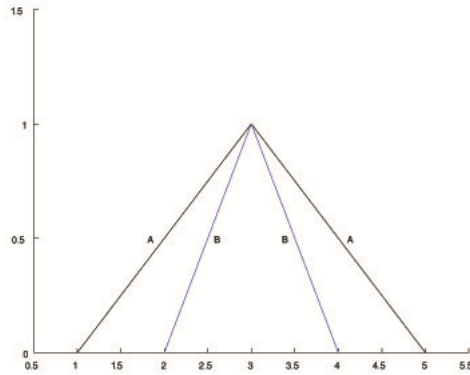


FIGURE 1. Fuzzy Numbers in Example 4.1

Example 4.2. [10] Consider two fuzzy numbers, $A = (-1, 1, 5, 5)$ and $B = (0, 2, 2)$, (see Figure 2). Using proposed method,

$$R(A, \gamma = 1) = -4.5, \quad R(B, \gamma = 1) = -1$$

therefore $B \succ A$. Obviously, the obtained result by the proposed method in [1], $A \sim B$, is unreasonable.

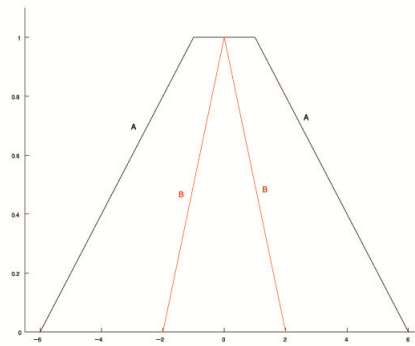


FIGURE 2. Fuzzy Numbers in Example 4.2

Example 4.3. Consider two fuzzy numbers, $A = (2, 4, 1, 1)$ and $B = (3, 2, 2)$, (see Figure 3). Using proposed method,

$$R(A, \gamma = 1) = 0.5, \quad R(B, \gamma = 1) = 2.$$

Therefore $B \succ A$. Obviously, the obtained result by the proposed methods in [1] and [10], $A \sim B$, is unreasonable.

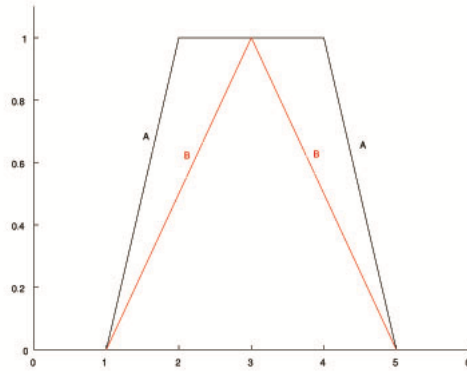


FIGURE 3. Fuzzy Numbers in Example 4.3

Example 4.4. Consider two fuzzy numbers, $A = (0, 2, 2)$ and $B = (-1, 1, 3, 3)$, (see Figure 4). Using proposed method,

$$R(A, \gamma = 1) = -1, \quad R(B, \gamma = 1) = -3.$$

therefore $A \succ B$. By using the proposed method in [1], we obtain $A \sim B$ that is an unreasonable result. Also, by using the introduced method in [10], $A \succ B$. From Corollary (3.3), we have:

$$R(-A, \gamma = 1) = -1,$$

$$R(-B, \gamma = 1) = -3.$$

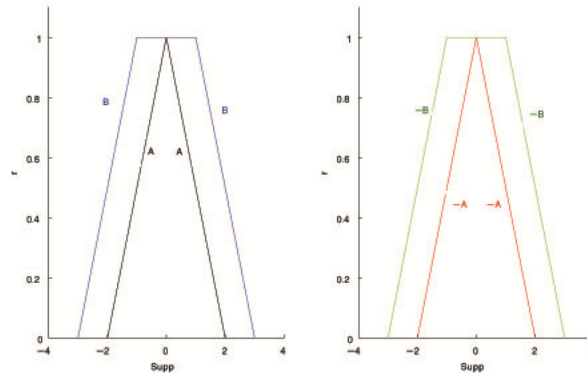


FIGURE 4. Fuzzy Numbers in Example 4.4

It is clear that

$$-A \succ -B.$$

Example 4.5. [17], Consider two fuzzy numbers A and B with the membership functions:

$$\mu_A(x) = \begin{cases} x - 1 & \text{when } x \in [1, 2], \\ \frac{5-x}{3} & \text{when } x \in [2, 5], \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_B(x) = \begin{cases} \sqrt{1 - (x - 2)^2} & \text{when } x \in [1, 2], \\ \sqrt{1 - \frac{1}{4}(x - 2)^2} & \text{when } x \in [2, 4], \\ 0 & \text{otherwise.} \end{cases}$$

(see Figure 5)

By employing Liou and Wang's ranking method [17], different rankings are produced for the same problem when applying different indices of optimism (α). For an optimistic decision maker, with ($\alpha = 1$): $A \prec B$; for a moderator decision maker, with ($\alpha = 0.5$): $A \succ B$; and for pessimistic decision maker, with ($\alpha = 0$): $A \succ B$. By applying the Sign Distance method [2] with $p = 1$, $d_p(A, A_0) = 5$, $d_p(B, A_0) = 4.78$, and with $p = 2$, $d_p(A, A_0) = 3.9157$, $d_p(B, A_0) = 3.8045$, the ranking order $A \succ B$ is obtained. By using Chu and Tsao's ranking method [6], there is $S(A) = 1.2445$ and $S(B) = 1.1821$, therefore, $A \succ B$. By using our proposed method, there is

$$R(A, \gamma = 0) = 7, \quad R(B, \gamma = 0) = 6.5.$$

Thus, the ranking order is $A \succ B$, too. Also, the result of the Distance Minimization method was similar to that of method. Obviously, this method can also rank fuzzy numbers other than triangular and trapezoidal ones. Comparably to Liou and Wang's method, and along with Chu and Tsao's method, our method produces a simpler ranking result.

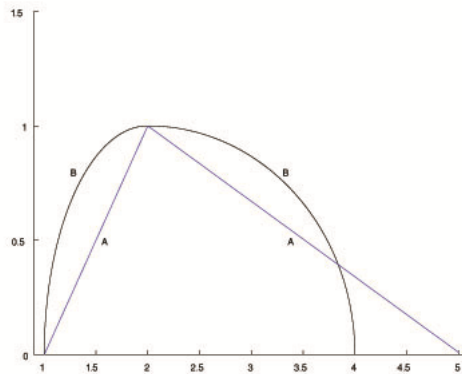


FIGURE 5. Fuzzy Numbers in Example 4.5

Example 4.6. Consider four fuzzy numbers A, B, C and D with parametric forms

$$A = \left(6 - 4(1 - r)^{\frac{1}{3}}, 8 + 2(1 - r)^3\right),$$

$$B = (1.5 + 3r, 10.5 - 3r),$$

$$C = \left(6 - 4(1 - r^2)^{\frac{1}{2}}, 6 + 4(1 - r^2)^{\frac{1}{2}}\right),$$

$$D = (1 + 5r, 11 - 5r),$$

(see Figure 6).

Using proposed method,

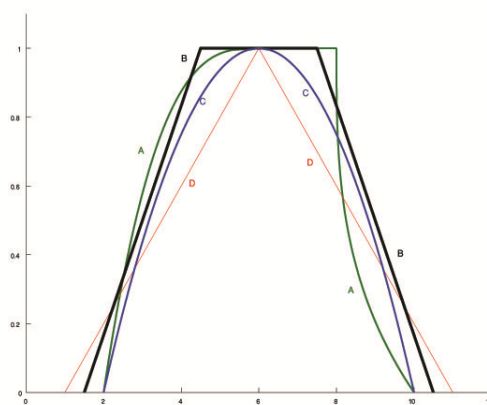


FIGURE 6. Fuzzy Numbers in Example 4.6.

$$R(A, \gamma = 1) = 2.8167,$$

$$R(B, \gamma = 1) = 1.5,$$

$$R(C, \gamma = 1) = 2.8584,$$

$$R(D, \gamma = 1) = 3.5.$$

It is clear that

$$D \succ C \succ A \succ B.$$

By using the proposed method in [10], the ranking order is:

$$A \succ D \succ B \succ C.$$

Example 4.7. Consider three fuzzy numbers, $A = (1, 0, 14)$, $B = (4, 6, 6)$ and $C = (2, 6, 2, 2)$, (see Figure 7). Using the proposed method,

$$R(A, \gamma) = 2.1667 + 5\gamma,$$

$$R(B, \gamma) = 4 + 5.5\gamma,$$

$$R(C, \gamma) = 4 + 3.5\gamma.$$

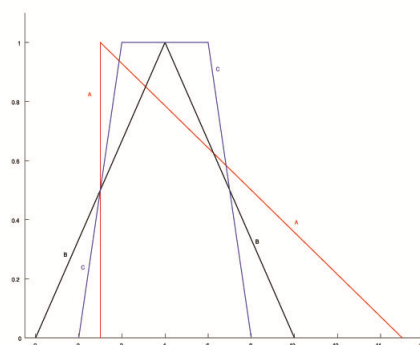


FIGURE 7. Fuzzy Numbers in Example 4.7

It is clear that

$$B \succ C \succ A.$$

From Corollary (3.3), we have:

$$R(-A, \gamma = -1) = -7.1667,$$

$$R(-B, \gamma = -1) = -9.5,$$

$$R(-C, \gamma = -1) = -7.5.$$

It is clear that

$$-A \succ -C \succ -B,$$

(see Figure 8).

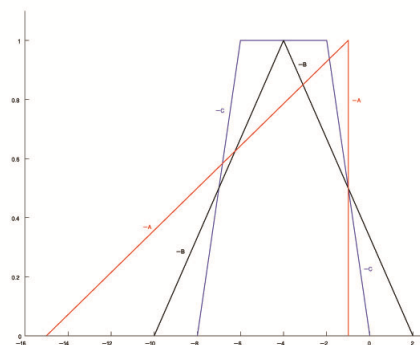


FIGURE 8. Fuzzy Numbers in example 4.7

By employing proposed method in [10], the ranking order is:

$$B \succ C \succ A \quad , \quad -A \succ -B \succ -C.$$

Also, by applying proposed method in [1], we have:

$$B \sim C \succ A \quad , \quad -B \sim -C \prec -A.$$

Example 4.8. Consider two fuzzy numbers, $A = (-4, -2, 1, 1)$ and $B = (-3, 2, 2)$, (see Figure 9). Using proposed method,

$$R(A, \gamma = -1) = -0.5, \quad R(B, \gamma = -1) = -2,$$

therefore $A \succ B$. Obviously, the obtained results based on the proposed method in [1] and [10], $A \sim B$, is unreasonable.

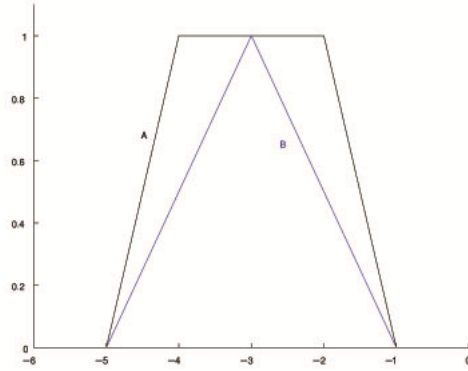


FIGURE 9. Fuzzy Numbers in Example 4.8

Example 4.9. Consider the following sets, see Yao and Wu [24].

Set 1: $A = (0.5, 0.1, 0.5)$, $B = (0.7, 0.3, 0.3)$, $C = (0.9, 0.5, 0.1)$, (see Figure 10).

Set 2: $A = (0.4, 0.7, 0.1, 0.2)$, $B = (0.7, 0.4, 0.2)$, $C = (0.7, 0.2, 0.2)$, (see Figure 11).

Set 3: $A = (0.5, 0.2, 0.2)$, $B = (0.5, 0.8, 0.2, 0.1)$, $C = (0.5, 0.2, 0.4)$, (see Figure 12).

Set 4: $A = (0.4, 0.7, 0.4, 0.1)$, $B = (0.5, 0.3, 0.4)$, $C = (0.6, 0.5, 0.2)$, (see Figure 13).

To compare the proposed approach with other methods, the reader is referred to Table 1.

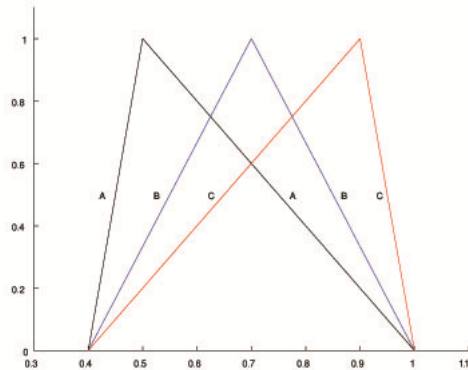


FIGURE 10. Fuzzy Numbers in Example 4.9

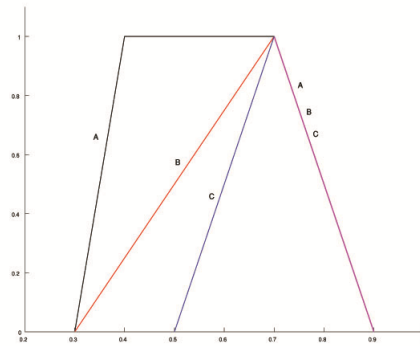


FIGURE 11. Fuzzy Numbers in Example 4.9

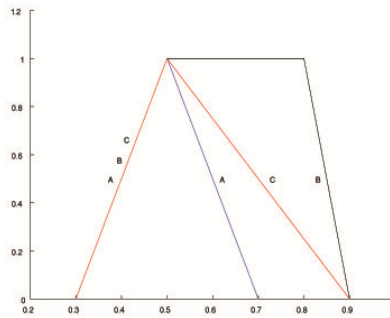


FIGURE 12. Fuzzy Numbers in Example 4.9

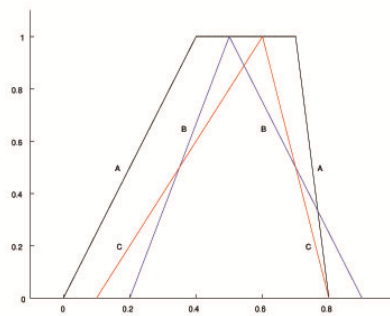


FIGURE 13. Fuzzy Numbers in Example 4.9

Note that, in Table 1 and in Set 4, for Sign Distance ($p=1$) [2], Distance Minimization [3], Chu-Tsao [6] and Yao-Wu [24] methods, the ranking order for fuzzy numbers B and C is $B \sim C$, which seems unreasonable regarding the figures.

Authors	Fuzzy number	Set1	Set2	Set3	Set4
Proposed method, Ezzati method and Abbasbandy method	A	0.5334	0.5584	0.5000	0.5250
	B	0.7000	0.6834	0.6416	0.5084
	C	0.8666	0.7000	0.5166	0.5750
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$B \prec A \prec C$
Sing Distance method with $p=1$	A	1.2000	1.1500	1.0000	0.0950
	B	1.4000	1.3000	1.2500	1.0500
	C	1.6000	1.4000	1.1000	1.0500
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
Sing Distance method with $p=2$	A	0.8869	0.8756	0.7257	0.7853
	B	1.0194	0.9522	0.9416	0.7958
	C	1.1605	1.0033	0.8165	0.8386
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Distance Minimization	A	0.6000	0.5750	0.5000	0.4750
	B	0.7000	0.6500	0.6250	0.5250
	C	0.9000	0.7000	0.5500	0.5250
	Result	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
Choobineh and Li	A	0.3333	0.5480	0.3330	0.5000
	B	0.5000	0.5830	0.4164	0.5833
	C	0.6670	0.6670	0.5417	0.6111
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Yager(1981)	A	0.6000	0.5750	0.5000	0.4500
	B	0.7000	0.6500	0.5500	0.5250
	C	0.8000	0.7000	0.6250	0.5500
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Chen(1985)	A	0.3375	0.4315	0.3750	0.5200
	B	0.5000	0.5625	0.4250	0.5700
	C	0.6670	0.6250	0.5500	0.6250
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Baldwin and Guild(1979)	A	0.3000	0.2700	0.2700	0.4000
	B	0.3300	0.2700	0.3700	0.4200
	C	0.4400	0.3700	0.4500	0.4200
	Results	$A \prec B \prec C$	$A \sim B \prec C$	$A \prec B \prec C$	$A \prec B \sim C$
Chu and Tsao	A	0.2990	0.2847	0.2500	0.2440
	B	0.3500	0.3247	0.3152	0.2624
	C	0.3993	0.3500	0.2747	0.2619
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Yao and Wu	A	0.6000	0.5750	0.5000	0.4750
	B	0.7000	0.6500	0.6250	0.5250
	C	0.8000	0.7000	0.5500	0.5250
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
Cheng distance	A	0.7900	0.7577	0.7071	0.7106
	B	0.8602	0.8149	0.8037	0.7256
	C	0.9268	0.8602	0.7458	0.7241
	Results	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Cheng CV uniform distribution	A	0.0272	0.0328	0.0133	0.0693
	B	0.0214	0.0246	0.0304	0.0385
	C	0.0225	0.0095	0.0275	0.0433
	Results	$A \prec C \prec B$	$A \prec B \prec C$	$B \prec C \prec A$	$A \prec C \prec B$

TABLE 1. Comparative Results for Example 4.9

5. Conclusions

In this paper, we presented a new approach for the ranking of all trapezoidal fuzzy numbers and consequently it was overcome the drawbacks of some related methods. All properties were studied in details and finally by using some numerical examples, we presented the advantage of the proposed method.

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R. EZZATI*, DEPARTMENT OF MATHEMATICS, KARAJ BRANCH, ISLAMIC AZAD UNIVERSITY,
31485 – 413, KARAJ, IRAN
E-mail address: ezati@kiau.ac.ir

S. KHEZERLOO, DEPARTMENT OF MATHEMATICS, ISLAMIC AZAD UNIVERSITY - SOUTH TEHRAN
BRANCH, TEHRAN, IRAN
E-mail address: s.khezerloo@azad.ac.ir

S. ZIARI, DEPARTMENT OF MATHEMATICS, FIROOZKOOH BRANCH, ISLAMIC AZAD UNIVERSITY,
FIROOZKOOH, IRAN

E-mail address: `szitari@iaufb.ac.ir`

*CORRESPONDING AUTHOR