NUMERICAL SOLUTIONS OF NONLINEAR FUZZY FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND KIND

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Abstract. In this paper, we use parametric form of fuzzy number, then an iterative approach for obtaining approximate solution for a class of nonlinear fuzzy Fredholm integro-differential equation of the second kind is proposed. This paper presents a method based on Newton-Cotes methods with positive coefficient. Then we obtain approximate solution of the nonlinear fuzzy integro-differential equations by an iterative approach.

1. Introduction

The solutions of integral equations have a major role in the field of science and engineering. A physical even can be modelled by the differential equation, an integral equation. Since few of these equations cannot be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [13, 29]. There are several numerical methods for solving linear Volterra integral equation [18, 37] and system of nonlinear Volterra integral equations [15]. Kauthen in [26] used a collocation method to solve the Volterra- Fredholm integral equation numerically. Borzabadi and Fard in [16] obtained a numerical solution of nonlinear Fredholm integral equations of the second kind.

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [40], Dubois and Prade [19]. We refer the reader to [24] for more information on fuzzy numbers and fuzzy arithmetic. The numerical solution of a fuzzy nonlinear equation by Newton's method and steepest descent method were considered [3, 5]. The topics of fuzzy integral equations (FIEs) and fuzzy differential equations (FDEs) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years [1, 2, 7, 8, 6, 9, 10, 33, 34]. They used the concept of H-differentiability which was introduced by Puri and Ralescu [35]. The concept of fuzzy random variable was proposed by Kwakernaak [28]. Then, the authors of [30, 31, 32] considered the random fuzzy differential equations where the two kinds of uncertainties (randomness and fuzziness) were incorporated. The fuzzy mapping function was introduced by Chang and Zadeh [17]. Later, Dubois and Prade [20] presented an elementary fuzzy calculus based on the extension principle also the concept of integration of fuzzy functions was first

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introduced by Dubois and Prade [20]. Babolian et al. and Abbasbandy et al. in [4, 12] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind. In this paper, we generalize the nonlinear fuzzy integral equations to the nonlinear fuzzy integro-differential equations

$$X'(s) = y(s) + \int_a^b k(s,t,X(t))dt.$$

In this paper, we present a novel and very simple numerical method based upon iterative methods for solving nonlinear fuzzy Fredholm integro-differential equations of the second kind.

2. **Preliminaries**

In this section the basic notations used in fuzzy operations are introduced. We start by defining the fuzzy number.

Definition 2.1. A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \longrightarrow I = [0,1]$ such that [27]:

i. u is upper semi-continuous;

ii. $u(x) = 0$ outside some interval $[a, d]$;

iii. There are real numbers b and c, $a \le b \le c \le d$, for which

1. $u(x)$ is monotonically increasing on $[a, b]$,
2. $u(x)$ is monotonically decreasing on $[c, d]$,
3. $u(x) = 1, b \le x \le c$.

The set of all the fuzzy numbers (as given in definition 1) is denoted by E^1.

An alternative definition which yields the same E^1 is given by Kaleva [25].

Definition 2.2. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

i. $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0, 1]$ and right continuous at 0;

ii. $\overline{u}(r)$ is a bounded monotonically decreasing, left continuous function on $(0, 1]$ and right continuous at 0;

iii. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

A crisp number r is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \le \alpha \le 1$. The set of all the fuzzy numbers is denoted by E^1. This fuzzy number space as shown in [39], can be embedded into the Banach space $B = \overline{C}[0,1] \times \overline{C}[0,1]$.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$ we define addition and multiplication by k as

$$\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r)),$$
$$\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r)),$$
$$\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\overline{u}(r),\ if\ k \ge 0,$$
$$\underline{ku}(r) = k\overline{u}(r), \overline{ku}(r) = k\underline{u}(r),\ if\ k < 0.$$

Definition 2.3. For arbitrary fuzzy numbers u, v, we use the distance [22]:

$$D(u,v) = sup_{0 \le r \le 1} max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}$$

and it is shown that (E^1, D) is a complete metric space [35].

Definition 2.4. Let $f : [a, b] \longrightarrow E^1$, for each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i], 1 \leq i \leq n$ suppose

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}), \Delta := max\{|t_i - t_{i-1}|, i = 1, 2, \ldots, n\}.$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)dt = lim_{\Delta \longrightarrow 0} R_p$$

provided that this limit exists in the metric D [21, 22].

If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [22] and also,

$$\underline{\int_a^b f(t; r)dt)} = \int_a^b \underline{f}(t; r)dt, \quad \overline{(\int_a^b f(t; r)dt)} = \int_a^b \overline{f}(t; r)dt.$$

Definition 2.5. Let $u, v \in E^1$. If there exists $w \in E^1$ such that $u = v + w$ then w is called the H-difference of u, v and it is denoted by $u - v$.

Definition 2.6. A function $f : (a, b) \longrightarrow E^1$ is called H-differentiable at $\hat{x} \in (a, b)$ if, for $h > 0$ sufficiently small, there exist the H-differences $f(\hat{x} + h) - f(\hat{t}), f(\hat{x}) - f(\hat{x} - h)$, and an element $f'(\hat{x}) \in E^1$ such that:

$$lim_{h \longrightarrow 0^+} D(\frac{f(\hat{x} + h) - f(\hat{x})}{h}, f'(\hat{x})) = lim_{h \longrightarrow 0^+} D(\frac{f(\hat{x}) - f(\hat{x} - h)}{h}, f'(\hat{x})) = 0.$$

Then $f'(\hat{x})$ is called the fuzzy derivative of f at $\hat{x}$.

3. **Fuzzy Integro-differential Equation**

The nonlinear Fredholm integro-differential equation of the second kind [23] is

$$X'(s) = y(s) + \int_a^b k(s, t, X(t))dt, \quad X(s_0) = X_0, \tag{1}$$

where k is an arbitrary given kernel function and $y(s)$ is a given function of $s \in [a, b]$. If X is a fuzzy function, $y(s)$ is a given fuzzy function of $s \in [a, b]$ and X' is the fuzzy derivative of X [36], this equation may only possess fuzzy solution. Sufficient condition for the existence equation of the second kind, is given in [14].

For solving equaton (1) we may replace equation (1) by the equivalent system

$$\underline{X}'(s) = \underline{y}(s) + \int_a^b \underline{k}(s, t, X(t))dt = \underline{y}(s) + \int_a^b F(s, t, \underline{X}, \overline{X}))dt, \quad \underline{X}(s_0) = \underline{X_0},$$
$$\overline{X}'(s) = \overline{y}(s) + \int_a^b \overline{k}(s, t, X(t))dt = \overline{y}(s) + \int_a^b G(s, t, \underline{X}, \overline{X}))dt, \quad \overline{X}(s_0) = \overline{X_0} \tag{2}$$

which possesses a unique solution $(\underline{X}, \overline{X}) \in B$ which is a fuzzy function, i.e. for each s, the pair $(\underline{X}(s; r), \overline{X}(s; r))$ is a fuzzy number.

The parametric form of equation (2) is given by

$$\underline{X}'(s; r) = \underline{y}(s; r) + \int_a^b F(s, t, \underline{X}(t; r), \overline{X}(t; r)))dt, \quad \underline{X}(s_0; r) = \underline{X_0}(r),$$

$$\overline{X}'(s;r) = \overline{y}(s;r) + \int_a^b G(s,t,\underline{X}(t;r),\overline{X}(t;r)))dt, \quad \overline{X}(s_0;r) = \overline{X_0}(r) \tag{3}$$

for $r \in [0,1]$. In most cases, however, analytical solution to equation (3) may not be found and a numerical approach must be considered.

4. The Numerical Approach

We replace the interval $[a,b]$ by a set of discrete equally spaced grid points

$$a = s_0 < s_1 < \ldots < s_N = b$$

at which the exact solution $(\underline{X}(s;r), \overline{X}(s;r))$ is approximated by some $(\underline{x}(s;r), \overline{x}(s;r))$. The exact and approximate solutions at s_i, $0 \le i \le N$ are denoted by $X_i(r) = (\underline{X}_i(r), \overline{X}_i(r))$ and $x_i(r) = (\underline{x}_i(r), \overline{x}_i(r))$, respectively. The grid points at which the solution is calculated are

$$s_i = s_0 + ih, \quad h = (b-a)/N; \quad 1 \le i \le N.$$

The first-order approximation of $\underline{X}'(s;r)$ and $\overline{X}'(s;r)$ is given by

$$Z'(s;r) \approx \frac{Z(s+h;r) - Z(s;r)}{h} \tag{4}$$

where $Z(s;r)$ is $\underline{X}(s;r)$ and $\overline{X}(s;r)$ alternatively. By virtue of equation (4) we obtain

$$\underline{X}_{i+1}(r) = \underline{X}_i(r) + h[\underline{y}_i(r) + \int_a^b F(s_i,t,\underline{X}(t;r),\overline{X}(t;r))dt] + \frac{h^2}{2}\underline{X}''(\underline{\zeta}_i),$$
$$\underline{X}(s_0;r) = \underline{X_0}(r),$$
$$\overline{X}_{i+1}(r) = \overline{X}_i(r) + h[\overline{y}_i(r) + \int_a^b G(s_i,t,\underline{X}(t;r),\overline{X}(t;r))dt] + \frac{h^2}{2}\overline{X}''(\overline{\zeta}_i),$$
$$\overline{X}(s_0;r) = \overline{X_0}(r), \quad i = 0,1,\ldots,N, \tag{5}$$

where $s_i < \underline{\zeta}_i, \overline{\zeta}_i < s_{i+1}$.

The Newton-Cotes method [11] is given by

$$\int_a^b Z(t)dt = \sum_{j=0}^N w_j Z(t_j) + O(h^\nu) \tag{6}$$

where Z is F and G alternatively and ν depends upon the used method of Newton-Cotes with positive coefficient for estimating of the integral in equation (6). By virtue of equation (6) we obtain

$$\underline{X}_{i+1}(r) = \underline{X}_i(r) + h[\underline{y}_i(r) + \sum_{j=0}^N w_j F(s_i,t_j,\underline{X}_j(r),\overline{X}_j(r)))]$$
$$+\frac{h^2}{2}\underline{X}''(\underline{\zeta}_i) + O(h^{\nu+1}), \quad \underline{X}(s_0;r) = \underline{X_0}(r),$$
$$\overline{X}_{i+1}(r) = \overline{X}_i(r) + h[\overline{y}_i(r) + \sum_{j=0}^N w_j G(s_i,t_j,\underline{X}_j(r),\overline{X}_j(r)))]$$
$$+\frac{h^2}{2}\overline{X}''(\overline{\zeta}_i) + O(h^{\nu+1}), \quad \overline{X}(s_0;r) = \overline{X_0}(r), \quad i = 0,1,\ldots,N. \tag{7}$$

Following equation (7) we define

$$\underline{x}_{i+1}(r) = \underline{x}_i(r) + h[\underline{y}_i(r) + \sum_{j=0}^{N} w_j F(s_i, t_j, \underline{x}_j(r), \overline{x}_j(r)))],$$

$$\underline{x}(s_0; r) = \underline{x_0}(r),$$

$$\overline{x}_{i+1}(r) = \overline{x}_i(r) + h[\overline{y}_i(r) + \sum_{j=0}^{N} w_j G(s_i, t_j, \underline{x}_j(r), \overline{x}_j(r)))],$$

$$\overline{x}(s_0; r) = \overline{x_0}(r), \ i = 0, 1, \ldots, N. \tag{8}$$

The polygon curves

$$\underline{x}(s; h; r) \triangleq \{[s_0, \underline{x}_0(r)], [s_1, \underline{x}_1(r)], \}, \ldots, [s_N, \underline{x}_N(r)]\},$$

$$\overline{x}(s; h; r) \triangleq \{[s_0, \overline{x}_0(r)], [s_1, \overline{x}_1(r)], \}, \ldots, [s_N, \overline{x}_N(r)]\} \tag{9}$$

are the approximates to $\underline{X}(s; r)$ and $\overline{X}(s; r)$, respectively, over the interval $s_0 \leq s \leq s_N$.

Let $F(s, t, u, v)$ and $G(s, t, u, v)$ be the functions F and G of equation (2) where u and v are constants and $u \leq v$. In other words $F(s, t, u, v)$ and $G(s, t, u, v)$ are obtained by substituting $X = (u, v)$ in equation (2). The domain where F and G are defined is therefore

$$B = \{(s, t, u, v) | a \leq s, t \leq b, -\infty < v < +\infty, -\infty < u \leq v\}.$$

Theorem 4.1. *Let $F(s, t, u, v)$ and $G(s, t, u, v)$ belong to $C^1(B)$, let the partial derivatives of F, G be bounded over B and $D(X_p, x_p) = max_{0 \leq i \leq N}\{D(X_i, x_i)\}$. Then, for arbitrary fixed $r : 0 \leq r \leq 1$,*

$$lim_{h \longrightarrow 0} \ \underline{x}_p(r) = \underline{X}_p(r), \quad lim_{h \longrightarrow 0} \ \overline{x}_p(r) = \overline{X}_p(r).$$

Proof. Let

$$\underline{X}_p(r) = \underline{X}_{p-1}(r) + h[\underline{y}_{p-1}(r) + \sum_{j=0}^{N} w_j F(s_{p-1}, t_j, \underline{X}_j(r), \overline{X}_j(r))$$

$$+ \frac{h^2}{2} \underline{X}''(\underline{\zeta}_{p-1}) + O(h^{\nu+1}),$$

$$\overline{X}_p(r) = \overline{X}_{p-1}(r) + h[\overline{y}_{p-1}(r) + \sum_{j=0}^{N} w_j G(s_{p-1}, t_j, \underline{X}_j(r), \overline{X}_j(r))$$

$$+ \frac{h^2}{2} \overline{X}''(\overline{\zeta}_{p-1}) + O(h^{\nu+1}) \tag{10}$$

and we have:

$$\underline{x}_p(r) = \underline{x}_{p-1}(r) + h[\underline{y}_{p-1}(r) + \sum_{j=0}^{N} w_j F(s_{p-1}, t_j, \underline{x}_j(r), \overline{x}_j(r))],$$

$$\overline{x}_p(r) = \overline{x}_{p-1}(r) + h[\overline{y}_{p-1}(r) + \sum_{j=0}^{N} w_j G(s_{p-1}, t_j, \underline{x}_j(r), \overline{x}_j(r))]. \tag{11}$$

Consequently

$$\underline{X}_p(r)-\underline{x}_p(r)=\underline{X}_{p-1}(r)-\underline{x}_{p-1}(r)+h[\sum_{j=0}^{N}w_j(F(s_{p-1},t_j,\underline{X}_j(r),\overline{X}_j(r))$$

$$-F(s_{p-1},t_j,\underline{x}_j(r),\overline{x}_j(r)))]+\frac{h^2}{2}\underline{X}''(\underline{\zeta}_{p-1})+O(h^{\nu+1}),$$

$$\overline{X}_p(r)-\overline{x}_p(r)=\overline{X}_{p-1}(r)-\overline{x}_{p-1}(r)+h[\sum_{j=0}^{N}w_j(G(s_{p-1},t_j,\underline{X}_j(r),\overline{X}_j(r))$$

$$-G(s_{p-1},t_j,\underline{x}_j(r),\overline{x}_j(r)))]+\frac{h^2}{2}\overline{X}''(\overline{\zeta}_{p-1})+O(h^{\nu+1}).$$

Denote $W_p=\underline{X}_p(r)-\underline{x}_p(r)$, $V_p=\overline{X}_p(r)-\overline{x}_p(r)$. Then

$$|W_p|\leq|W_{p-1}|+2Lh(b-a)D(X_p,x_p)+\frac{h^2}{2}\underline{M}+O(h^{\nu+1}),$$

$$|V_p|\leq|V_{p-1}|+2Lh(b-a)D(X_p,x_p)+\frac{h^2}{2}\overline{M}+O(h^{\nu+1}),$$

$$\underline{M}=max_{s_0\leq s\leq s_N}\underline{X}''(s;r),\quad \overline{M}=max_{s_0\leq s\leq s_N}\overline{X}''(s;r)$$

and $L>0$ is a bound for the partial derivatives of F,G. Thus, we have

$$|W_p|\leq|W_0|+p2Lh(b-a)D(X_p,x_p)+p\frac{h^2}{2}\underline{M}+O(h^{\nu+1}),$$

$$|V_p|\leq|V_0|+p2Lh(b-a)D(X_p,x_p)+p\frac{h^2}{2}\overline{M}+O(h^{\nu+1}).$$

Since $W_0=V_0=0$ we obtain

$$|W_p|\leq p2Lh(b-a)D(X_p,x_p)+p\frac{h^2}{2}\underline{M}+O(h^{\nu+1}),$$

$$|V_p|\leq p2Lh(b-a)D(X_p,x_p)+p\frac{h^2}{2}\overline{M}+O(h^{\nu+1})$$

and if $h\longrightarrow 0$ we get $W_p\longrightarrow 0$, $V_p\longrightarrow 0$ which concludes the proof. □

So far, we came to the nonlinear equation system (8) with a special form that let us offer a numerical approach for obtaining the approximate solution.

Iterative methods are widely used for finding approximate solution of nonlinear equations systems [38]. The nonlinear equations system (8) also has a structure that permits to approximate its solution by an iterative method. For this purpose, we apply a successive substitution, similar to Jacobi method of solving linear equations systems and therefore define an iterative process leading to the sequence of vectors $\underline{x}^{(k)}$ and $\overline{x}^{(k)}$, where the components of the vectors satisfy the iteration formulas,

$$\underline{x}_{i+1}^{(k+1)}(r)=\underline{x}_i^{(k)}(r)+h[\underline{y}_i(r)+\sum_{j=0}^{N}w_jF(s_i,t_j,\underline{x}_j^{(k)}(r),\overline{x}_j^{(k)}(r))],$$

$$\underline{x}(s_0;r)=\ \underline{x_0}(r),$$

$$\overline{x}_{i+1}^{(k+1)}(r) = \overline{x}_i^{(k)}(r) + h[\overline{y}_i(r) + \sum_{j=0}^{N} w_j G(s_i, t_j, \underline{x}_j^{(k)}(r), \overline{x}_j^{(k)}(r))],$$
$$\overline{x}(s_0; r) = \overline{x_0}(r), \quad i = 0, 1, \ldots, N, \ k = 0, 1, \ldots, K. \tag{12}$$

However, we should first study the conditions that guarantee the convergence of the approximate solution.

Theorem 4.2. *Considering assumptions of Theorem 1 and*

$$D(x_p^{(k)}, x_p^*) = max_{0 \le i \le N}\{D(x_i^{(k)}, x_i^*),\}$$

the produced sequence $x^{(k)}$ from the iteration process (12) tends to the exact solution of (8), say x^*, for any arbitrary fuzzy initial vector $x^{(0)}$ with $x^{(k)}(s_0; r) = x_0(r)$ for all k.

Proof. By (8) and (12) we have,

$$|\underline{x}_p^{(k+1)}(r) - \underline{x}_p^*(r)| \le |\underline{x}_{p-1}^{(k)}(r) - \underline{x}_{p-1}^*(r)| +$$
$$h \sum_{j=0}^{n} w_j |F(s_{p-1}, t_j, \underline{x}_j^{(k)}(r), \overline{x}_j^{(k)}(r)) - F(s_{p-1}, t_j, \underline{x}_j^*(r), \overline{x}_j^*(r))|,$$
$$|\overline{x}_p^{(k+1)}(r) - \overline{x}_p^*(r)| \le |\overline{x}_{p-1}^{(k)}(r) - \overline{x}_{p-1}^*(r)| +$$
$$h \sum_{j=0}^{n} w_j |G(s_{p-1}, t_j, \underline{x}_j^{(k)}(r), \overline{x}_j^{(k)}(r)) - G(s_{p-1}, t_j, \underline{x}_j^*(r), \overline{x}_j^*(r))|$$

and according to the conditions of theorem 4.1,

$$|\underline{x}_p^{(k+1)}(r) - \underline{x}_p^*(r)| \le |\underline{x}_{p-1}^{(k)}(r) - \underline{x}_{p-1}^*(r)| + 2Lh(b-a)D(x_p^{(k)}, x_p^*),$$
$$|\overline{x}_p^{(k+1)}(r) - \overline{x}_p^*(r)| \le |\overline{x}_{p-1}^{(k)}(r) - \overline{x}_{p-1}^*(r)| + 2Lh(b-a)D(x_p^{(k)}, x_p^*).$$

Denote $W_p^{(k+1)} = \underline{x}_p^{(k+1)}(r) - \underline{x}_p^*(r)$, $V_p^{(k+1)} = \overline{x}_p^{(k+1)}(r) - \overline{x}_p(r)$. Then

$$|W_p^{(k+1)}| \le |W_{p-1}^{(k)}| + 2Lh(b-a)D(x_p^{(k)}, x_p^*),$$
$$|V_p^{(k+1)}| \le |V_{p-1}^{(k)}| + 2Lh(b-a)D(x_p^{(k)}, x_p^*).$$

Thus, we have

$$|W_p^{(k+1)}| \le |W_0^{(k)}| + 2Lph(b-a)D(x_p^{(k)}, x_p^*),$$
$$|V_p^{(k+1)}| \le |V_0^{(k)}| + 2pLh(b-a)D(x_p^{(k)}, x_p^*).$$

Since $W_0^{(k)} = V_0^{(k)} = 0$ for all k we obtain

$$|W_p^{(k+1)}| \le 2Lph(b-a)D(x_p^{(k)}, x_p^*),$$
$$|V_p^{(k+1)}| \le 2pLh(b-a)D(x_p^{(k)}, x_p^*).$$

and if $h \longrightarrow 0$ we get $W_p^{(k+1)} \longrightarrow 0$, $V_p^{(k+1)} \longrightarrow 0$ for all k which concludes the proof. □

5. **Numerical Examples**

To illustrate the technique proposed in this paper, consider the following examples. In this examples we take $max_{0\leq i\leq n}\{D(x_i^{(k+1)}, x_i^k)\} < 10^{-4}$.

Example 5.1. Consider the following nonlinear fuzzy Fredholm integro-differential equation

$$X'(s) = (r - \frac{r^2}{8}, \frac{12-4r-r^2}{8}) + \int_0^1 \frac{t}{2} X^2(t)dt,$$
$$X(0) = 0; \quad 0 \leq s, t \leq 1, \quad 0 \leq r \leq 1.$$

The parametric equations are

$$\underline{X}'(s;r) = (r - \frac{r^2}{8}) + \int_0^1 \frac{t}{2} min(p(t;r))dt,$$
$$\overline{X}'(s;r) = (\frac{12-4r-r^2}{8}) + \int_0^1 \frac{t}{2} max(p(t;r))dt,$$
$$\underline{X}(0;r) = 0, \ \overline{X}(0;r) = 0, \ 0 \leq s, t \leq 1, 0 \leq r \leq 1,$$

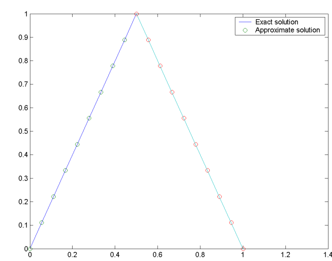

FIGURE 1. Compares the Exact Solution and Obtained Solution at $s = 0.5$

where $p(t;r) = \{\underline{X}(t;r)\underline{X}(t;r), \underline{X}(t;r)\overline{X}(t;r), \overline{X}(t;r)\overline{X}(t;r)\}$. The exact solution in this case is given by $\underline{X}(s;r) = rs, \quad \overline{X}(s;r) = (2-r)s$.

The exact and obtained solution of nonlinear fuzzy Fredholm integro-differential equation in this example at $s = 0.5$ is shown in Figure 1.

Example 5.2. Consider the following nonlinear fuzzy Fredholm integro-differential equation

$$X'(s) = y(s) + \int_0^1 -2ste^{X(t)}dt, \quad X(0) = 0, \quad 0 \leq s, t \leq \ 1,$$
$$\underline{y}(s;r) = 2s(0.25 + 0.5r) + \frac{s}{1.25 - 0.5r}(e^{1.25-0.5r} - 1),$$
$$\overline{y}(s;r) = 2s(1.25 - 0.5r) + \frac{s}{0.25 + 0.5r}(e^{0.25+0.5r} - 1), \quad 0 \leq r \leq 1.$$

The exact solution in this case is given by

$$\underline{X}(s;r) = (0.25 + 0.5r)s^2,$$

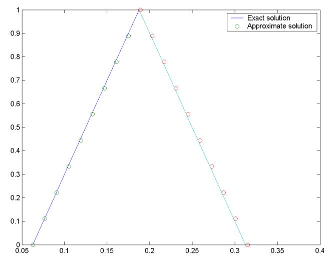

Figure 2. Compares the Exact Solution and Obtained Solution

$$\overline{X}(s;r) = (1.25 - 0.5r)s^2, \ 0 \leq r \leq 1.$$

The parametric equations are

$$\underline{X}'(s;r)(s) = \underline{y}(s;r) + \int_0^1 -2ste^{\overline{X}(t;r)}dt, \quad \underline{X}(0;r) = 0,$$
$$\overline{X}'(s;r)(s) = \overline{y}(s;r) + \int_0^1 -2ste^{\underline{X}(t;r)}dt, \quad \overline{X}(0;r) = 0, \ 0 \leq s,t \leq \ 1.$$

The exact and obtained solution of nonlinear fuzzy Fredholm integro-differential equation in this example at $s = 0.5$ is shown in Figure 2.

6. Conclusions

We propose a general numerical procedure for treating nonlinear fuzzy Fredholm integro-differential equations of the second kind. The original nonlinear fuzzy Fredholm integro-differential equation is replaced by two parametric nonlinear Fredholm integro-differential equations which are then solved numerically using classical algorithm. In this paper the standard Newton-Cotes method is designed for approximating integral. Also we can execute this method in a computer simply.

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