

CVAR REDUCED FUZZY VARIABLES AND THEIR SECOND ORDER MOMENTS

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ABSTRACT. Based on credibilistic value-at-risk (CVaR) of regular fuzzy variable, we introduce a new CVaR reduction method for type-2 fuzzy variables. The reduced fuzzy variables are characterized by parametric possibility distributions. We establish some useful analytical expressions for mean values and second order moments of common reduced fuzzy variables. The convex properties of second order moments with respect to parameters are also discussed. Finally, we take second order moment as a new risk measure, and develop a mean-moment model to optimize fuzzy portfolio selection problems. According to the analytical formulas of second order moments, the mean-moment optimization model is equivalent to parametric quadratic convex programming problems, which can be solved by general-purpose optimization software. The solution results reported in the numerical experiments demonstrate the credibility of the proposed optimization method.

1. Introduction

As a natural extension of type-1 fuzzy set, type-2 fuzzy set was first introduced by Zadeh [19]. Since then, type-2 fuzzy set theory has been further explored in the literature. Dubois and Prade [3] investigated the operations in a fuzzy-valued logic. Mizumoto and Tanaka [14] discussed what kinds of algebraic structures the grades of type-2 fuzzy sets form under join, meet and negation. Mendel [12] summarized the developments and applications of type-2 fuzzy sets before the year 2001. Mendel and John [13] introduced some basic concepts to characterize type-2 fuzzy set, including the type-2 membership function, the secondary membership function, the footprint of uncertainty and so on. Hu and Wang [5] introduced interval-valued type-2 fuzzy sets and interval-valued type-2 fuzzy relations and discussed their properties. In the literature, type-2 fuzzy numbers have been used to indicate the similarity degree of fuzzy sets [1, 7]. At the same time, type-2 fuzzy sets have been applied successfully to many application-oriented fields. For example, Kundu et al. [8] considered two fixed charge transportation problems with type-2 fuzzy parameters. Huang et al. [6] proposed a novel dynamic optimal path planning method, which employed a type-2 fuzzy logic inference system to path analysis for each cell established in the cellular automata algorithm. For recent developments about type-2 fuzzy theory,

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the interested readers may refer to Hao and Mendel [4], Ngan [16], Moharrer et al. [15] and Torshizi and Zarandi [17].

Defuzzification is one of the critical steps involved in type-2 fuzzy system. By reduction, all computations in three-dimensional space are degenerated into calculations in two-dimensional plane so as to reduce greatly the computing complexity. Several reduction methods for type-2 fuzzy sets have been presented in the literature. Coupland and John [2] presented a geometric-based defuzzification method for type-2 fuzzy sets. Liu [10] gave a centroid type-reduction strategy for interval type-2 fuzzy sets. Motivated by the work mentioned above, the present paper introduces a new reduction method in fuzzy possibility theory. First, we define the CVaR for regular fuzzy variables based on credibility measure. Then, we develop CVaR reduction method for secondary possibility distributions. The idea of the proposed method is to reduce uncertainty in secondary possibility distributions, and retains the most important information in the parametric possibility distributions of reduced fuzzy variables. There are two types of parameters included in the obtained parametric possibility distributions. The first type of parameters is to describe the degree of uncertainty that a type-2 fuzzy variable takes on its values, while the second type of parameters is to represent the credibility level in the support of type-2 fuzzy variable. From the geometrical viewpoint, the first type of parameters determines the shape of the support of a type-2 fuzzy variable, while the second type of parameters determines the location of possibility distribution in the support of the type-2 fuzzy variable. From this viewpoint, our CVaR reduction method has some advantages over other existing methods by introducing the location parameter in the possibility distribution. Since the CVaR reduced fuzzy variables have parametric possibility distributions, the computation about their numerical characteristics is an interesting issue for research. In this paper, we first derive the analytical expressions of mean values for common reduced fuzzy variables. Then we derive the analytical expressions of the second order moments to measure the variations of parametric possibility distributions with respect to mean values. Finally, we take mean value and second order moment as two optimization indexes, and apply them to fuzzy portfolio selection problems.

The rest of this paper is organized as follows. Section 2 introduces some concepts in fuzzy theory. Section 3 defines the CVaR for regular fuzzy variable, and derives some useful CVaR formulas for common regular fuzzy variables. In Section 4, we develop the CVaR reduction method for type-2 fuzzy variables. For common reduced fuzzy variables, Section 5 discusses the computation of mean values, and Section 6 establishes the analytical expressions of the second order moments. Section 7 provides a practical application example about portfolio selection problem, in which the proposed second order moment is taken as a new measure to gauge the risk resulted from fuzzy uncertainty. Finally, Section 8 concludes the paper.

2. Fundamental Concepts

Let $(\Gamma, \mathcal{A}, \text{Pos})$ be a possibility space, where Γ is the universe of discourse, \mathcal{A} an ample field on Γ that is closed under arbitrary unions, intersections and complement, and Pos a possibility measure on \mathcal{A} . Using possibility measure, the credibility

measure of an event $A \in \mathcal{A}$ was defined as

$$\text{Cr}(A) = \frac{1}{2} (1 + \text{Pos}(A) - \text{Pos}(A^c)),$$

where $A^c = \Gamma \setminus A$ is the complementary event of A .

Definition 2.1. [18] Let (Γ, \mathcal{A}) be an ample space. A function $\xi : \Gamma \rightarrow \mathfrak{R}$ is called a fuzzy variable if

$$\{\gamma \mid \xi(\gamma) \leq r\} \in \mathcal{A}$$

for any $r \in \mathfrak{R}$.

Let $(\Gamma, \mathcal{A}, \text{Pos})$ be a possibility space. An m -ary regular fuzzy vector $\xi = (\xi_1, \dots, \xi_m)$ is defined as a measurable map from Γ to the space $[0, 1]^m$ in the sense that for every $r = (r_1, \dots, r_m) \in [0, 1]^m$, one has

$$\{\gamma \in \Gamma \mid \xi(\gamma) \leq r\} = \{\gamma \in \Gamma \mid \xi_1(\gamma) \leq r_1, \xi_2(\gamma) \leq r_2, \dots, \xi_m(\gamma) \leq r_m\} \in \mathcal{A}.$$

When $m = 1$, ξ is called a regular fuzzy variable (RFV).

In this paper, we denote $\mathcal{R}([0, 1])$ as the collection of all RFVs on $[0, 1]$.

If $\xi = (r_1, r_2, r_3)$ with $0 \leq r_1 < r_2 < r_3 \leq 1$, then ξ is a triangular RFV. Similarly, if $\xi = (r_1, r_2, r_3, r_4)$ with $0 \leq r_1 < r_2 \leq r_3 < r_4 \leq 1$, then ξ is a trapezoidal RFV.

Definition 2.2. [9] Let ξ be a fuzzy variable defined on a possibility space $(\Gamma, \mathcal{A}, \text{Pos})$. The credibility distribution function of ξ is defined by

$$G_\xi(r) = \text{Cr}\{\gamma \in \Gamma \mid \xi(\gamma) \leq r\}, \quad r \in \mathfrak{R}.$$

Example 2.3. Let $\xi = (0.2, 0.4, 0.5, 0.85)$ be a trapezoidal RFV. The possibility distribution of ξ is shown in Figure 1. The credibility distribution of ξ is computed by

$$\text{Cr}\{\xi \leq r\} = \begin{cases} 0, & \text{if } x < 0.2 \\ \frac{5r-1}{2}, & \text{if } 0.2 \leq r \leq 0.4 \\ \frac{1}{2}, & \text{if } 0.4 \leq r \leq 0.5 \\ \frac{20r-3}{14}, & \text{if } 0.5 \leq r \leq 0.85 \\ 1, & \text{if } x > 0.85, \end{cases}$$

which is plotted in Figure 2.

Let $\tilde{\text{Pos}} : \mathcal{A} \rightarrow \mathcal{R}([0, 1])$ be a set function defined on \mathcal{A} such that $\{\tilde{\text{Pos}}(A) \mid A \in \mathcal{A} \text{ atom}\}$ is a family of mutually independent RFVs. We call $\tilde{\text{Pos}}$ a fuzzy possibility measure if it satisfies the following conditions:

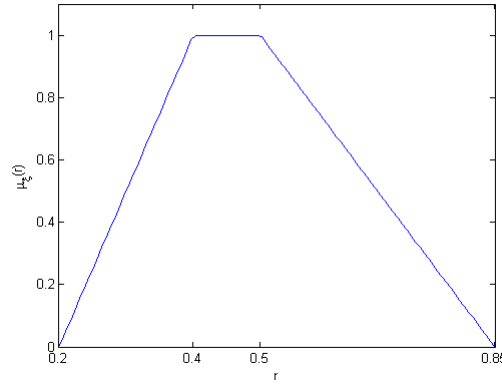
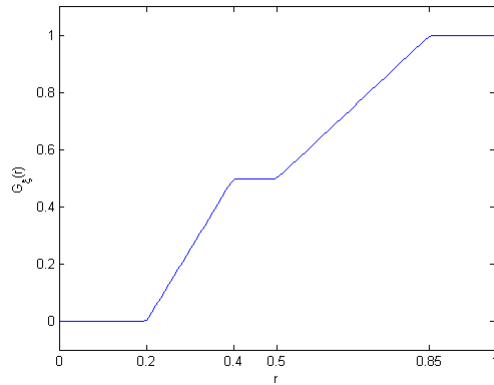
$$(\tilde{P}1): \tilde{\text{Pos}}(\emptyset) = \tilde{0};$$

$$(\tilde{P}2): \text{For any subclass } \{A_i \mid i \in I\} \text{ of } \mathcal{A} \text{ (finite, countable or uncountable)}$$

$$\tilde{\text{Pos}}\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \tilde{\text{Pos}}(A_i).$$

Moreover, if $\mu_{\tilde{\text{Pos}}(\Gamma)}(1) = 1$, then we call $\tilde{\text{Pos}}$ a normalized fuzzy possibility measure.

The triplet $(\Gamma, \mathcal{A}, \tilde{\text{Pos}})$ is referred to as a fuzzy possibility space (FPS).

FIGURE 1. The Possibility Distribution of ξ FIGURE 2. The Credibility Distribution of ξ

Remark 2.4. Fuzzy possibility measure is a generalization of (non-fuzzy) possibility measure in the literature. That is, if for any $A \in \mathcal{A}$, $\tilde{\text{Pos}}(A)$ is a crisp number in $[0, 1]$ instead of a fuzzy number in $[0, 1]$, then $\tilde{\text{Pos}}$ is just a (non-fuzzy) possibility measure, and denoted by Pos . For the sake of clarity, we provide the following example to explain the difference between possibility measure and fuzzy possibility measure.

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $\mathcal{A} = \mathcal{P}(\Gamma)$, the power set of Γ . Define a set function Pos on $\mathcal{P}(\Gamma)$ as follows:

$$\text{Pos}\{\gamma_1\} = 0.2, \text{Pos}\{\gamma_2\} = 1, \text{Pos}\{\gamma_3\} = 0.6,$$

and for any other set $A \in \mathcal{P}(\Gamma)$, $\text{Pos}(A) = \max_{\gamma_i \in A} \text{Pos}\{\gamma_i\}$. Then Pos is a possibility measure, and $(\Gamma, \mathcal{A}, \text{Pos})$ is a possibility space.

On the other hand, if we define a set function $\tilde{\text{Pos}} : \mathcal{P}(\Gamma) \rightarrow \mathcal{R}([0, 1])$ as

$$\text{Pos}\{\gamma_1\} = (0.15, 0.2, 0.4), \text{Pos}\{\gamma_2\} = \tilde{1}, \text{Pos}\{\gamma_3\} = (0.5, 0.6, 0.7),$$

and for any other set $A \in \mathcal{P}(\Gamma)$, $\tilde{\text{Pos}}(A) = \max_{\gamma_i \in A} \tilde{\text{Pos}}\{\gamma_i\}$, then $\tilde{\text{Pos}}$ is a fuzzy possibility measure, and $(\Gamma, \mathcal{A}, \tilde{\text{Pos}})$ is a fuzzy possibility space.

Let $(\Gamma, \mathcal{A}, \tilde{\text{Pos}})$ be an FPS. A map $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_m) : \Gamma \rightarrow \mathfrak{R}^m$ is called an m -ary type-2 fuzzy vector if for any $r = (r_1, r_2, \dots, r_m) \in \mathfrak{R}^m$, the set $\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) \leq r\}$ is an element of \mathcal{A} , i.e.,

$$\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) \leq r\} = \{\gamma \in \Gamma \mid \tilde{\xi}_1(\gamma) \leq r_1, \tilde{\xi}_2(\gamma) \leq r_2, \dots, \tilde{\xi}_m(\gamma) \leq r_m\} \in \mathcal{A}.$$

As $m = 1$, the map $\tilde{\xi} : \Gamma \rightarrow \mathfrak{R}$ is called a type-2 fuzzy variable.

Let $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_m)$ be a type-2 fuzzy vector. The secondary possibility distribution function $\tilde{\mu}_{\tilde{\xi}}(x)$ of $\tilde{\xi}$ is a map $\mathfrak{R}^m \rightarrow \mathcal{R}([0, 1])$ such that

$$\tilde{\mu}_{\tilde{\xi}}(x) = \tilde{\text{Pos}}\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) = x\}, \quad x \in \mathfrak{R}^m,$$

while the type-2 possibility distribution function $\mu_{\tilde{\xi}}(x)$ of $\tilde{\xi}$ is a map $\mathfrak{R}^m \times J_x \rightarrow [0, 1]$ such that

$$\mu_{\tilde{\xi}}(x, u) = \text{Pos}\{\tilde{\mu}_{\tilde{\xi}}(x) = u\}, \quad (x, u) \in \mathfrak{R}^m \times J_x,$$

where Pos is the possibility measure induced by the distribution of $\tilde{\mu}_{\tilde{\xi}}(x)$, and $J_x \subset [0, 1]$ is the support of $\tilde{\mu}_{\tilde{\xi}}(x)$, i.e., $J_x = \{u \in [0, 1] \mid \tilde{\mu}_{\tilde{\xi}}(x, u) > 0\}$.

The support of a type-2 fuzzy vector $\tilde{\xi}$ is defined as

$$\text{supp } \tilde{\xi} = \{(x, u) \in \mathfrak{R}^m \times [0, 1] \mid \mu_{\tilde{\xi}}(x, u) > 0\},$$

where $\mu_{\tilde{\xi}}(x, u)$ is the type-2 possibility distribution function of $\tilde{\xi}$.

Example 2.5. Let $\tilde{\xi} = (\tilde{3}, \tilde{5}, \tilde{9}; 0.5, 0.8)$ be a type-2 triangular fuzzy variable. The support of $\tilde{\xi}$ is shown in Figure 3.

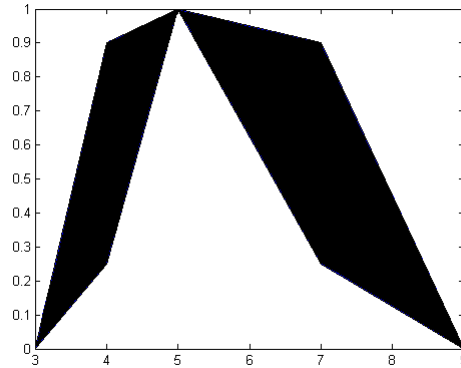


FIGURE 3. The Support of Type-2 Triangular Fuzzy Variable $\tilde{\xi}$

3. The CVaRs for Regular Fuzzy Variables

If ξ is a regular fuzzy variable, then the CVaR of ξ , denoted by $\text{CVaR}_\alpha(\xi)$, is defined by

$$\text{CVaR}_\alpha(\xi) = \inf\{r \mid \text{Cr}\{\xi \leq r\} \geq \alpha\}, \alpha \in (0, 1].$$

The CVaR of ξ is different from the lower VaR and upper VaR defined by possibility measure for a regular fuzzy variable. When a regular fuzzy variable ξ has continuous possibility distribution, we have the following relations among the CVaR, lower VaR and upper VaR. For any $\alpha \in (0, 1)$, we have $\text{CVaR}_{\frac{\alpha}{2}}(\xi) = \text{VaR}_\alpha^L(\xi)$, and $\text{CVaR}_{1-\frac{\alpha}{2}}(\xi) = \text{VaR}_\alpha^U(\xi)$.

In the following, we derive some useful VaR formulas for common regular fuzzy variables.

Theorem 3.1. *If ξ is a triangular regular fuzzy variable, then we have*

$$\text{CVaR}_\alpha(\xi) = \begin{cases} r_1 + 2\alpha(r_2 - r_1), & \text{if } \alpha \in (0, 0.5] \\ 2r_2 - r_3 + 2\alpha(r_3 - r_2), & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

Proof. According to the possibility distribution of ξ , we have the following credibility distribution of ξ ,

$$\text{Cr}\{\xi \leq x\} = \begin{cases} 0, & \text{if } x < r_1 \\ \frac{x-r_1}{2(r_2-r_1)}, & \text{if } r_1 \leq x \leq r_2 \\ \frac{r_3-2r_2+x}{2(r_3-r_2)}, & \text{if } r_2 \leq x \leq r_3 \\ 1, & \text{if } x > r_3. \end{cases}$$

If $\alpha \in (0, 0.5]$, then $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$\frac{x - r_1}{2(r_2 - r_1)} - \alpha = 0.$$

Therefore, we have $\text{CVaR}_\alpha(\xi) = r_1 + 2\alpha(r_2 - r_1)$.

On the other hand, if $\alpha \in (0.5, 1]$, then $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$\frac{r_3 - 2r_2 + x}{2(r_3 - r_2)} - \alpha = 0.$$

Thus, we have $\text{CVaR}_\alpha(\xi) = 2r_2 - r_3 + 2\alpha(r_3 - r_2)$. The proof of theorem is complete. \square

As a consequence of Theorem 3.1, we have the following results about the relations among the VaRs, upper mean value E^* , lower mean value E_* and mean value E of regular triangular fuzzy variable.

Corollary 3.2. *Let $\xi = (r_0 - \theta_l, r_0, r_0 + \theta_r)$ be a triangular regular fuzzy variable with $\theta_l, \theta_r > 0$.*

- (i) *If $\alpha = 3/4$, then $\text{CVaR}_\alpha(\xi) = E^*[\xi]$;*
- (ii) *If $\alpha = 1/4$, then $\text{CVaR}_\alpha(\xi) = E_*[\xi]$;*
- (iii) *If $\theta_l \leq \theta_r$ and $\alpha = (5\theta_r - \theta_l)/8\theta_r$, then $\text{CVaR}_\alpha(\xi) = E[\xi]$;*
- (iv) *If $\theta_l \geq \theta_r$ and $\alpha = (3\theta_l + \theta_r)/8\theta_l$, then $\text{CVaR}_\alpha(\xi) = E[\xi]$.*

Theorem 3.3. *If ξ is a trapezoidal regular fuzzy variable, then we have*

$$\text{CVaR}_\alpha(\xi) = \begin{cases} r_1 + 2\alpha(r_2 - r_1), & \text{if } \alpha \in (0, 0.5] \\ 2r_3 - r_4 + 2\alpha(r_4 - r_3), & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

Proof. First, the credibility distribution of fuzzy variable ξ is calculated by

$$\text{Cr}\{\xi \leq x\} = \begin{cases} 0, & \text{if } x < r_1 \\ \frac{x-r_1}{2(r_2-r_1)}, & \text{if } r_1 \leq x \leq r_2 \\ \frac{1}{2}, & \text{if } r_2 \leq x \leq r_3 \\ \frac{r_4-2r_3+x}{2(r_4-r_3)}, & \text{if } r_3 \leq x \leq r_4 \\ 1, & \text{if } x > r_4. \end{cases}$$

If $\alpha \in (0, 0.5]$, then $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$\frac{x - r_1}{2(r_2 - r_1)} - \alpha = 0.$$

Thus, we have $\text{CVaR}_\alpha(\xi) = r_1 + 2\alpha(r_2 - r_1)$.

On the other hand, if $\alpha \in (0.5, 1]$, then $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$\frac{r_4 - 2r_3 + x}{2(r_4 - r_3)} - \alpha = 0.$$

Thus, we have $\text{CVaR}_\alpha(\xi) = 2r_3 - r_4 + 2\alpha(r_4 - r_3)$. The proof of theorem is complete. \square

Theorem 3.4. *Let ξ be a normal regular fuzzy variable with the following possibility distribution*

$$\mu_\xi(x) = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in [0, 1], \mu \in [0, 1].$$

If we denote $a = \exp(-\mu^2/2\sigma^2)$ and $b = \exp(-(1-\mu)^2/2\sigma^2)$, then we have

$$\text{CVaR}_\alpha(\xi) = \begin{cases} \mu - \sqrt{-2\sigma^2 \ln 2\alpha}, & \text{if } \alpha \in [\frac{a}{2}, 0.5] \\ \mu + \sqrt{-2\sigma^2 \ln 2(1-\alpha)}, & \text{if } \alpha \in (0.5, 1 - \frac{b}{2}]. \end{cases}$$

Proof. The credibility distribution of fuzzy variable ξ is the following function

$$\text{Cr}\{\xi \leq x\} = \begin{cases} \frac{1}{2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } 0 \leq x \leq \mu \\ 1 - \frac{1}{2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } \mu \leq x \leq 1. \end{cases}$$

If $\alpha \in (a/2, 0.5]$, then $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$\frac{1}{2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - \alpha = 0, \quad 0 \leq x \leq \mu.$$

Hence, we have $\text{CVaR}_\alpha(\xi) = \mu - \sqrt{-2\sigma^2 \ln 2\alpha}$.

On the other hand, if $\alpha \in (0.5, 1 - b/2]$, then $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$1 - \frac{1}{2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - \alpha = 0, \quad \mu \leq x \leq 1.$$

Thus, we have $\text{CVaR}_\alpha(\xi) = \mu + \sqrt{-2\sigma^2 \ln 2(1-\alpha)}$. The proof of theorem is complete. \square

Theorem 3.5. *Let ξ be a gamma regular fuzzy variable with the following possibility distribution*

$$\mu_\xi(x) = \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), \quad x \in [0, 1],$$

where r is a positive integer, $0 < \lambda \leq 1/r$, and denote $c = (1/\lambda r)^r \exp(r - 1/\lambda)$.

(i) *If $\alpha \in (0, 0.5]$, then $\text{CVaR}_\alpha(\xi)$ of ξ is the solution of the following equation*

$$\frac{1}{2} \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \alpha = 0, \quad x \in [0, \lambda r];$$

(ii) *If $\alpha \in (0.5, 1 - c/2]$, then $\text{CVaR}_\alpha(\xi)$ of ξ is the solution of the following equation*

$$1 - \frac{1}{2} \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \alpha = 0, \quad x \in [\lambda r, 1].$$

Proof. We only prove assertion (i), and assertion (ii) can be proved similarly. According to the possibility distribution of ξ , we have

$$\text{Cr}\{\xi \leq x\} = \begin{cases} \frac{1}{2} \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } 0 \leq x \leq \lambda r \\ 1 - \frac{1}{2} \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } \lambda r \leq x \leq 1. \end{cases}$$

By the definition of $\text{CVaR}_\alpha(\xi)$, we know that $\text{CVaR}_\alpha(\xi)$ is the solution of the following equation

$$\frac{1}{2} \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \alpha = 0, \quad x \in [0, \lambda r],$$

which completes the proof of assertion (i). \square

4. A New CVaR Reduction Method

Let $(\Gamma, \mathcal{A}, \tilde{\text{Pos}})$ be a fuzzy possibility space, and $\tilde{\xi}$ a type-2 fuzzy variable with secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$. To reduce the uncertainty in $\tilde{\mu}_{\tilde{\xi}}(x)$, we employ the CVaR of $\tilde{\text{Pos}}\{\tilde{\xi} = x\}$ as the representing value of $\tilde{\mu}_{\tilde{\xi}}(x)$. The method is referred to as the CVaR reduction. The reduced fuzzy variable obtained by CVaR reduction method is denoted by ξ .

There are two kinds of parameters included in the possibility distributions of CVaR reduced fuzzy variables. The first type parameters θ_l and θ_r represent the degree of uncertainty that a type-2 fuzzy variable $\tilde{\xi}$ takes on its value x , while the second type parameter α means the credibility level in the support of a type-2 fuzzy variable. From the geometrical viewpoint, the parameters θ_l and θ_r determine the lower and upper boundaries of possibility distribution, while α determines the location of possibility distribution between the lower boundary and upper boundary. When parameter α varies in the interval $(0, 1]$, the possibility distribution varies between the lower and upper boundaries. As a consequence, our CVaR reduction method is more flexible than other existing reduction methods by introducing the parameter α in possibility distributions.

In the following, we discuss the CVaR reduction for common type-2 fuzzy variables.

Theorem 4.1. Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable, and $\theta = (\theta_l, \theta_r)$.

(i) If $\alpha \in (0, 0.5]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-2\alpha\theta_l)x-(1-2\alpha)\theta_l r_2-r_1}{r_2-r_1}, & \text{if } x \in [\frac{r_1+r_2}{2}, r_2] \\ \frac{-(1+\theta_l-2\alpha\theta_l)x+(1-2\alpha)\theta_l r_2+r_3}{r_3-r_2}, & \text{if } x \in [r_2, \frac{r_2+r_3}{2}] \\ (1 - \theta_l + 2\alpha\theta_l) \frac{r_3-x}{r_3-r_2}, & \text{if } x \in [\frac{r_2+r_3}{2}, r_3]. \end{cases}$$

(ii) If $\alpha \in (0.5, 1]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_r + 2\alpha\theta_r) \frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_r-2\alpha\theta_r)x-(1-2\alpha)\theta_r r_2-r_1}{r_2-r_1}, & \text{if } x \in [\frac{r_1+r_2}{2}, r_2] \\ \frac{-(1+\theta_r-2\alpha\theta_r)x+(1-2\alpha)\theta_r r_2+r_3}{r_3-r_2}, & \text{if } x \in [r_2, \frac{r_2+r_3}{2}] \\ (1 - \theta_r + 2\alpha\theta_r) \frac{r_3-x}{r_3-r_2}, & \text{if } x \in [\frac{r_2+r_3}{2}, r_3]. \end{cases}$$

Proof. We only prove the second assertion, and the first can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_\xi(x)$ of $\tilde{\xi}$ is the following triangular regular fuzzy variable

$$\left(\frac{x-r_1}{r_2-r_1} - \theta_l \min \left\{ \frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1} \right\}, \frac{x-r_1}{r_2-r_1}, \frac{x-r_1}{r_2-r_1} + \theta_r \min \left\{ \frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1} \right\} \right)$$

for any $x \in [r_1, r_2]$, and

$$\left(\frac{r_3-x}{r_3-r_2} - \theta_l \min \left\{ \frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2} \right\}, \frac{r_3-x}{r_3-r_2}, \frac{r_3-x}{r_3-r_2} + \theta_r \min \left\{ \frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2} \right\} \right)$$

for any $x \in [r_2, r_3]$. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, we have

$$\begin{aligned} \mu_\xi(x; \theta, \alpha) &= \text{Pos}\{\xi = x\} \\ &= \begin{cases} \frac{x-r_1}{r_2-r_1} - (1-2\alpha)\theta_l \min \left\{ \frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1} \right\}, & \text{if } x \in [r_1, r_2] \\ \frac{r_3-x}{r_3-r_2} - (1-2\alpha)\theta_l \min \left\{ \frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2} \right\}, & \text{if } x \in [r_2, r_3] \end{cases} \\ &= \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-2\alpha\theta_l)x-(1-2\alpha)\theta_l r_2-r_1}{r_2-r_1}, & \text{if } x \in [\frac{r_1+r_2}{2}, r_2] \\ \frac{-(1+\theta_l-2\alpha\theta_l)x+(1-2\alpha)\theta_l r_2+r_3}{r_3-r_2}, & \text{if } x \in [r_2, \frac{r_2+r_3}{2}] \\ (1 - \theta_l + 2\alpha\theta_l) \frac{r_3-x}{r_3-r_2}, & \text{if } x \in [\frac{r_2+r_3}{2}, r_3], \end{cases} \end{aligned}$$

which completes the proof of assertion (ii). \square

The following corollary shows that the E^* , E_* and E reduction methods are the special cases of the CVaR reduction method for type-2 triangular fuzzy variable.

Corollary 4.2. Let $\tilde{\xi}$ be a type-2 triangular fuzzy variable and ξ_1, ξ_2 and ξ_3 be the reduced fuzzy variables obtained by E^* , E_* and E reduction methods respectively.

(i) For E^* reduction method, $\mu_\xi(x; \theta, \frac{3}{4}) = \mu_{\xi_1}(x; \theta)$;

- (ii) For E_* reduction method, $\mu_\xi(x; \theta, \frac{1}{4}) = \mu_{\xi_2}(x; \theta)$;
 (iii) For E reduction method, if $\theta_l \leq \theta_r$, then $\mu_\xi\left(x; \theta, \frac{5\theta_r - \theta_l}{8\theta_r}\right) = \mu_{\xi_3}(x; \theta)$;
 (iv) For E reduction method, if $\theta_l \geq \theta_r$, then $\mu_\xi\left(x; \theta, \frac{3\theta_l + \theta_r}{8\theta_l}\right) = \mu_{\xi_3}(x; \theta)$.

Example 4.3. Let $\tilde{\xi} = (\tilde{2}, \tilde{3}, \tilde{4}; 0.5, 1)$ be a type-2 triangular fuzzy variable. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. By the CVaR reduction method, if $\alpha \in (0, 0.5]$, the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (0.5 + \alpha)x - 2\alpha - 1, & \text{if } x \in [2, \frac{5}{2}] \\ (1.5 - \alpha)x + 3\alpha - 3.5, & \text{if } x \in [\frac{5}{2}, 3] \\ -(1.5 - \alpha)x - 3\alpha + 5.5, & \text{if } x \in [3, \frac{7}{2}] \\ -(0.5 + \alpha)x + 4\alpha + 2, & \text{if } x \in [\frac{7}{2}, 4], \end{cases}$$

and if $\alpha \in (0.5, 1]$, the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} 2\alpha x - 4\alpha, & \text{if } x \in [2, \frac{5}{2}] \\ (2 - 2\alpha)x + 6\alpha - 5, & \text{if } x \in [\frac{5}{2}, 3] \\ -(2 - 2\alpha)x - 6\alpha + 7, & \text{if } x \in [3, \frac{7}{2}] \\ -2\alpha x + 8\alpha, & \text{if } x \in [\frac{7}{2}, 4]. \end{cases}$$

Theorem 4.4. Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable, and $\theta = (\theta_l, \theta_r)$.

(i) If $\alpha \in (0, 0.5]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \frac{x - r_1}{r_2 - r_1}, & \text{if } x \in [r_1, \frac{r_1 + r_2}{2}] \\ \frac{(1 + \theta_l - 2\alpha\theta_l)x - (1 - 2\alpha)\theta_l r_2 - r_1}{r_2 - r_1}, & \text{if } x \in [\frac{r_1 + r_2}{2}, r_2] \\ 1, & \text{if } x \in [r_2, r_3] \\ \frac{-(1 + \theta_l - 2\alpha\theta_l)x + (1 - 2\alpha)\theta_l r_3 + r_4}{r_4 - r_3}, & \text{if } x \in [r_3, \frac{r_3 + r_4}{2}] \\ (1 - \theta_l + 2\alpha\theta_l) \frac{r_4 - x}{r_4 - r_3}, & \text{if } x \in [\frac{r_3 + r_4}{2}, r_4]. \end{cases}$$

(ii) If $\alpha \in (0.5, 1]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_r + 2\alpha\theta_r) \frac{x - r_1}{r_2 - r_1}, & \text{if } x \in [r_1, \frac{r_1 + r_2}{2}] \\ \frac{(1 + \theta_r - 2\alpha\theta_r)x - (1 - 2\alpha)\theta_r r_2 - r_1}{r_2 - r_1}, & \text{if } x \in [\frac{r_1 + r_2}{2}, r_2] \\ 1, & \text{if } x \in [r_2, r_3] \\ \frac{-(1 + \theta_r - 2\alpha\theta_r)x + (1 - 2\alpha)\theta_r r_3 + r_4}{r_4 - r_3}, & \text{if } x \in [r_3, \frac{r_3 + r_4}{2}] \\ (1 - \theta_r + 2\alpha\theta_r) \frac{r_4 - x}{r_4 - r_3}, & \text{if } x \in [\frac{r_3 + r_4}{2}, r_4]. \end{cases}$$

Proof. We only prove assertion (i), and the rest can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_\xi(x)$ of $\tilde{\xi}$ is the following triangular regular fuzzy variable

$$\left(\frac{x - r_1}{r_2 - r_1} - \theta_l \min \left\{ \frac{x - r_1}{r_2 - r_1}, \frac{r_2 - x}{r_2 - r_1} \right\}, \frac{x - r_1}{r_2 - r_1}, \frac{x - r_1}{r_2 - r_1} + \theta_r \min \left\{ \frac{x - r_1}{r_2 - r_1}, \frac{r_2 - x}{r_2 - r_1} \right\} \right)$$

for any $x \in [r_1, r_2]$, the regular fuzzy variable $\tilde{1}$ for any $x \in [r_2, r_3]$, and

$$\left(\frac{r_4 - x}{r_4 - r_3} - \theta_l \min \left\{ \frac{r_4 - x}{r_4 - r_3}, \frac{x - r_3}{r_4 - r_3} \right\}, \frac{r_4 - x}{r_4 - r_3}, \frac{r_4 - x}{r_4 - r_3} + \theta_r \min \left\{ \frac{r_4 - x}{r_4 - r_3}, \frac{x - r_3}{r_4 - r_3} \right\} \right)$$

for any $x \in [r_3, r_4]$. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, we have

$$\begin{aligned} \mu_\xi(x; \theta, \alpha) &= \text{Pos}\{\xi = x\} \\ &= \begin{cases} \frac{x-r_1}{r_2-r_1} - (1-2\alpha)\theta_l \min \left\{ \frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1} \right\}, & \text{if } x \in [r_1, r_2] \\ 1, & \text{if } x \in [r_2, r_3] \\ \frac{r_4-x}{r_4-r_3} - (1-2\alpha)\theta_l \min \left\{ \frac{r_4-x}{r_4-r_3}, \frac{x-r_3}{r_4-r_3} \right\}, & \text{if } x \in [r_3, r_4] \end{cases} \\ &= \begin{cases} (1-\theta_l + 2\alpha\theta_l) \frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-2\alpha\theta_l)x - (1-2\alpha)\theta_l r_2 - r_1}{r_2-r_1}, & \text{if } x \in [\frac{r_1+r_2}{2}, r_2] \\ 1, & \text{if } x \in [r_2, r_3] \\ \frac{-(1+\theta_l-2\alpha\theta_l)x + (1-2\alpha)\theta_l r_3 + r_4}{r_4-r_3}, & \text{if } x \in [r_3, \frac{r_3+r_4}{2}] \\ (1-\theta_l + 2\alpha\theta_l) \frac{r_4-x}{r_4-r_3}, & \text{if } x \in [\frac{r_3+r_4}{2}, r_4], \end{cases} \end{aligned}$$

which completes the proof of assertion (i). \square

Example 4.5. Let $\tilde{\xi} = (\tilde{1}, \tilde{2}, \tilde{3}, \tilde{5}; 0.6, 0.8)$ be a type-2 trapezoidal fuzzy variable. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. By the CVaR reduction method, if $\alpha \in (0, 0.5]$, the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (0.4 + 1.2\alpha)x - 1.2\alpha - 0.4, & \text{if } x \in [1, \frac{3}{2}] \\ (1.6 - 1.2\alpha)x + 2.4\alpha - 2.2, & \text{if } x \in [\frac{3}{2}, 2] \\ 1, & \text{if } x \in [2, 3] \\ -(0.8 - 0.6\alpha)x - 1.8\alpha + 3.4, & \text{if } x \in [3, 4] \\ -(0.2 + 0.6\alpha)x + 3\alpha + 1, & \text{if } x \in [4, 5], \end{cases}$$

and if $\alpha \in (0.5, 1]$, the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (0.2 + 1.6\alpha)x - 1.6\alpha - 0.2, & \text{if } x \in [1, \frac{3}{2}] \\ (1.8 - 1.6\alpha)x + 3.2\alpha - 2.6, & \text{if } x \in [\frac{3}{2}, 2] \\ 1, & \text{if } x \in [2, 3] \\ -(0.9 - 0.8\alpha)x - 2.4\alpha + 3.7, & \text{if } x \in [3, 4] \\ -(0.1 + 0.8\alpha)x + 4\alpha + 0.5, & \text{if } x \in [4, 5]. \end{cases}$$

Theorem 4.6. Let $\tilde{\xi} = \tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$ be a type-2 normal fuzzy variable, and $\theta = (\theta_l, \theta_r)$.

(i) If $\alpha \in (0, 0.5]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } x \leq \mu - \sigma\sqrt{2 \ln 2} \text{ or } x \geq \mu + \sigma\sqrt{2 \ln 2} \\ (1 + \theta_l - 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1 - 2\alpha)\theta_l, & \text{if } \mu - \sigma\sqrt{2 \ln 2} \leq x \leq \mu + \sigma\sqrt{2 \ln 2}. \end{cases}$$

(ii) If $\alpha \in (0.5, 1]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_r + 2\alpha\theta_r) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \\ \text{if } x \leq \mu - \sigma\sqrt{2\ln 2} \text{ or } x \geq \mu + \sigma\sqrt{2\ln 2} \\ (1 + \theta_r - 2\alpha\theta_r) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1 - 2\alpha)\theta_r, \\ \text{if } \mu - \sigma\sqrt{2\ln 2} \leq x \leq \mu + \sigma\sqrt{2\ln 2}. \end{cases}$$

Proof. We only prove assertion (i), and the rest can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_\xi(x)$ of $\tilde{\xi}$ is the following triangular regular fuzzy variable

$$\begin{aligned} & \left(\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - \theta_l \min\left\{1 - \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}, \right. \\ & \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \\ & \left. \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + \theta_r \min\left\{1 - \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\} \right). \end{aligned}$$

Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, we have

$$\begin{aligned} \mu_\xi(x; \theta, \alpha) &= \text{Pos}\{\xi = x\} \\ &= \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1 - 2\alpha)\theta_l \times \min\left\{1 - \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\} \\ &= \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \\ \text{if } x \leq \mu - \sigma\sqrt{2\ln 2} \text{ or } x \geq \mu + \sigma\sqrt{2\ln 2} \\ (1 + \theta_l - 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1 - 2\alpha)\theta_l, \\ \text{if } \mu - \sigma\sqrt{2\ln 2} \leq x \leq \mu + \sigma\sqrt{2\ln 2}, \end{cases} \end{aligned}$$

which completes the proof of assertion (i). \square

The following corollary shows that the E^* , E_* and E reduction methods are the special cases of the CVaR reduction method for type-2 normal fuzzy variable.

Corollary 4.7. Let $\tilde{\xi}$ be a type-2 normal fuzzy variable and η_1, η_2 and η_3 be the reduced fuzzy variables obtained by E^* , E_* and E reduction methods respectively.

- (i) For E^* reduction method, $\mu_\xi(x; \theta, \frac{3}{4}) = \mu_{\eta_1}(x; \theta)$;
- (ii) For E_* reduction method, $\mu_\xi(x; \theta, \frac{1}{4}) = \mu_{\eta_2}(x; \theta)$;
- (iii) For E reduction method, if $\theta_l \leq \theta_r$, then $\mu_\xi(x; \theta, \frac{5\theta_r - \theta_l}{8\theta_r}) = \mu_{\eta_3}(x; \theta)$;
- (iv) For E reduction method, if $\theta_l \geq \theta_r$, then $\mu_\xi(x; \theta, \frac{3\theta_l + \theta_r}{8\theta_l}) = \mu_{\eta_3}(x; \theta)$.

Example 4.8. Let $\tilde{\xi} = \tilde{n}(2, 0.5; 0.3, 0.7)$ be a type-2 normal fuzzy variable. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. By the CVaR reduction method, if $\alpha \in (0, 0.5]$, then the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_\xi(x; \theta, \alpha) =$$

$$\begin{cases} (0.7 + 0.6\alpha) \exp(-(x-2)^2), & \text{if } x \leq 2 - \sqrt{\ln 2} \text{ or } x \geq 2 + \sqrt{\ln 2} \\ (1.3 - 0.6\alpha) \exp(-(x-2)^2) - 0.3(1-2\alpha), & \text{if } 2 - \sqrt{\ln 2} \leq x \leq 2 + \sqrt{\ln 2}, \end{cases}$$

and if $\alpha \in (0.5, 1]$, then the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (0.3 + 1.4\alpha) \exp(-(x-2)^2), & \text{if } x \leq 2 - \sqrt{\ln 2} \text{ or } x \geq 2 + \sqrt{\ln 2} \\ (1.7 - 1.4\alpha) \exp(-(x-2)^2) + 0.7(2\alpha - 1), & \text{if } 2 - \sqrt{\ln 2} \leq x \leq 2 + \sqrt{\ln 2}. \end{cases}$$

Theorem 4.9. Let $\tilde{\xi} = \tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$ be a type-2 gamma fuzzy variable, and $\theta = (\theta_l, \theta_r)$.

(i) If $\alpha \in (0, 0.5]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) \leq \frac{1}{2} \\ (1 + \theta_l - 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_l, & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) > \frac{1}{2}. \end{cases}$$

(ii) If $\alpha \in (0.5, 1]$, then the reduced fuzzy variable ξ has the following parametric possibility distribution

$$\mu_\xi(x; \theta, \alpha) = \begin{cases} (1 - \theta_r + 2\alpha\theta_r) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) \leq \frac{1}{2} \\ (1 + \theta_r - 2\alpha\theta_r) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_r, & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) > \frac{1}{2}. \end{cases}$$

Proof. We only prove assertion (i), and the rest can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_\xi(x)$ of $\tilde{\xi}$ is the following triangular regular fuzzy variable

$$\begin{aligned} & \left(\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \theta_l \min\left\{1 - \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right)\right\}, \right. \\ & \left. \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), \right. \\ & \left. \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) + \theta_r \min\left\{1 - \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right)\right\} \right). \end{aligned}$$

Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, we have

$$\begin{aligned} \mu_\xi(x; \theta, \alpha) &= \text{Pos}\{\xi = x\} = \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) \\ &- (1 - 2\alpha)\theta_l \times \min\left\{1 - \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right)\right\} \\ &= \begin{cases} (1 - \theta_l + 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) \leq \frac{1}{2} \\ (1 + \theta_l - 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_l, & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) > \frac{1}{2}, \end{cases} \end{aligned}$$

which completes the proof of assertion (i). \square

The following corollary illustrates that the E^* , E_* and E reduction methods are the special cases of the CVaR reduction method for type-2 gamma fuzzy variable.

Corollary 4.10. Let $\tilde{\xi}$ be a type-2 gamma fuzzy variable and ζ_1, ζ_2 and ζ_3 be the reduced fuzzy variables obtained by E^* , E_* and E reduction methods respectively.

- (i) For E^* reduction method, $\mu_{\xi}(x; \theta, \frac{3}{4}) = \mu_{\zeta_1}(x; \theta)$;
- (ii) For E_* reduction method, $\mu_{\xi}(x; \theta, \frac{1}{4}) = \mu_{\zeta_2}(x; \theta)$;
- (iii) For E reduction method, if $\theta_l \leq \theta_r$, then $\mu_{\xi}(x; \theta, \frac{5\theta_r - \theta_l}{8\theta_r}) = \mu_{\zeta_3}(x; \theta)$;
- (iv) For E reduction method, if $\theta_l \geq \theta_r$, then $\mu_{\xi}(x; \theta, \frac{3\theta_l + \theta_r}{8\theta_l}) = \mu_{\zeta_3}(x; \theta)$.

Example 4.11. Let $\tilde{\xi} = \tilde{\gamma}(5, 2; 0.5, 0.8)$ be a type-2 gamma fuzzy variable. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. By the CVaR reduction method, if $\alpha \in (0, 0.5]$, then the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_{\xi}(x; \theta, \alpha) = \begin{cases} \frac{1}{200}(1 + 2\alpha)x^2 \exp(2 - \frac{x}{5}), & \text{if } (\frac{x}{10})^2 \exp(2 - \frac{x}{5}) \leq \frac{1}{2} \\ \frac{1}{200}(3 - 2\alpha)x^2 \exp(2 - \frac{x}{5}) - \frac{1}{2}(1 - 2\alpha), & \text{if } (\frac{x}{10})^2 \exp(2 - \frac{x}{5}) > \frac{1}{2}, \end{cases}$$

and if $\alpha \in (0.5, 1]$, then the reduced fuzzy variable ξ of $\tilde{\xi}$ has the following possibility distribution

$$\mu_{\xi}(x; \theta, \alpha) = \begin{cases} \frac{1}{500}(1 + 8\alpha)x^2 \exp(2 - \frac{x}{5}), & \text{if } (\frac{x}{10})^2 \exp(2 - \frac{x}{5}) \leq \frac{1}{2} \\ \frac{1}{500}(9 - 8\alpha)x^2 \exp(2 - \frac{x}{5}) + \frac{4}{5}(2\alpha - 1), & \text{if } (\frac{x}{10})^2 \exp(2 - \frac{x}{5}) > \frac{1}{2}. \end{cases}$$

5. The Mean Values of CVaR Reduced Fuzzy Variables

For common CVaR reduced fuzzy variables, this section will derive the analytical expressions of mean values.

Theorem 5.1. Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable. Then the mean value of the CVaR reduced fuzzy variable ξ is

$$E[\xi] = \begin{cases} \frac{r_1 + 2r_2 + r_3}{4} - \frac{r_1 - 2r_2 + r_3}{8}(1 - 2\alpha)\theta_l, & \text{if } \alpha \in (0, 0.5] \\ \frac{r_1 + 2r_2 + r_3}{4} - \frac{r_1 - 2r_2 + r_3}{8}(1 - 2\alpha)\theta_r, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

Proof. We only prove the assertion in the case $\alpha \in (0, 0.5]$. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi}(x; \theta, \alpha)$ is given in Theorem 4.1. Thus

$$E[\xi] = \frac{1}{2} \int_0^1 (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta.$$

Note that $\mu_{\xi}((r_1 + r_2)/2) = \mu_{\xi}((r_2 + r_3)/2) = (1 - \theta_l + 2\alpha\theta_l)/2$. If $\beta \in (0, (1 - \theta_l + 2\alpha\theta_l)/2]$, then $\xi_{\inf}(\beta)$ and $\xi_{\sup}(\beta)$ are the solutions of the following equations respectively,

$$(1 - \theta_l + 2\alpha\theta_l) \frac{x - r_1}{r_2 - r_1} = \beta$$

and

$$(1 - \theta_l + 2\alpha\theta_l) \frac{r_3 - x}{r_3 - r_2} = \beta.$$

As a consequence, we have the following solutions

$$\xi_{\inf}(\beta) = \frac{(r_2 - r_1)\beta + r_1(1 - \theta_l + 2\alpha\theta_l)}{1 - \theta_l + 2\alpha\theta_l}$$

and

$$\xi_{\sup}(\beta) = \frac{-(r_3 - r_2)\beta + r_3(1 - \theta_l + 2\alpha\theta_l)}{1 - \theta_l + 2\alpha\theta_l}.$$

Therefore, when $\beta \in (0, (1 - \theta_l + \alpha\theta_l)/2]$, we have

$$\xi_{\inf}(\beta) + \xi_{\sup}(\beta) = \frac{(2r_2 - r_1 - r_3)\beta + (1 - \theta_l + 2\alpha\theta_l)(r_1 + r_3)}{1 - \theta_l + 2\alpha\theta_l}.$$

On the other hand, when $\beta \in ((1 - \theta_l + \alpha\theta_l)/2, 1]$, it is similar to prove the following result

$$\xi_{\inf}(\beta) + \xi_{\sup}(\beta) = \frac{(2r_2 - r_1 - r_3)\beta + 2(1 - 2\alpha)\theta_l r_2 + (r_1 + r_3)}{1 + \theta_l - 2\alpha\theta_l}.$$

Hence, the mean value of ξ is computed by

$$\begin{aligned} \mathbb{E}[\xi] &= \frac{1}{2} \left(\int_0^{\frac{1-\theta_l+2\alpha\theta_l}{2}} (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta + \int_{\frac{1-\theta_l+2\alpha\theta_l}{2}}^1 (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta \right) \\ &= \frac{r_1 + 2r_2 + r_3}{4} - \frac{r_1 - 2r_2 + r_3}{8}(1 - 2\alpha)\theta_l. \end{aligned}$$

The proof of theorem is complete. \square

Example 5.2. Let $\tilde{\xi}$ be the type-2 triangular fuzzy variable defined in Example 4.3. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 5.1, the mean value of ξ is $\mathbb{E}[\xi] = 3$.

Theorem 5.3. Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable. Then the mean value of the CVaR reduced fuzzy variable ξ is

$$\mathbb{E}[\xi] = \begin{cases} \frac{r_1+r_2+r_3+r_4}{4} - \frac{r_1-r_2-r_3+r_4}{8}(1-2\alpha)\theta_l, & \text{if } \alpha \in (0, 0.5] \\ \frac{r_1+r_2+r_3+r_4}{4} - \frac{r_1-r_2-r_3+r_4}{8}(1-2\alpha)\theta_r, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

Proof. We only prove the assertion in the case $\alpha \in (0, 0.5]$. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.4.

Note that $\mu_\xi((r_1 + r_2)/2) = \mu_\xi((r_3 + r_4)/2) = (1 - \theta_l + 2\alpha\theta_l)/2$. If $\beta \in (0, (1 - \theta_l + 2\alpha\theta_l)/2]$, then $\xi_{\inf}(\beta)$ and $\xi_{\sup}(\beta)$ are the solutions of the following equations respectively,

$$(1 - \theta_l + 2\alpha\theta_l) \frac{x - r_1}{r_2 - r_1} = \beta$$

and

$$(1 - \theta_l + 2\alpha\theta_l) \frac{r_4 - x}{r_4 - r_3} = \beta.$$

As a consequence, we have the following solutions

$$\xi_{\inf}(\beta) = \frac{(r_2 - r_1)\beta + r_1(1 - \theta_l + 2\alpha\theta_l)}{1 - \theta_l + 2\alpha\theta_l}$$

and

$$\xi_{\sup}(\beta) = \frac{-(r_4 - r_3)\beta + r_4(1 - \theta_l + 2\alpha\theta_l)}{1 - \theta_l + 2\alpha\theta_l}.$$

Therefore, when $\beta \in (0, (1 - \theta_l + \alpha\theta_l)/2]$, we have

$$\xi_{\inf}(\beta) + \xi_{\sup}(\beta) = \frac{(r_2 + r_3 - r_1 - r_4)\beta + (1 - \theta_l + 2\alpha\theta_l)(r_1 + r_4)}{1 - \theta_l + 2\alpha\theta_l}.$$

On the other hand, when $\beta \in ((1 - \theta_l + \alpha\theta_l)/2, 1]$, we have the following computational result

$$\xi_{\inf}(\beta) + \xi_{\sup}(\beta) = \frac{(r_2 + r_3 - r_1 - r_4)\beta + (1 - 2\alpha)(r_2 + r_3)\theta_l + (r_1 + r_4)}{1 + \theta_l - 2\alpha\theta_l}.$$

Hence, the mean value of ξ is computed by

$$\begin{aligned} E[\xi] &= \frac{1}{2} \left(\int_0^{\frac{1-\theta_l+2\alpha\theta_l}{2}} (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta + \int_{\frac{1-\theta_l+2\alpha\theta_l}{2}}^1 (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta \right) \\ &= \frac{r_1 + r_2 + r_3 + r_4}{4} - \frac{r_1 - r_2 - r_3 + r_4}{8} (1 - 2\alpha)\theta_l. \end{aligned}$$

The proof of theorem is complete. \square

Example 5.4. Let $\tilde{\xi}$ be the type-2 trapezoidal fuzzy variable defined in Example 4.5. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 5.3, the mean value of ξ is computed by

$$E[\xi] = \begin{cases} 2.675 + 0.15\alpha, & \text{if } \alpha \in (0, 0.5] \\ 2.65 + 0.2\alpha, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

Theorem 5.5. Let $\tilde{\xi} = \tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$ be a type-2 normal fuzzy variable. Then the mean value of the CVaR reduced fuzzy variable ξ is equal to μ .

Proof. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.6.

Note that $\mu_\xi(\mu - \sigma\sqrt{2\ln 2}) = \mu_\xi(\mu + \sigma\sqrt{2\ln 2}) = (1 - \theta_l + 2\alpha\theta_l)/2$. If $\beta \in (0, (1 - \theta_l + 2\alpha\theta_l)/2]$, then $\xi_{\inf}(\beta)$ and $\xi_{\sup}(\beta)$ are the solutions of the following equation,

$$(1 - \theta_l + 2\alpha\theta_l) \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) = \beta.$$

As a consequence, we have the following solutions

$$\xi_{\inf}(\beta) = \mu - \sqrt{-2\sigma^2 \ln \frac{\beta}{1 - \theta_l + 2\alpha\theta_l}}$$

and

$$\xi_{\sup}(\beta) = \mu + \sqrt{-2\sigma^2 \ln \frac{\beta}{1 - \theta_l + 2\alpha\theta_l}}.$$

Therefore, when $\beta \in (0, (1 - \theta_l + 2\alpha\theta_l)/2]$, we have $\xi_{\inf}(\beta) + \xi_{\sup}(\beta) = 2\mu$.

On the other hand, when $\beta \in ((1 - \theta_l + 2\alpha\theta_l)/2, 1]$, we have the result $\xi_{\inf}(\beta) + \xi_{\sup}(\beta) = 2\mu$.

Hence, the mean value of ξ is computed by

$$\begin{aligned} E[\xi] &= \frac{1}{2} \left(\int_0^{\frac{1-\theta_l+2\alpha\theta_l}{2}} (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta + \int_{\frac{1-\theta_l+2\alpha\theta_l}{2}}^1 (\xi_{\inf}(\beta) + \xi_{\sup}(\beta)) d\beta \right) \\ &= \mu. \end{aligned}$$

The proof of theorem is complete. \square

Example 5.6. Let $\tilde{\xi}$ be the type-2 normal fuzzy variable defined in Example 4.8. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 5.5, the mean value of ξ is $E[\xi] = 2$.

Theorem 5.7. Let $\tilde{\xi} = \tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$ be a type-2 gamma fuzzy variable. Suppose that $x_1, x_2 \in \mathfrak{R}^+$ satisfy

$$\left(\frac{x_1}{\lambda r}\right)^r \exp\left(r - \frac{x_1}{\lambda}\right) = \frac{1}{2}, \quad \left(\frac{x_2}{\lambda r}\right)^r \exp\left(r - \frac{x_2}{\lambda}\right) = \frac{1}{2}.$$

(i) If $\alpha \in (0, 0.5]$, then the mean value of the CVaR reduced fuzzy variable ξ has the following parametric possibility distribution

$$\begin{aligned} E[\xi] &= \lambda r - \frac{\lambda r!}{2r^r} \exp(r) + \lambda \sum_{n=0}^r \frac{r!}{r^n (r-n)!} \\ &- \frac{(1-2\alpha)\theta_l}{2} \left\{ x_1 + x_2 - 2\lambda r - \frac{\lambda r!}{r^r} \exp(r) + \lambda \sum_{n=0}^r \frac{r!}{(r-n)!} \left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n} \right) \right\}. \end{aligned}$$

(ii) If $\alpha \in (0.5, 1]$, then the mean value of the CVaR reduced fuzzy variable ξ has the following parametric possibility distribution

$$\begin{aligned} E[\xi] &= \lambda r - \frac{\lambda r!}{2r^r} \exp(r) + \lambda \sum_{n=0}^r \frac{r!}{r^n (r-n)!} \\ &- \frac{(1-2\alpha)\theta_r}{2} \left\{ x_1 + x_2 - 2\lambda r - \frac{\lambda r!}{r^r} \exp(r) + \lambda \sum_{n=0}^r \frac{r!}{(r-n)!} \left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n} \right) \right\}. \end{aligned}$$

Proof. We only prove the first assertion. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.9. As a consequence, the credibility distribution of ξ is

$$\begin{aligned} \text{Cr}\{\xi \geq x\} &= \\ &\begin{cases} 1 - \frac{1}{2}(1 - \theta_l + 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } 0 \leq x \leq x_1 \\ 1 - \frac{1}{2}[(1 + \theta_l - 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_l], & \text{if } x_1 \leq x \leq \lambda r \\ \frac{1}{2}[(1 + \theta_l - 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_l], & \text{if } \lambda r \leq x \leq x_2 \\ \frac{1}{2}(1 - \theta_l + 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } x \geq x_2, \end{cases} \end{aligned}$$

where $x_1, x_2 \in \mathfrak{R}^+$ satisfy

$$\left(\frac{x_1}{\lambda r}\right)^r \exp\left(r - \frac{x_1}{\lambda}\right) = \frac{1}{2}, \quad \left(\frac{x_2}{\lambda r}\right)^r \exp\left(r - \frac{x_2}{\lambda}\right) = \frac{1}{2}.$$

Therefore, the mean value of ξ is computed as follows

$$\begin{aligned} E[\xi] &= \int_0^{x_1} \text{Cr}\{\xi \geq x\} dx + \int_{x_1}^{\lambda r} \text{Cr}\{\xi \geq x\} dx \\ &+ \int_{\lambda r}^{x_2} \text{Cr}\{\xi \geq x\} dx + \int_{x_2}^{+\infty} \text{Cr}\{\xi \geq x\} dx = \lambda r - \frac{\lambda r!}{2r^r} \exp(r) + \lambda \sum_{n=0}^r \frac{r!}{r^n(r-n)!} \\ &- \frac{(1-2\alpha)\theta_l}{2} \left\{ x_1 + x_2 - 2\lambda r - \frac{\lambda r!}{r^r} \exp(r) + \lambda \sum_{n=0}^r \frac{r!}{(r-n)!} \left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n} \right) \right\}. \end{aligned}$$

The proof of assertion (i) is complete. \square

Example 5.8. Let $\tilde{\xi}$ be the type-2 gamma fuzzy variable defined in Example 4.11. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 5.7, the mean value of ξ is computed by

$$E[\xi] = \begin{cases} 12.1408 + 2.2458\alpha, & \text{if } \alpha \in (0, 0.5] \\ 11.4671 + 3.5933\alpha, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

6. The Second Order Moments of CVaR Reduced Fuzzy Variables

Let ξ be a reduced fuzzy variable with a parametric possibility distribution. To measure the variation of the parametric possibility distribution about its mean value, we adopt the following n th moment of ξ ,

$$M_n[\xi] = \int_{(-\infty, +\infty)} (x - E[\xi])^n d(\text{Cr}\{\xi \leq x\}),$$

where the credibility distribution is defined by the parametric possibility distribution of ξ ,

$$\text{Cr}\{\xi \leq x\} = \frac{1}{2} \left(\sup_{t \in \mathfrak{R}} \mu_\xi(t; \theta, \alpha) + \sup_{t \leq x} \mu_\xi(t; \theta, \alpha) - \sup_{t > x} \mu_\xi(t; \theta, \alpha) \right).$$

When $n = 2$, $M_2[\xi]$ is called the second order moment of ξ . In the following, we will establish the analytical expressions of second order moments for common reduced fuzzy variables. Firstly, we have the following results about type-2 triangular fuzzy variable.

Theorem 6.1. Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable, and ξ its CVaR reduced fuzzy variable.

(i) If $\alpha \in (0, 0.5)$, then the second order moment of ξ is

$$\begin{aligned} M_2[\xi] &= \frac{1}{48} (5r_1^2 + 4r_2^2 + 5r_3^2 - 4r_1r_2 - 6r_1r_3 - 4r_2r_3) - \frac{1}{16} (r_1 - r_3)^2 (1 - 2\alpha)\theta_l \\ &- \frac{1}{64} (r_1 - 2r_2 + r_3)^2 (1 - 2\alpha)^2 \theta_l^2, \end{aligned}$$

which is equivalent to the following parametric matrix form

$$M_2[\xi] = \frac{1}{2}r^T P^1 r,$$

where $r = (r_1, r_2, r_3)^T$, and the elements of the symmetric matrix P^1 are

$$\begin{aligned} P_{11}^1 &= P_{33}^1 = \frac{-(1-2\alpha)^2\theta_l^2}{32} - \frac{(1-2\alpha)\theta_l}{8} + \frac{5}{24}, \\ P_{12}^1 &= P_{23}^1 = \frac{(1-2\alpha)^2\theta_l^2}{16} - \frac{1}{12}, \\ P_{13}^1 &= \frac{-(1-2\alpha)^2\theta_l^2}{32} + \frac{(1-2\alpha)\theta_l}{8} - \frac{1}{8}, \\ P_{22}^1 &= \frac{-(1-2\alpha)^2\theta_l^2}{8} + \frac{1}{6}. \end{aligned}$$

(ii) If $\alpha \in (0.5, 1]$, then the second order moment of ξ is

$$\begin{aligned} M_2[\xi] &= \frac{1}{48}(5r_1^2 + 4r_2^2 + 5r_3^2 - 4r_1r_2 - 6r_1r_3 - 4r_2r_3) - \frac{1}{16}(r_1 - r_3)^2(1-2\alpha)\theta_r \\ &\quad - \frac{1}{64}(r_1 - 2r_2 + r_3)^2(1-2\alpha)^2\theta_r^2, \end{aligned}$$

which is equivalent to the following parametric matrix form

$$M_2[\xi] = \frac{1}{2}r^T P^2 r,$$

where $r = (r_1, r_2, r_3)^T$, and the elements of the symmetric matrix P^2 are

$$\begin{aligned} P_{11}^2 &= P_{33}^2 = \frac{-(1-2\alpha)^2\theta_r^2}{32} - \frac{(1-2\alpha)\theta_r}{8} + \frac{5}{24}, \\ P_{12}^2 &= P_{23}^2 = \frac{(1-2\alpha)^2\theta_r^2}{16} - \frac{1}{12}, \\ P_{13}^2 &= \frac{-(1-2\alpha)^2\theta_r^2}{32} + \frac{(1-2\alpha)\theta_r}{8} - \frac{1}{8}, \\ P_{22}^2 &= \frac{-(1-2\alpha)^2\theta_r^2}{8} + \frac{1}{6}. \end{aligned}$$

Moreover, the second order moment $M_2[\xi]$ is parametric quadratic convex function with respect to vector $r \in \mathfrak{R}^3$.

Proof. We only prove the first assertion. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.1. As a consequence, the credibility distribution of ξ is computed by

$$\text{Cr}\{\xi \leq x\} = \begin{cases} 0, & \text{if } x < r_1 \\ \frac{(1-\theta_l+2\alpha\theta_l)(x-r_1)}{2(r_2-r_1)}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-2\alpha\theta_l)x-(1-2\alpha)\theta_l r_2-r_1}{2(r_2-r_1)}, & \text{if } x \in [\frac{r_1+r_2}{2}, r_2] \\ 1 - \frac{-(1+\theta_l-2\alpha\theta_l)x+(1-2\alpha)\theta_l r_2+r_3}{2(r_3-r_2)}, & \text{if } x \in [r_2, \frac{r_2+r_3}{2}] \\ 1 - \frac{(1-\theta_l+2\alpha\theta_l)(r_3-x)}{2(r_3-r_2)}, & \text{if } x \in [\frac{r_2+r_3}{2}, r_3] \\ 1, & \text{if } x > r_3, \end{cases}$$

and the mean value of ξ is

$$E[\xi] = \frac{r_1 + 2r_2 + r_3}{4} - \frac{r_1 - 2r_2 + r_3}{8}(1 - 2\alpha)\theta_l.$$

Therefore, the second order moment of ξ is calculated as follows

$$\begin{aligned} M_2[\xi] &= \int_{(-\infty, +\infty)} (x - E[\xi])^2 dCr\{\xi \leq x\} \\ &= \frac{1 - \theta_l + 2\alpha\theta_l}{2(r_2 - r_1)} \int_{r_1}^{\frac{r_1+r_2}{2}} (x - E[\xi])^2 dx + \frac{1 + \theta_l - 2\alpha\theta_l}{2(r_2 - r_1)} \int_{\frac{r_1+r_2}{2}}^{r_2} (x - E[\xi])^2 dx \\ &\quad + \frac{1 + \theta_l - 2\alpha\theta_l}{2(r_3 - r_2)} \int_{r_2}^{\frac{r_2+r_3}{2}} (x - E[\xi])^2 dx + \frac{1 - \theta_l + 2\alpha\theta_l}{2(r_3 - r_2)} \int_{\frac{r_2+r_3}{2}}^{r_3} (x - E[\xi])^2 dx \\ &= \frac{1}{48}(5r_1^2 + 4r_2^2 + 5r_3^2 - 4r_1r_2 - 6r_1r_3 - 4r_2r_3) - \frac{1}{64}(r_1 - 2r_2 + r_3)^2(1 - 2\alpha)^2\theta_l^2 \\ &\quad - \frac{1}{16}(r_1 - r_3)^2(1 - 2\alpha)\theta_l = \frac{1}{2}r^T P^1 r. \end{aligned}$$

On the other hand, the integrand $(x - E[\xi])^2$ and the credibility distribution $Cr\{\xi \leq x\}$ are both nonnegative, so $M_2[\xi] \geq 0$ holds for any vector $r \in \mathfrak{R}^3$. In addition, P^1 is a 3×3 symmetric parametric matrix. Therefore, $M_2[\xi]$ is a positive semidefinite quadratic form. In other words, for any parameters θ_l, θ_r and α , the second order moment $M_2[\xi]$ is a parametric quadratic convex function with respect to vector $r \in \mathfrak{R}^3$. The proof of the assertion is complete. \square

Example 6.2. Let $\tilde{\xi}$ be the type-2 triangular fuzzy variable defined in Example 4.3. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 6.1, the second order moment of ξ is computed by

$$M_2[\xi] = \begin{cases} 0.2083 + 0.25\alpha, & \text{if } \alpha \in (0, 0.5] \\ 0.0833 + 0.5\alpha, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

For the reduced fuzzy variable of a type-2 trapezoidal fuzzy variable, we obtain the analytical expression of second order moment in the following theorem.

Theorem 6.3. Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable, and ξ be its CVaR reduced fuzzy variable.

(i) If $\alpha \in (0, 0.5]$, then the second order moment of ξ is

$$\begin{aligned} M_2[\xi] &= \\ &\frac{1}{48}(5r_1^2 + 5r_2^2 + 5r_3^2 + 5r_4^2 + 2r_1r_2 - 6r_1r_3 - 6r_1r_4 - 6r_2r_3 - 6r_2r_4 + 2r_3r_4) - \\ &\frac{(1 - 2\alpha)\theta_l}{16}(r_1^2 - r_2^2 - r_3^2 + r_4^2 - 2r_1r_4 + 2r_2r_3) - \frac{(1 - 2\alpha)^2\theta_l^2}{64}(r_1 - r_2 - r_3 + r_4)^2, \end{aligned}$$

which is equivalent to the following parametric matrix form

$$M_2[\xi] = \frac{1}{2}r^T Q^1 r,$$

where $r = (r_1, r_2, r_3, r_4)^T$, and the elements of the symmetric matrix Q^1 are

$$\begin{aligned} Q_{11}^1 &= Q_{44}^1 = \frac{-(1-2\alpha)^2\theta_l^2}{32} - \frac{(1-2\alpha)\theta_l}{8} + \frac{5}{24}, \\ Q_{12}^1 &= Q_{34}^1 = \frac{(1-2\alpha)^2\theta_l^2}{32} + \frac{1}{24}, \\ Q_{13}^1 &= Q_{24}^1 = \frac{(1-2\alpha)^2\theta_l^2}{32} - \frac{1}{8}, \\ Q_{14}^1 &= \frac{-(1-2\alpha)^2\theta_l^2}{32} + \frac{(1-2\alpha)\theta_l}{8} - \frac{1}{8}, \\ Q_{22}^1 &= Q_{33}^1 = \frac{-(1-2\alpha)^2\theta_l^2}{32} + \frac{(1-2\alpha)\theta_l}{8} + \frac{5}{24}, \\ Q_{23}^1 &= \frac{-(1-2\alpha)^2\theta_l^2}{32} - \frac{(1-2\alpha)\theta_l}{8} - \frac{1}{8}. \end{aligned}$$

(ii) If $\alpha \in (0.5, 1]$, then the second order moment of ξ is

$$\begin{aligned} M_2[\xi] &= \\ &= \frac{1}{48}(5r_1^2 + 5r_2^2 + 5r_3^2 + 5r_4^2 + 2r_1r_2 - 6r_1r_3 - 6r_1r_4 - 6r_2r_3 - 6r_2r_4 + 2r_3r_4) - \\ &= \frac{(1-2\alpha)\theta_r}{16}(r_1^2 - r_2^2 - r_3^2 + r_4^2 - 2r_1r_4 + 2r_2r_3) - \frac{(1-2\alpha)^2\theta_r^2}{64}(r_1 - r_2 - r_3 + r_4)^2, \end{aligned}$$

which is equivalent to the following parametric matrix form

$$M_2[\xi] = \frac{1}{2}r^T Q^2 r,$$

where $r = (r_1, r_2, r_3, r_4)^T$, and the elements of the symmetric matrix Q^2 are

$$\begin{aligned} Q_{11}^2 &= Q_{44}^2 = \frac{-(1-2\alpha)^2\theta_r^2}{32} - \frac{(1-2\alpha)\theta_r}{8} + \frac{5}{24}, \\ Q_{12}^2 &= Q_{34}^2 = \frac{(1-2\alpha)^2\theta_r^2}{32} + \frac{1}{24}, \\ Q_{13}^2 &= Q_{24}^2 = \frac{(1-2\alpha)^2\theta_r^2}{32} - \frac{1}{8}, \\ Q_{14}^2 &= \frac{-(1-2\alpha)^2\theta_r^2}{32} + \frac{(1-2\alpha)\theta_r}{8} - \frac{1}{8}, \\ Q_{22}^2 &= Q_{33}^2 = \frac{-(1-2\alpha)^2\theta_r^2}{32} + \frac{(1-2\alpha)\theta_r}{8} + \frac{5}{24}, \\ Q_{23}^2 &= \frac{-(1-2\alpha)^2\theta_r^2}{32} - \frac{(1-2\alpha)\theta_r}{8} - \frac{1}{8}. \end{aligned}$$

Moreover, the second order moment $M_2[\xi]$ is a parametric quadratic convex function with respect to vector $r \in \mathfrak{R}^4$.

Proof. We only prove the first assertion. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.4. As a consequence, the credibility distribution of ξ is calculated by

$$\text{Cr}\{\xi \leq x\} = \begin{cases} 0, & \text{if } x < r_1 \\ \frac{(1-\theta_l+2\alpha\theta_l)(x-r_1)}{2(r_2-r_1)}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-2\alpha\theta_l)x-(1-2\alpha)\theta_l r_2-r_1}{2(r_2-r_1)}, & \text{if } x \in [\frac{r_1+r_2}{2}, r_2] \\ \frac{1}{2}, & \text{if } x \in [r_2, r_3] \\ 1 - \frac{-(1+\theta_l-2\alpha\theta_l)x+(1-2\alpha)\theta_l r_3+r_4}{2(r_4-r_3)}, & \text{if } x \in [r_3, \frac{r_3+r_4}{2}] \\ 1 - \frac{(1-\theta_l+2\alpha\theta_l)(r_4-x)}{2(r_4-r_3)}, & \text{if } x \in [\frac{r_3+r_4}{2}, r_4] \\ 1, & \text{if } x > r_4, \end{cases}$$

and the mean value of ξ is

$$\text{E}[\xi] = \frac{r_1 + r_2 + r_3 + r_4}{4} - \frac{r_1 - r_2 - r_3 + r_4}{8}(1 - 2\alpha)\theta_l.$$

Therefore, the second order moment of ξ is computed as follows

$$\begin{aligned} \text{M}_2[\xi] &= \int_{(-\infty, +\infty)} (x - \text{E}[\xi])^2 d\text{Cr}\{\xi \leq x\} \\ &= \frac{1 - \theta_l + 2\alpha\theta_l}{2(r_2 - r_1)} \int_{r_1}^{\frac{r_1+r_2}{2}} (x - \text{E}[\xi])^2 dx + \frac{1 + \theta_l - 2\alpha\theta_l}{2(r_2 - r_1)} \int_{\frac{r_1+r_2}{2}}^{r_2} (x - \text{E}[\xi])^2 dx \\ &\quad + \frac{1 + \theta_l - 2\alpha\theta_l}{2(r_4 - r_3)} \int_{r_3}^{\frac{r_3+r_4}{2}} (x - \text{E}[\xi])^2 dx + \frac{1 - \theta_l + 2\alpha\theta_l}{2(r_4 - r_3)} \int_{\frac{r_3+r_4}{2}}^{r_4} (x - \text{E}[\xi])^2 dx \\ &= \frac{1}{48}(5r_1^2 + 5r_2^2 + 5r_3^2 + 5r_4^2 + 2r_1r_2 - 6r_1r_3 - 6r_1r_4 - 6r_2r_3 - 6r_2r_4 + 2r_3r_4) \\ &\quad - \frac{1}{16}(r_1^2 - r_2^2 - r_3^2 + r_4^2 - 2r_1r_4 + 2r_2r_3)(1 - 2\alpha)\theta_l \\ &\quad - \frac{1}{64}(r_1 - r_2 - r_3 + r_4)^2(1 - 2\alpha)^2\theta_l^2 = \frac{1}{2}r^T Q^1 r. \end{aligned}$$

On the other hand, the integrand $(x - \text{E}[\xi])^2$ and the credibility distribution $\text{Cr}\{\xi \leq x\}$ are both nonnegative, so $\text{M}_2[\xi] \geq 0$ holds for any vector $r \in \mathfrak{R}^4$. In addition, Q^1 is a 4×4 symmetric parametric matrix. Therefore, $\text{M}_2[\xi]$ is a positive semidefinite quadratic form. In other words, for any parameters θ_l, θ_r and α , the second moment $\text{M}_2[\xi]$ is a parametric quadratic convex function with respect to vector $r \in \mathfrak{R}^4$. The proof of the assertion is complete. \square

Example 6.4. Let $\tilde{\xi}$ be the type-2 trapezoidal fuzzy variable defined in Example 4.5. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 6.3, the second order moment of ξ is computed by

$$\text{M}_2[\xi] = \begin{cases} 0.5277 + 2.4974\alpha - 0.0224\alpha^2, & \text{if } \alpha \in (0, 0.5] \\ 0.1108 + 3.34\alpha - 0.04\alpha^2, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

For the reduced fuzzy variable of a type-2 normal fuzzy variable, the following theorem obtains the analytical expression of second order moment.

Theorem 6.5. Let $\tilde{\xi} = \tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$ be a type-2 normal fuzzy variable, and ξ be its CVaR reduced fuzzy variable. Then the second order moment of ξ is

$$M_2[\xi] = \begin{cases} 2\sigma^2 - (1 - 2\alpha)\theta_l\sigma^2 \ln 2, & \text{if } \alpha \in (0, 0.5] \\ 2\sigma^2 - (1 - 2\alpha)\theta_r\sigma^2 \ln 2, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

Moreover, the second order moment $M_2[\xi]$ is a parametric quadratic convex function on \Re with respect to σ .

Proof. We only prove the first equation in the case $\alpha \in (0, 0.5]$. Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.6. As a consequence, the credibility distribution of ξ is computed by

$$\text{Cr}\{\xi \leq x\} = \begin{cases} \frac{1}{2}(1 - \theta_l + 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } x \leq \mu - \sigma\sqrt{2\ln 2} \\ \frac{1}{2}[(1 + \theta_l - 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1 - 2\alpha)\theta_l], & \text{if } \mu - \sigma\sqrt{2\ln 2} \leq x \leq \mu \\ 1 - \frac{1}{2}[(1 + \theta_l - 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1 - 2\alpha)\theta_l], & \text{if } \mu \leq x \leq \mu + \sigma\sqrt{2\ln 2} \\ 1 - \frac{1}{2}(1 - \theta_l + 2\alpha\theta_l) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } x \geq \mu + \sigma\sqrt{2\ln 2}, \end{cases}$$

and the mean value of ξ is $E[\xi] = \mu$. Therefore, the second moment of ξ is calculated by

$$\begin{aligned} M_2[\xi] &= \int_{(-\infty, +\infty)} (x - \mu)^2 d\text{Cr}\{\xi \leq x\} \\ &= \int_{(-\infty, \mu - \sigma\sqrt{2\ln 2})} (x - \mu)^2 d\text{Cr}\{\xi \leq x\} + \int_{(\mu - \sigma\sqrt{2\ln 2}, \mu)} (x - \mu)^2 d\text{Cr}\{\xi \leq x\} \\ &\quad + \int_{(\mu, \mu + \sigma\sqrt{2\ln 2})} (x - \mu)^2 d\text{Cr}\{\xi \leq x\} + \int_{(\mu + \sigma\sqrt{2\ln 2}, +\infty)} (x - \mu)^2 d\text{Cr}\{\xi \leq x\} \\ &= 2\sigma^2 - 2(1 - 2\alpha)\theta_l\sigma^2 \ln 2. \end{aligned}$$

Moreover, the coefficient $2\sigma^2 - 2(1 - 2\alpha)\theta_l\sigma^2 \ln 2 > 0$ for any $\theta_l, \alpha \in [0, 1]$, so $M_2[\xi] \geq 0$ holds for any vector $\sigma > 0$. Thus, the second order moment $M_2[\xi]$ is a parametric quadratic convex function on \Re with respect to σ for any θ_l and α . The proof of the assertion is complete. \square

Example 6.6. Let $\tilde{\xi}$ be the type-2 normal fuzzy variable defined in Example 4.8. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 6.5, the second order moment of ξ is computed by

$$M_2[\xi] = \begin{cases} 1 - 0.15(1 - 2\alpha) \ln 2, & \text{if } \alpha \in (0, 0.5] \\ 1 - 0.35(1 - 2\alpha) \ln 2, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

For the reduced fuzzy variable of a type-2 gamma fuzzy variable, we obtain the analytical formula of the second order moment in the next theorem.

Theorem 6.7. Let $\tilde{\xi} = \tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$ be a type-2 gamma fuzzy variable, and ξ be its CVaR reduced fuzzy variable. Suppose that $x_1, x_2 \in \Re^+$ satisfy

$$\left(\frac{x_1}{\lambda r}\right)^r \exp\left(r - \frac{x_1}{\lambda}\right) = \frac{1}{2}, \quad \left(\frac{x_2}{\lambda r}\right)^r \exp\left(r - \frac{x_2}{\lambda}\right) = \frac{1}{2}.$$

(i) If $\alpha \in (0, 0.5]$, then the second order moment of ξ is

$$\begin{aligned} M_2[\xi] &= (\lambda r - E[\xi])^2 - \frac{\lambda r!}{r^r} (\lambda r + \lambda - E[\xi]) e^r + 2\lambda^2 r \sum_{n=0}^{r+1} \frac{(r+1)!}{r^n (r-n+1)!} \\ &- 2\lambda \sum_{n=0}^r \frac{E[\xi] r!}{r^n (r-n)!} - \frac{(1-2\alpha)\theta_l}{2} \left\{ x_1^2 + x_2^2 - 2\lambda^2 r^2 - 2(x_1 + x_2 - 2\lambda r) E[\xi] \right. \\ &- \frac{2\lambda r!}{r^r} (\lambda r + \lambda - E[\xi]) \exp(r) + 2\lambda^2 \sum_{n=0}^{r+1} \frac{(r+1)!}{(r-n+1)!} \left(\frac{\lambda^{n-1}}{x_1^{n-1}} + \frac{\lambda^{n-1}}{x_2^{n-1}} - \frac{2}{r^{n-1}} \right) \\ &\left. - 2\lambda E[\xi] \sum_{n=0}^r \frac{r!}{(r-n)!} \left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n} \right) \right\}. \end{aligned}$$

(ii) If $\alpha \in (0.5, 1]$, then the second order moment of ξ is

$$\begin{aligned} M_2[\xi] &= (\lambda r - E[\xi])^2 - \frac{\lambda r!}{r^r} (\lambda r + \lambda - E[\xi]) e^r + 2\lambda^2 r \sum_{n=0}^{r+1} \frac{(r+1)!}{r^n (r-n+1)!} \\ &- 2\lambda \sum_{n=0}^r \frac{E[\xi] r!}{r^n (r-n)!} - \frac{(1-2\alpha)\theta_r}{2} \left\{ x_1^2 + x_2^2 - 2\lambda^2 r^2 - 2(x_1 + x_2 - 2\lambda r) E[\xi] \right. \\ &- \frac{2\lambda r!}{r^r} (\lambda r + \lambda - E[\xi]) \exp(r) + 2\lambda^2 \sum_{n=0}^{r+1} \frac{(r+1)!}{(r-n+1)!} \left(\frac{\lambda^{n-1}}{x_1^{n-1}} + \frac{\lambda^{n-1}}{x_2^{n-1}} - \frac{2}{r^{n-1}} \right) \\ &\left. - 2\lambda E[\xi] \sum_{n=0}^r \frac{r!}{(r-n)!} \left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n} \right) \right\}. \end{aligned}$$

Proof. We only prove assertion (i). Since ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_\xi(x; \theta, \alpha)$ is given in Theorem 4.9. As a consequence, the credibility distribution of ξ is calculated by

$$\text{Cr}\{\xi \leq x\} = \begin{cases} \frac{1}{2}(1 - \theta_l + 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } 0 \leq x \leq x_1 \\ \frac{1}{2}[(1 + \theta_l - 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_l], & \text{if } x_1 \leq x \leq \lambda r \\ 1 - \frac{1}{2}[(1 + \theta_l - 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1 - 2\alpha)\theta_l], & \text{if } \lambda r \leq x \leq x_2 \\ 1 - \frac{1}{2}(1 - \theta_l + 2\alpha\theta_l) \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } x \geq x_2, \end{cases}$$

where $x_1, x_2 \in \mathfrak{R}^+$ satisfy

$$\left(\frac{x_1}{\lambda r}\right)^r \exp\left(r - \frac{x_1}{\lambda}\right) = \frac{1}{2}, \quad \left(\frac{x_2}{\lambda r}\right)^r \exp\left(r - \frac{x_2}{\lambda}\right) = \frac{1}{2}.$$

The mean value $E[\xi]$ of ξ is calculated in Theorem 5.7. Therefore, the second order moment of ξ is calculated by

$$\begin{aligned} M_2[\xi] &= \int_{(-\infty, +\infty)} (x - E[\xi])^2 d\text{Cr}\{\xi \leq x\} \\ &= \int_{(0, x_1)} (x - E[\xi])^2 d\text{Cr}\{\xi \leq x\} + \int_{(x_1, \lambda r)} (x - E[\xi])^2 d\text{Cr}\{\xi \leq x\} \end{aligned}$$

$$\begin{aligned}
& + \int_{(\lambda r, x_2)} (x - E[\xi])^2 dCr\{\xi \leq x\} + \int_{(x_2, +\infty)} (x - E[\xi])^2 dCr\{\xi \leq x\} \\
& = (\lambda r - E[\xi])^2 - \frac{\lambda r!}{r^r} (\lambda r + \lambda - E[\xi]) \exp(r) + 2\lambda^2 r \sum_{n=0}^{r+1} \frac{(r+1)!}{r^n (r-n+1)!} - 2\lambda \sum_{n=0}^r \frac{E[\xi]^n r!}{r^n (r-n)!} \\
& \quad - \frac{(1-2\alpha)\theta_l}{2} \left\{ x_1^2 + x_2^2 - 2(x_1 + x_2 - 2\lambda r)E[\xi] - \frac{2\lambda r!}{r^r} (\lambda r + \lambda - E[\xi]) \exp(r) - 2\lambda^2 r^2 \right. \\
& \quad \left. + 2\lambda^2 \sum_{n=0}^{r+1} \frac{(r+1)!}{(r-n+1)!} \left(\frac{\lambda^{n-1}}{x_1^{n-1}} + \frac{\lambda^{n-1}}{x_2^{n-1}} - \frac{2}{r^{n-1}} \right) - 2\lambda E[\xi] \sum_{n=0}^r \frac{r!}{(r-n)!} \left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n} \right) \right\}.
\end{aligned}$$

The proof of assertion (i) is complete. \square

Example 6.8. Let $\tilde{\xi}$ be the type-2 gamma fuzzy variable defined in Example 4.11. Suppose ξ is the CVaR reduced fuzzy variable of $\tilde{\xi}$. According to Theorem 6.7, the second order moment of ξ is computed by

$$M_2[\xi] = \begin{cases} 78.1836 + 90.1238\alpha - 5.0435\alpha^2, & \text{if } \alpha \in (0, 0.5] \\ 50.6926 + 149.0398\alpha - 12.9112\alpha^2, & \text{if } \alpha \in (0.5, 1]. \end{cases}$$

In this example, since the reduced fuzzy variable ξ has variable possibility distribution with respect to parameter α instead of fixed possibility distribution, its second order moment also depends on the parameter α . According to above formula, we can calculate the second order moment of ξ for any given $\alpha \in (0, 1]$. For instance, if $\alpha = 0.25$, then the second order moment is $M_2[\xi] = 78.1836 + 90.1238 \times 0.25 - 5.0435 \times 0.25^2 = 100.3993$.

7. A Mean-moment Optimization Model

In this section, we present an application example about portfolio selection problem, which was first proposed by Markowitz [11] in stochastic environment. Different from the existing method in the literature, we construct a new mean-moment optimization model for fuzzy portfolio selection problems, in which the *mean value* and *second order moment* discussed in the above sections are used as two important optimization indexes.

Given a collection of potential investments indexed from 1 to n , let $\tilde{\xi}_i$ denote the return in the next time period on investment i , $i = 1, 2, \dots, n$. In the current development, we assume that $\tilde{\xi}_i$ is characterized by a type-2 fuzzy variable with known secondary possibility distribution.

A portfolio is determined by specifying what fraction of one's assets to put into each investment. That is, a portfolio is a collection of nonnegative numbers x_i , $i = 1, 2, \dots, n$ such that $\sum_{i=1}^n x_i = 1$. As a consequence, the return one would obtain using a given portfolio is determined by equation (1).

$$\sum_{i=1}^n x_i \tilde{\xi}_i = \tilde{\xi}^T x \tag{1}$$

where $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n)^T$ and $x = (x_1, x_2, \dots, x_n)^T$. Given a portfolio x , $\tilde{\xi}^T x$ is also a type-2 fuzzy variable, and we denote its reduced fuzzy variable as $r(\tilde{\xi}^T x)$.

One the basis of expectation criterion, the reward associated with such a portfolio can be defined as the expected return determined by equation (2).

$$\mathbb{E} \left[r \left(\sum_{i=1}^n x_i \tilde{\xi}_i \right) \right] = \mathbb{E} \left[r(\tilde{\xi}^T x) \right] \quad (2)$$

If the investor only concerns the reward, then it is simple for him to put all his assets in the investment with the highest expected return. However, it is known that investments with high rewards usually result in a high level of risk. Therefore, there is a need to define a risk measure for fuzzy return $r(\tilde{\xi}^T x)$. In this section, we employ the second order moment to define the risk associated with the investment by equation (3).

$$\mathbb{M}_2 \left[r \left(\sum_{i=1}^n x_i \tilde{\xi}_i \right) \right] = \mathbb{M}_2 \left[r(\tilde{\xi}^T x) \right] \quad (3)$$

which is a quadratic deviation from the mean value $\mathbb{E}[r(\tilde{\xi}^T x)]$.

Using the notations above, we next build a new mean-moment optimization model (4) for fuzzy portfolio selection problems.

$$\begin{aligned} \min \quad & \mathbb{M}_2 \left[r \left(\sum_{i=1}^n x_i \tilde{\xi}_i \right) \right] \\ \text{s.t.} \quad & \mathbb{E} \left[r \left(\sum_{i=1}^n x_i \tilde{\xi}_i \right) \right] \geq \psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0 \quad i = 1, 2, \dots, n \end{aligned} \quad (4)$$

When $\tilde{\xi}_i = (r_{i,1}, r_{i,2}, r_{i,3}, r_{i,4}; \theta_{i,l}, \theta_{i,r}), i = 1, 2, \dots, n$, are mutually independent type-2 trapezoidal fuzzy returns. In the case of $\alpha \leq 0.5$, according to Theorem 6.3, the second order moment of fuzzy return $r(\tilde{\xi}^T x)$ is computed by

$$\mathbb{M}_2 \left[r(\tilde{\xi}^T x) \right] = \frac{1}{2} x^T H^1 x,$$

where $x = (x_1, x_2, \dots, x_n)^T$, $H^1 = R^T Q^1 R$, and the coefficient matrix defined by equation (5) is the knowledge about security returns.

$$R = \begin{pmatrix} r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & r_{22} & \cdots & r_{n2} \\ r_{13} & r_{23} & \cdots & r_{n3} \\ r_{14} & r_{24} & \cdots & r_{n4} \end{pmatrix} \quad (5)$$

In the case of $\alpha > 0.5$, the second order moment of fuzzy return $r(\tilde{\xi}^T x)$ is calculated by

$$\mathbb{M}_2 \left[r(\tilde{\xi}^T x) \right] = \frac{1}{2} x^T H^2 x$$

where $H^2 = R^T Q^2 R$, and R is determined by equation (5).

We next consider the equivalent parametric form of $E[r(\tilde{\xi}^T x)]$. In the case of $\alpha \leq 0.5$, according to Theorem 5.3, the mean value of fuzzy return $r(\tilde{\xi}^T x)$ is computed by

$$E[r(\tilde{\xi}^T x)] = C_1^T x,$$

where $x = (x_1, x_2, \dots, x_n)^T$, $C_1 = (c_{1,1}, c_{1,2}, \dots, c_{1,n})^T$,

$$c_{1,i} = \frac{r_{i,1} + r_{i,2} + r_{i,3} + r_{i,4}}{4} - \frac{r_{i,1} - r_{i,2} - r_{i,3} + r_{i,4}}{8} (1 - 2\alpha)\theta_l,$$

and $\theta_l = \max_{1 \leq i \leq n} \theta_{i,l}$. In the case of $\alpha > 0.5$, the mean value of fuzzy return $r(\tilde{\xi}^T x)$ is calculated by

$$E[r(\tilde{\xi}^T x)] = C_2^T x,$$

where $C_2 = (c_{2,1}, c_{2,2}, \dots, c_{2,n})^T$, and

$$c_{2,i} = \frac{r_{i,1} + r_{i,2} + r_{i,3} + r_{i,4}}{4} - \frac{r_{i,1} - r_{i,2} - r_{i,3} + r_{i,4}}{8} (1 - 2\alpha)\theta_r,$$

and $\theta_r = \min_{1 \leq i \leq n} \theta_{i,r}$.

As a consequence, in the case of $\alpha \leq 0.5$, model (4) is equivalent to the following parametric programming model (6).

$$\begin{aligned} \min \quad & \frac{1}{2} x^T H^1 x \\ \text{s.t.} \quad & C_1^T x \geq \psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0 \quad i = 1, 2, \dots, n \end{aligned} \quad (6)$$

In the case of $\alpha > 0.5$, model (4) is equivalent to the following parametric programming model (7).

$$\begin{aligned} \min \quad & \frac{1}{2} x^T H^2 x \\ \text{s.t.} \quad & C_2^T x \geq \psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0 \quad i = 1, 2, \dots, n \end{aligned} \quad (7)$$

Example 7.1. Consider an investor intends to invest his fund in twenty-two securities. Let x_i be the investment proportion to security i , and $\tilde{\xi}_i$'s mutually independent type-2 trapezoidal fuzzy returns for $i = 1, 2, \dots, 22$. The parametric distributions of $\tilde{\xi}_i, i = 1, 2, \dots, 22$ are represented as follows.

$$\begin{aligned} \tilde{\xi}_1 &= (\widetilde{0.9946}, \widetilde{0.9967}, \widetilde{1.0012}, \widetilde{1.0016}; \theta_{1,l}, \theta_{1,r}), \\ \tilde{\xi}_2 &= (\widetilde{1.0011}, \widetilde{1.0020}, \widetilde{1.0061}, \widetilde{1.0092}; \theta_{2,l}, \theta_{2,r}), \\ \tilde{\xi}_3 &= (\widetilde{0.9986}, \widetilde{1.0073}, \widetilde{1.0081}, \widetilde{1.0094}; \theta_{3,l}, \theta_{3,r}), \\ \tilde{\xi}_4 &= (\widetilde{0.9983}, \widetilde{1.0096}, \widetilde{1.0122}, \widetilde{1.0263}; \theta_{4,l}, \theta_{4,r}), \\ \tilde{\xi}_5 &= (\widetilde{1.0033}, \widetilde{1.0122}, \widetilde{1.0262}, \widetilde{1.0310}; \theta_{5,l}, \theta_{5,r}), \end{aligned}$$

$$\begin{aligned}
\tilde{\xi}_6 &= (\widetilde{1.0146}, \widetilde{1.0159}, \widetilde{1.0248}, \widetilde{1.0499}; \theta_{6,l}, \theta_{6,r}), \\
\tilde{\xi}_7 &= (\widetilde{1.0209}, \widetilde{1.0225}, \widetilde{1.0416}, \widetilde{1.0553}; \theta_{7,l}, \theta_{7,r}), \\
\tilde{\xi}_8 &= (\widetilde{1.0291}, \widetilde{1.0299}, \widetilde{1.0468}, \widetilde{1.0679}; \theta_{8,l}, \theta_{8,r}), \\
\tilde{\xi}_9 &= (\widetilde{1.0259}, \widetilde{1.0468}, \widetilde{1.0618}, \widetilde{1.0709}; \theta_{9,l}, \theta_{9,r}), \\
\tilde{\xi}_{10} &= (\widetilde{1.0350}, \widetilde{1.0514}, \widetilde{1.0671}, \widetilde{1.0830}; \theta_{10,l}, \theta_{10,r}), \\
\tilde{\xi}_{11} &= (\widetilde{1.0388}, \widetilde{1.0469}, \widetilde{1.0702}, \widetilde{1.0851}; \theta_{11,l}, \theta_{11,r}), \\
\tilde{\xi}_{12} &= (\widetilde{1.0385}, \widetilde{1.0629}, \widetilde{1.0758}, \widetilde{1.0986}; \theta_{12,l}, \theta_{12,r}), \\
\tilde{\xi}_{13} &= (\widetilde{1.0414}, \widetilde{1.0569}, \widetilde{1.0770}, \widetilde{1.1024}; \theta_{13,l}, \theta_{13,r}), \\
\tilde{\xi}_{14} &= (\widetilde{1.0511}, \widetilde{1.0529}, \widetilde{1.0769}, \widetilde{1.1116}; \theta_{14,l}, \theta_{14,r}), \\
\tilde{\xi}_{15} &= (\widetilde{1.0422}, \widetilde{1.0766}, \widetilde{1.0877}, \widetilde{1.1168}; \theta_{15,l}, \theta_{15,r}), \\
\tilde{\xi}_{16} &= (\widetilde{1.0373}, \widetilde{1.0914}, \widetilde{1.0972}, \widetilde{1.1171}; \theta_{16,l}, \theta_{16,r}), \\
\tilde{\xi}_{17} &= (\widetilde{1.0460}, \widetilde{1.0932}, \widetilde{1.1048}, \widetilde{1.1269}; \theta_{17,l}, \theta_{17,r}), \\
\tilde{\xi}_{18} &= (\widetilde{1.0640}, \widetilde{1.0760}, \widetilde{1.1130}, \widetilde{1.1300}; \theta_{18,l}, \theta_{18,r}), \\
\tilde{\xi}_{19} &= (\widetilde{1.0615}, \widetilde{1.0785}, \widetilde{1.1155}, \widetilde{1.1275}; \theta_{19,l}, \theta_{19,r}), \\
\tilde{\xi}_{20} &= (\widetilde{1.0456}, \widetilde{1.0986}, \widetilde{1.1221}, \widetilde{1.1257}; \theta_{20,l}, \theta_{20,r}), \\
\tilde{\xi}_{21} &= (\widetilde{1.0549}, \widetilde{1.0896}, \widetilde{1.1279}, \widetilde{1.1293}; \theta_{21,l}, \theta_{21,r}), \\
\tilde{\xi}_{22} &= (\widetilde{1.0619}, \widetilde{1.0992}, \widetilde{1.1257}, \widetilde{1.1533}; \theta_{22,l}, \theta_{22,r}).
\end{aligned}$$

This problem was considered in the literature, but the parameters $\theta_{i,l}$ and $\theta_{i,r}$ included in the secondary distributions are the same for different fuzzy returns $\tilde{\xi}_i, i = 1, 2, \dots, 22$. In this section, we extend this problem by assuming that the parameters $\theta_{i,l}$ and $\theta_{i,r}$ are different and take on their values in the interval $[0, 1]$. We build this portfolio selection problem as model (4). In this case, the portfolio selection problem is equivalent to the following parametric quadratic convex programming model (8).

$$\begin{aligned}
\min \quad & \frac{1}{2}x^T H^j x \\
\text{s.t.} \quad & C_j^T x \geq \psi \\
& \sum_{i=1}^n x_i = 1 \\
& x_i \geq 0 \quad i = 1, 2, \dots, 22
\end{aligned} \tag{8}$$

where $H^j = R^T Q^j R, j = 1, 2$, the matrix R is defined by equation (5), and the parametric matrix Q^j about θ_l and θ_r is given in Theorem 6.3. We next solve the convex programming model (8) by Lingo software.

In our numerical experiments, we set the parameters

$$(\theta_{1,l}, \theta_{2,l}, \dots, \theta_{22,l}) =$$

(0.2058, 0.3134, 0.0975, 0.3469, 0.2649, 0.3706, 0.0854, 0.1419, 0.3157, 0.2595, 0.0357, 0.2340, 0.3577, 0.1922, 0.1712, 0.0318, 0.0462, 0.2235, 0.3171, 0.0344, 0.1816, 0.2952),

and

$$(\theta_{1,r}, \theta_{2,r}, \dots, \theta_{22,r}) =$$

(0.8147, 0.1275, 0.6324, 0.2785, 0.9575, 0.1576, 0.9572, 0.8003, 0.4218, 0.7922, 0.6557, 0.8491, 0.6787, 0.7431, 0.6555, 0.7060, 0.2769, 0.1971, 0.6948, 0.9502, 0.4387, 0.7655).

Thus, we have $\theta_l = \max_{1 \leq i \leq n} \theta_{i,l} = 0.3706$, and $\theta_r = \min_{1 \leq i \leq n} \theta_{i,r} = 0.1275$.

In the case of $\alpha = 0.1449$, for various values of ψ , we report the obtained optimal allocation proportions to 22 securities in Table 1.

ψ	x_1	x_3	x_{10}	x_{17}	x_{22}
0.9985	96.80463	3.195369	0	0	0
1.0129	0	87.16909	12.83091	0	0
1.0278	0	67.50335	19.51263	12.98403	0
1.0399	0	53.68988	19.45708	26.85304	0
1.0587	0	32.02779	19.87874	48.09347	0
1.0886	0	0	14.38793	85.61207	0
1.0958	0	0	0	86.60654	13.39346
1.1099	0	0	0	2.644669	97.35532

TABLE 1. The Proportions to the 22 Securities with $\alpha = 0.1449$

In the case of $\alpha = 0.8232$, for different values of ψ , we report the obtained optimal allocation proportions to 22 securities in Table 2.

ψ	x_1	x_2	x_3	x_{10}	x_{17}	x_{18}	x_{22}
0.9985	100	0	0	0	0	0	0
1.0129	0	29.31099	62.39864	0	0	8.290371	0
1.0278	0	0	74.43365	2.700466	0	22.86589	0
1.0399	0	0	50.80296	27.71073	0	21.48632	0
1.0587	0	0	14.64149	65.20998	0	20.14853	0
1.0886	0	0	0	17.97689	18.20904	63.81407	0
1.0958	0	0	0	0	8.81141	89.11859	2.070005
1.1099	0	0	0	0	0	0.1775088	99.82249

TABLE 2. The Proportions to the 22 Securities with $\alpha = 0.8232$

From Tables 1 and 2, we observe that model (4) can provide diversification investments to securities. If we use a fixed value 0.1449 or 0.8232 of parameter α and change the values of parameter ψ , then the invested securities are changed accordingly. On the other hand, if we use a fixed value of ψ such as 1.0129 and take the values of α from the set $\{0.1449, 0.8232\}$, then the invested securities are changed from securities 3 and 10 to securities 2, 3 and 18. Even if the invested securities are the same, the invested proportions to them are often different. As

a consequence, the computational results demonstrate that the invested securities and their invested proportions in our portfolio selection problem depend heavily on the possibility distributions of fuzzy returns. By the definition of parameter α , it determines the possibility distributions of fuzzy returns.

In summary, the computational results demonstrate that parametric possibility distributions have some advantages over fixed possibility distributions when we employ them to model fuzzy portfolio selection problems.

8. Conclusions and Future Research

In this paper, we presented a new reduction method for type-2 fuzzy variables, and obtained the following major new results.

- (i) We defined the CVaR for regular fuzzy variable, and established some useful CVaR formulas for common regular fuzzy variables.
- (ii) We proposed the CVaR reduction method for type-2 fuzzy variables. For the reduced triangular, trapezoidal, normal and gamma fuzzy variables, we derived their useful parametric possibility distributions.
- (iii) According to the parametric possibility distributions of the reduced triangular, trapezoidal, normal and gamma fuzzy variables, we established some useful analytical formulas of mean values.
- (iv) For reduced fuzzy variables, the n th moments were first defined to gauge the variations of parametric possibility distributions with respect to their mean values. Then, we established some useful analytical formulas of second order moments for common reduced fuzzy variables.
- (v) Applying the second order moment as a new risk measure, we developed a mean-moment optimization method for fuzzy portfolio selection problems. The solution results reported in the numerical experiments demonstrated that parametric possibility distributions have some advantages over fixed possibility distributions when we employ them to model fuzzy portfolio selection problems.

Future research might address the following two aspects. First, this paper established some useful analytical expressions about mean values and second order moments of common reduced fuzzy variables. The analytical expressions about higher order moments of reduced fuzzy variables and their sums should be studied in the near future. Second, as far as the practical applications are concerned, this paper suggested to use the second order moment of reduced fuzzy variable as a measure to gauge the risk resulted from fuzzy uncertainty. The theoretical results obtained in this paper have potential applications in other practical risk management and engineering optimization problems, including transportation problem, location and allocation problem, data envelopment analysis, emergency supplies prepositioning problem and so on.

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