FUNCTORIAL SEMANTICS OF TOPOLOGICAL THEORIES

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Abstract. Following the categorical approach to universal algebra through algebraic theories, proposed by F. W. Lawvere in his PhD thesis, this paper aims at introducing a similar setting for general topology. The cornerstone of the new framework is the notion of *categorically-algebraic* (*catalg*) *topological theory*, whose models induce a category of topological structures. We introduce the quasicategory of catalg topological theories and consider its functorial relationships with the quasicategory of the categories of models, in order to provide convenient means for studying topological structures via the properties of their corresponding theories.

1. Introduction

The notion of (L-)*fuzzy set* introduced by L. A. Zadeh [70] and J. A. Goguen [27] initiated reconsideration of the whole of mathematics in the light of fuzziness. In particular, fuzzy analogue of general topology, called *lattice-valued* or *many-valued topology*, has been developing rapidly, starting from the pioneering papers of C. L. Chang [12], J. A. Goguen [28] and R. Lowen [43]. Time passing, the following approaches began to dominate the others: *point-set lattice-theoretic* (*poslat*) *topology* of S. E. Rodabaugh [49] (recently, in collaboration with J. T. Denniston and A. Melton [17]), (L, M)*-fuzzy topology* of T. Kubiak and A. Šostak [41] (the idea stemming from U. Höhle [34]), *monadic topology* of W. Gähler [25] (recently, in collaboration with P. Eklund *et al.* [23]) as well as *categorical fuzzy topology* of P. Eklund [22] (backed extensively by S. E. Rodabaugh [51, 53]). In view of the diversity in the settings of the authors, making them prove the standard topological properties for every new theory in play, there has recently been various attempts to introduce a common unified framework, which would provide convenient means of interaction between different theories. The most notorious results in this area came from S. E. Rodabaugh, who presented in [52, 54] a setting, based on category theory and originating from *algebraic theories* (*in clone form*) of E. G. Manes [45]. Due to "reasonable" requirements on doing topology, his setting, on one hand, is more flexible than Gähler's monadic topology, which employs unnecessarily cumbersome language of monads, and, on the other hand, provides more tools than Eklund's categorical topology, which essentially is restricted to the standard category-theoretic

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inventory, the latter, because of its generality, being not always suitable for solving particular topological problems. On a closer inspection, however, it appeared that the proposed framework was unable to incorporate several important topological developments. For example, the well-known case of *closure spaces* [5, 20] was outside the scope of the approach. The reason for the deficiency is simple: paying much attention to the category-theoretic background of the framework, the author neglected completely its algebraic aspects. More precisely, the topological structures resulting from the approach are based explicitly in particular algebras called *semi-quantales*, which generalize the algebraic structure of topology on a set, namely, *frame* or *locale* [39, 48]. On the other hand, the underlying algebras of closure spaces are $\bigwedge$-*semilattices* with the singled out bottom element.

In order to remove the drawback, we introduced the concept of *categorically-algebraic* (*catalg*) *topology* [57], which is based on both category theory and universal algebra, relying more on the former. The acronym "catalg" is motivated by the well-known "poslat" (see above) and reminds of the background of the theory (objects of categories instead of sets and algebras instead of lattices). While the approach of S. E. Rodabaugh is an extension (or, rather, simplification) of the algebraic theories of E. G. Manes (essentially, monads), the catalg setting originates from a particular outlook on the classical set-theoretic topology, relying on three bedrocks (encountered already in Rodabaugh's framework, but in a quite disguised manner).

(1) *Powerset theory*, which is the functor $\mathbf{Set} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op}$ from the category of sets to the dual category of the variety of complete Boolean algebras, defined by $(X \xrightarrow{f} Y)^{\leftarrow} = \mathbf{2}^X \xrightarrow{(f^{\leftarrow})^{op}} \mathbf{2}^Y$, $f^{\leftarrow}(\alpha) = \alpha \circ f$ (actually, the well-known contravariant powerset functor of, e.g., [2, Examples 3.20(9)], written according to the traditions of the fuzzy community, i.e., representing the powerset $\mathcal{P}(X)$ of a given set X through the set of all maps $X \xrightarrow{\alpha} \mathbf{2}$ to the two-element Boolean algebra $\mathbf{2}$).

(2) *Topological theory*, which is the functor $\mathbf{Set} \xrightarrow{\mathcal{T}} \mathbf{Frm}^{op} = \mathbf{Set} \xrightarrow{(-)^{\leftarrow}} \mathbf{CBAlg}^{op} \xrightarrow{\|-\|^{op}} \mathbf{Frm}^{op}$, obtained by the composition of the powerset theory with the forgetful functor $\mathbf{CBAlg} \xrightarrow{\|-\|} \mathbf{Frm}$ to the variety of frames, whose dual category $\mathbf{Frm}^{op}$ is usually denoted $\mathbf{Loc}$.

(3) *Topological structures*, which is the category $\mathbf{Top}$ (concrete over $\mathbf{Set}$), whose objects (*topological spaces*) are pairs (X, τ), where τ is a subframe of $\mathcal{T}X$ (*topology*), and whose morphisms (*continuous maps*) $(X, \tau) \xrightarrow{f} (Y, \sigma)$ are maps $X \xrightarrow{f} Y$ such that $(\mathcal{T}f)^{op}(\alpha) \in \tau$ for every $\alpha \in \sigma$ (*continuity*).

Catalg topology extends the first two of the above items to the following concepts.

(1′) *Catalg powerset theory*, which is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual category of a variety of algebras $\mathbf{A}$.

(2′) *Catalg topological theory*, which is a functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op} = \mathbf{X} \xrightarrow{P} \mathbf{A}^{op} \xrightarrow{\|-\|^{op}} \mathbf{B}^{op}$, obtained by the composition of a given catalg powerset theory P and a reduct **B** of **A** (in the standard algebraic sense, meaning the dropping of some primitive operations), whose forgetful functor is given by $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$.

The respective third item relies on a catalg topological theory T instead of the classical $\mathcal{T}$ (replacing subframes with subalgebras), providing *catalg topological structures* as the objects of the category $\mathbf{Top}(T)$. The main advantages of the new setting are its simplicity and flexibility. The latter means the ability to incorporate the most important approaches to fuzzy topology, e.g., (poslat) categorical topology of S. E. Rodabaugh, and (L, M)-fuzzy topology of T. Kubiak and A. Šostak (requires *lattice-valued catalg topology* [58, 63]) together with its extension done by C. Guido [30] as well as J. T. Denniston, A. Melton and S. E. Rodabaugh [17]. Moreover, the missing case of closure spaces is also included in the framework (more distant topological settings like, e.g., that of D. Hofmann [33] are currently being checked). On the other hand, the setting is extremely non-demanding, e.g., starting from a handful of notions (basically, two functors), catalg topology adds an additional tool or requirement iff there is a real necessity for it, propagating the so-called plug-and-play approach. To compare, the monadic setting of W. Gähler starts with the heavyweight (and not always justified) theory of partially ordered monads, liable to frighten the potential user of the framework. It is one of the main purposes of this paper to show another advantage of our approach: the ability to provide convenient means of interaction between different topological settings.

The motivation for the developments stems from the PhD thesis of F. W. Lawvere [42], which introduced a categorical approach to universal algebra through the so-called *algebraic theories*, initiating *categorical algebra*, in which varieties of algebras are formalized without details of equational presentations. In nearly half a century that followed Lawvere's introduction, his initial idea has undergone numerous generalizations, ramifications and applications in different areas such as, e.g., algebraic geometry, topology and computer science, including generalization from one-sorted to many-sorted algebras. The cornerstone of the setting is the category of algebraic theories and their morphisms, the promoted slogan being simple: study not the algebraic structure but its respective algebraic theory. As an example, one can mention the extension of the famous Morita problem to categorical algebra, running as follows: find the conditions on two theories, so that their respective categories of algebraic structures are equivalent. Inspired by K. Morita, who studied the issue for the categories of left modules over a ring R in the 1950s [46], the problem has been successfully approached in categorical algebra for both one-sorted [10, 21] and many-sorted algebraic theories [4]. Following the idea of F. W. Lawvere, this paper aims at introducing a categorical approach to (lattice-valued) general topology through the above-mentioned *catalg topological theories*. In particular, we define the quasicategory of topological theories and consider its relationships to the quasicategory of the categories of the form $\mathbf{Top}(T)$ (see above), whose objects are topological structures generated by the theories, which can be also called their *models*. For instance, one can easily restate the above-mentioned Morita problem for the new framework, thereby starting a new line of research. Our main purpose lies in providing convenient means of studying topological structures through the properties of their respective topological theories. This paper is bound to make the first step in this direction (a brief account of the obtained results has already appeared in [59], some parts of it announced in [24, 58]).

We notice that the idea of topological theory employed in this paper goes back up to the construction of A. Grothendieck, who considered functors from the dual of a category $\mathbf{X}$ (*index category*) to the category $\mathbf{CAT}$ of categories (the so-called *indexed categories*; also notice that we do not bother ourselves with the difference between quasicategories and categories). Given an indexed category $\mathbf{X}^{op} \xrightarrow{T} \mathbf{CAT}$, one defines the category $T^{\#}$, whose objects are pairs (X, A), where X is an $\mathbf{X}$-object and A is a TX-object, and whose morphisms $(X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)$ comprise an $\mathbf{X}$-morphism $X_1 \xrightarrow{f} X_2$ and a TX_1-morphism $A_1 \xrightarrow{\varphi} Tf^{op}(A_2)$. Moreover, there exists the projection functor $T^{\#} \xrightarrow{P} \mathbf{X}$, which is defined by $P((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)) = X_1 \xrightarrow{f} X_2$. The above machinery is known as *Grothendieck construction* [19], which provides a correspondence between indexed categories T over $\mathbf{X}$ and fibrations $(T^{\#}, P)$ over $\mathbf{X}$. The obtained fibrations become faithful on replacing the category $\mathbf{CAT}$ with the category $\mathbf{Pos}$ of partially ordered sets. Further simplification is provided by topological theories of O. Wyler [68, 69], who considers functors $\mathbf{X}^{op} \xrightarrow{T} \mathbf{CSLat}(\bigwedge)$ (the category of $\bigwedge$-semilattices), which describe precisely the topological categories over $\mathbf{X}$. In [66, Chapter I(2)], W. Tholen mentions the notion of morphism between topological theories of O. Wyler as a natural transformation (or a "weakly commutative" diagram), which has the form

$$\begin{array}{ccc} \mathbf{X}_1^{op} & \xrightarrow{\quad F \quad} & \mathbf{X}_2^{op} \\ {\scriptstyle T_1}\big\downarrow & \overset{\gamma}{\Longrightarrow} & \big\downarrow{\scriptstyle T_2} \\ \mathbf{CSLat}(\bigwedge) & \xrightarrow[1_{\mathbf{CSLat}(\bigwedge)}]{} & \mathbf{CSLat}(\bigwedge), \end{array}$$

and shows that the category of topological theories and their morphisms is equivalent to the category of topological categories and (initial source)-preserving functors. The present manuscript simplifies the approach of A. Grothendieck and, simultaneously, extends the notion of topological theory morphism of W. Tholen, to present a new approach to the topic, suitable for doing lattice-valued topology.

The paper is based on both category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [2, 44, 45]. For algebraic notions we recommend [13, 29]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e.g., with that of an adjoint situation.

2. Categorically-algebraic Topology

2.1. **Algebraic and Categorical Preliminaries.** We start with the algebraic and categorical preliminaries crucial for the fruitful perusal of the paper. Experienced readers can easily skip the matter, consulting the subsection for the notations of the author only. At the bottom of our approach lies the notion of *algebra*, which is thought of as a set with a family of operations defined on it, satisfying certain identities. The theory of universal algebra of, e.g., [13] calls a class of finitary algebras (induced by a set of finitary operations), which is closed under the formation of homomorphic images, subalgebras and direct products, a *variety*. Motivated by the algebraic structures employed in lattice-valued topology (where set-theoretic unions are replaced by arbitrary joins), we consider infinitary algebraic theories, extending the approach of varieties to cover our needs.

Definition 2.1. Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly, proper or empty) class of cardinal numbers. An Ω-*algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ-*ary primitive operations* on A). An Ω-*homomorphism* $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that the diagram

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{\varphi^{n_\lambda}} & B^{n_\lambda} \\ {\scriptstyle \omega_\lambda^A}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^B} \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

commutes for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ stands for the construct of Ω-algebras and Ω-homomorphisms.

Every concrete category of this paper is supposed to have the underlying functor $|-|$ to its respective ground category, the latter mentioned explicitly in each case.

Definition 2.2. Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of* Ω-*algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$, closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).

Definition 2.3. Given a variety $\mathbf{A}$, a *reduct* of $\mathbf{A}$ is a pair $(\| - \|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$, where $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ is a concrete functor.

The following constructs are simple examples of varieties: $\mathbf{CSLat}(\Xi)$ of Ξ-*semilattices* (partially ordered sets having arbitrary Ξ, where $\Xi \in \{\bigwedge, \bigvee\}$), **SQuant** of *semi-quantales* ($\bigvee$-semilattices, equipped with a binary operation $\otimes$, called *multiplication* or *tensor product*), **Quant** of *quantales* (semi-quantales, whose multiplication is associative and distributes across $\bigvee$ from both sides), **UQuant** of *unital quantales* (quantales, whose multiplication has the unit $\imath$), **Frm** of *frames* (unital quantales, whose multiplication coincides with the meet operation), **DmFrm** of *DeMorgan frames* (frames, equipped with an order-reversing involution), **CBAlg** of *complete Boolean algebras* (DeMorgan frames, whose involution provides the complement) [39, 40, 52, 54, 55]. Taken in the reverse order, the categories provide a sequence of reducts in the sense of Definition 2.3. Moreover, an important reduct of **Frm** is the variety **SFrm** (also denoted **CSLF** [50]) of *semi-frames* (unital semi-quantales, whose multiplication coincides with the meet operation) [52, 54]. Out of the scope of these examples lies the variety **CSL** of *closure semilattices*

($\bigwedge$-semilattices, with the singled out bottom element $\bot$), which has already been mentioned (implicitly) in the Introduction.

Some words are due the notations of the paper which differ from those used in the previous manuscripts on the topic (this article follows the notations of category theory, whereas its predecessors adhered to the respective ones of the fuzzy community). From now on, varieties are denoted **A**, **B**, **C**, with **S** standing for their subcategories. The categorical dual of a variety **A** is denoted $\mathbf{A}^{op}$, whose objects (resp. morphisms) are called *op-algebras* (resp. *op-homomorphisms*). The dual of **Frm** uses the already accepted notation **Loc** [7, 39] (adding "**S**" in front in case of semi-frames). Given a homomorphism φ, the corresponding op-one is denoted φ^{op} and vice versa. Every algebra A of a variety **A** gives rise to the subcategory $\mathbf{S}_A$ of $\mathbf{A}^{op}$, whose only morphism is the identity $A \xrightarrow{1_A} A$. Two algebraic notations should be mentioned as well. Given an **A**-homomorphism $A_1 \xrightarrow{\varphi} A_2$, the restriction of φ to its image is denoted $\overline{\varphi}$. Given a subalgebra B of an **A**-algebra A, $B \overset{e_B}{\hookrightarrow} A$ stands for the corresponding set-theoretic inclusion.

2.2. **Categorically-algebraic Topology.** Having the necessary preliminaries in hand, this section serves as an introduction into the theory of *categorically-algebraic* (*catalg*) *topology*. The reader is strongly advised to recall the (poslat) powerset theories of S. E. Rodabaugh [52, Definition 3.4, 3.5] and topological theories of J. Adámek *et al.* [2, Exercise 22B], the extension of which provides the main starting point for the new setting.

Every set map $X \xrightarrow{f} Y$ provides two operators: *image operator* $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, $f^{\rightarrow}(S) = \{f(x) \mid x \in S\}$ and *preimage operator* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, $f^{\leftarrow}(T) = \{x \mid f(x) \in T\}$. The latter one (already mentioned in the introductory section in a functorial way) can be extended to a more general setting as follows (the respective generalization of the former is described elsewhere).

Definition 2.4. A *catalg backward powerset theory* (*cabp-theory*) in a category **X** (*ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ to the dual category of some variety **A**.

The following example illustrates the concept, extending the standard fixed- and variable-basis approaches of lattice-valued topology [35, 51] (recall that **Set** is the category of sets and maps).

Example 2.5. Given a variety **A**, every subcategory **S** of $\mathbf{A}^{op}$ induces a functor $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{S}=(-)^{\leftarrow}} \mathbf{A}^{op}$, $(X_1, A_1) \xrightarrow{(f,\varphi)^{\leftarrow}} (X_2, A_2) = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$, $(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$. The case $\mathbf{S} = \mathbf{S}_A$ is denoted $\mathcal{S}_A = (-)^{\leftarrow}_A$ and is called *fixed-basis approach*, whereas all other cases are referred to as *variable-basis approach*. The functor $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^{\leftarrow}_{\mathbf{2}}} \mathbf{CBAlg}^{op}$ (recall from the Introduction that **2** is the two-element Boolean algebra) gives the above-mentioned preimage operator (and the functor mentioned in the Introduction), whereas the functors $\mathbf{Set} \times \mathbf{S}_{\mathbb{I}} \xrightarrow{\mathcal{Z}=(-)^{\leftarrow}_{\mathbb{I}}} \mathbf{DmLoc}$ ($\mathbb{I}$ is the unit interval $[0, 1]$, equipped with the standard algebraic structure), $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}=(-)^{\leftarrow}_L} \mathbf{UQuant}^{op}$ and $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{R}_1=(-)^{\leftarrow}}$

$\mathbf{USQuant}^{op}$, $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{R}_2=(-)^{\leftarrow}} \mathbf{SLoc}$, $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{R}_3=(-)^{\leftarrow}} \mathbf{Loc}$ provide the operators of L. A. Zadeh [70], J. A. Goguen [28] and S. E. Rodabaugh [52], [50], [18]. The case $\mathbf{X} \xrightarrow{\mathcal{R}} \mathbf{SQuant}^{op}$ is precisely the backward powerset theory considered in the categorical topology of S. E. Rodabaugh [52, Definition 3.5(2)].

The next step introduces topological theories, based on powerset theories, which ultimately give rise to catalg topological structures. The notion has no analogue in the categorical setting of S. E. Rodabaugh, which passes directly from powerset theories to topological structures. In order not to obscure the main point of this paper, we provide a slightly simplified definition of the concept.

Definition 2.6. Let $\mathbf{X}$ be a category and let $\mathcal{T} = (P, (\|-\|, \mathbf{B}))$ comprise a cabp-theory $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ and a reduct $(\|-\|, \mathbf{B})$ of $\mathbf{A}$. The *catalg topological theory* (*cat-theory*) *in* $\mathbf{X}$ *induced by* $\mathcal{T}$ is the functor $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, which is given by the equality $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op} = \mathbf{X} \xrightarrow{P} \mathbf{A}^{op} \xrightarrow{\|-\|^{op}} \mathbf{B}^{op}$.

The last step provides the category of topological structures, which is generated by a given topological theory. Following the ideas of F. W. Lawvere, the objects of this category are called *models* of the theory.

Definition 2.7. Let T be a cat-theory in a category $\mathbf{X}$. $\mathbf{Top}(T)$ is the concrete category over $\mathbf{X}$, whose objects (*catalg topological spaces* or T*-spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is a subalgebra of TX (*catalg topology* or T*-topology* on X), and whose morphisms (*catalg continuous* or T*-continuous* $\mathbf{X}$*-morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ with the property that $((Tf)^{op})^{\rightarrow}(\tau_2) \subseteq \tau_1$ (*catalg continuity* or T*-continuity*).

The following example illustrates Definition 2.7, thereby justifying the fruitfulness of the proposed catalg approach.

Example 2.8. The case of the ground category $\mathbf{X} = \mathbf{Set} \times \mathbf{S}$ is called *variety-based topology*. In particular, $\mathbf{Top}((\mathcal{S}_A, \mathbf{B}))$ provides the category $A_{\mathbf{B}}$-$\mathbf{Top}$, which is the framework for *fixed-basis variety-based topology*, whereas $\mathbf{Top}((\mathcal{S}, \mathbf{B}))$ gives the category $(\mathbf{S}, \mathbf{B})$-$\mathbf{Top}$ (the case $\mathbf{A} = \mathbf{B}$ is shortened to $\mathbf{S}$-$\mathbf{Top}$), which is the framework for *variable-basis variety-based topology*. More specific, $\mathbf{Top}((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the category $\mathbf{Top}$ of topological spaces, whereas $\mathbf{Top}((\mathcal{P}, \mathbf{CSL}))$ is isomorphic to the category $\mathbf{Cls}$ of closure spaces of D. Aerts [6]. Moreover, $\mathbf{Top}((\mathcal{Z}, \mathbf{Frm}))$ is isomorphic to the category $\mathbb{I}$-$\mathbf{Top}$ of fixed-basis fuzzy topological spaces of C. L. Chang [12], $\mathbf{Top}((\mathcal{G}, \mathbf{UQuant}))$ is isomorphic to the category L-$\mathbf{Top}$ of fixed-basis L-fuzzy topological spaces of J. A. Goguen [28], whereas $\mathbf{Top}((\mathcal{R}_1, \mathbf{USQuant}))$, $\mathbf{Top}((\mathcal{R}_2, \mathbf{SFrm}))$, $\mathbf{Top}((\mathcal{R}_3, \mathbf{Frm}))$ are isomorphic to the categories $\mathbf{S}$-$\mathbf{Top}_i$, $i \in \{1, 2, 3\}$ for variable-basis poslat topology of S. E. Rodabaugh [52], [50], [18]. The case $\mathbf{Top}((\mathcal{R}, \mathbf{SQuant}))$ provides the category $\mathbf{T}_{\mathbf{X}\mathcal{R}}$ of topological structures of S. E. Rodabaugh [52, Definition 3.7], which is called the topological theory of $\mathcal{R}$.

Example 2.8 backs our claim from the Introduction that the catalg approach provides a common framework for many (lattice-valued) topological settings. Moreover, its last item shows the crucial difference between our catalg setting and the categorical approach of S. E. Rodabaugh, whose topological theories are defined as categories (and not functors) of the form $\mathbf{Top}((\mathcal{R}, \mathbf{SQuant}))$, thereby mixing theories and their models. The reader should also notice that categories of the form $\mathbf{Top}(T)$ considered by J. Adámek *et al.* [2, Exercise 22B], despite certain similarity in the notations, are defined in a more general way. Having another goal, we will not consider further the relationships between the two settings, leaving the topic for our subsequent research (see Problem 6.4 in the concluding section of the paper).

The subsequent developments will need a simple property of catalg topological spaces, namely, a generalization of the classical result of general topology that continuity of a map can be checked on the elements of a subbase. This important feature has already been extended to (poslat) categorical topology by S. E. Rodabaugh [51, Theorem 3.2.6], [52, Theorem 3.10].

Definition 2.9. Let A be an $\mathbf{A}$-algebra, let S be a subset of A and let Ω be a subclass of $\Omega_{\mathbf{A}}$. The smallest Ω-subreduct of A containing S is denoted $\langle S\rangle_{\Omega}$ (or $\langle S\rangle$ if $\Omega = \Omega_{\mathbf{A}}$). Let $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ be a cat-theory, let (X, τ) be a T-space, let S be a subset of TX and let Ω be a subclass of $\Omega_{\mathbf{B}}$. S is called an Ω-*base* of τ provided that $\tau = \langle S\rangle_{\Omega}$. $\Omega_{\mathbf{B}}$-bases are called *subbases*.

Example 2.10. The setting of the category $\mathbf{S\text{-}Top}_1$ (recall Example 2.8) provides the well-known $\{\bigvee\}$-bases (resp. $\{\bigvee, \otimes, 1\}$-bases or subbases) of poslat topology [53]. The category $\mathbf{Top}((\mathcal{R}, \mathbf{SQuant}))$ provides a more sophisticated recent categorical approach of S. E. Rodabaugh [52, Theorem 3.10]. The setting of the category $\mathbf{Top}$ gives rise to the classical definition of base (resp. subbase), where elements of the topology are unions (resp. unions of finite intersections) of elements of the base (resp. subbase).

The next lemma (see [60, 61] for the proof) shows a simple (and, at the same time, a very important) relation between (pre)image operators and subreducts.

Lemma 2.11. *Let $A_1 \xrightarrow{\varphi} A_2$ be an $\mathbf{A}$-homomorphism, let Ω be a subclass of $\Omega_{\mathbf{A}}$.*

(1) *For every Ω-subreduct B of A_2, $\varphi^{\leftarrow}(B)$ is an Ω-subreduct of A_1.*
(2) *For every subset $S \subseteq A_1$, $\varphi^{\rightarrow}(\langle S\rangle_{\Omega}) = \langle \varphi^{\rightarrow}(S)\rangle_{\Omega}$.*

The proof of the next result is based on Lemma 2.11 and can be found in, e.g., [61] (or conducted by the reader as an easy exercise, to get into the topic of the paper).

Lemma 2.12. *Let T be a cat-theory in $\mathbf{X}$ and let (X_1, τ_1), (X_2, τ_2) be T-spaces such that $\tau_2 = \langle S\rangle_{\Omega}$. An $\mathbf{X}$-morphism $X \xrightarrow{f} Y$ is T-continuous iff $((Tf)^{op})^{\rightarrow}(S) \subseteq \tau_1$.*

The next subsection recalls a particular concept, originally due to S. Vickers [67], whose importance has been boosted up recently in connection with its applications to the theory of lattice-valued topology.

2.3. **Categorically-algebraic Topological Systems.** Having introduced the theory of catalg topology, we recall basic elements of one of the most crucial of its developments, namely, the theory of *catalg topological systems* (see [62, 64] for a full discussion of its various aspects). The reader is advised to refresh his knowledge on *comma categories* [2, 44].

Definition 2.13. Given a cat-theory $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, $\mathbf{TopSys}(T)$ is the comma category $(T \downarrow 1_{\mathbf{B}^{op}})$, concrete over the product category $\mathbf{X} \times \mathbf{B}^{op}$, whose objects (X, κ, B) (resp. morphisms $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)$) are called *catalg topological systems* or *T-systems* (resp. *catalg continuous* or *T-continuous morphisms*).

The following example illustrates Definition 2.13, thereby justifying our introduction of the new concept.

Example 2.14. The case of the ground category $\mathbf{X} = \mathbf{Set} \times \mathbf{S}$ is called *variety-based approach*. In particular, $\mathbf{TopSys}((\mathcal{S}_A, \mathbf{B}))$ provides the category $A_{\mathbf{B}}$-$\mathbf{TopSys}$, which is the framework for *fixed-basis variety-based topological systems*, whereas $\mathbf{TopSys}((\mathcal{S}, \mathbf{B}))$ gives the category $(\mathbf{S}, \mathbf{B})$-$\mathbf{TopSys}$ (the case $\mathbf{A} = \mathbf{B}$ is shortened to $\mathbf{S}$-$\mathbf{TopSys}$), which is the framework for *variable-basis variety-based topological systems*. More specific, $\mathbf{TopSys}((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the category $\mathbf{TopSys}$ of topological systems of S. Vickers [67], whereas $\mathbf{TopSys}((\mathcal{P}, \mathbf{Set}))$ is isomorphic to the ground category for the categories of *interchange systems* of J. T. Denniston, A. Melton, S. E. Rodabaugh [15, 16]. $\mathbf{TopSys}((\mathcal{R}_3, \mathbf{Frm}))$ with $\mathbf{S} = \mathbf{Loc}$ is isomorphic to the category $\mathbf{Loc}$-$\mathbf{TopSys}$ of lattice-valued topological systems of J. T. Denniston *et al.* [14].

One of the main results of the theory of catalg systems is the possibility of representing the category $\mathbf{Top}(T)$ as a full subcategory (with convenient properties) of the category $\mathbf{TopSys}(T)$ [64]. The reader is to recall the algebraic notation "$e_{(-)}$" for inclusion of subalgebras, mentioned at the end of Subsection 2.1.

Theorem 2.15.

(1) *There exists a full embedding* $\mathbf{Top}(T) \xhookrightarrow{\mathrm{E}} \mathbf{TopSys}(T)$, $\mathrm{E}((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, e^{op}_{\tau_1}, \tau_1) \xrightarrow{(f,\varphi)} (X_2, e^{op}_{\tau_2}, \tau_2)$*, where* φ^{op} *is the restriction* $\tau_2 \xrightarrow{(Tf)^{op}|^{\tau_1}_{\tau_2}} \tau_1$.

(2) *There exists a functor* $\mathbf{TopSys}(T) \xrightarrow{\mathrm{Spat}} \mathbf{Top}(T)$*,* $\mathrm{Spat}((X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)) = (X_1, (\kappa_1^{op})^{\rightarrow}(B_1)) \xrightarrow{f} (X_2, (\kappa_2^{op})^{\rightarrow}(B_2))$.

(3) Spat *is a right-adjoint-left-inverse to* E*, the respective co-universal arrows being regular monomorphisms.*

The functor Spat of Theorem 2.15 extends the *system spatialization procedure* of S. Vickers [67], which makes a topological space of a topological system (spatialize), and from which the acronym "Spat" stems.

Corollary 2.16. $\mathbf{Top}(T)$ *is isomorphic to a full (regular mono)-coreflective subcategory of* $\mathbf{TopSys}(T)$.

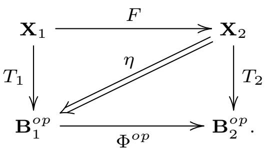
In [64], we showed that the category $\mathbf{Top}(T)$ (resp. $\mathbf{TopSys}(T)$) is topological (resp. essentially algebraic, under certain conditions on the theory T) over its ground category. Moreover, a particular case of the variety-based developments (cf. Example 2.8, 2.14), namely, lattice-valued approach over the variety $\mathbf{Frm}$ of frames, has been extensively studied in the series of short communications by J. T. Denniston, A. Melton, S. E. Rodabaugh [14, 15, 16] and also by C. Guido [31]. For example, the nature of the categories **Top**(T) and **TopSys**(T) (topological and algebraic, respectively) triggered the construction of a specific variable-basis extension of (L, M)-fuzzy topology of T. Kubiak and A. Šostak, in order to incorporate the category of lattice-valued topological systems, thereby providing the embedding opposite to the one given in Theorem 2.15(1) [17]. The general case, however, looks even more attractive. Without the standard monadic requirements on the setting in play (cf., e.g., monadic topology of W. Gähler [26] or D. Hofmann [33]), the category **TopSys**(T) provides an analogue of the category of algebras in our catalg approach (cf., e.g., the category of (*lax*) *Eilenberg-Moore algebras* in the framework of W. Gähler (resp. D. Hofmann)), whose particular subcategory gives rise to the setting of topological structures. Moreover, **TopSys**(T) is not the category **Alg**(T) of T-*algebras* w.r.t. an endofunctor $\mathbf{X} \xrightarrow{T} \mathbf{X}$ [2, Definition 5.37], since the catalg setting is based on a functor whose domain and codomain are different. In other words, even in the highly non-monadic developments, we are confronted with the notion of algebra w.r.t. to a functor. More than that, when looking closely at the category **TopSys**(T), one realizes that its objects are rather analogues of T-*coalgebras* [1, Definition 3.4], as one might expect working in the general theory of systems following J. J. M. M. Rutten [56] and H. P. Gumm [36]. It will be the topic of our further research to study the category **TopSys**(T) as an extension of the universal theory of coalgebras and its relationships to topology.

3. Topological Theories Versus Their Generated Structures

With the required preliminaries in hand, this section develops the main topic of the paper, namely, an approach to topological structures through their corresponding topological theories. In the first step, we introduce a specific quasicategory, which will accommodate the theories.

3.1. **Quasicategory of Topological Theories.** Following [2], we distinguish between categories and quasicategories, the latter defined similarly to the former except that their objects do not necessarily form a class and their hom-families are not necessarily sets. The notion of functor is extended to the case of quasicategories to quasifunctor. Notice that unlike the standard meaning of "quasiness" in the literature, quasifunctors must have all the properties of a functor except that their (co)domains can be quasicategories.

Definition 3.1. **TpThr** is the quasicategory, whose objects are cat-theories $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, and whose morphisms $T_1 \xrightarrow{(F,\Phi,\eta)} T_2$ (for the sake of convenience, occasionally, shortened to η) comprise two functors $\mathbf{X}_1 \xrightarrow{F} \mathbf{X}_2$, $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ and a natural transformation $T_2 F \xrightarrow{\eta} \Phi^{op} T_1$, or, more specifically,



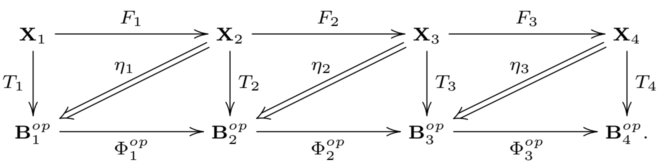
Given two cat-theories $T_1 \xrightarrow{\eta_1} T_2$ and $T_2 \xrightarrow{\eta_2} T_3$, their composition is defined by $T_3F_2F_1 \xrightarrow{\eta_2 \odot \eta_1} \Phi_2^{op}\Phi_1^{op}T_1 = T_3F_2F_1 \xrightarrow{\eta_2 F_1} \Phi_2^{op}T_2F_1 \xrightarrow{\Phi_2^{op}\eta_1} \Phi_2^{op}\Phi_1^{op}T_1$. The identity on a cat-theory T is provided by the identity natural transformation $T \xrightarrow{1_T} T$.

As was already mentioned in the Introduction, Definition 3.1 was motivated by the categorical approach to universal algebra of F. W. Lawvere, started in his PhD thesis and later on taken up by many researchers. An experienced reader, however, will notice immediately the difference between the algebraic theories of F. W. Lawvere and topological theories of our approach. Recall from [3, Definition 1.1] that an *algebraic theory* is a small category $\mathcal{T}$ with finite products. Given two algebraic theories $\mathcal{T}_1$ and $\mathcal{T}_2$, a functor $\mathcal{T}_1 \xrightarrow{M} \mathcal{T}_2$ is called a *morphism of algebraic theories* provided that it preserves finite products [3, Definition 9.1]. In such a way, one obtains the *2-category Th of theories*, with objects – algebraic theories, 1-cells (or morphisms) – morphisms of algebraic theories and 2-cells – natural transformations [3, Definition 9.11]. In our setting, topological theories are functors, whereas their morphisms are natural transformations with no additional conditions (see, however, the discussion at the end of Subsection 3.2). Moreover, topological theories need not be "small" (neither the domain nor the codomain of a theory should be a small category). This results in a quasicategory instead of a category, whose structure is more complicated than in the algebraic case. The distance between the settings gets even bigger, when we pass to the notion of algebra for an algebraic theory. Indeed, an *algebra* for the theory $\mathcal{T}$ is a functor $\mathcal{T} \xrightarrow{A} \mathbf{Set}$ preserving finite products. $Alg\,\mathcal{T}$ is then the category of algebras of $\mathcal{T}$, with morphisms, called *homomorphisms*, being natural transformations. Taking into account that the framework of F. W. Lawvere is concrete, i.e., **Set**-based, we could turn for a moment to variety-based topology, whose category $(\mathbf{S},\mathbf{B})$-**Top** falls short of $Alg\,\mathcal{T}$. The discussion has probably convinced the reader of the huge gap between categorical algebra and catalg topology (the crucial point being our wish not to rely exclusively on category theory, but also to have the useful tools of universal algebra in hand, or, briefly speaking, to build topology from algebra with the help of category theory). Without further delay, we proceed to the development of the latter, postponing the comparison of the two settings till our subsequent papers.

The next lemma shows that the construction of Definition 3.1 is a quasicategory.

Lemma 3.2. *The composition law of Definition 3.1 is associative and the identities are correct.*

Proof. Consider three theory morphisms $T_1 \xrightarrow{\eta_1} T_2$, $T_2 \xrightarrow{\eta_2} T_3$ and $T_3 \xrightarrow{\eta_3} T_4$, or, more specifically, the following diagram:



Straightforward computations show that

$$T_4F_3F_2F_1 \xrightarrow{(\eta_3\odot\eta_2)\odot\eta_1} \Phi_3^{op}\Phi_2^{op}\Phi_1^{op}T_1 =$$

$$T_4F_3F_2F_1 \xrightarrow{(\eta_3\odot\eta_2)_{F_1}} \Phi_3^{op}\Phi_2^{op}T_2F_1 \xrightarrow{\Phi_3^{op}\Phi_2^{op}\eta_1} \Phi_3^{op}\Phi_2^{op}\Phi_1^{op}T_1 =$$

$$T_4F_3F_2F_1 \xrightarrow{\eta_{3\,F_2F_1}} \Phi_3^{op}T_3F_2F_1 \xrightarrow{\Phi_3^{op}\eta_{2\,F_1}} \Phi_3^{op}\Phi_2^{op}T_2F_1 \xrightarrow{\Phi_3^{op}\Phi_2^{op}\eta_1} \Phi_3^{op}\Phi_2^{op}\Phi_1^{op}T_1 =$$

$$T_4F_3F_2F_1 \xrightarrow{\eta_{3\,F_2F_1}} \Phi_3^{op}(T_3F_2F_1 \xrightarrow{\eta_{2\,F_1}} \Phi_2^{op}T_2F_1 \xrightarrow{\Phi_2^{op}\eta_1} \Phi_2^{op}\Phi_1^{op}T_1) =$$

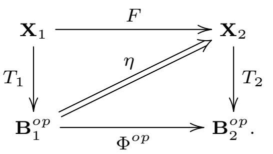
$$T_4F_3F_2F_1 \xrightarrow{\eta_{3\,F_2F_1}} \Phi_3^{op}T_3F_2F_1 \xrightarrow{\Phi_3^{op}(\eta_2\odot\eta_1)} \Phi_3^{op}\Phi_2^{op}\Phi_1^{op}T_1 =$$

$$T_4F_3F_2F_1 \xrightarrow{\eta_3\odot(\eta_2\odot\eta_1)} \Phi_3^{op}\Phi_2^{op}\Phi_1^{op}T_1,$$

the last statement of the lemma being clear. □

Notice that the composition law of the quasicategory **TpThr** resembles the *star product* of [32, Definition 13.10], but does not coincide with it. The next definition provides a partial dualization of the quasicategory of theories, which is reflected in its notation "*d*".

Definition 3.3. $\mathbf{TpThr}^d$ is the quasicategory, whose objects are cat-theories $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, and whose morphisms $T_1 \xrightarrow{(F,\Phi,\eta)} T_2$ (occasionally, shortened to η) comprise two functors $\mathbf{X}_1 \xrightarrow{F} \mathbf{X}_2$, $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ and a natural transformation $\Phi^{op}T_1 \xrightarrow{\eta} T_2F$, or, more specifically,



Given two theory morphisms $T_1 \xrightarrow{\eta_1} T_2$ and $T_2 \xrightarrow{\eta_2} T_3$, their composition is defined by $\Phi_2^{op}\Phi_1^{op}T_1 \xrightarrow{\eta_2\boxdot\eta_1} T_3F_2F_1 = \Phi_2^{op}\Phi_1^{op}T_1 \xrightarrow{\Phi_2^{op}\eta_1} \Phi_2^{op}T_2F_1 \xrightarrow{\eta_{2\,F_1}} T_3F_2F_1$. The identity on a theory T is provided by the identity natural transformation $T \xrightarrow{1_T} T$.

Similar to Lemma 3.2, the reader can verify that the definition of the quasicategory $\mathbf{TpThr}^d$ is correct w.r.t. the composition and the identities. The reason for introducing an additional setting for theories will become clear from the next two subsections of the paper. To end the current one, let us notice that the quasicategories **TpThr** and $\mathbf{TpThr}^d$ provide two ways of interaction between different topological settings, the need for which was indicated in the Introduction. To be more general, we introduced natural transformations in the setting of the theory morphisms, instead of restricting ourselves to the simple case of commutativity, which, in the current framework, is reflected through the identity natural transformation. The (temporary) lack of the examples to back the decision (pointed out by P. Waszkiewicz (Jagiellonian University, Poland) during the presentation of the results at "Applications of Algebra XV", Zakopane, Poland, March 7 - 13, 2011) is compensated by the high flexibility of the obtained setting, shown in the forthcoming developments of the paper. Being abstractly defined, however, at the moment, the morphisms of topological theories do not look convincing enough for their potential users. It is the purpose of the next subsection to show their fruitfulness in the setting of topological structures generated by the theories.

3.2. **From Theories to Structures.** To begin with, we introduce the quasicategory of models of catalg topological theories.

Definition 3.4. **TpSpc** is the quasicategory, whose objects are categories of the form $\mathbf{Top}(T)$ and whose morphisms are functors between them.

Continuing the comparison with the case of categorical algebra of F. W. Lawvere, started in the previous subsection, we recall from [3, Definition 9.11] the quasicategory ALG, whose objects are *algebraic categories* (categories, equivalent to $Alg\,\mathcal{T}$ for some algebraic theory $\mathcal{T}$) and whose morphisms are *algebraic functors* (functors, preserving limits and sifted colimits). Since the notion of *topological category* is already well-established in the literature [2, Definition 21.7], we do not define topological categories as those categories, which are equivalent to categories of the form $\mathbf{Top}(T)$. The simple reason being the fact that while all the categories $\mathbf{Top}(T)$ are topological [64], we do not know, at the moment, whether it is possible to provide the respective catalg topological theory T for an arbitrary topological category (see Problems 6.1, 6.4). Moreover, the current paper imposes no restrictions on the morphisms of the quasicategory **TpSpc**, though one of the possible conditions could be preservation of initial sources. We are still unable to formulate the precise requirements, applicability of the existing notion of *topological functor* [2, Definition 21.1] to our setting being yet the subject for verification.

One of the crucial moments of the categorical algebra in the sense of F. W. Lawvere is the quasifunctor $Th^{op} \xrightarrow{Alg} ALG$, which assigns to every algebraic theory $\mathcal{T}$ the category $Alg\,\mathcal{T}$ and to every functor $\mathcal{T}_1 \xrightarrow{M} \mathcal{T}_2$ the functor $Alg\,M = (-)M$ [3, Definition 9.13] (notice the important analogy with the above-mentioned preimage operator). The functor Alg has many interesting properties, considered in, e.g., [3, Chapter 9] and related to the so-called *canonical algebraic theories* [3, Definition 8.11]. An attentive reader will notice immediately that the construction of Alg is highly dependant on the definition of algebraic theory morphism, which is the main contribution of the PhD thesis of F. W. Lawvere. To provide a topological counterpart of the developments, the following theorem shows a possibility of going from topological theories to their generated categories of models, translating the respective theory morphisms into functors. A preliminary version of the result has already appeared in [24], the paper in question introducing an extension of the concept of variety-based topological system.

Theorem 3.5. *There exists the correspondence* $\mathbf{TpThr} \xrightarrow{\mathrm{Top}} \mathbf{TpSpc}$, $\mathrm{Top}(T_1 \xrightarrow{\eta} T_2) = \mathbf{Top}(T_1) \xrightarrow{\mathrm{Top}\,\eta} \mathbf{Top}(T_2)$, *defined on morphisms by* $\mathrm{Top}\,\eta((X,\tau) \xrightarrow{f} (Y,\sigma)) = (FX, (\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow}(\Phi\tau)) \xrightarrow{Ff} (FY, (\eta_Y^{op} \circ \Phi e_\sigma)^{\rightarrow}(\Phi\sigma))$, *which preserves identities.*

Proof. To show the correctness of $\mathrm{Top}\,\eta$, we have to verify its preservation of continuity. Given a $\mathbf{Top}(T_1)$-morphism $(X,\tau) \xrightarrow{f} (Y,\sigma)$, it follows that

$$\begin{aligned}
&(((T_2Ff)^{op})^{\rightarrow} \circ (\eta_Y^{op} \circ \Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&(((T_2Ff)^{op})^{\rightarrow} \circ (\eta_Y^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&(((\eta_Y \circ T_2Ff)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) \overset{(\dagger)}{=} \\
&(((\Phi^{op}T_1f \circ \eta_X)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op})^{\rightarrow} \circ ((\Phi^{op}T_1f)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op})^{\rightarrow} \circ (\Phi(T_1f)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op})^{\rightarrow} \circ (\Phi(T_1f)^{op} \circ \Phi e_\sigma)^{\rightarrow})(\Phi\sigma) \overset{(\dagger\dagger)}{=} \\
&((\eta_X^{op})^{\rightarrow} \circ (\Phi e_\tau \circ \Phi\widetilde{(T_1f)}^{op})^{\rightarrow})(\Phi\sigma) = \\
&((\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow} \circ (\Phi\widetilde{(T_1f)}^{op})^{\rightarrow})(\Phi\sigma) \overset{(\dagger\dagger\dagger)}{\subseteq} \\
&(\eta_X^{op} \circ \Phi e_\tau)^{\rightarrow}(\Phi\tau),
\end{aligned}$$

where (†) uses commutativity of the diagram (η is a natural transformation)

$$\begin{array}{ccc}
T_2FX & \xrightarrow{\eta_X} & \Phi^{op}T_1X \\
{\scriptstyle T_2Ff}\downarrow & & \downarrow{\scriptstyle \Phi^{op}T_1f} \\
T_2FY & \xrightarrow[\eta_Y]{} & \Phi^{op}T_1Y,
\end{array}$$

(††) relies on the application of the functor Φ to the commutative rectangle

$$\begin{array}{ccc}
\sigma & \xrightarrow{\widetilde{(T_1f)}^{op}=(T_1f)^{op}|_\sigma^\tau} & \tau \\
{\scriptstyle e_\sigma}\downarrow & & \downarrow{\scriptstyle e_\tau} \\
T_1Y & \xrightarrow[(T_1f)^{op}]{} & T_1X,
\end{array}$$

whereas († † †) employs the $\mathbf{B}_2$-homomorphism $\Phi\sigma \xrightarrow{\Phi\widetilde{(T_1f)}^{op}} \Phi\tau$. The proof of the last statement of the theorem is straightforward. □

Following the analogy with the categorical algebra, an immediate question arises on whether Top is actually a quasifunctor, i.e., does it preserve the composition. At the moment, we are able neither to prove nor to disprove the property. On the other hand, there exists a particular (and not unnatural) subquasicategory of **TpThr**, the restriction to which the correspondence Top gives rise to a quasifunctor.

Definition 3.6. $\mathbf{TpThr}_s$ is the subquasicategory of **TpThr**, which has the same objects, and whose morphisms $T_1 \xrightarrow{(F,\Phi,\eta)} T_2$ additionally require the functor $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ to preserve surjective homomorphisms (which is reflected in the index "s" in the quasicategory name).

The actual property required for the preservation of the composition is slightly different and runs as follows. Let $\mathbf{A} \xrightarrow{\Phi} \mathbf{B}$ be a functor between varieties. Given an $\mathbf{A}$-homomorphism $A_1 \xrightarrow{\varphi} A_2$, there exists the obvious factorization $A_1 \xrightarrow{\varphi} A_2 = A_1 \xrightarrow{\overline{\varphi}} \varphi^{\rightarrow}(A_1) \xhookrightarrow{e_{\varphi^{\rightarrow}(A)}} A_2$ (the reader should recall the algebraic notation

"$\overline{(-)}$" for restricted homomorphisms, mentioned at the end of Subsection 2.1), translated by the functor Φ into $\Phi A_1 \xrightarrow{\Phi\varphi} \Phi A_2 = \Phi A_1 \xrightarrow{\Phi\overline{\varphi}} \Phi\varphi^{\rightarrow}(A_1) \xrightarrow{\Phi e_{\varphi^{\rightarrow}(A)}} \Phi A_2$. The looked for condition is then

$$(\mathcal{J}) \qquad (\Phi e_{\varphi^{\rightarrow}(A_1)})^{\rightarrow}(\Phi\varphi^{\rightarrow}(A_1)) \subseteq (\Phi\varphi)^{\rightarrow}(\Phi A_1),$$

which holds trivially in the case of Φ being the identity functor. The next lemma shows that $(\mathcal{J})$ can be rewritten in the way Definition 3.6 does.

Lemma 3.7. *Let* $\mathbf{A} \xrightarrow{\Phi} \mathbf{B}$ *be a functor between varieties. Then* Φ *satisfies* $(\mathcal{J})$ *iff it preserves surjective homomorphisms.*

Proof. The necessity: Given a surjective $\mathbf{A}$-homomorphism $A_1 \xrightarrow{\varphi} A_2$, the above-mentioned factorization translates into $A_1 \xrightarrow{\varphi} A_2 = A_1 \xrightarrow{\overline{\varphi}} A_2 \xrightarrow{1_{A_2}} A_2$ and $\Phi A_1 \xrightarrow{\Phi\varphi} \Phi A_2 = \Phi A_1 \xrightarrow{\Phi\overline{\varphi}} \Phi A_2 \xrightarrow{\Phi 1_{A_2}} \Phi A_2$. Then $\Phi A_2 = (\Phi 1_{A_2})^{\rightarrow}(\Phi A_2) \overset{(\mathcal{J})}{\subseteq} (\Phi\varphi)^{\rightarrow}(\Phi A_1) \subseteq \Phi A_2$, which implies $(\Phi\varphi)^{\rightarrow}(\Phi A_1) = \Phi A_2$.

The sufficiency: In the factorization before the lemma, $\Phi\overline{\varphi}$ is surjective. If $b \in (\Phi e_{\varphi^{\rightarrow}(A_1)})^{\rightarrow}(\Phi\varphi^{\rightarrow}(A_1))$, then there exists $c \in \Phi\varphi^{\rightarrow}(A_1)$ such that $\Phi e_{\varphi^{\rightarrow}(A_1)}(c) = b$. Since $\Phi\overline{\varphi}$ is surjective, there exists $d \in \Phi A_1$ such that $\Phi\overline{\varphi}(d) = c$ and, therefore, $b = \Phi e_{\varphi^{\rightarrow}(A_1)} \circ \Phi\overline{\varphi}(d) = \Phi\varphi(d) \in (\Phi\varphi)^{\rightarrow}(\Phi A_1)$. □

All preliminaries in their places, we can present the promised result.

Theorem 3.8. *The restriction* $\mathbf{TpThr}_s \xrightarrow{\text{Top}} \mathbf{TpSpc}$ *preserves the composition, providing a quasifunctor.*

Proof. Given two topological theory morphisms $T_1 \xrightarrow{(F_1,\Phi_1,\eta_1)} T_2$ and $T_2 \xrightarrow{(F_2,\Phi_2,\eta_2)} T_3$, on the one hand,

$$\begin{aligned}
&\text{Top}((F_2,\Phi_2,\eta_2)\circ(F_1,\Phi_1,\eta_1))((X,\tau) \xrightarrow{f} (Y,\sigma)) = \\
&\text{Top}((F_2F_1,\Phi_2\Phi_1,\eta_2\odot\eta_1))((X,\eta) \xrightarrow{f} (Y,\sigma)) = \\
&(F_2F_1X,((\eta_2\odot\eta_1)^{op}_X\circ\Phi_2\Phi_1 e_\tau)^{\rightarrow}(\Phi_2\Phi_1\tau)) \xrightarrow{F_2F_1f} \\
&(F_2F_1Y,((\eta_2\odot\eta_1)^{op}_Y\circ\Phi_2\Phi_1 e_\sigma)^{\rightarrow}(\Phi_2\Phi_1\sigma)) \overset{(\dagger)}{=} \\
&(F_2F_1X,(\eta_2{}^{op}_{F_1X}\circ\Phi_2(\eta_1{}^{op}_X\circ\Phi_1 e_\tau))^{\rightarrow}(\Phi_2\Phi_1\tau)) \xrightarrow{F_2F_1f} \\
&(F_2F_1Y,(\eta_2{}^{op}_{F_1Y}\circ\Phi_2(\eta_1{}^{op}_Y\circ\Phi_1 e_\sigma))^{\rightarrow}(\Phi_2\Phi_1\sigma)) = \\
&(F_2F_1X,\tau_{21}) \xrightarrow{F_2F_1f} (F_2F_1Y,\sigma_{21}),
\end{aligned}$$

where $(\dagger)$ employs $(\eta_2\odot\eta_1)^{op}_X\circ\Phi_2\Phi_1 e_\tau = (\Phi_2^{op}\eta_1\circ\eta_{2F_1})^{op}_X\circ\Phi_2\Phi_1 e_\tau = \eta_2{}^{op}_{F_1X}\circ\Phi_2\eta_1{}^{op}_X\circ\Phi_2\Phi_1 e_\tau = \eta_2{}^{op}_{F_1X}\circ\Phi_2(\eta_1{}^{op}_X\circ\Phi_1 e_\tau)$. On the other hand, the reader could easily perform the following computations:

$$\begin{aligned}
&\text{Top}(F_2,\Phi_2,\eta_2)\circ\text{Top}(F_1,\Phi_1,\eta_1)((X,\tau) \xrightarrow{f} (Y,\sigma)) = \\
&\text{Top}(F_2,\Phi_2,\eta_2)((F_1X,(\eta_1{}^{op}_X\circ\Phi_1 e_\tau)^{\rightarrow}(\Phi_1\tau)) \xrightarrow{F_1f} (F_1Y,(\eta_1{}^{op}_Y\circ\Phi_1 e_\sigma)^{\rightarrow}(\Phi_1\sigma))) = \\
&\text{Top}(F_2,\Phi_2,\eta_2)((F_1X,\tau_1) \xrightarrow{F_1f} (F_1Y,\sigma_1)) = \\
&(F_2F_1X,(\eta_2{}^{op}_{F_1X}\circ\Phi_2 e_{\tau_1})^{\rightarrow}(\Phi_2\tau_1)) \xrightarrow{F_2F_1f} (F_2F_1Y,(\eta_2{}^{op}_{F_1Y}\circ\Phi_2 e_{\sigma_1})^{\rightarrow}(\Phi_2\sigma_1)) = \\
&(F_2F_1X,\tau_2) \xrightarrow{F_2F_1f} (F_2F_1Y,\sigma_2).
\end{aligned}$$

The attentive reader could now easily see that for the conclusion of the proof, it will be enough to show that topology τ_{21} equals topology τ_2.

$\tau_{21} \subseteq \tau_2$: There exists the factorization

$$\Phi_1\tau \xrightarrow{\eta_1{}^{op}_X \circ \Phi_1 e_\tau} T_2F_1X = \Phi_1\tau \xrightarrow{\overline{\eta_1{}^{op}_X \circ \Phi_1 e_\tau}} \tau_1 \overset{e_{\tau_1}}{\hookrightarrow} T_2F_1X,$$

which gives

$$\Phi_2\Phi_1\tau \xrightarrow{\Phi_2(\eta_1{}^{op}_X \circ \Phi_1 e_\tau)} \Phi_2T_2F_1X \xrightarrow{\eta_2{}^{op}_{F_1X}} T_3F_2F_1X =$$
$$\Phi_2\Phi_1\tau \xrightarrow{\Phi_2(\overline{\eta_1{}^{op}_X \circ \Phi_1 e_\tau})} \Phi_2\tau_1 \xrightarrow{\Phi_2 e_{\tau_1}} \Phi_2T_2F_1X \xrightarrow{\eta_2{}^{op}_{F_1X}} T_3F_2F_1X.$$

If $b \in \tau_{21}$, then there exists $c \in \Phi_2\Phi_1\tau$ such that $\eta_2{}^{op}_{F_1X} \circ \Phi_2(\eta_1{}^{op}_X \circ \Phi_1 e_\tau)(c) = b$ and, therefore, $d = \Phi_2(\overline{\eta_1{}^{op}_X \circ \Phi_1 e_\tau})(c) \in \Phi_2\tau_1$. It follows that $b = \eta_2{}^{op}_{F_1X} \circ \Phi_2(\eta_1{}^{op}_X \circ \Phi_1 e_\tau)(c) = \eta_2{}^{op}_{F_1X} \circ \Phi_2 e_{\tau_1} \circ \Phi_2(\overline{\eta_1{}^{op}_X \circ \Phi_1 e_\tau})(c) = \eta_2{}^{op}_{F_1X} \circ \Phi_2 e_{\tau_1}(d) \in \tau_2$.

$\tau_2 \subseteq \tau_{21}$: If $b \in \tau_2$, then there exists $c \in \Phi_2\tau_1$ with $b = \eta_2{}^{op}_{F_1X} \circ \Phi_2 e_{\tau_1}(c)$. By $(\mathcal{J})$, there exists $d \in \Phi_2\Phi_1\tau$ with $\Phi_2(\eta_1{}^{op}_X \circ \Phi_1 e_\tau)(d) = \Phi_2 e_{\tau_1}(c)$ and thus, $b = \eta_2{}^{op}_{F_1X} \circ \Phi_2 e_{\tau_1}(c) = \eta_2{}^{op}_{F_1X} \circ \Phi_2(\eta_1{}^{op}_X \circ \Phi_1 e_\tau)(d) \in \tau_{21}$. □

It is interesting to notice that the inclusion $\tau_{21} \subseteq \tau_2$ in Theorem 3.8 requires no additional condition, which opens the possibility to obtain a kind of *lax quasifunctor* [8, Definition 7.5.1] on the whole quasicategory **TpThr**. The rigid development of this opportunity (e.g., the required structure of 2-quasicategory on **TpThr**) is still under consideration.

Having done with the case of the category **TpThr**, we turn to its dualized version $\mathbf{TpThr}^d$. The respective analogue of the correspondence Top in the new setting is provided as follows.

Theorem 3.9. *There is a correspondence* $\mathbf{TpThr}^d \xrightarrow{\mathrm{Top}^d} \mathbf{TpSpc}$, $\mathrm{Top}^d(T_1 \xrightarrow{\eta} T_2) = \mathbf{Top}(T_1) \xrightarrow{\mathrm{Top}^d\,\eta} \mathbf{Top}(T_2)$, *given on morphisms by* $\mathrm{Top}^d\,\eta((X,\tau) \xrightarrow{f} (Y,\sigma)) = (FX, (\eta^{op}_X)^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau)) \xrightarrow{Ff} (FY, (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma))$, *and preserving identities.*

Proof. Similar to Theorem 3.5, we restrict ourselves to showing that $\mathrm{Top}^d\,\eta$ preserves continuity. Given a $\mathbf{Top}(T_1)$-morphism $(X,\tau) \xrightarrow{f} (Y,\sigma)$, it follows that

$$((T_2Ff)^{op})^{\rightarrow} \circ (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) \overset{(\dagger)}{\subseteq}$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\eta^{op}_X)^{\rightarrow} \circ ((T_2Ff)^{op})^{\rightarrow} \circ (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) =$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\eta^{op}_X \circ (T_2Ff)^{op})^{\rightarrow} \circ (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) =$$
$$(\eta^{op}_X)^{\leftarrow} \circ ((T_2Ff \circ \eta_X)^{op})^{\rightarrow} \circ (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) \overset{(\dagger\dagger)}{=}$$
$$(\eta^{op}_X)^{\leftarrow} \circ ((\eta_Y \circ \Phi^{op}T_1f)^{op})^{\rightarrow} \circ (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) =$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\Phi(T_1f)^{op})^{\rightarrow} \circ (\eta^{op}_Y)^{\rightarrow} \circ (\eta^{op}_Y)^{\leftarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) \overset{(\dagger)}{\subseteq}$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\Phi(T_1f)^{op})^{\rightarrow} \circ (\Phi e_\sigma)^{\rightarrow}(\Phi\sigma) =$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\Phi(T_1f)^{op} \circ \Phi e_\sigma)^{\rightarrow}(\Phi\sigma) \overset{(\dagger\dagger\dagger)}{=}$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\Phi e_\tau \circ \Phi(\widetilde{T_1f})^{op})^{\rightarrow}(\Phi\sigma) =$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow} \circ (\Phi(\widetilde{T_1f})^{op})^{\rightarrow}(\Phi\sigma) \overset{(\dagger\dagger\dagger\dagger)}{\subseteq}$$
$$(\eta^{op}_X)^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau),$$

where (†) use the standard property of (pre)image operators w.r.t. to the maps $T_2FX \xrightarrow{\eta_X^{op}} \Phi T_1 X$ and $T_2FY \xrightarrow{\eta_Y^{op}} \Phi T_1 Y$, (††) relies on commutativity of the diagram (η is a natural transformation)

$$\begin{array}{ccc} \Phi^{op}T_1X & \xrightarrow{\eta_X} & T_2FX \\ {\scriptstyle \Phi^{op}T_1 f}\downarrow & & \downarrow{\scriptstyle T_2Ff} \\ \Phi^{op}T_1Y & \xrightarrow[\eta_Y]{} & T_2FY, \end{array}$$

(† † †) exploits the application of the functor Φ to the commutative diagram

$$\begin{array}{ccc} \sigma & \xrightarrow{(\widetilde{T_1 f})^{op}=(T_1 f)^{op}|_{\sigma}^{\tau}} & \tau \\ {\scriptstyle e_\sigma}\downarrow & & \downarrow{\scriptstyle e_\tau} \\ T_1Y & \xrightarrow[(T_1 f)^{op}]{} & T_1X, \end{array}$$

whereas († † ††) takes into account the $\mathbf{B}_2$-homomorphism $\Phi\sigma \xrightarrow{\Phi(\widetilde{T_1 f})^{op}} \Phi\tau$. □

Similar to the case of the quasicategory **TpThr**, we are unable to answer the question on whether the correspondence Top^d is a quasifunctor. However, the restriction of its domain results again in the required preservation of the composition.

Definition 3.10. $\mathbf{TpThr}^d_{pi}$ is the subquasicategory of $\mathbf{TpThr}^d$, which has the same objects, and whose morphisms $T_1 \xrightarrow{(F,\Phi,\eta)} T_2$ additionally require the functor $\mathbf{B}_1 \xrightarrow{\Phi} \mathbf{B}_2$ to preserve pullbacks and injective homomorphisms (which is reflected in the index "*pi*" in the quasicategory name).

The next result is an analogue of Theorem 3.8 in the setting of $\mathbf{TpThr}^d$.

Theorem 3.11. *The restriction* $\mathbf{TpThr}^d_{pi} \xrightarrow{\mathrm{Top}^d} \mathbf{TpSpc}$ *preserves the composition, providing a quasifunctor.*

Proof. We start with one preliminary remark. Suppose we are given a $\mathbf{TpThr}^d$-morphism $T_1 \xrightarrow{(F,\Phi,\eta)} T_2$ and a $\mathbf{Top}(T_1)$-morphism $(X,\tau) \xrightarrow{f} (Y,\sigma)$. Since every variety is closed in its respective category $\mathbf{Alg}(\Omega)$ under the formation of subalgebras and products (cf. Definition 2.2), there exists the pullback

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\pi_1} & T_2FX \\ {\scriptstyle \pi_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \eta_X^{op}} \\ \Phi\tau & \xrightarrow[\Phi e_\tau]{} & \Phi T_1 X, \end{array}$$

where $\mathfrak{B} = \{(b_1, b_2) \in T_2FX \times \Phi\tau \mid \eta_X^{op}(b_1) = \Phi e_\tau(b_2)\}$ and $\pi_i(b_1,b_2) = b_i$. Moreover, $\pi_1^{\rightarrow}(\mathfrak{B}) = (\eta_X^{op})^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau)$ (recall the formula for the correspondence Top^d from Theorem 3.9), since $b_1 \in (\eta_X^{op})^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau)$ iff there exists $b_2 \in \Phi\tau$

such that $\eta_X^{op}(b_1) = \Phi e_\tau(b_2)$ iff there exists $b_2 \in \Phi\tau$ such that $(b_1, b_2) \in \mathfrak{B}$ iff $b_1 \in \pi_1^{\rightarrow}(\mathfrak{B})$. That remark having been made, suppose $T_1 \xrightarrow{(F_1,\Phi_1\eta_1)} T_2$ and $T_2 \xrightarrow{(F_2,\Phi_2\eta_2)} T_3$ are two theory morphisms. On the one hand,

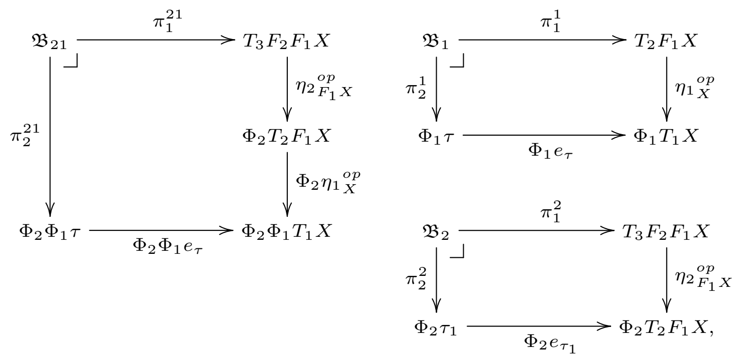
$$\begin{aligned}
&\mathrm{Top}^d((F_2,\Phi_2,\eta_2)\circ(F_1,\Phi_1,\eta_1))((X,\tau)\xrightarrow{f}(Y,\sigma)) = \\
&\mathrm{Top}^d((F_2F_1,\Phi_2\Phi_1,\eta_2\boxdot\eta_1))((X,\tau)\xrightarrow{f}(Y,\sigma)) = \\
&(F_2F_1X,((\eta_2\boxdot\eta_1)_X^{op})^{\leftarrow}\circ(\Phi_2\Phi_1e_\tau)^{\rightarrow}(\Phi_2\Phi_1\tau)) \xrightarrow{F_2F_1f} \\
&(F_2F_1Y,((\eta_2\boxdot\eta_1)_Y^{op})^{\leftarrow}\circ(\Phi_2\Phi_1e_\sigma)^{\rightarrow}(\Phi_2\Phi_1\sigma)) \overset{(\dagger)}{=} \\
&(F_2F_1X,(\eta_2{}_{F_1X}^{op})^{\leftarrow}\circ(\Phi_2\eta_1{}_X^{op})^{\leftarrow}\circ(\Phi_2\Phi_1e_\tau)^{\rightarrow}(\Phi_2\Phi_1\tau)) \xrightarrow{F_2F_1f} \\
&(F_2F_1Y,(\eta_2{}_{F_1Y}^{op})^{\leftarrow}\circ(\Phi_2\eta_1{}_Y^{op})^{\leftarrow}\circ(\Phi_2\Phi_1e_\sigma)^{\rightarrow}(\Phi_2\Phi_1\sigma)) = \\
&(F_2F_1X,\tau_{21}) \xrightarrow{F_2F_1f} (F_2F_1Y,\sigma_{21}),
\end{aligned}$$

where $(\dagger)$ employs $((\eta_2\boxdot\eta_1)_X^{op})^{\leftarrow} = ((\eta_2{}_{F_1X}\circ\Phi_2^{op}\eta_1{}_X)^{op})^{\leftarrow} = (\Phi_2\eta_1{}_X^{op}\circ\eta_2{}_{F_1X}^{op})^{\leftarrow} = (\eta_2{}_{F_1X}^{op})^{\leftarrow}\circ(\Phi_2\eta_1{}_X^{op})^{\leftarrow}$. On the other hand,

$$\begin{aligned}
&\mathrm{Top}^d(F_2,\Phi_2,\eta_2)\circ\mathrm{Top}^d(F_1,\Phi_1,\eta_1)((X,\tau)\xrightarrow{f}(Y,\sigma)) = \\
&\mathrm{Top}^d(F_2,\Phi_2,\eta_2)((F_1X,(\eta_1{}_X^{op})^{\leftarrow}\circ(\Phi_1e_\tau)^{\rightarrow}(\Phi_1\tau)) \xrightarrow{F_1f} \\
&(F_1Y,(\eta_1{}_Y^{op})^{\leftarrow}\circ(\Phi_1e_\sigma)^{\rightarrow}(\Phi_1\sigma))) = \\
&\mathrm{Top}^d(F_2,\Phi_2,\eta_2)((F_1X,\tau_1)\xrightarrow{F_1f}(F_1Y,\sigma_1)) = \\
&(F_2F_1X,(\eta_2{}_{F_1X}^{op})^{\leftarrow}\circ(\Phi_2e_{\tau_1})^{\rightarrow}(\Phi_2\tau_1)) \xrightarrow{F_2F_1f} \\
&(F_2F_1Y,(\eta_2{}_{F_1Y}^{op})^{\leftarrow}\circ(\Phi_2e_{\sigma_1})^{\rightarrow}(\Phi_2\sigma_1)) = \\
&(F_2F_1X,\tau_2) \xrightarrow{F_2F_1f} (F_2F_1Y,\sigma_2).
\end{aligned}$$

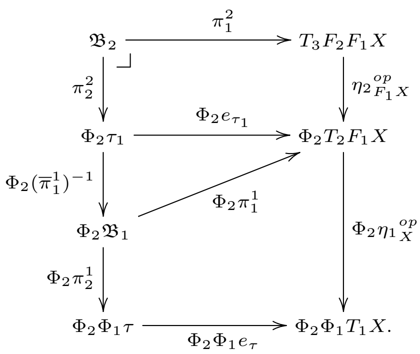
It will be enough to show that $\tau_{21} = \tau_2$.

$\tau_2 \subseteq \tau_{21}$: By the remark at the beginning of the proof, one has the pullbacks



where $\tau_{21} = (\pi_1^{21})^{\rightarrow}(\mathfrak{B}_{21})$, $\tau_1 = (\pi_1^1)^{\rightarrow}(\mathfrak{B}_1)$ and $\tau_2 = (\pi_1^2)^{\rightarrow}(\mathfrak{B}_2)$, respectively. Since Φ_1 preserves injective homomorphisms (by the definition of the category $\mathbf{TpThr}^d_{pi}$), $\Phi_1 e_\tau$ in the middle pullback is injective. Straightforward calculations show that π_1^1 is then injective as well ($\pi_1^1(b_1, b_2) = \pi_1^1(b_1', b_2')$ implies $b_1 = b_1'$ implies $\Phi_1 e_\tau(b_2) = {\eta_1}^{op}_X(b_1) = {\eta_1}^{op}_X(b_1') = \Phi_1 e_\tau(b_2')$ implies $b_2 = b_2'$) and thus, $\mathfrak{B} \xrightarrow{\overline{\pi}_1^1} \tau_1$ is an isomorphism. As a result, the standard factorization $\mathfrak{B} \xrightarrow{\pi_1^1} T_2F_1X = \mathfrak{B} \xrightarrow{\overline{\pi}_1^1} \tau_1 \xhookrightarrow{e_{\tau_1}} T_2F_1X$ can be rewritten as $\tau_1 \xhookrightarrow{e_{\tau_1}} T_2F_1X = \tau_1 \xrightarrow{(\overline{\pi}_1^1)^{-1}} \mathfrak{B} \xrightarrow{\pi_1^1} T_2F_1X$. This factorization and the last two of the above rectangles provide the following commutative diagram:

(1)



By the pullback property of the first rectangles above, there exists a factorization $\mathfrak{B}_2 \xrightarrow{\pi_1^2} T_3F_2F_1X = \mathfrak{B}_2 \xrightarrow{\varphi} \mathfrak{B}_{21} \xrightarrow{\pi_1^{21}} T_3F_2F_1X$ and, therefore, $\tau_2 = (\pi_1^2)^{\rightarrow}(\mathfrak{B}_2) = (\pi_1^{21} \circ \varphi)^{\rightarrow}(\mathfrak{B}_2) \subseteq (\pi_1^{21})^{\rightarrow}(\mathfrak{B}_{21}) = \tau_{21}$.

$\tau_{21} \subseteq \tau_2$: Since Φ_1 preserves injective homomorphisms and Φ_2 preserves pullbacks (by the definition of $\mathbf{TpThr}^d_{pi}$), Diagram (1) above is a pullback. Commutativity of the first pullback diagram of the above-mentioned three, provides then a factorization $\mathfrak{B}_{21} \xrightarrow{\pi_1^{21}} T_3F_2F_1X = \mathfrak{B}_{21} \xrightarrow{\psi} \mathfrak{B}_2 \xrightarrow{\pi_1^2} T_3F_2F_1X$ and, therefore, $\tau_{21} = (\pi_1^{21})^{\rightarrow}(\mathfrak{B}_{21}) = (\pi_1^2 \circ \psi)^{\rightarrow}(\mathfrak{B}_{21}) \subseteq (\pi_1^2)^{\rightarrow}(\mathfrak{B}_2) = \tau_2$. □

We believe it is important to remark that unlike the case of Theorem 3.8, both inclusions $\tau_{21} \subseteq \tau_2$ and $\tau_2 \subseteq \tau_{21}$ in Theorem 3.11 require some additional conditions. An attentive reader will also quite easily notice that the cited theorems take our topological framework somewhat closer to the algebraic developments of F. W. Lawvere. Recall from Subsection 3.1 that algebraic theory morphisms are not arbitrary functors, but necessarily the (finite) product-preserving ones. Our initial setting imposed no conditions on the topological theory morphisms. The obtained results, however, clearly show that the morphisms in question should respect the internal structure of topological theories, Theorems 3.8, 3.11 providing the first attempt to describe those preservation properties. In due time, we hope to clarify the requirements completely, which, unlike the algebraic case, do not seem straightforward (to the present author, at least) to define. Some of the subsequent results of this paper will contribute to the desired solution as well.

The above-mentioned theorems also make clear the promised (in the introductory section) more concrete interaction means between different topological settings, which can be directly used in (however, theoretic) applications. It will be the topic of our subsequent manuscripts to consider concrete applications of the new theory in (probably, not only lattice-valued) topology. Our current goal, however, is rather more theoretic. Having introduced a possible way of shifting from topological theories to the categories of their induced topological structures, we would like to turn the attention of the reader to the respective categories of topological systems. Surprisingly enough, it appears that there exists a nicer (which means, simpler) analogue of the developed machinery in the case of topological systems. The following subsection provides an outlook on the obtained results.

3.3. **From Theories to Systems.** Similar to the case of topological structures, we begin with the definition of the quasicategory to incorporate the categories of topological systems, generated by catalg topological theories.

Definition 3.12. **TpStm** is the quasicategory, whose objects are categories of the form $\mathbf{TopSys}(T)$ and whose morphisms are functors between them.

In view of the prolonged discussion after Definition 3.4, we would like only to draw the attention of the reader to the important fact that there are no additional conditions on the morphisms of the category **TpStm**, which, taking into account the remarks, which were made at the end of Subsection 2.3, is a coalgebraic analogue of the category ALG of algebraic categories. The following theorem opens up a convenient way of translating topological theory morphisms into functors between their respective categories of systems.

Theorem 3.13. *There exists a quasifunctor* $\mathbf{TpThr} \xrightarrow{\mathrm{TopSys}} \mathbf{TpStm}$,

$$\mathrm{TopSys}(T_1 \xrightarrow{\eta} T_2) = \mathbf{TopSys}(T_1) \xrightarrow{\mathrm{TopSys}\,\eta} \mathbf{TopSys}(T_2),$$

defined on morphisms by the formula $\mathrm{TopSys}\,\eta((X,\kappa,B) \xrightarrow{(f,\varphi)} (X',\kappa',B')) = (FX, \Phi^{op}\kappa \circ \eta_X, \Phi^{op}B) \xrightarrow{(Ff,\Phi^{op}\varphi)} (FX', \Phi^{op}\kappa' \circ \eta_{X'}, \Phi^{op}B')$.

Proof. We begin with the verification that $\mathrm{TopSys}\,\eta$ preserves continuity. Given a $\mathbf{TopSys}(T_1)$-morphism $(X,\kappa,B) \xrightarrow{(f,\varphi)} (X',\kappa',B')$, the diagram

$$\begin{array}{ccc} T_1X & \xrightarrow{T_1f} & T_1X' \\ \kappa\downarrow & & \downarrow\kappa' \\ B & \xrightarrow[\varphi]{} & B' \end{array}$$

commutes and that provides the commutativity of the next one:

$$\begin{array}{ccc} T_2FX & \xrightarrow{T_2Ff} & T_2FX' \\ \eta_X\downarrow & & \downarrow\eta_{X'} \\ \Phi^{op}T_1X & \xrightarrow{\Phi^{op}T_1f} & \Phi^{op}T_1X' \\ \Phi^{op}\kappa\downarrow & & \downarrow\Phi^{op}\kappa' \\ \Phi^{op}B & \xrightarrow[\Phi^{op}\varphi]{} & \Phi^{op}B'. \end{array}$$

It follows that $(FX, \Phi^{op}\kappa \circ \eta_X, \Phi^{op}B) \xrightarrow{(Ff,\Phi^{op}\varphi)} (FX', \Phi^{op}\kappa' \circ \eta_{X'}, \Phi^{op}B')$ is a $\mathbf{TopSys}(T_2)$-morphism.

The challenging issue of the preservation of the composition by TopSys can be dealt with in the new framework in a very simple way. Given two theory morphisms $T_1 \xrightarrow{(F_1,\Phi_1,\eta_1)} T_2$ and $T_2 \xrightarrow{(F_2,\Phi_2,\eta_2)} T_3$,

$$\begin{aligned}
&\mathrm{TopSys}((F_2,\Phi_2,\eta_2)\circ(F_1,\Phi_1,\eta_1))((X,\kappa,B) \xrightarrow{(f,\varphi)} (X',\kappa',B')) = \\
&\mathrm{TopSys}(F_2F_1,\Phi_2\Phi_1,\eta_2\odot\eta_1)((X,\kappa,B) \xrightarrow{(f,\varphi)} (X',\kappa',B'))) = \\
&(F_2F_1X, \Phi_2^{op}\Phi_1^{op}\kappa\circ(\eta_2\odot\eta_1)_X, \Phi_2^{op}\Phi_1^{op}B,) \xrightarrow{(F_2F_1f,\Phi_2^{op}\Phi_1^{op}\varphi)} \\
&\qquad (F_2F_1X', \Phi_2^{op}\Phi_1^{op}\kappa'\circ(\eta_2\odot\eta_1)_{X'}, \Phi_2^{op}\Phi_1^{op}B') \overset{(\dagger)}{=} \\
&(F_2F_1X, \Phi_2^{op}(\Phi_1^{op}\kappa\circ\eta_{1X})\circ\eta_{2F_1X}, \Phi_2^{op}\Phi_1^{op}B) \xrightarrow{(F_2F_1f,\Phi_2^{op}\Phi_1^{op}\varphi)} \\
&\qquad (F_2F_1X', \Phi_2^{op}(\Phi_1^{op}\kappa'\circ\eta_{1X'})\circ\eta_{2F_1X'}, \Phi_2^{op}\Phi_1^{op}B') = \\
&\mathrm{TopSys}(F_2,\Phi_2,\eta_2)((F_1X, \Phi_1^{op}\kappa\circ\eta_{1X}, \Phi_1^{op}B) \xrightarrow{(F_1f,\Phi_1\varphi)} \\
&\qquad (F_1X', \Phi_1^{op}\kappa'\circ\eta_{1X'}, \Phi_1^{op}B')) = \\
&\mathrm{TopSys}(F_2,\Phi_2,\eta_2)\circ\mathrm{TopSys}(F_1,\Phi_2,\eta_1)((X,\kappa,B) \xrightarrow{(f,\varphi)} (X',\kappa',B')),
\end{aligned}$$

where (†) relies on the fact that $\Phi_2^{op}\Phi_1^{op}\kappa\circ(\eta_2\odot\eta_1)_X = \Phi_2^{op}\Phi_1^{op}\kappa\circ\Phi_2^{op}\eta_{1X}\circ\eta_{2F_1X} = \Phi_2^{op}(\Phi_1^{op}\kappa\circ\eta_{1X})\circ\eta_{2F_1X}$.

Preservation of the identities is straightforward. □

The reader should notice the advantage over the case of topological structures that the domain of the new quasifunctor is the whole quasicategory **TpThr**, i.e., we do not need any restriction to a specific subquasicategory of **TpThr**. A significant disadvantage, however, is the fact that, at the moment, we are unable to provide an analogue of the procedure for the case of the category $\mathbf{TpThr}^d$.

We have already noticed that the category **TpStm** is much closer to the category *ALG* than **TpSpc** is. It would be a challenging task to compare the properties of the former two categories, providing a kind of functorial semantics of coalgebraic theories. Since the topic is rather off the topological goal of the paper and, moreover, requires much effort, we will not pursue it further.

4. Some Properties of the Quasifunctors Top, Top^d and TopSys

Having provided a way from theories to both topological structures and systems through the quasifunctors Top, Top^d and TopSys, in the next step, we would like to consider some of the properties of the latter. The following two subsections provide an account of the obtained results.

4.1. **An Approach to Morita Problem.** The famous Morita problem in categorical algebra has already been mentioned in the Introduction. The setting of this manuscript provides a way for the straightforward translation of the issue into the realm of topology. Following the pattern of [9, Section 3.12], the extended problem can be stated as follows (number "1" in its title will become clear in the second part of this subsection).

Problem 4.1 (Morita problem 1)**.** Find conditions on two topological theories T_1, T_2, so that their corresponding categories $\mathbf{Top}(T_1)$ and $\mathbf{Top}(T_2)$ are equivalent.

Following [3, Definition 15.2], we call two topological theories T_1, T_2 *Morita equivalent* provided that their corresponding categories $\mathbf{Top}(T_1)$ and $\mathbf{Top}(T_2)$ are equivalent. At the moment, we are unable to provide a general solution to Problem 4.1. However, as the first approach to the question, we are able to deal with a much simpler task, namely, to provide the necessary and sufficient conditions on two theories, so that their respective categories of topological structures are equal. The next theorem does the job (one should recall the notations of Definition 2.9).

Theorem 4.2. *Given topological theories T_1 and T_2, $\mathbf{Top}(T_1) = \mathbf{Top}(T_2)$ iff the following conditions hold:*

(1) $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}$*;*
(2) $T_1X = T_2X = TX$ *for every* $\mathbf{X}$*-object* X*;*
(3) $\langle (T_1f)^{op}(\alpha)\rangle = \langle (T_2f)^{op}(\alpha)\rangle$ *for any* $\mathbf{X}$*-morphism* $X \xrightarrow{f} Y$ *and any* $\alpha \in TY$*.*

Proof. The necessity: To show Item (1), notice first that if $X \xrightarrow{f} Y$ is an $\mathbf{X}_1$-morphism, then $(X, T_1X) \xrightarrow{f} (Y, T_1Y)$ is a $\mathbf{Top}(T_1)$-morphism and thus, a $\mathbf{Top}(T_2)$-morphism as well, which implies $X \xrightarrow{f} Y$ is an $\mathbf{X}_2$-morphism. In other words, $\mathbf{X}_1$ is a subcategory of $\mathbf{X}_2$ and similarly the converse. To show Item (2), notice that given an $\mathbf{X}$-object X, (X, T_1X) is a T_1-space and thus, a T_2-space as well, which implies that T_1X is a subalgebra of T_2X. The converse statement is similar, and then, T_1X and T_2X are equal as algebras. Item (3) is a bit more demanding. Given an $\mathbf{X}$-morphism $X \xrightarrow{f} Y$ and $\alpha \in T_1Y$, by Lemma 2.12, $(X, \langle (T_1f)^{op}(\alpha)\rangle) \xrightarrow{f} (Y, \langle\alpha\rangle)$ is a $\mathbf{Top}(T_1)$-morphism, i.e., a $\mathbf{Top}(T_2)$-morphism. Then $(T_2f)^{op}(\alpha) \in \langle (T_1f)^{op}(\alpha)\rangle$, implying $\langle (T_2f)^{op}(\alpha)\rangle \subseteq \langle (T_1f)^{op}(\alpha)\rangle$. The converse inclusion is similar.

The sufficiency: Given a $\mathbf{Top}(T_1)$-morphism $(X, \tau) \xrightarrow{f} (Y, \sigma)$, both (X, τ) and (Y, σ) are T_2-spaces by Items (1), (2) and the closure of varieties under subalgebras. To show that f is T_2-continuous, fix $\alpha \in \sigma$. Since $(T_1f)^{op}(\alpha) \in \tau$, $(T_2f)^{op}(\alpha) \in \langle (T_2f)^{op}(\alpha)\rangle \overset{\text{Item (3)}}{=} \langle (T_1f)^{op}(\alpha)\rangle \subseteq \tau$. As a result, $(X, \tau) \xrightarrow{f} (Y, \sigma)$ is a $\mathbf{Top}(T_2)$-morphism, i.e., $\mathbf{Top}(T_1)$ is a subcategory of $\mathbf{Top}(T_2)$ and similarly the converse. □

It will be the topic of our further research to replace the equality of the categories $\mathbf{Top}(T_1)$ and $\mathbf{Top}(T_2)$ in Theorem 4.2 by equivalence, adjusting the respective requirements on the theories accordingly. On the other hand, the setting of the manuscript provides another line of research, namely, the case of topological systems. Following our discussion at the end of Subsection 2.3, the latter structures provide an analogue of algebras (better, coalgebras) induced by topological theories, showing to which extent topology is (co)algebraic. The respective Morita problem is easy to state, which, together with its first analogue for topological structures (Problem 4.1), constitutes the general Morita problem of catalg topology.

Problem 4.3 (Morita problem 2)**.** Find conditions on two topological theories T_1, T_2, so that their respective categories $\mathbf{TopSys}(T_1)$, $\mathbf{TopSys}(T_2)$ are equivalent.

Similar to the just considered case of the categories of the form $\mathbf{Top}(T)$, we are unable (at the moment) to provide a general answer. On the other hand, the shift from equivalence to equality has an easy solution.

Theorem 4.4. *For topological theories T_1, T_2, $\mathbf{TopSys}(T_1) = \mathbf{TopSys}(T_2)$ iff the following conditions hold:*

(1) $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}$*;*

(2) $T_1 f = T_2 f = Tf$ *for every* $\mathbf{X}$*-morphism* $X \xrightarrow{f} Y$*;*

(3) $B \xrightarrow{\varphi} TX$ *is a* $\mathbf{B}_1$*-morphism iff* $B \xrightarrow{\varphi} TX$ *is a* $\mathbf{B}_2$*-morphism, for every* $\mathbf{X}$*-object* X*.*

Proof. The necessity: To show Item (1), notice first that given an $\mathbf{X}_1$-morphism $X \xrightarrow{f} Y$, it follows that $(X, 1_{T_1 X}, T_1 X) \xrightarrow{(f, T_1 f)} (Y, 1_{T_1 Y}, T_1 Y)$ is a $\mathbf{TopSys}(T_1)$-morphism and thus, a $\mathbf{TopSys}(T_2)$-morphism as well, which implies that $X \xrightarrow{f} Y$ is an $\mathbf{X}_2$-morphism. In other words, $\mathbf{X}_1$ is a subcategory of $\mathbf{X}_2$ and similarly the converse. To show Item (2), notice that given an $\mathbf{X}$-morphism $X \xrightarrow{f} Y$, by the proof of Item (1), $(X, 1_{T_1 X}, T_1 X) \xrightarrow{(f, T_1 f)} (Y, 1_{T_1 Y}, T_1 Y)$ is a $\mathbf{TopSys}(T_2)$-morphism, providing the commutative diagram:

$$\begin{array}{ccc} T_2 X & \xrightarrow{T_2 f} & T_2 Y \\ {\scriptstyle 1_{T_1 X}} \downarrow & & \downarrow {\scriptstyle 1_{T_1 Y}} \\ T_1 X & \xrightarrow[T_1 f]{} & T_1 Y. \end{array}$$

It follows that $T_1 f = T_2 f$ and, in particular, $T_1 X = T_2 X$, $T_1 Y = T_2 Y$. To show Item (3), fix an $\mathbf{X}$-object X. Then, $B \xrightarrow{\varphi} TX$ is a $\mathbf{B}_1$-morphism iff (X, φ^{op}, B) is a $\mathbf{TopSys}(T_1)$-object iff (X, φ^{op}, B) is a $\mathbf{TopSys}(T_2)$-object iff $B \xrightarrow{\varphi} TX$ is a $\mathbf{B}_2$-morphism.

The sufficiency: Given a $\mathbf{TopSys}(T_1)$-morphism $(X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)$, Items (1)-(3) imply that both (X_1, κ_1, B_1) and (X_2, κ_2, B_2) are $\mathbf{TopSys}(T_2)$-objects and, moreover, by Item (2), the next diagram commutes

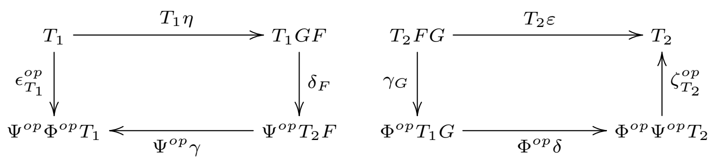
$$\begin{array}{ccc} TX & \xrightarrow{T_2 f} & TY \\ {\scriptstyle \kappa_1} \downarrow & & \downarrow {\scriptstyle \kappa_2} \\ B_1 & \xrightarrow[\varphi]{} & B_2. \end{array}$$

As a result, $(X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)$ is a $\mathbf{TopSys}(T_2)$-morphism. □

Comparing the requirement items of Theorems 4.2 and 4.4, one sees that the first of them is shared, whereas the second one is strictly stronger in the setting of systems. Moreover, it is easy to see that the first two items of Theorem 4.4 imply all three items of Theorem 4.2. More precise relationships between the conditions in question are yet to be clarified, and that will be the topic of our further research. However, the next subsections will help us (and the interested reader) to approach the solution of the two general Morita problems introduced in this subsection.

4.2. **Lifting Adjunctions from Theories to Structures and Systems.** This and the subsequent subsections deal with the somewhat truncated Morita problems, namely, with providing "reasonable" sufficient conditions on two topological theories, so that their generated categories of structures (or systems) are equivalent. More precisely, we are interested in the possibility of lifting "equivalences" between topological theories (defined in a quite obvious way) to equivalences between their induced categories of the form $\mathbf{Top}(T)$ or $\mathbf{TopSys}(T)$. To explore the matter in a general way, we begin by lifting adjoint situations instead of equivalences. The initial setting is rather simple. Suppose we are given two topological theories $T_1 \xrightarrow{(F,\Phi,\gamma)} T_2$ and $T_2 \xrightarrow{(G,\Psi,\delta)} T_1$ such that there exist two adjoint situations: $(\eta, \varepsilon) : F \dashv G : \mathbf{X}_2 \to \mathbf{X}_1$, $(\zeta, \epsilon) : \Psi \dashv \Phi : \mathbf{B}_2 \to \mathbf{B}_1$. In view of Theorem 3.8, we get two functors, for the sake of shortness, having more compact notations: $T_1 \xrightarrow{(F,\Phi,\gamma)} T_2 \mapsto \mathbf{Top}(T_1) \xrightarrow{H} \mathbf{Top}(T_2)$ and $T_2 \xrightarrow{(G,\Psi,\delta)} T_1 \mapsto \mathbf{Top}(T_2) \xrightarrow{K} \mathbf{Top}(T_1)$. The general question is whether the functors H and K make an adjoint situation. At the moment, we are able to provide a sufficient condition only, which runs as follows.

$(\mathcal{A})$ The following diagrams

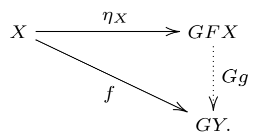


commute.

The next theorem presents the respective result obtained so far.

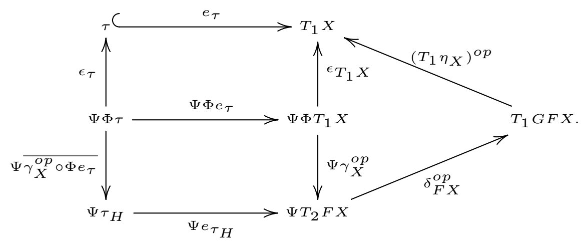
Theorem 4.5. *In the framework of the category* $\mathbf{TpThr}_s$*, if* $(\mathcal{A})$ *holds, then there exists the adjoint situation* $(\eta, \varepsilon) : H \dashv K : \mathbf{Top}(T_2) \to \mathbf{Top}(T_1)$.

Proof. Given a T_1-space (X, τ), the $\mathbf{X}_1$-morphism $X \xrightarrow{\eta_X} GFX$ is a G-universal arrow for X, i.e., for every $\mathbf{X}_1$-morphism $X \xrightarrow{f} GY$ there exists a unique $\mathbf{X}_2$-morphism $FX \xrightarrow{g} Y$, which makes the next triangle commute



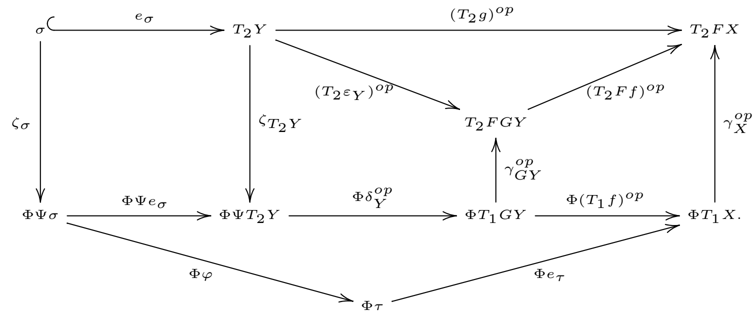
We show that $(X, \tau) \xrightarrow{\eta_X} (KH(X, \tau) = K(FX, (\gamma_X^{op} \circ \Phi e_\tau)^{\to}(\Phi\tau)) = K(FX, \tau_H) = (GFX, (\delta_{FX}^{op} \circ \Psi e_{\tau_H})^{\to}(\Psi\tau_H)) = (GFX, \tau_{KH}))$ is a K-universal arrow for (X, τ).

To begin with, we verify T_1-continuity of η_X. Notice that $((T_1\eta_X)^{op})^{\to}(\tau_{KH}) = ((T_1\eta_X)^{op})^{\to} \circ (\delta_{FX}^{op} \circ \Psi e_{\tau_H})^{\to}(\Psi\tau_H) = ((T_1\eta_X)^{op} \circ \delta_{FX}^{op} \circ \Psi e_{\tau_H})^{\to}(\Psi\tau_H)$. Moreover, the left-hand rectangle of $(\mathcal{A})$ gives the following commutative diagram:



If $b \in ((T_1\eta_X)^{op})^{\rightarrow}(\tau_{KH})$, then there exists $c \in \Psi\tau_H$ such that $b = (T_1\eta_X)^{op} \circ \delta^{op}_{FX} \circ \Psi e_{\tau_H}(c)$. Since Ψ preserves surjective homomorphisms (recall Definition 3.6), there exists $d \in \Psi\Phi\tau$ such that $\Psi\overline{\gamma^{op}_X \circ \Phi e_\tau}(d) = c$. It follows that $\epsilon_\tau(d) = a \in \tau$ and, moreover, $a = e_\tau(a) = e_\tau \circ \epsilon_\tau(d) = \epsilon_{T_1X} \circ \Psi\Phi e_\tau(d) = (T_1\eta_X)^{op} \circ \delta^{op}_{FX} \circ \Psi\gamma^{op}_X \circ \Psi\Phi e_\tau(d) = (T_1\eta_X)^{op} \circ \delta^{op}_{FX} \circ \Psi e_{\tau_H} \circ \Psi\overline{\gamma^{op}_X \circ \Phi e_\tau}(d) = (T_1\eta_X)^{op} \circ \delta^{op}_{FX} \circ \Psi e_{\tau_H}(c) = b$.

To show the universal property of η_X w.r.t. to K, we will use its universal property w.r.t. to G. Given a $\mathbf{Top}(T_1)$-morphism $(X,\tau) \xrightarrow{f} (K(Y,\sigma) = (GY, (\delta^{op}_Y \circ \Psi e_\sigma)^{\rightarrow}(\Psi\sigma)))$, we check that the above-mentioned $\mathbf{X}_2$-morphism $FX \xrightarrow{g} Y$ is T_2-continuous, i.e., $(H(X,\tau) = (FX, (\gamma^{op}_X \circ \Phi e_\tau)^{\rightarrow}(\Psi\tau))) \xrightarrow{g} (Y,\sigma)$ is a $\mathbf{Top}(T_2)$-morphism. Since f is T_1-continuous, there is the factorization $\Psi\sigma \xrightarrow{\Psi e_\sigma} \Psi T_2 Y \xrightarrow{\delta^{op}_Y} T_1GY \xrightarrow{(T_1f)^{op}} T_1X = \Psi\sigma \xrightarrow{\varphi} \tau \xhookrightarrow{e_\tau} T_1X$. This factorization together with the right-hand rectangle of $(\mathcal{A})$ provides the next commutative (and complex) diagram:



If $b \in ((T_2g)^{op})^{\rightarrow}(\sigma)$, then there exists $c \in \sigma$ such that $(T_2g)^{op}(c) = b$. From the above diagram, it follows that $\Phi\varphi \circ \zeta_\sigma(c) = d \in \Phi\tau$ and $b = (T_2g)^{op} \circ e_\sigma(c) = \gamma^{op}_X \circ \Phi e_\tau \circ \Phi\varphi \circ \zeta_\sigma(c) = \gamma^{op}_X \circ \Phi e_\tau(d) \in (\gamma^{op}_X \circ \Phi e_\tau)^{\rightarrow}(\Phi\tau)$.

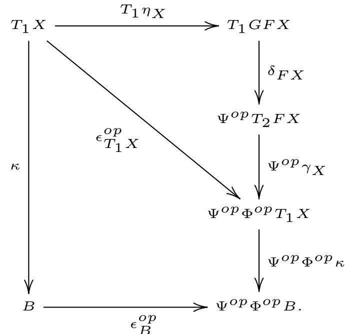
Standard methods of category theory give the adjoint situation in question. □

To continue the line, we show that a similar result holds for the categories of the form $\mathbf{TopSys}(T)$. By Theorem 3.13, we get two functors, for the sake of shortness, having more compact notations: $T_1 \xrightarrow{(F,\Phi,\gamma)} T_2 \mapsto \mathbf{TopSys}(T_1) \xrightarrow{H} \mathbf{TopSys}(T_2)$ and $T_2 \xrightarrow{(G,\Psi,\delta)} T_1 \mapsto \mathbf{TopSys}(T_2) \xrightarrow{K} \mathbf{TopSys}(T_1)$.

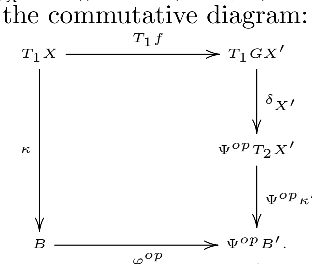
Theorem 4.6. *In the framework of the category* $\mathbf{TpThr}$, *if* $(\mathcal{A})$ *holds, then there is the adjoint situation* $((\eta, \epsilon^{op}), (\varepsilon, \zeta^{op})) : H \dashv K : \mathbf{TopSys}(T_2) \to \mathbf{TopSys}(T_1)$.

Proof. Given a T_1-system (X, κ, B), $X \xrightarrow{\eta_X} GFX$ is a G-universal arrow for X and $\Psi\Phi B \xrightarrow{\epsilon_B} B$ is a Ψ-co-universal arrow for B. We show that $(X, \kappa, B) \xrightarrow{(\eta_X, \epsilon_B^{op})} (KH(X, \kappa, B) = K(FX, \Phi^{op}\kappa \circ \gamma_X, \Phi^{op}B) = (GFX, \Psi^{op}(\Phi^{op}\kappa \circ \gamma_X) \circ \delta_{FX}, \Psi^{op}\Phi^{op}B))$ is a K-universal arrow for (X, κ, B).

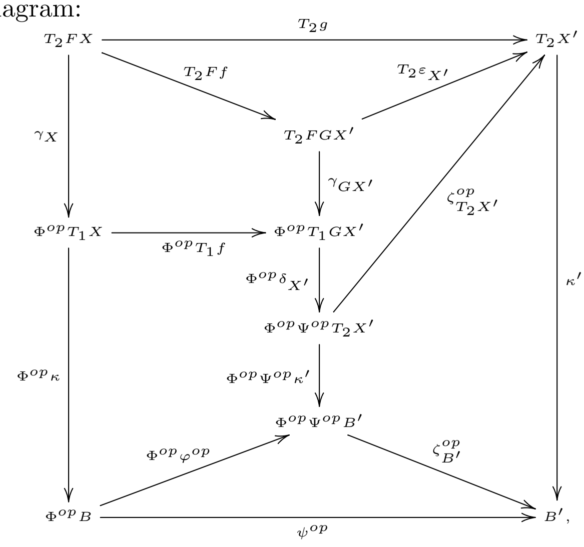
The required T_1-continuity of the pair follows from the left-hand rectangle in $(\mathcal{A})$, employed in the following commutative diagram:



Given a **TopSys**(T_1)-morphism $(X, \kappa, B) \xrightarrow{(f,\varphi)} (K(X', \kappa', B') = (GX', \Psi^{op}\kappa' \circ \delta_{X'}, \Psi^{op}B'))$, there exists a unique $\mathbf{X}_2 \times \mathbf{B}_2^{op}$-morphism $(FX, \Phi^{op}B) \xrightarrow{(g,\psi)} (X', B')$, which fits the factorization $(X, B) \xrightarrow{(f,\varphi)} (GX', \Psi^{op}B') = (X, B) \xrightarrow{(\eta_X, \epsilon_B^{op})} (GFX, \Psi^{op}\Phi^{op}B) \xrightarrow{(Gg, \Psi^{op}\psi)} (GX', \Psi^{op}B')$. We check that $(H(X, \kappa, B) = (FX, \Phi^{op}\kappa \circ \gamma_X, \Phi^{op}B)) \xrightarrow{(g,\psi)} (X', \kappa', B')$ is T_2-continuous. Since (f, φ) is T_1-continuous, there exists the commutative diagram:



Taken together with the right-hand rectangle of $(\mathcal{A})$, it gives rise to the following commutative diagram:



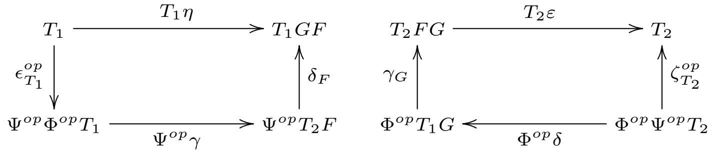
which proves the continuity in question.

Standard methods of category theory give the adjoint situation in question. □

Having done with the category **TpThr**, we turn our attention to its partly dualized version $\mathbf{TpThr}^d$. The sufficient condition in this case undergoes a modification and can be written as follows.

The following diagrams

($\mathcal{B}$)

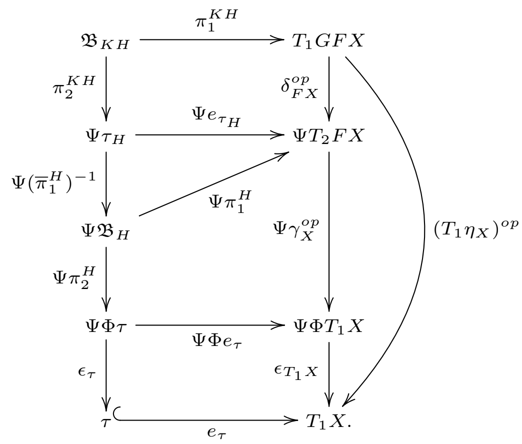


commute.

Theorem 4.7. *In the framework of the category* $\mathbf{TpThr}^d_{pi}$*, if* ($\mathcal{B}$) *holds, then there exists the adjoint situation* $(\eta,\varepsilon): H \dashv K : \mathbf{Top}(T_2) \to \mathbf{Top}(T_1)$.

Proof. Given a T_1-space (X,τ), as in Theorem 4.5, we show that the $\mathbf{X}_1$-morphism $(X,\tau) \xrightarrow{\eta_X} (KH(X,\tau) = K(FX, (\gamma^{op}_X)^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau)) = K(FX,\tau_H) = (GFX, (\delta^{op}_{FX})^{\leftarrow} \circ (\Psi e_{\tau_H})^{\rightarrow}(\Psi\tau_H)) = (GFX,\tau_{KH}))$ is a K-universal arrow for (X,τ). T_1-continuity of η_X is provided by the left-hand rectangle of ($\mathcal{B}$), employed in a slight modification of Diagram (1) in the proof of Theorem 3.11 (recall its notations):

(2)



With the diagram in mind, we proceed as follows. If $c \in ((T_1\eta_X)^{op})^{\rightarrow}(\tau_{KH})$, then there exists $(b_1,b_2) \in \mathfrak{B}_{KH}$ such that $(T_1\eta_X)^{op}(b_1) = c$ (recall from the proof of Theorem 3.11 that $\tau_{KH} = (\pi_1^{KH})^{\rightarrow}(\mathfrak{B}_{KH})$). Moreover, since $(b_1,b_2) \in \mathfrak{B}_{KH}$, $\delta^{op}_{FX}(b_1) = \Psi e_{\tau_H}(b_2)$ and thus, $\Psi\gamma^{op}_X \circ \delta^{op}_{FX}(b_1) = \Psi\gamma^{op}_X \circ \Psi e_{\tau_H}(b_2) = \Psi\Phi e_\tau \circ \Psi\pi_2^H \circ \Psi(\overline{\pi}_1^H)^{-1}(b_2)$, which implies that $\epsilon_{T_1X} \circ \Psi\gamma^{op}_X \circ \delta^{op}_{FX}(b_1) = \epsilon_{T_1X} \circ \Psi\Phi e_\tau \circ \Psi\pi_2^H \circ \Psi(\overline{\pi}_1^H)^{-1}(b_2) = e_\tau \circ \epsilon_\tau \circ \Psi\pi_2^H \circ \Psi(\overline{\pi}_1^H)^{-1}(b_2) \in \tau$. Altogether, $c = (T_1\eta_X)^{op}(b_1) = \epsilon_{T_1X} \circ \Psi\gamma^{op}_X \circ \delta^{op}_{FX}(b_1) \in \tau$.

For the property of K-universality, take some $\mathbf{Top}(T_1)$-morphism $(X,\tau) \xrightarrow{f} (K(Y,\sigma) = (GY, (\delta^{op}_Y)^{\leftarrow} \circ (\Psi e_\sigma)^{\rightarrow}(\Psi\sigma)))$. We show that the respective $\mathbf{X}_2$-morphism

$FX \xrightarrow{g} Y$, obtained through the G-universal property of η_X, provides T_2-continuous morphism $(H(X,\tau) = (FX, (\gamma_X^{op})^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau))) \xrightarrow{g} (Y,\sigma)$. From the pullback (recall our remark at the beginning of Theorem 3.11)

$$\begin{array}{ccc}
\mathfrak{B}_K & \xrightarrow{\pi_1^K} & T_1GY \\
{\scriptstyle \pi_2^K}\downarrow & \lrcorner & \downarrow{\scriptstyle \delta_Y^{op}} \\
\Psi\sigma & \xrightarrow[\Psi e_\sigma]{} & \Psi T_2 Y,
\end{array}$$

and the fact that Φ preserves pullbacks (recall Definition 3.10), we get the pullback

$$\begin{array}{ccc}
\Phi\mathfrak{B}_K & \xrightarrow{\Phi\pi_1^K} & \Phi T_1GY \\
{\scriptstyle \Phi\pi_2^K}\downarrow & \lrcorner & \downarrow{\scriptstyle \Phi\delta_Y^{op}} \\
\Phi\Psi\sigma & \xrightarrow[\Phi\Psi e_\sigma]{} & \Phi\Psi T_2 Y.
\end{array}$$

However, the right-hand rectangle of $(\mathcal{B})$ gives the following commutative diagram:

$$\begin{array}{ccccccc}
\sigma & \xhookrightarrow{e_\sigma} & T_2Y & \xrightarrow{(T_2\varepsilon_Y)^{op}} & T_2FGY & \xrightarrow{\gamma_{GY}^{op}} & \Phi T_1GY \\
{\scriptstyle \zeta_\sigma}\downarrow & & & \searrow^{\zeta_{T_2Y}} & & & \downarrow{\scriptstyle \Phi\delta_Y^{op}} \\
\Phi\Psi\sigma & & & \xrightarrow[\Phi\Psi e_\sigma]{} & & & \Phi\Psi T_2Y
\end{array}$$

and, therefore, in view of the above pullback, there exists a $\mathbf{B}_2$-homomorphism $\sigma \xrightarrow{\psi} \Phi\mathfrak{B}_K$, making the next diagram commute:

$$\begin{array}{ccccc}
\sigma & \xrightarrow{e_\sigma} & T_2Y & \xrightarrow{(T_2\varepsilon_Y)^{op}} & T_2FGY \\
{\scriptstyle \psi}\downarrow & & & & \downarrow{\scriptstyle \gamma_{GY}^{op}} \\
\Phi\mathfrak{B}_K & & \xrightarrow[\Phi\pi_1^K]{} & & \Phi T_1GY.
\end{array}$$

The last diagram is a crucial building block of the subsequent one, which also relies on the obvious factorization (recall T_1-continuity of f) $\mathfrak{B}_K \xrightarrow{\pi_1^K} T_1GY \xrightarrow{(T_1f)^{op}} T_1X = \mathfrak{B}_K \xrightarrow{\varphi} \tau \xhookrightarrow{e_\tau} T_1X$:

$$\begin{array}{ccccc}
\sigma & \xrightarrow{e_\sigma} & T_2Y & \xrightarrow{(T_2g)^{op}} & T_2FX \\
 & & \searrow^{(T_2\varepsilon_Y)^{op}} & \nearrow_{(T_2Ff)^{op}} & \\
{\scriptstyle \psi}\downarrow & & T_2FGY & & \downarrow{\scriptstyle \gamma_X^{op}} \\
 & & \downarrow{\scriptstyle \gamma_{GY}^{op}} & & \\
\Phi\mathfrak{B}_K & \xrightarrow[\Phi\pi_1^K]{} & \Phi T_1GY & & \\
{\scriptstyle \Phi\varphi}\downarrow & & & \searrow^{\Phi(T_1f)^{op}} & \\
\Phi\tau & & \xrightarrow[\Phi e_\tau]{} & & \Phi T_1X.
\end{array}$$

The last diagram and the pullback

$$\begin{array}{ccc} \mathfrak{B}_H & \xrightarrow{\pi_1^H} & T_2FX \\ {\scriptstyle \pi_2^H}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle \gamma_X^{op}} \\ \Phi\tau & \xrightarrow[\Phi e_\tau]{} & \Phi T_1X \end{array}$$

provide a $\mathbf{B}_2$-homomorphism $\sigma \xrightarrow{\phi} \mathfrak{B}_H$ making the diagram

$$\begin{array}{ccc} \sigma & \xrightarrow{e_\sigma} & T_2Y \\ {\scriptstyle \phi}\Big\downarrow & & \Big\downarrow{\scriptstyle (T_2g)^{op}} \\ \mathfrak{B}_H & \xrightarrow[\pi_1^H]{} & T_2FX \end{array}$$

commute. It follows that $((T_2g)^{op})^{\rightarrow}(\sigma) \subseteq (\pi_1^H)^{\rightarrow}(\mathfrak{B}_H) = (\gamma_X^{op})^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau)$ and that was to show.

Standard methods of category theory give the adjoint situation in question. □

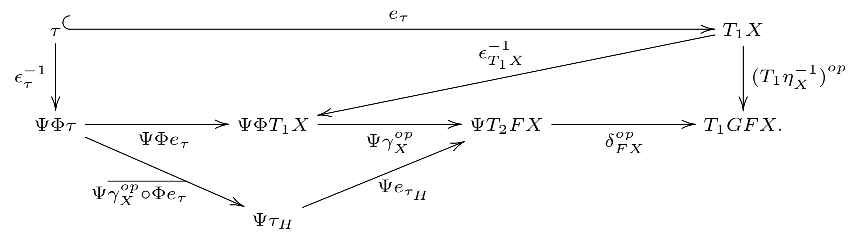
An attentive reader will recall from Subsection 3.3 that, at the moment, there is no way to relate the quasicategories $\mathbf{TpThr}^d$ and $\mathbf{TpStm}$. On the other hand, the just obtained results look interesting enough. For example, it appears that the same condition $(\mathcal{A})$ is applicable to both topological structures and systems, being the main reason for putting both structures and systems into one subsection. Taking into account the results of the previous subsection, where the structure and system approaches to Morita problem appeared to be different, it seems reasonable to differentiate the approaches of this subsection as well. It will be the topic of our further research to do the job. Currently, however, we proceed to the just mentioned Morita problem, providing the promised sufficient condition on two topological theories, so that their respective categories of structures or systems are equivalent.

4.3. **Lifting Equivalences from Theories to Structures and Systems.** In the previous subsection, we considered the possibility of lifting adjoint situations from the setting of theories to their respective framework of topological structures or systems. As a natural generalization of the topic, this subsection considers the lift of equivalences. The initial setting deviates from the already considered framework in just one respect, namely, we assume that there exist two equivalences: $(\eta, \varepsilon) : F \dashv G : \mathbf{X}_2 \to \mathbf{X}_1$ and $(\zeta, \epsilon) : \Psi \dashv \Phi : \mathbf{B}_2 \to \mathbf{B}_1$. It appears that the crucial conditions $(\mathcal{A})$ and $(\mathcal{B})$ can stay unchanged, the respective results provided in the following theorems.

Theorem 4.8. *In the setting of the category* $\mathbf{TpThr}_s$, *if* $(\mathcal{A})$ *holds, then there exists the equivalence* $(\eta, \varepsilon) : H \dashv K : \mathbf{Top}(T_2) \to \mathbf{Top}(T_1)$.

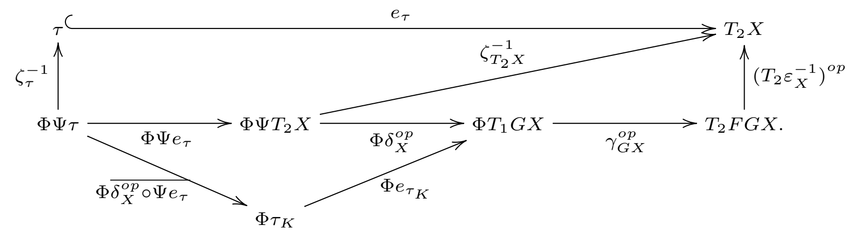
Proof. By Theorem 4.5, we show only that the obtained universal arrows are isomorphisms, which results from verification of continuity of their respective inverses.

To show that $(KH(X,\tau) = K(FX, (\gamma_X^{op} \circ \Phi e_\tau)^{\rightarrow}(\Phi\tau)) = K(FX, \tau_H) = (GFX, (\delta_{FX}^{op} \circ \Psi e_{\tau_H})^{\rightarrow}(\Psi\tau_H)) = (GFX, \tau_{KH})) \xrightarrow{\eta_X^{-1}} (X, \tau)$ is T_1-continuous, consider the following commutative diagram, which employs the left-hand rectangle of $(\mathcal{A})$ and the fact that we are dealing with equivalences instead of adjunctions:



Taking the diagram into account, $((T_1\eta_X^{-1})^{op})^{\rightarrow}(\tau) \subseteq (\delta_{FX}^{op} \circ \Psi e_{\tau_H})^{\rightarrow}(\Psi\tau_H) = \tau_{KH}$.

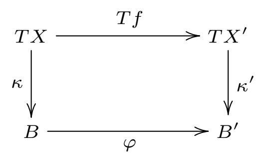
To show that $(X,\tau) \xrightarrow{\varepsilon_X^{-1}} (HK(X,\tau) = H(GX, (\delta_X^{op} \circ \Psi e_\tau)^{\rightarrow}(\Psi\tau)) = H(GX, \tau_K) = (FGX, (\gamma_{GX}^{op} \circ \Phi e_{\tau_K})^{\rightarrow}(\Phi\tau_K)) = (FGX, \tau_{HK}))$ is T_2-continuous, we consider the following commutative diagram, which employs the right-hand rectangle of $(\mathcal{A})$ and the fact that the adjunctions are substituted by the equivalences:



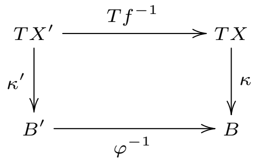
If $b \in ((T_2\varepsilon_X^{-1})^{op})^{\rightarrow}(\tau_{HK})$, then there exists $c \in \Phi\tau_K$ such that $(T_2\varepsilon_X^{-1})^{op} \circ \gamma_{GX}^{op} \circ \Phi e_{\tau_K}(c) = b$. Since Φ preserves surjective homomorphisms (recall Definition 3.6), there exists $d \in \Phi\Psi\tau$ such that $\Phi\overline{\delta_X^{op} \circ \Psi e_\tau}(d) = c$, which implies that $\zeta_\tau^{-1}(d) = a \in \tau$ and $a = \zeta_{T_2X}^{-1} \circ \Phi\Psi e_\tau(d) = (T_2\varepsilon_X^{-1})^{op} \circ \gamma_{GX}^{op} \circ \Phi\delta_X^{op} \circ \Phi\Psi e_\tau(d) = (T_2\varepsilon_X^{-1})^{op} \circ \gamma_{GX}^{op} \circ \Phi e_{\tau_K} \circ \Phi\overline{\delta_X^{op} \circ \Psi e_\tau}(d) = (T_2\varepsilon_X^{-1})^{op} \circ \gamma_{GX}^{op} \circ \Phi e_{\tau_K}(c) = b$. □

Theorem 4.9. *In the framework of the category* **TpThr***, if* $(\mathcal{A})$ *holds, then there exists the equivalence* $((\eta, \epsilon^{op}), (\varepsilon, \zeta^{op})) : H \dashv K : \mathbf{TopSys}(T_2) \rightarrow \mathbf{TopSys}(T_1)$.

Proof. The proof follows from the simple observation that given a $\mathbf{TopSys}(T)$-morphism $(X, \kappa, B) \xrightarrow{(f,\varphi)} (X', \kappa', B')$ such that (f, φ) is an $\mathbf{X} \times \mathbf{B}^{op}$-isomorphism, (f, φ) is a $\mathbf{TopSys}(T)$-isomorphism. Indeed, commutativity of the diagram



provides the commutative diagram



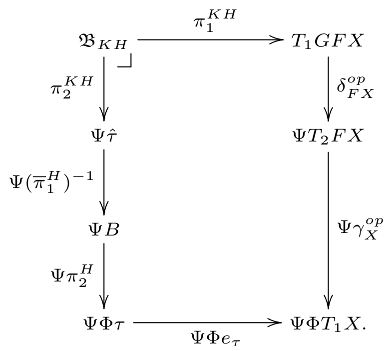
and that was to show. □

The case of the category $\mathbf{TpThr}^d$ is equally straightforward.

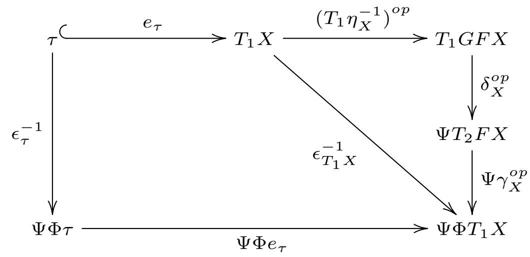
Theorem 4.10. *In the framework of the category* $\mathbf{TpThr}^d_{pi}$*, if* $(\mathcal{B})$ *holds, then there exists the equivalence* $(\eta, \varepsilon) : H \dashv K : \mathbf{Top}(T_2) \to \mathbf{Top}(T_1)$*.*

Proof. We follow the pattern of Theorem 4.8 and show that the inverse of the universal arrows constructed in Theorem 4.7 are continuous.

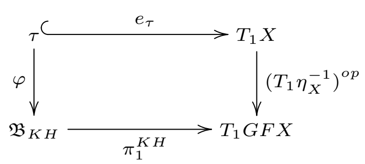
To show that $(KH(X,\tau) = K(FX, (\gamma_X^{op})^{\leftarrow} \circ (\Phi e_\tau)^{\rightarrow}(\Phi\tau)) = K(FX, \tau_H) = (GFX, (\delta_{FX}^{op})^{\leftarrow} \circ (\Psi e_{\tau_H})^{\rightarrow}(\Psi\tau_H)) = (GFX, \tau_{KH})) \xrightarrow{\eta_X^{-1}} (X,\tau)$ is T_1-continuous, we employ the following pullback, which is a truncated version of Diagram (2) in the proof of Theorem 4.7:



Commutativity of the next diagram (which employs the left-hand rectangle of $(\mathcal{B})$)



provides a $\mathbf{B}_1$-homomorphism $\tau \xrightarrow{\varphi} \mathfrak{B}_{KH}$ (recall the above pullback), which makes the following diagram

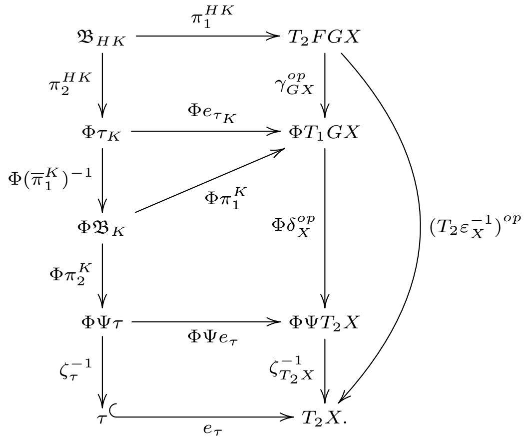


commute. Having the above diagram in hand, the reader can easily perform the following easy line of calculations:

$$((T_1\eta_X^{-1})^{op})^{\rightarrow}(\tau) \subseteq (\pi_1^{KH})^{\rightarrow}(\mathfrak{B}_{KH}) = \tau_{KH},$$

which was to show.

To get T_2-continuity of $(X,\tau) \xrightarrow{\varepsilon_X^{-1}} (HK(X,\tau) = H(GX, (\delta_X^{op})^{\leftarrow} \circ (\Psi e_\tau)^{\rightarrow}(\Psi\tau)) = H(GX,\tau_K) = (FGX, (\gamma_{GX}^{op})^{\leftarrow} \circ (\Phi e_{\tau_K})^{\rightarrow}(\Phi\tau_K)) = (FGX,\tau_{HK}))$, we use a slight modification of Diagram (2) in the proof of Theorem 4.7, which employs the right-hand rectangle of $(\mathcal{B})$:



It then follows that $((T_2\varepsilon_X^{-1})^{op})^{\rightarrow}(\tau_{HK}) \subseteq \tau$ (recall that $\tau_{HK} = (\pi_1^{HK})^{\rightarrow}(\mathfrak{B}_{HK})$) and that was to show. □

The theorems of this subsection provide the promised sufficient conditions for the Morita problems of this paper (Problems 4.1, 4.3). As the reader can see, at the bottom of the machinery lie requirements $(\mathcal{A})$, $(\mathcal{B})$. Being essential in the developments, both $(\mathcal{A})$ and $(\mathcal{B})$ are potential subject for research. It will be the topic of our further papers to conduct it in full generality, while here, we dwell upon their modified (and stronger) versions, which are the topic of the next subsection.

4.4. **Relation Between the Conditions for Adjunction Lifting.** Looking closer at requirement $(\mathcal{A})$ of Subsection 4.2, we see that it employs two commutative diagrams. A natural question arises on the relationships between them, i.e., whether the commutativity of one of the rectangles implies the property for the other. At the moment, we are able neither to prove nor to disprove the statement. Moreover, since every adjoint situation involves two identities relating its unit and co-unit [2, Definition 19.3], the number of diagrams seems to be reasonable. Despite that, in this subsection, we show a relation between the rectangles in question. The main point is to introduce two additional requirements, running as follows (a mono- (or epi-) natural transformation consists of monomorphisms (or epimorphisms)):

The natural transformation θ, given by the commutativity of the diagram

$$(\mathcal{C}_1)\qquad \begin{CD} T_2FG @>\gamma_G>> \Phi^{op}T_1G @>\Phi^{op}T_1\eta_G>> \Phi^{op}T_1GFG \\ @V\theta VV @. @VV\Phi^{op}\delta_{FG}V \\ T_2FG @<<< @<\zeta^{op}_{T_2FG}<< \Phi^{op}\Psi^{op}T_2FG \end{CD}$$

is a mono- (or epi-) natural transformation.

The natural transformation ϑ, given by the commutativity of the diagram:

$$(\mathcal{C}_2)\qquad \begin{CD} \Psi^{op}\Phi^{op}T_1 @>\Psi^{op}\Phi^{op}T_1\eta>> \Psi^{op}\Phi^{op}T_1GF @>\Psi^{op}\Phi^{op}\delta_F>> \Psi^{op}\Phi^{op}\Psi^{op}T_2F \\ @V\vartheta VV @. @VV\Psi^{op}\zeta^{op}_{T_2F}V \\ \Psi^{op}\Phi^{op}T_1 @<<< @<\Psi^{op}\gamma<< \Psi^{op}T_2F \end{CD}$$

is a mono- (or epi-) natural transformation.

The respective results are listed in the following theorem.

Theorem 4.11.

(1) *The left-hand rectangle of* $(\mathcal{A})$ *together with* $(\mathcal{C}_1)$ *provides the right-hand rectangle of* $(\mathcal{A})$.

(2) *The right-hand rectangle of* $(\mathcal{A})$ *together with* $(\mathcal{C}_2)$ *provides the left-hand rectangle of* $(\mathcal{A})$.

Proof. The proof of Item (1) is based in the following (rather complex) diagram, whose inner building blocks except for the most outer rectangle commute:

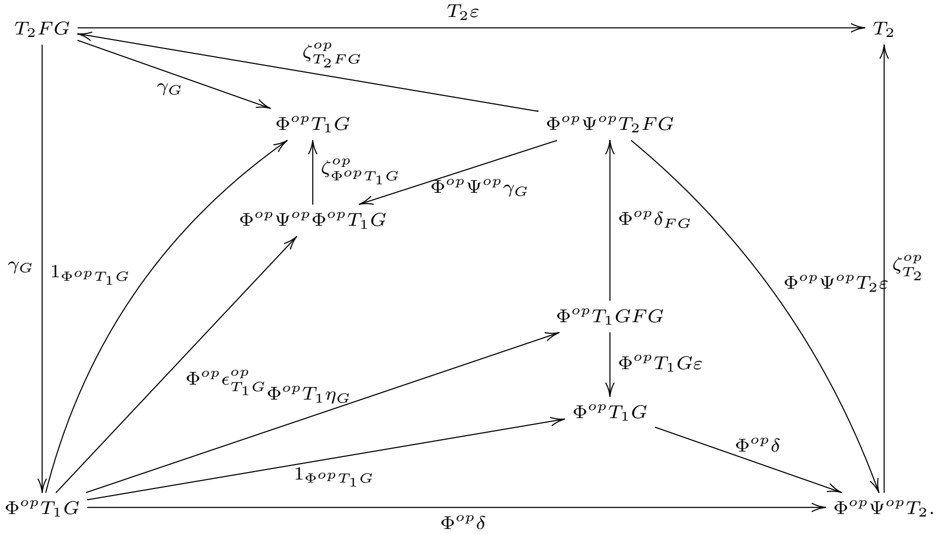


Diagram chasing provides the following sequences. On the one hand, $\zeta^{op}_{T_2}\circ\Phi^{op}\delta\circ\gamma_G = T_2\varepsilon\circ\zeta^{op}_{T_2FG}\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\gamma_G = T_2\varepsilon\circ\theta$ and, on the other hand, $\theta = \zeta^{op}_{T_2FG}\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\gamma_G \overset{(\dagger)}{=} \zeta^{op}_{T_2FG}\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\zeta^{op}_{\Phi^{op}T_1G}\circ\Phi^{op}\epsilon^{op}_{T_1G}\circ\gamma_G \overset{(\dagger\dagger)}{=} \zeta^{op}_{T_2FG}\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\zeta^{op}_{\Phi^{op}T_1G}\circ\Phi^{op}\Psi^{op}\gamma_G\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\gamma_G \overset{(\dagger\dagger\dagger)}{=} \zeta^{op}_{T_2FG}\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\gamma_G\circ\zeta^{op}_{T_2FG}\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G\circ\gamma_G = \theta\circ\theta$, where (†) uses $1_{\Phi^{op}T_1G} = \zeta^{op}_{\Phi^{op}T_1G}\circ\Phi^{op}\epsilon^{op}_{T_1G}$, (††) relies on $\Phi^{op}\epsilon^{op}_{T_1G} = \Phi^{op}\Psi^{op}\gamma_G\circ\Phi^{op}\delta_{FG}\circ\Phi^{op}T_1\eta_G$, whereas († † †) employs $\zeta^{op}_{\Phi^{op}T_1G}\circ\Phi^{op}\Psi^{op}\gamma_G = \gamma_G\circ\zeta^{op}_{T_2FG}$. The condition of the theorem on θ implies $\theta = 1_{T_2FG}$ and, therefore, $\zeta^{op}_{T_2}\circ\Phi^{op}\delta\circ\gamma_G = T_2\varepsilon$.

The proof of Item (2) is based on the next diagram, whose inner building blocks except for the most outer rectangle are commutative:

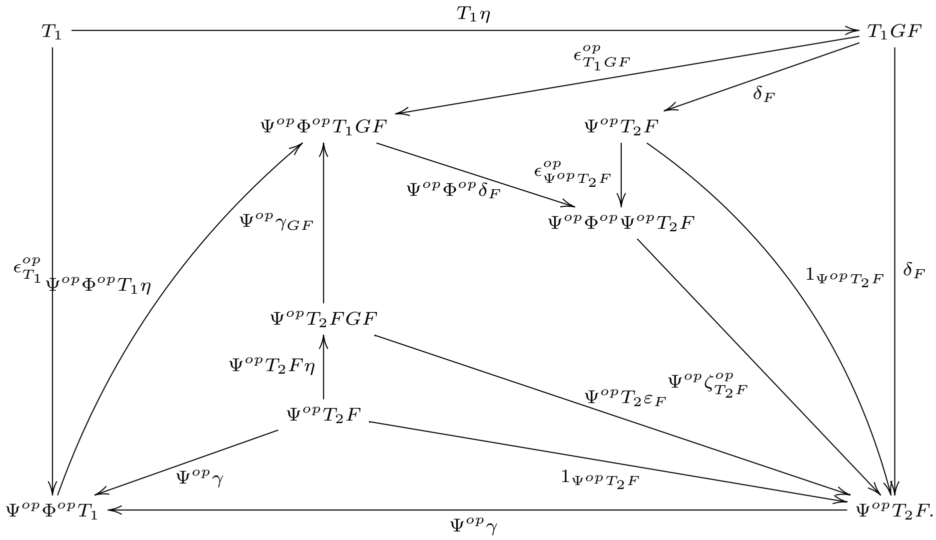


Diagram chasing once again provides the following. On the one hand, $\Psi^{op}\gamma\circ\delta_F\circ T_1\eta = \Psi^{op}\gamma\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\Phi^{op}T_1\eta\circ\epsilon^{op}_{T_1} = \vartheta\circ\epsilon^{op}_{T_1}$ and, on the other hand, $\vartheta = \Psi^{op}\gamma\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\Phi^{op}T_1\eta \overset{(\dagger)}{=} \Psi^{op}\gamma\circ\Psi^{op}T_2\varepsilon_F\circ\Psi^{op}T_2F\eta\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\Phi^{op}T_1\eta \overset{(\dagger\dagger)}{=} \Psi^{op}\gamma\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\gamma_{GF}\circ\Psi^{op}T_2F\eta\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\Phi^{op}T_1\eta \overset{(\dagger\dagger\dagger)}{=} \Psi^{op}\gamma\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\Phi^{op}T_1\eta\circ\Psi^{op}\gamma\circ\Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\Phi^{op}T_1\eta = \vartheta\circ\vartheta$, where (†) uses $1_{\Psi^{op}T_2F} = \Psi^{op}T_2\varepsilon_F\circ\Psi^{op}T_2F\eta$, (††) relies on $\Psi^{op}T_2\varepsilon_F = \Psi^{op}\zeta^{op}_{T_2F}\circ\Psi^{op}\Phi^{op}\delta_F\circ\Psi^{op}\gamma_{GF}$, whereas († † †) employs $\Psi^{op}\gamma_{GF}\circ\Psi^{op}T_2F\eta = \Psi^{op}\Phi^{op}T_1\eta\circ\Psi^{op}\gamma$. The condition of the theorem on ϑ implies $\vartheta = 1_{\Psi^{op}\Phi^{op}T_1}$ and, therefore, $\Psi^{op}\gamma\circ\delta_F\circ T_1\eta = \epsilon^{op}_{T_1}$. □

An attentive reader will notice that Item (1) of Theorem 4.11 avoided in its formulation (but not in its proof) the use of the natural transformation ε, whereas Item (2) avoided ϵ (in a similar manner). It is still unclear to us, whether it is possible to exclude more transformations, rewriting the requirements accordingly.

5. **Powerset Theories Versus Topological Theories**

In the previous two sections, we concentrated our attention entirely on the properties of catalg topological theories. The reader will recall, however, that the framework of catalg topology starts with the particularly defined powerset theories (see Definition 2.4). One would certainly wish to develop a similar machinery for the latter notion as well. The simplest solution would be to use powerset theories instead of topological ones in the already obtained procedures. A bit of thinking, though, tells us that topological theories have one essential feature, namely, they reduce a part of algebraic structure of powerset theories, before generating their respective topological structures. In one word, to obtain a more thorough machinery for powerset theories, we should proceed in two steps. In the first step, we pass from powerset theories to topological ones, whereas, in the second step, we use the procedures developed earlier in the paper. It is the main purpose of this section, to describe explicitly the first (and very important) step.

5.1. **Quasicategory of Powerset Theories and Its Related Notions.** Similar to the setting of topological theories, we begin by the definition of the quasicategory of powerset theories. The new definition essentially reiterates the steps of Definition 3.1 and thus, is shortened. All omitted details can be found in Subsection 3.1.

Definition 5.1. **PwThr** is the quasicategory, whose objects are cabp-theories $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$, and whose morphisms $P_1 \xrightarrow{(F,\Phi,\eta)} P_2$ (or just η) comprise two functors $\mathbf{X}_1 \xrightarrow{F} \mathbf{X}_2$, $\mathbf{A}_1 \xrightarrow{\Phi} \mathbf{A}_2$ and a natural transformation $P_2F \xrightarrow{\eta} \Phi^{op}P_1$. Composition and identities are those of the category **TpThr** of Definition 3.1.

Due to the already considerable size of the paper, we will not introduce an analogue of the quasicategory $\mathbf{TpThr}^d$, leaving the subject till its subsequent development. On the other hand, to provide for the reduction of the algebraic structure, we present an additional quasicategory of variety reducts.

Definition 5.2. **RdVar** is the quasicategory, whose objects are variety reducts $\mathbf{A} \xrightarrow{\|-\|} \mathbf{B}$ (in the sense of Definition 2.3), and whose morphisms $\|-\|_1 \xrightarrow{(\Phi,\Psi,\varepsilon)} \|-\|_2$ (or just ε) comprise two functors $\mathbf{A}_1 \xrightarrow{\Phi} \mathbf{A}_2$, $\mathbf{B}_1 \xrightarrow{\Psi} \mathbf{B}_2$ and a natural transformation $\|-\|_2^{op}\Phi^{op} \xrightarrow{\varepsilon} \Psi^{op}\|-\|_1^{op}$, or, more specifically,

$$\begin{array}{ccc} \mathbf{A}_1^{op} & \xrightarrow{\Phi^{op}} & \mathbf{A}_2^{op} \\ {\scriptstyle \|-\|_1^{op}}\downarrow & \overset{\varepsilon}{\swarrow} & \downarrow{\scriptstyle \|-\|_2^{op}} \\ \mathbf{B}_1^{op} & \xrightarrow[\Psi^{op}]{} & \mathbf{B}_2^{op}. \end{array}$$

Composition and identities are those of the category **TpThr** of Definition 3.1.

We notice that variety reduct morphisms employ the dual categories of the varieties in question. In the following, we are going to describe categorically the procedure of reducing algebraic structure, when passing from powerset theories to topological theories. At the bottom of the approach, lies the following category, which is the so-called "glueing" of the categories **PwThr** and **RdVar** defined above.

Definition 5.3. $\mathbf{PwThr} \bowtie \mathbf{RdVar}$ is the quasicategory, whose objects are pairs $(P, \| - \|)$, comprising a powerset theory P and a variety reduct $\| - \|$ such that $cod\, P = dom\, \| - \|^{op}$, and whose morphisms $(P_1, \| - \|_1) \xrightarrow{((F,\Phi,\eta),(\Psi,\Theta,\varepsilon))} (P_2, \|-\|_2)$ consist of a powerset theory morphism $P_1 \xrightarrow{(F,\Phi,\eta)} P_2$ and a variety reduct morphism $\| - \|_1 \xrightarrow{(\Psi,\Theta,\varepsilon)} \| - \|_2$ such that $\Phi = \Psi$. Composition and identities are defined componentwise in the respective categories.

It is easy to see that $\mathbf{PwThr} \bowtie \mathbf{RdVar}$ is a full subcategory of the product category $\mathbf{PwThr} \times \mathbf{RdVar}$. We have already mentioned the word "glueing" before its definition. The term is quite inofficial (nothing in common with, e.g., *Artin glueing* along a functor considered in [11, 38, 65], which is a particular instance of comma categories), the main motivation for its use given by the following theorems.

Theorem 5.4. *There is a non-full embedding* $\mathbf{PwThr} \overset{E}{\hookrightarrow} \mathbf{PwrThr} \bowtie \mathbf{RdVar}$ *defined by* $E(P_1 \xrightarrow{(F,\Phi,\eta)} P_2) = (P_1, 1_{\mathbf{A}_1}) \xrightarrow{((F,\Phi,\eta),(\Phi,\Phi,1_{\Phi^{op}}))} (P_2, 1_{\mathbf{A}_2})$.

Proof. To show that the embedding E preserves the composition, notice that given two powerset theory morphisms $P_1 \xrightarrow{(F_1,\Phi_1,\eta_1)} P_2$ and $P_2 \xrightarrow{(F_2,\Phi_2,\eta_2)} P_3$, $E((F_2, \Phi_2, \eta_2) \circ (F_1, \Phi_1, \eta_1)) = E(F_2F_1, \Phi_2\Phi_1, \eta_2 \odot \eta_1) = ((F_2F_1, \Phi_2\Phi_1, \eta_2 \odot \eta_1), (\Phi_2\Phi_1, \Phi_2\Phi_1, 1_{\Phi_2^{op}\Phi_1^{op}})) = ((F_2, \Phi_2, \eta_2), (\Phi_2, \Phi_2, 1_{\Phi_2^{op}})) \circ ((F_1, \Phi_1, \eta_1), (\Phi_1, \Phi_1, 1_{\Phi_1^{op}})) = E(F_2, \Phi_2, \eta_2) \circ E(F_1, \Phi_1, \eta_1)$. Of all the other properties, we will show non-fullness only.

Let $\mathbf{A}$ be a variety with an initial object I (e.g., the variety $\mathbf{CSLat}(\bigvee)$ or $\mathbf{Frm}$). There exists the constant functor $\mathbf{A} \xrightarrow{\Psi} \mathbf{A}$, which takes everything to the identity morphism $I \xrightarrow{1_I} I$. Moreover, there exists the natural transformation $\Psi \xrightarrow{\varepsilon} 1_{\mathbf{A}}$ defined by $\Psi A \xrightarrow{\varepsilon_A} 1_{\mathbf{A}}A = I \xrightarrow{!} A$, where ! is the unique morphism induced by I. Altogether, we obtain an $\mathbf{RdVar}$-morphism $1_{\mathbf{A}} \xrightarrow{(1_{\mathbf{A}},\Psi,\varepsilon^{op})} 1_{\mathbf{A}}$ and we can easily assume that $1_{\mathbf{A}} \neq \Psi$ (the variety $\mathbf{A}$ has something apart from the identity $I \xrightarrow{1_I} I$). Since $1_{\mathbf{A}^{op}} \xrightarrow{(1_{\mathbf{A}^{op}},1_{\mathbf{A}},1_{1_{\mathbf{A}^{op}}})} 1_{\mathbf{A}^{op}}$ is a $\mathbf{PwThr}$-morphism, it follows that $(1_{\mathbf{A}^{op}}, 1_{\mathbf{A}}) \xrightarrow{((1_{\mathbf{A}^{op}},1_{\mathbf{A}},1_{1_{\mathbf{A}^{op}}}),(1_{\mathbf{A}},\Psi,\varepsilon^{op}))} (1_{\mathbf{A}^{op}}, 1_{\mathbf{A}})$ is in $\mathbf{PwThr} \bowtie \mathbf{RdVar}$ and is not in the image of E (recall that $1_{\mathbf{A}} \neq \Psi$). □

Theorem 5.5. *There is a non-full embedding* $\mathbf{RdVar} \overset{M}{\hookrightarrow} \mathbf{PwrThr} \bowtie \mathbf{RdVar}$ *given by* $M(\|-\|_1 \xrightarrow{(\Phi,\Psi,\varepsilon)} \|-\|_2) = (1_{\mathbf{A}_1^{op}}, \|-\|_1) \xrightarrow{((\Phi^{op},\Phi,1_{\Phi^{op}}),(\Phi,\Psi,\varepsilon))} (1_{\mathbf{A}_2^{op}}, \|-\|_2)$.

Proof. In view of the previous theorem, we show that the embedding is non-full only. Let $\mathbf{A}$ be a variety with a terminal object T (e.g., the variety $\mathbf{CSLat}(\bigvee)$ or $\mathbf{Frm}$). There exists the constant functor $\mathbf{A} \xrightarrow{\Phi} \mathbf{A}$ taking everything to the identity morphism $T \xrightarrow{1_T} T$. Moreover, there exists the natural transformation $1_{\mathbf{A}} \xrightarrow{\eta} \Phi$ defined by $1_{\mathbf{A}}A \xrightarrow{\eta_A} \Phi A = A \xrightarrow{!} T$, where ! is the unique morphism induced by T. The two together provide a $\mathbf{PwThr}$-morphism $1_{\mathbf{A}^{op}} \xrightarrow{(\Phi^{op},1_{\mathbf{A}},\eta^{op})} 1_{\mathbf{A}^{op}}$ and we can easily assume that $1_{\mathbf{A}} \neq \Phi$. Since $1_{\mathbf{A}} \xrightarrow{(1_{\mathbf{A}},1_{\mathbf{A}},1_{1_{\mathbf{A}^{op}}})} 1_{\mathbf{A}}$ is an $\mathbf{RdVar}$-morphism,

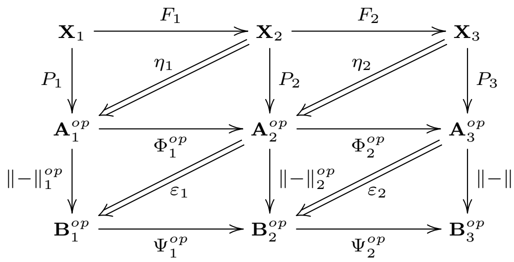
$(1_{\mathbf{A}^{op}}, 1_{\mathbf{A}}) \xrightarrow{(\Phi^{op}, 1_{\mathbf{A}}, \eta^{op}), (1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{1_{\mathbf{A}^{op}}})} (1_{\mathbf{A}^{op}}, 1_{\mathbf{A}})$ is in $\mathbf{PwThr} \bowtie \mathbf{RdVar}$ and is not in the image of M (recall that $1_{\mathbf{A}} \neq \Phi$). $\square$

Theorems 5.4, 5.5, being simple, show that the category $\mathbf{PwThr} \bowtie \mathbf{RdVar}$ gives a common framework for both **PwThr** and **RdVar**. In the next subsection, we show how it relates to the procedure of reducing redundant algebraic properties.

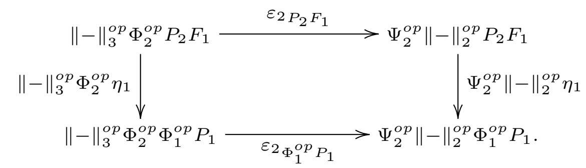
5.2. **From Powerset Theories to Topological Theories.** The main purpose of this section is to show a simple way of producing topological theories from powerset theories and their respective variety reducts.

Theorem 5.6. *There exists a functor* $\mathbf{PwrThr} \bowtie \mathbf{RdVar} \xrightarrow{\mathrm{T}} \mathbf{TpThr}$*, which is defined by the formula* $\mathrm{T}((P_1, \| - \|_1) \xrightarrow{((F,\Phi,\eta),(\Phi,\Psi,\varepsilon))} (P_2, \| - \|_2)) = T_1 \xrightarrow{(F,\Psi,\varepsilon \circledcirc \eta)} T_2$*, where* $T_i = \| - \|_i^{op} P_i$ *and* $\| - \|_2^{op} P_2 F \xrightarrow{\varepsilon \circledcirc \eta} \Psi^{op} \| - \|_1^{op} P_1 = \| - \|_2^{op} P_2 F \xrightarrow{\| - \|_2^{op} \eta} \| - \|_2^{op} \Phi^{op} P_1 \xrightarrow{\varepsilon_{P_1}} \Psi^{op} \| - \|_1^{op} P_1$.

Proof. The only real challenge is to show the preservation of composition. Given two $\mathbf{PwrThr} \bowtie \mathbf{RdVar}$-morphisms $(P_1, \| - \|_1) \xrightarrow{((F_1,\Phi_1,\eta_1),(\Phi_1,\Psi_1,\varepsilon_1))} (P_2, \| - \|_2)$ and $(P_2, \| - \|_2) \xrightarrow{((F_2,\Phi_2,\eta_2),(\Phi_2,\Psi_2,\varepsilon_2))} (P_3, \| - \|_3)$, on the one hand, we get that $\mathrm{T}(((F_2, \Phi_2, \eta_2), (\Phi_2, \Psi_2, \varepsilon_2)) \circ ((F_1, \Phi_1, \eta_1), (\Phi_1, \Psi_1, \varepsilon_1))) = \mathrm{T}((F_2 F_1, \Phi_2 \Phi_1, \eta_2 \odot \eta_1), (\Phi_2 \Phi_1, \Psi_2 \Psi_1, \varepsilon_2 \odot \varepsilon_1)) = (F_2 F_1, \Psi_2 \Psi_1, (\varepsilon_2 \odot \varepsilon_1) \circledcirc (\eta_2 \odot \eta_1))$, whereas, on the other hand, we can obtain $\mathrm{T}((F_2, \Phi_2, \eta_2), (\Phi_2, \Psi_2, \varepsilon_2)) \circ \mathrm{T}((F_1, \Phi_1, \eta_1), (\Phi_1, \Psi_1, \varepsilon_1)) = (F_2, \Psi_2, \varepsilon_2 \circledcirc \eta_2) \circ (F_1, \Psi_1, \varepsilon_1 \circledcirc \eta_1) = (F_2 F_1, \Psi_2 \Psi_1, (\varepsilon_2 \circledcirc \eta_2) \odot (\varepsilon_1 \circledcirc \eta_1))$. In what follows, we use the diagram



and the procedure similar to the interchange law of the star product (already mentioned in this paper) of natural transformations given in [32, Theorem 13.12]. $(\varepsilon_2 \odot \varepsilon_1) \circledcirc (\eta_2 \odot \eta_1) = (\varepsilon_2 \odot \varepsilon_1)_{P_1} \circ \| - \|_3^{op} (\eta_2 \odot \eta_1) = (\Psi_2^{op} \varepsilon_1 \circ \varepsilon_{2\Phi_1^{op}})_{P_1} \circ \| - \|_3^{op} (\Phi_2^{op} \eta_1 \circ \eta_{2F_1}) = \Psi_2^{op} \varepsilon_{1P_1} \circ \varepsilon_{2\Phi_1^{op} P_1} \circ \| - \|_3^{op} \Phi_2^{op} \eta_1 \circ \| - \|_3^{op} \eta_{2F_1} \overset{(\dagger)}{=} \Psi_2^{op} \varepsilon_{1P_1} \circ \Psi_2^{op} \| - \|_2^{op} \eta_1 \circ \varepsilon_{2P_2F_1} \circ \| - \|_3^{op} \eta_{2F1} = \Psi_2^{op} (\varepsilon_{1P_1} \circ \| - \|_2^{op} \eta_1) \circ (\varepsilon_{2P_2} \circ \| - \|_3^{op} \eta_2)_{F_1} = \Psi_2^{op} (\varepsilon_1 \circledcirc \eta_1) \circ (\varepsilon_2 \circledcirc \eta_2)_{F_1} = (\varepsilon_2 \circledcirc \eta_2) \odot (\varepsilon_1 \circledcirc \eta_1)$, in which (†) relies on commutativity of the diagram



□

Combining the functor T of Theorem 5.6 with the functors Top of Theorem 3.8 (notice that the respective subquasicategory of **PwrThr** ⋈ **RdVar** is required) and TopSys of Theorem 3.13, one obtains the possibility to move from powerset theories to the categories of topological structures generated by their corresponding topological theories. It will be the topic of our further research to study this opening thoroughly. In particular, the properties of the functor T itself provide a nice challenge for research. One can easily see that T is surjective on objects. Other properties like, e.g., faithfulness, fullness, *etc.* are still in progress.

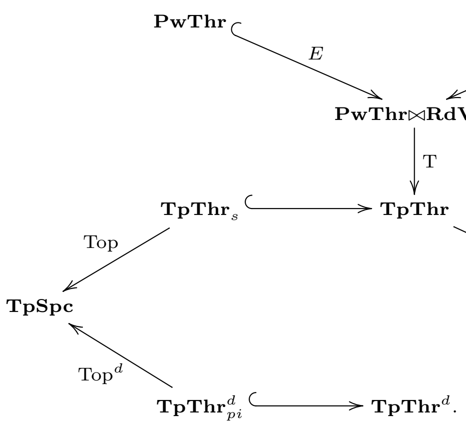
The theorem of this subsection is the last result on the topic of the paper obtained so far. The next concluding section of the manuscript provides a brief summary of the available results and mentions open problems related to the new setting.

6. **Conclusion**

In this paper, we presented a new approach to (lattice-valued) topology based on topological theories, which in their turn rely on category theory and universal algebra. At the bottom of the proposed framework lies categorically-algebraic (catalg) topology, which provides a new viewpoint on topological structures and their related concepts such as, e.g., topological systems. The main achievements of the manuscript are

(1) the definition of the quasicategory **TpThr** of catalg topological theories and the study of its functorial relationships with the quasicategory of the categories of topological structures generated by the theories;
(2) the definition of the quasicategory **PwThr** of catalg powerset theories and the study of its functorial relationships with the quasicategory **TpThr**.

Our obtained most important functorial relations are given in the next diagram:



An interested reader will find many ways in which the above diagram can (and should) be completed. Moreover, the framework of catalg topological theories allowed us to formulate a topological analogue of Morita problem (Problems 4.1, 4.3) well-known in categorical algebra, providing an opening for developing a topological counterpart of *Morita theory* [9, Section 3.12]. Being unable to provide a general solution for the topological Morita problem, we have presented an answer to its reduced form. Our main (and ambitious) goal is to develop the setting of topological theories up to the extent of algebraic theories of [3], [9, Chapter 3]. To give a starting point for the research, below we list some other open problems, which appear of importance to the already existing developments (the reader is also invited to address numerous open questions scattered throughout the paper).

6.1. **From Topological Structures to Their Generated Theories.** We have already mentioned topological theories of J. Adámek *et al.* [2, Exercise 22B] and their respective categories of models of the form $\mathbf{Top}(T)$, which, despite coincidence in the notations, are different from the similar categories of this paper. In particular, there exists the so-called *fibre-functor*, which associates with every topological category $(\mathbf{D}, U)$ over $\mathbf{X}$ a topological theory $\mathbf{X} \xrightarrow{T} \mathbf{CSLat}(\bigvee)$ (in the sense of J. Adámek *et al.*) such that its respective category $\mathbf{Top}(T)$ is concretely isomorphic to $(\mathbf{D}, U)$. Reconsidering the result in our setting, we get the next problem.

Problem 6.1. Does there exist a functor from **TpSpc** (resp. **TpStm**) to **TpThr**?

In case of a positive answer, following [3, Theorem 9.15], which provides a biequivalence between the categories Th_c^{op} (canonical algebraic theories, already mentioned in Subsection 3.2) and ALG, one could question the existence of an adjoint situation between the quasicategories **TpSpc** (resp. **TpStm**) and some subquasicategory of **TpThr**. As a necessary criteria, one should check preservation of (co)limits by Top (resp. TopSys), opening the problem on their existence in **TpThr**. At the moment, we are unable to construct even binary (co)products in the latter category, the main sticking point being the varietal description of (co)product of varieties. On the other hand, there is a possibility to check preservation of monomorphisms or epimorphisms by Top (resp. TopSys), posing the problem on their characterization in the category **TpThr**. In one word, the categorical structure of **TpThr** is the subject of immediate study.

6.2. **Cartesian Closedness of the Categories of Topological Structures.** It is well-known that the category **Top** is not cartesian closed [2, Example 27.3(2)]. To remove the deficiency, suitable supercategories of **Top** are considered in the literature, e.g., the category of *convergence spaces* of G. Jäger [37] or G. Preuß [47]. The setting of topological theories developed in this paper suggests the following problem (originally, posed by W. Tholen (York University, Canada), during the poster presentation of the author at the “ 91st Peripatetic Seminar on Sheaves and Logic”, Amsterdam, The Netherlands, November 27 - 28, 2010).

Problem 6.2. Describe those topological theories T, which give cartesian closed categories $\mathbf{Top}(T)$.

A general solution to this problem (if obtainable), could be of extreme importance to the setting of (lattice-valued) topology.

6.3. **Catalg Extension of Topological Properties.** An important point of the catalg setting is that which topological properties (e.g., separation, compactness, connectedness, *etc.*) are applicable in the case of objects of categories of the form $\mathbf{Top}(T)$. For example, it is widely-known that the category of compact Hausdorff spaces is *monadic*, i.e., isomorphic to the category of Eilenberg-Moore algebras for a monad on the category of sets [44, Section VI.9]. As a result, there exists a particular "algebraic" theory (i.e., a monad, which is a more general approach to categorical algebra than that of F. W. Lawvere), whose models (Eilenberg-Moore algebras) are precisely the compact Hausdorff spaces. In view of the fact, we formulate the next open problem of this paper (also suggested by W. Tholen).

Problem 6.3. Given a topological property, characterize those theories T, whose corresponding categories $\mathbf{Top}(T)$ contain precisely the spaces with this property.

6.4. **Topological Theories of J. Adámek *et al.* Versus the Approach of This Paper.** The first problem of this section was motivated by the already frequently mentioned topological theories of J. Adámek *et al.* [2, Exercise 22B]. Moreover, their topological theories gave rise to our catalg approach. Recently, S. E. Rodabaugh [52, Section 3.6] completely resolved the relationship between his categorical approach to topology and the respective one of J. Adámek *et al.*, failing, however, to decide which of the two is more general (even despite the fact that both approaches are based in a fixed variety of algebras, the respective one of J. Adámek *et al.* being more general than that of S. E. Rodabaugh). In view of the result, we can postulate the last problem of this paper.

Problem 6.4. What are the relationships between the topological theories of this paper and the respective ones of J. Adámek *et al.*? In particular, does catalg topology incorporate the latter approach?

All the problems formulated in this paper will be addressed in our subsequent manuscripts on the topic.

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