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A NEW APPROACH TO CARISTI'S FIXED POINT THEOREM ON NON-ARCHIMEDEAN FUZZY METRIC SPACES

S. SEDGHI, N. SHOBKOLAEI AND I. ALTUN

ABSTRACT. In the present paper, we give a new approach to Caristi's fixed point theorem on non-Archimedean fuzzy metric spaces. For this we define an ordinary metric d using the non-Archimedean fuzzy metric M on a nonempty set X and we establish some relationship between (X, d) and $(X, M, *)$. Hence, we prove our result by considering the original Caristi's fixed point theorem.

1. Introduction and preliminaries

After the definition of the concept of fuzzy metric space by some authors [4], [8], [9], the fixed point theory on these spaces has been developing (see, [1], [2], [3], [5], [6], [7]). Generally, this theory on fuzzy metric space are done for contractive or contractive type mappings (see [8], [10], [11], [12] and references therein). In this paper we introduce the concept of M-Caristi mapping on non-Archimedean fuzzy metric space and give a fixed point theorem.

For the sake of completeness, we briefly recall some notions from the theory of fuzzy metric spaces used in this paper.

Definition 1.1. [14] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Two basic examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2. [4] A fuzzy metric space in the sense of George and Veeramani is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following properties, for all $x, y, z \in X$ and $t, s > 0$:

(F-1) $M(x, y, t) > 0$,
(F-2) $M(x, y, t) = 1$ if and only if $x = y$,
(F-3) $M(x, y, t) = M(y, x, t)$,
(F-4) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous,
(F-5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

If, in the above definition, the triangular inequality (F-5) is replaced by

(F-6) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$,

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then the triple $(X, M, *)$ is called a *non-Archimedean fuzzy metric space*. It is easy to check that the triangular inequality (F-6) implies (F-5), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 1.3. Let (X, d) be an ordinary metric space and θ be a nondecreasing and continuous function from $(0, \infty)$ into $(0, 1)$ such that $\lim_{t\longrightarrow\infty} \theta(t) = 1$. Some examples of these functions are $\theta(t) = \dfrac{t}{t+1}$, $\theta(t) = 1 - e^{-t}$ and $\theta(t) = e^{-\frac{1}{t}}$. Let $a * b \leq ab$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = [\theta(t)]^{d(x,y)}$$

for all $x, y \in X$. It is easy to see that $(X, M, *)$ is a non-Archimedean fuzzy metric space.

Example 1.4. Let $X = (0, \infty)$, $a * b = ab$ and M be the fuzzy set on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}},$$

for all $x, y \in X$. Then $(X, M, *)$ is a non-Archimedean fuzzy metric space.

Example 1.5. Let (X, d) be an ordinary metric space and $a * b = ab$. If M be the fuzzy set on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$, then $(X, M, *)$ is a non-Archimedean fuzzy metric space.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. This definition of Cauchy sequence is identical with that given by George and Veeramani [4]. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

2. Main Results

In this section we give a new approach to Caristi's fixed point theorem on non-Archimedean fuzzy metric space.

Let Φ denote the family of all mappings $\phi : (0, 1] \to [0, \infty)$ satisfying,

(1) ϕ is continuous and decreasing,
(2) $\phi(t) = 0 \Leftrightarrow t = 1$ and $\phi(ts) \leq \phi(t) + \phi(s)$, for every $t, s \in (0, 1]$.

Since the function ϕ defined by $\phi(t) = \log_a t$, for $0 < a < 1$ belonging to Φ, then it is nonempty.

Lemma 2.1. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with $a*b \geq ab$ for all $a, b \in [0,1]$. If we define $d : X^2 \to [0, \infty)$ by*

$$d(x,y) = \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \phi(M(x,y,t))dt,$$

then d is a metric on X.

Proof. It is clear from the definition that d is well defined and symmetric. Also,

$$\begin{aligned} d(x,y) &= 0 \Leftrightarrow \phi(M(x,y,t)) = 0 \text{ for all } t > 0 \\ &\Leftrightarrow M(x,y,t) = 1 \text{ for all } t > 0 \\ &\Leftrightarrow x = y. \end{aligned}$$

Finally, since $M(x,y,t) \geq M(x,z,t) * M(z,y,t) \geq M(x,z,t)M(z,y,t)$ and $\phi \in \Phi$ it follows that

$$\begin{aligned} d(x,y) &= \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \phi(M(x,y,t))dt \\ &\leq \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \phi(M(x,z,t)M(z,y,t))dt \\ &\leq \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \phi(M(x,z,t))dt + \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \phi(M(z,y,t))dt \\ &= d(x,z) + d(z,y). \end{aligned}$$

This proves that d is a metric on X. □

Remark 2.2. Considering $\phi(t) = \log_a t$, for $0 < a < 1$ we can say from Lemma 2.1 that, if $(X, M, *)$ be a fuzzy metric space with $a * b \geq ab$, the function d defined by

$$d(x,y) = \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a M(x,y,t)dt$$

is a metric on X.

Lemma 2.3. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with $a*b \geq ab$ for every $a, b \in [0,1]$ and $\psi : X \times (0, \infty) \to (0, \infty)$. Define the relation $\preceq$ on X as follows:*

$$x \preceq y \Leftrightarrow M(x,y,t) \geq \frac{\psi(x,t)}{\psi(y,t)}.$$

Then $\preceq$ is a (partial) order on X, called the partial order induced by ψ.

Proof. For all $x \in X$ and $t > 0$, $M(x,x,t) = 1 = \frac{\psi(x,t)}{\psi(x,t)}$, then $x \preceq x$, that is, $\preceq$ is reflexive. Again for $x, y \in X$, let $x \preceq y$ and $y \preceq x$. Then for all $t > 0$,

$$M(x,y,t) \geq \frac{\psi(x,t)}{\psi(y,t)} \text{ and } M(y,x,t) \geq \frac{\psi(y,t)}{\psi(x,t)}.$$

Therefore

$$\begin{aligned} 1 &= M(x,x,t) \geq M(x,y,t) * M(y,x,t) \\ &\geq M(x,y,t)M(y,x,t) \geq 1 \end{aligned}$$

from which it follows that $M(x,y,t) = M(y,x,t) = 1$, i.e. $x = y$. Thus $\preceq$ is antisymmetric. Now for $x,y,z \in X$, let $x \preceq y$ and $y \preceq z$, then for all $t > 0$,

$$M(x,y,t) \geq \frac{\psi(y,t)}{\psi(x,t)} \text{ and } M(y,z,t) \geq \frac{\psi(y,t)}{\psi(z,t)}.$$

Therefore

$$\begin{aligned} M(x,z,t) &\geq M(x,y,t) * M(y,z,t) \\ &\geq M(x,y,t)M(y,z,t) \geq \frac{\psi(x,t)}{\psi(z,t)}, \end{aligned}$$

This shows that $x \preceq z$. □

Example 2.4. Let $X = (0,\infty)$, $a * b = ab$ and

$$M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}.$$

Then $(X,M,*)$ is a non-Archimedean fuzzy metric space, which is also complete (see [13]). Let $\psi : X \times (0,\infty) \to (0,\infty)$, $\psi(x,t) = x$. Then for $x,y \in X$,

$$\begin{aligned} x \preceq y &\Leftrightarrow M(x,y,t) \geq \tfrac{\psi(x,t)}{\psi(y,t)} \\ &\Leftrightarrow M(x,y,t) \geq \frac{x}{y} \\ &\Leftrightarrow \tfrac{\min\{x,y\}}{\max\{x,y\}} \geq \frac{x}{y} \\ &\Leftrightarrow x \leq y. \end{aligned}$$

Therefore $\preceq$ is a well known order on X.

Definition 2.5. Let $(X,M,*)$ be a non-Archimedean fuzzy metric space and $\psi : X \times (0,\infty) \to (0,\infty)$. Let $T : X \to X$ be a mapping such that

$$M(x,Tx,t) \geq \frac{\psi(x,t)}{\psi(Tx,t)},$$

for all $x \in X$ and $t > 0$, then T is called M-Caristi map w.r.t ψ on $(X,M,*)$.

Remark 2.6. Let (X,d) be a metric space and $\varphi : X \to \mathbb{R}$ be a function. Define the relation $\preceq$ on X through

$$x \preceq y \Leftrightarrow d(x,y) \leq \varphi(x) - \varphi(y).$$

Then it is well known that $\preceq$ is a partial order on X, called the partial order induced by φ. Also we know that the mapping $T : X \to X$ is called a Caristi map w.r.t φ, if $d(x,Tx) \leq \varphi(x) - \varphi(Tx)$.

Lemma 2.7. *Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with $a*b \geq ab$ for all $a,b \in [0,1]$ and $\psi : X \times (0,\infty) \to (0,\infty)$. Let d be a metric on X as given in Remark 2.2 and $\varphi(x) = \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a \psi(x,t)dt$. Then the following hold:*

1. *The relation $\preceq$ is a partial order on $(X,M,*)$ induced by ψ if and only if it is partial order on (X,d) induced by φ.*
2. *If ψ is upper semi-continuous w.r.t first variable and bounded from above if and only if φ is lower semi-continuous and bounded from below,*

(3) $(X, M, *)$ *is a complete non-Archimeden fuzzy metric space if and only if* (X, d) *is complete metric space,*

(4) $T : X \to X$ *is* M*-Caristi map w.r.t* ψ *on* $(X, M, *)$ *if and only if it is Caristi map w.r.t* φ *on* (X, d).

Proof. (1) For all $x, y \in X$, we have

$$\begin{aligned} M(x,y,t) \geq \tfrac{\psi(x,t)}{\psi(y,t)} \quad &\Leftrightarrow \quad \log_a M(y,z,t) \leq \log_a \psi(x,t) - \log_a \psi(y,t) \\ &\Leftrightarrow \quad d(x,y) \leq \varphi(x) - \varphi(y), \end{aligned}$$

that is, $\preceq$ is a partial order on $(X, M, *)$ induced by ψ if and only if it is partial order on (X, d) induced by φ.

(2) Let $x \in X$ and $\{x_n\}$ be a sequence in X. By cosidering the definition of d, we see that $\lim_{n\to\infty} M(x_n, x, t) = 1$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. Also we have (since $0 < a < 1$)

$$\begin{aligned} \limsup_{n\to\infty} \psi(x_n,t) \quad &\leq \quad \psi(x,t) \Leftrightarrow \log_a(\limsup_{n\to\infty} \psi(x_n,t)) \geq \log_a \psi(x,t) \\ &\Leftrightarrow \quad \limsup_{n\to\infty} (\log_a \psi(x_n,t)) \geq \log_a \psi(x,t) \\ &\Leftrightarrow \quad \sup_{\alpha\in(0,1)} \int_\alpha^1 \liminf_{n\to\infty} (\log_a \psi(x_n,t))dt \geq \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a \psi(x,t)dt \\ &\Leftrightarrow \quad \liminf_{n\to\infty} \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a \psi(x_n,t)dt \geq \varphi(x) \\ &\Leftrightarrow \quad \liminf_{n\to\infty} \varphi(x_n) \geq \varphi(x), \end{aligned}$$

that is, ψ is upper semi-continuous w.r.t first variable if and only if φ is lower semi-continuous. Similarly, it is easy to see that ψ is bounded from above if and only if φ is bounded from below.

Also, (3) and (4) follows immediately from definitions. □

Theorem 2.8. *Let* $(X, M, *)$ *be a complete fuzzy metric space such that* $a * b \geq ab$ *for every* $a, b \in [0, 1]$ *and* $\psi : X \times (0, \infty) \to (0, \infty)$ *be an upper semi-continuous w.r.t first variable and bounded from above. If* $T : X \to X$ *is a* M*-Caristi map w.r.t* ψ*, then* T *has a fixed point.*

Proof. Define a metric $d : X^2 \to [0, \infty)$ by

$$d(x,y) = \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a M(x,y,t)dt$$

and

$$\varphi(x) = \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a \psi(x,t)dt,$$

then by Lemma 2.7 we know that T is a Caristi map w.r.t φ on (X, d) and (X, d) is complete. Hence, T has a fixed point. □

Example 2.9. Let $X = (0, \infty)$, $a * b = ab$ and

$$M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}.$$

Then $(X, M, *)$ is a complete non-Archimedean fuzzy metric space (see [13]). Define $T : X \to X$, by

$$Tx = \begin{cases} \frac{1}{x} & , \quad x \leq 1 \\ 1 & , \quad x > 1 \end{cases}$$

and $\psi : X \times (0, \infty) \to (0, \infty)$ by

$$\psi(x,t) = \begin{cases} x^4 & , \quad x \leq 1 \\ \frac{1}{x} & , \quad x > 1 \end{cases},$$

then ψ is continuous and bounded. Now we consider the following cases:

Case 1. Let $x \leq 1$, then

$$M(x, Tx, t) = \frac{\min\{x, \frac{1}{x}\}}{\max\{x, \frac{1}{x}\}} = x^2 \geq \frac{x^4}{x} = \frac{\psi(x,t)}{\psi(\frac{1}{x}, t)} = \frac{\psi(x,t)}{\psi(Tx,t)}.$$

Case 2. Let $x > 1$, then

$$M(x, Tx, t) = \frac{\min\{x, 1\}}{\max\{x, 1\}} = \frac{1}{x} \geq \frac{\frac{1}{x}}{1} = \frac{\psi(x,t)}{\psi(1,t)} = \frac{\psi(x,t)}{\psi(Tx,t)}.$$

Therefore T is a M-Caristi map w.r.t. ψ. Hence by Theorem 2.8, T has a fixed point in X.

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S. Sedghi, Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran

E-mail address: sedghi.gh@qaemshahriau.ac.ir

N. Shobkolaei, Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran

E-mail address: nabi_shobe@yahoo.comg

I. Altun*, Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

E-mail address: ishakaltun@yahoo.com

*Corresponding author