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NON-NEWTONIAN FUZZY NUMBERS AND RELATED APPLICATIONS

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ABSTRACT. Although there are many excellent ways presenting the principle of the classical calculus, the novel presentations probably leads most naturally to the development of the non-Newtonian calculus. The important point to note is that the non-Newtonian calculus is a self-contained system independent of any other system of calculus. Since this self-contained work is intended for a wide audience, including engineers, scientists and mathematicians. The main purpose of the present paper is to construct of fuzzy numbers with respect to the non-Newtonian calculus and is to give the necessary and sufficient conditions according to the generalization of the notion of fuzzy numbers by using the generating functions. Also we introduce the concept of non-Newtonian fuzzy distance and give some properties regarding convergence of sequences and series of fuzzy numbers with some illustrative examples.

1. Introduction

Due to the rapid development of the fuzzy theory [16], however, some of these basic concepts have been modified and improved. One of them set mapping operations to the case of interval valued fuzzy sets. To accomplish this we need to introduce the idea of the level sets of interval fuzzy sets and the related formulation of a representation of an interval valued fuzzy set in terms of its level sets. Once having these structures we then can provide the desired extension to interval valued fuzzy sets. Also the effectiveness of level sets comes from not only their required memory capacity for fuzzy sets, but also their two valued nature. This nature contributes to an effective derivation of the fuzzy-inference algorithm based on the families of the level sets.

In the period from 1967 till 1972, Grossman and Katz [3] introduced the non-Newtonian calculus consisting of the branches of geometric, bigeometric, quadratic and biquadratic calculus etc. Also Grossman extended this notion to the other fields in [4, 5]. All these calculi can be described simultaneously within the framework of a general theory. We prefer to use the name *non-Newtonian* to indicate any of calculi other than the classical calculus.

Many authors have extensively developed the notion of multiplicative calculus. The complete mathematical description of multiplicative calculus was given

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by Bashirov et al. [1]. Based on the multiplicative differentiation, various numerical approximation techniques were proposed and discussed. In some cases, for example wage-rate (in dollars, euro, etc.) and related problems, the use of bigeometric (multiplicative) calculus is advocated instead of a traditional Newtonian one. Mısırlı and Gürefe have examined multiplicative Adams-Bashforth-Moulton methods for differential equations with numerical examples in [10]. Also some authors have also worked on the classical sequence spaces and related topics by using non-Newtonian calculus [2, 15]. Kadak et al. [6, 7] have introduced matrix transformations between certain sequence spaces over the non-Newtonian complex field and have generalized Runge-Kutta method by using a new kind differentiation i.e. $*$-differentiation.

According to Zadeh's extension principle, M. Stojakovic and Z. Stojakovic investigated the convergence of series of fuzzy numbers in [11] and they gave some results which complete their previous results in [12]. Additionally; Talo and Basar [13, 14] have extended the main results related to the sequence spaces and matrix transformations on the real or complex field to the fuzzy numbers with the level sets. Following this idea Kadak and Basar [8] have studied Fourier series of fuzzy numbers.

2. **Preliminaries, Background and Notation**

Arithmetic is any system that satisfies the whole of the ordered field axioms whose domain is a subset of $\mathbb{R}$. There are infinitely many types of arithmetic, all of which are isomorphic, that is, structurally equivalent.

A *generator* is a one-to-one function whose domain is $\mathbb{R}$ and whose range is a subset $\mathbb{R}_\alpha$ of $\mathbb{R}$ where $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$. Each generator generates exactly one arithmetic, and conversely each arithmetic is generated by exactly one generator. If $I(x) = x$ for all $x \in \mathbb{R}$, then I is called *identity function* whose inverse is itself. In the special cases $\alpha = I$ and $\alpha = exp$, α generates the classical and geometric arithmetics, respectively. By α-*arithmetic*, we mean the arithmetic whose domain is $\mathbb{R}$ and whose operations are defined as follows: For $x, y \in \mathbb{R}_\alpha$ and any generator α,

$$\begin{array}{lccl} \alpha - addition & x\dot{+}y & = & \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} \\ \alpha - substraction & x\dot{-}y & = & \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\} \\ \alpha - multiplication & x\dot{\times}y & = & \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\} \\ \alpha - division & x\dot{/}y & = & \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\} \\ \alpha - order & x\dot{<}y & \Leftrightarrow & \alpha^{-1}(x) < \alpha^{-1}(y) \end{array}$$

As an example choosing the generator $\alpha : \mathbb{R} \longrightarrow \mathbb{R}_{\exp}$, $\alpha(x) = e^x$ and α-arithmetic turns out to be Geometric arithmetic:

$$\begin{array}{lccccl} \alpha - addition & x\dot{+}y & = & e^{\{\ln x + \ln y\}} & = & x \cdot y \\ \alpha - substraction & x\dot{-}y & = & e^{\{\ln x - \ln y\}} & = & x \div y \\ \alpha - multiplication & x\dot{\times}y & = & e^{\{\ln x \ln y\}} & = & x^{\ln y} = y^{\ln x} \\ \alpha - division & x\dot{/}y & = & e^{\{\ln x / \ln y\}} & = & x^{\frac{1}{\ln y}}. \end{array}$$

Following Grosmann and Katz [5] we give the infinitely many q-arithmetics, of which the quadratic and harmonic arithmetics are special cases for $p = 2$ and $p = -1$, respectively. The generator function $q : \mathbb{R} \to \mathbb{R}_q$ and its inverse q^{-1} are defined as follows:

$$q(x) = \begin{cases} x^{1/p} & , \quad x > 0 \\ 0 & , \quad x = 0 \\ -(-x)^{1/p} & , \quad x < 0. \end{cases} , \; q^{-1}(x) = \begin{cases} x^{p} & , \quad x > 0 \\ 0 & , \quad x = 0, \\ -(-x)^{p} & , \quad x < 0. \end{cases} (p \in \mathbb{R} \setminus \{0\}).$$

If $p = 1$ then the q-calculus is reduced to the classical calculus.

The notion of α-summation can be interpreted as:

$$_{\alpha}\sum_{k=1}^{n} x_k = \alpha\left\{ \sum_{k=1}^{n} \alpha^{-1}(x_k) \right\} = \alpha\{\alpha^{-1}(x_1) + \cdots + \alpha^{-1}(x_n)\} \text{ for all } x_k \in \mathbb{R}_\alpha.$$

The α-square of a number x in $\mathbb{R}_\alpha$ is denoted by $x \dot{\times} x$ which will be denoted by x^{2_N}. For each nonnegative number t, the symbol $\sqrt{x}^N$ will be used to denote $t = \alpha\{\sqrt{\alpha^{-1}(x)}\}$ which is the unique nonnegative number whose α-square is equal to x, which means that $t^{2_N} = x$. Through out this section we denote the p-th α-exponent and the q-th α-root of $x \in \mathbb{R}_\alpha$ by x^{p_N} and $\sqrt[q]{x}^N$, respectively. Therefore, we write

$$\begin{aligned} x^{2_N} = x \dot{\times} x &= \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(x)\} = \alpha\{[\alpha^{-1}(x)]^2\} \\ x^{3_N} = x^{2_N} \dot{\times} x &= \alpha\{\alpha^{-1}\{\alpha[\alpha^{-1}(x) \times \alpha^{-1}(x)]\} \times \alpha^{-1}(x)\} = \alpha\{[\alpha^{-1}(x)]^3\} \\ &\vdots \\ x^{p_N} = x^{(p-1)_N} \dot{\times} x &= \alpha\{[\alpha^{-1}(x)]^p\}. \end{aligned}$$

The α-absolute value of a number x in $\mathbb{R}_\alpha$ is defined as $\alpha|\alpha^{-1}(x)|$ and is denoted by $|x|_\alpha$. For each number x in $\mathbb{R}_\alpha$, $\sqrt{x^{2_N}}^N = |x|_\alpha$ where

$$|x|_\alpha = \begin{cases} x & , \quad x \dot{>} 0 \\ \dot{0} & , \quad x = \dot{0} \\ 0 \dot{-} x & , \quad x \dot{<} 0 \end{cases} = \alpha\{|\alpha^{-1}(x)|\}.$$

For any numbers r and s in $\mathbb{R}_\alpha$, if $r \dot{<} s$, then the set of all numbers $x \in \mathbb{R}_\alpha$ such that $r \dot{<} x \dot{<} s$ is called an α-interval, is denoted by $[\dot{r}, \dot{s}]$. Also we use the notation of α-zero and α-one by $\dot{0}$ and $\dot{1}$, respectively.

Definition 2.1. [2] Let X be a non-empty set and $d_\alpha : X \times X \to \mathbb{R}_\alpha$ be a function such that for all x, y, $z \in X$, the following axioms hold:

(M1) $d_\alpha(x, y) = \dot{0}$ if and only if $x = y$,
(M2) $d_\alpha(x, y) = d_\alpha(y, x)$,
(M3) $d_\alpha(x, y) \dot{\leq} d_\alpha(x, z) \dot{+} d_\alpha(z, y)$.

Then, the pair (X, d_α) and d_α are called an *α-metric space* and an *α-metric* on X, respectively.

Theorem 2.2. [2] *$(\mathbb{R}_\alpha, d_\alpha)$ is an α-metric space.*

Definition 2.3. [3] Let (u_n) be an infinite sequence of the elements in $\mathbb{R}_\alpha$. Then there is at most one element $u \in \mathbb{R}_\alpha$ such that every α-interval with u in its α-interior contains all but finitely many terms of (u_n). If there is such a number u, then (u_n) is said to be α-convergent to u, which is called the α-limit of (u_n). It is trivial that $\alpha = I$, α-convergence reduces to the classical convergence. Also (u_n) is said to be α-bounded if there exists an element $M \in \mathbb{R}_\alpha^+$ such that $|x_n|_\alpha \dot{\leq} M$ for every $n \in \mathbb{N}$.

2.1. **Non-Newtonian Complex Field and Basic Definitions.** In the present section we give a new type of calculus denoted by $*$-calculus which represents general structure of non-Newtonian calculus [3]. Since all arithmetics are isomorphic, one can easily obtain all arithmetics by using an unique function from *alpha* to the *beta* arithmetic. Suppose that α and β be two arbitrarily selected generators and also be the ordered pair of arithmetics (α-arithmetic, β-arithmetic). The sets $(\mathbb{R}_\alpha, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{<})$ and $(\mathbb{R}_\beta, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{<})$ are complete ordered field. Also *beta*(*alpha*)-generator generates *beta*(*alpha*)-arithmetic [4]. Definitions and properties given for α-arithmetic are also valid for β-arithmetic.

In $*$-calculus, α-arithmetic is used for arguments and β-arithmetic is used for values; in particular, changes in arguments and values are measured by α-differences and β-differences, respectively. The operators of this calculus type are applied only to functions with arguments in $\mathbb{R}_\alpha$ and values in $\mathbb{R}_\beta$. The $*$-limit of a function with two generator α and β defined by

$$^{*}\lim_{x \to a} f(x) = b \Leftrightarrow \forall \varepsilon \ddot{>} 0, \exists \delta \dot{>} 0 \ni |f(x) \ddot{-} b|_\beta \ddot{\leq} \varepsilon forallx \in \mathbb{R}_\alpha, |x \dot{-} a|_\alpha \dot{<} \delta,$$

for $\varepsilon \in \mathbb{R}_\beta$ and $\delta \in \mathbb{R}_\alpha$. A function f is $*$-continuous at a point a in $\mathbb{R}_\alpha$ if and only if a is an argument of f and $^{*}\lim_{x\to a} f(x) = f(a)$. When α and β are the identity function I, the concepts of $*$-limit and $*$-continuity are identical with those of classical limit and classical continuity, but that is possible even when α and β do not equal I. (see [15])

The isomorphism from α-arithmetic to β-arithmetic is a unique function ι (iota) with the following properties:

(i) ι is one to one.
(ii) ι is from $\mathbb{R}_\alpha$ onto $\mathbb{R}_\beta$.
(iii) For any numbers u and v in $\mathbb{R}_\alpha$,

$$\begin{aligned} \iota(u \dot{+} v) &= \iota(u) \ddot{+} \iota(v),\ \iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v),\ \iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v), \\ \iota(u \dot{/} v) &= \iota(u) \ddot{/} \iota(v), u \dot{\leq} v \Leftrightarrow \iota(u) \ddot{\leq} \iota(v). \end{aligned}$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every x in $\mathbb{R}_\alpha$, and that $\iota(\dot{n}) = \ddot{n}$ for every integer n. Since, for example $u \dot{+} v = \iota^{-1}\{\iota(u) \ddot{+} \iota(v)\}$, it should be clear that any statement in α-arithmetic can readily be transformed into a statement in β-arithmetic.

Let a and b be arbitrarily chosen elements. The ordered pair $(a, b) \in \mathbb{R}_\alpha \times \mathbb{R}_\beta \subseteq \mathbb{R}^2$ is called as a $*$-point. The set of all $*$-points is called the set of $*$-complex numbers and is denoted by $\mathbb{C}^*$, i.e,

$$\mathbb{C}^* := \{z^* = (a, b) \mid a \in \mathbb{R}_\alpha \subseteq \mathbb{R},\ b \in \mathbb{R}_\beta \subseteq \mathbb{R}\}.$$

Define the binary operations addition ($\oplus : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$) and multiplication ($\odot : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$) of ${z_1}^* = (a_1, b_1)$ and ${z_2}^* = (a_2, b_2)$

$$ {z_1}^* \oplus {z_2}^* = (a_1 \dot{+} a_2, b_1 \ddot{+} b_2) = \big(\alpha\{\alpha^{-1}(a_1) + \alpha^{-1}(a_2)\}, \beta\{\beta^{-1}(b_1) + \beta^{-1} b_2\}\big) $$

$$ {z_1}^* \odot {z_2}^* = \big(\alpha\{\alpha^{-1}(a_1)\alpha^{-1}(a_2) - \beta^{-1}(b_1)\beta^{-1}(b_2)\}, \beta\{\alpha^{-1}(a_1)\beta^{-1}(b_2) + \beta^{-1}(b_1)\alpha^{-1}(a_2)\}\big) $$

for $a_1, a_2 \in \mathbb{R}_\alpha$ and $b_1, b_2 \in \mathbb{R}_\beta$.

Theorem 2.4. [15] $(\mathbb{C}^*, \oplus, \odot)$ *is a field.*

Definition 2.5. Let X be a non-empty set and $d^* : X \times X \to \mathbb{R}_\beta$ be a function such that for all x, y, $z \in X$, then the following axioms hold:

(NM1) $d^*(x, y) = \ddot{0}$ if and only if $x = y$,
(NM2) $d^*(x, y) = d^*(y, x)$,
(NM3) $d^*(x, y) \ddot{\leq} d^*(x, z) \ddot{+} d^*(z, y)$.

Then, the pair (X, d^*) and d^* are called a non-Newtonian metric($*$-metric) space and a $*$-metric on X, respectively.

The $*$-distance d^* between two arbitrar elements $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ of the set $\mathbb{C}^*$ is defined by

$$ \begin{aligned} d^*(z_1, z_2) &= \sqrt{[\iota(a_1 \dot{-} a_2)]^{2_N} \ddot{+} (b_1 \ddot{-} b_2)^{2_N}}^{N} \\ &= \beta\big\{\sqrt{(\alpha^{-1}\{a_1\} - \alpha^{-1}\{a_2\})^2 + (\beta^{-1}\{b_1\} - \beta^{-1}\{b_2\})^2}\big\} \end{aligned} $$

where β-square $\sqrt{x}^N$ is $\beta(\sqrt{\beta^{-1}(x)})$ for $a_1, a_2 \in \mathbb{R}_\alpha$ and $b_1, b_2 \in \mathbb{R}_\beta$.

Definition 2.6. Given a point $x_0 \in X$. Then, for a positive β-real number r,

$$ B^*(x_0; r) = \{x \in X \mid d^*(x, x_0) \ddot{<} r\} \text{ and } B^*[x_0; r] = \{x \in X \mid d^*(x, x_0) \ddot{\leq} r\} $$

are $*$-neighborhood (or $*$-open(closed) ball) of centre x_0 and radius r, respectively.

We see that an $*$-open ball of radius r is the set of all points in X whose beta-distance from the center of the ball is less than r and we say directly from the definition that every $*$-neighborhood of x_0 contains x_0; in other words, x_0 is a point of each of its $*$-neighborhoods.

Definition 2.7. Let (X, d^*) be a $*$-metric space. Then the followings are valid:

(i) $G \subset X$ is called $*$-open set if and only if every $*$-point of G has a $*$-neighborhood contained in G. Also $G \subset X$ is called $*$-closed set if and only if its complement is $*$-open.
(ii) The $*$-interior G^o is the largest $*$-open set contained in G and the $*$-closure $\overline{G}$ is the smallest $*$-closed set contained in G.
(iii) $E \subseteq X$. We say that the family of $*$-open sets $\{G_n\}_{n \in \mathcal{A}}$ is an $*$-open covering of E iff $E \subseteq \bigcup_n G_n$.

Proposition 2.8.

(i) *A subset K of a $*$-metric space is said to be $*$-compact if every $*$-open covering of K contains a finite subcovering.*

(ii) *A subset $K \subset \mathbb{R}_\alpha$ of a $*$-metric space is $*$-compact if and only if it is α-closed and α-bounded.*

Proof. Proof is straightforward. □

Definition 2.9. The following statements hold:

(i) Let x, y be two points in $\mathbb{C}^* \subseteq \mathbb{R}^2$. The set

$$[x,y] = \left\{ z = (\mu_1 \odot x \oplus \mu_2 \odot y) \in C^* \;\middle|\; \mu_1 \dot{+} \mu_2 = \dot{1} \text{ for all } \mu_1, \mu_2 \in [\dot{0}, \dot{1}] \right\}$$

is called a $*$-segment with the endpoints x and y.

(ii) A subset M of C^* is called $*$-convex, if it contains, every $*$-point on the $*$-segment connecting x and y is in C^*.

Definition 2.10. (usual $*$-topology) Consider the set of numbers of $\mathbb{C}^*$ with

$$\tau = \{ S \subseteq \mathbb{C}^* \mid \forall x \in S,\ \exists r \in \mathbb{R}_\beta^+ \ \ni B^*(x_0; r) \subseteq S \}$$

for all $x_0 \in \mathbb{C}^*$ and $r \in \mathbb{R}_\beta$. Then $(\mathbb{C}^*, \tau)$ is a topological space with respect to the distance d^* and is called usual $*$-topology on $\mathbb{C}^*$.

Definition 2.11. The following statements are valid:

(i) The $*$-points P_1, P_2 and P_3 are $*$-collinear provided that at least one of the following holds:

$$\begin{aligned} d^*(P_2, P_1) \ddot{+} d^*(P_1, P_3) &= d^*(P_2, P_3), \\ d^*(P_1, P_2) \ddot{+} d^*(P_2, P_3) &= d^*(P_1, P_3), \\ d^*(P_1, P_3) \ddot{+} d^*(P_3, P_2) &= d^*(P_1, P_2). \end{aligned}$$

(ii) A $*$-line is a set L of at least two distinct $*$-points such that for any distinct $*$-points P_1 and P_2 in L, a $*$-point P_3 is in L if and only if P_1, P_2 and P_3 are $*$-collinear. Taking $\alpha = \beta = I$, the $*$-lines turn into the straight lines of Euclidean analytic geometry in two dimensions. (see [4])

(iii) The $*$-slope of a $*$-line through the $*$-points (a_1, b_1) and (a_2, b_2) is given by

$$m^* = \frac{b_2 \ddot{-} b_1}{\iota(a_2 \dot{-} a_1)} := \beta \left\{ \frac{\beta^{-1}(b_2) - \beta^{-1}(b_1)}{\alpha^{-1}(a_2) - \alpha^{-1}(a_1)} \right\}, (a_1 \neq a_2)$$

for $a_1, a_2 \in \mathbb{R}_\alpha$ and $b_1, b_2 \in \mathbb{R}_\beta$.

(iv) The equation of a $*$-line in the form $y \ddot{-} y_0 = m^* \ddot{\times} \iota(x \dot{-} x_0)$ where m^* is the $*$-slope of the $*$-line and (x_0, y_0) are the coordinates of a given $*$-point on the $*$-line. For example given $*$-points $(a_1, \ddot{0})$ and $(a_2, \ddot{1})$ the equation of a $*$-line can be written as follows:

$$y = \beta \left\{ \frac{\alpha^{-1}(x) - \alpha^{-1}(a_1)}{\alpha^{-1}(a_2) - \alpha^{-1}(a_1)} \right\}$$

where α and β are arbitrary generators and $a_1 \neq a_2$ for every $a_1, a_2 \in \mathbb{R}_\alpha$.

3. Non-Newtonian Fuzzy Numbers

We begin by giving some required definitions and statements of theorems, propositions, and lemmas.

Definition 3.1. A *non-Newtonian fuzzy number(∗-fuzzy number)* is a fuzzy set on $\mathbb{R}_\alpha$, i.e. a mapping from $\mathbb{R}_\alpha$ to β-interval as $u : \mathbb{R}_\alpha \subseteq \mathbb{R} \to [\ddot{0}, \ddot{1}] \subseteq \mathbb{R}$ which satisfies the following four conditions:

(i) u is ∗-normal, i.e. there exists an $x_0 \in \mathbb{R}_\alpha$ such that $u(x_0) = \ddot{1}$.

(ii) u is fuzzy ∗-convex in Definition 2.9, i.e.

$$u(\mu_1 \dot{\times} x \dot{+} \mu_2 \dot{\times} y) \ddot{\geq} \min\{u(x), u(y)\}, \;\; \mu_1 \dot{+} \mu_2 = \dot{1}$$

holds for all $\mu_1, \mu_2 \in [\dot{0}, \dot{1}]$ and $x, y \in \mathbb{R}_\alpha$.

(iii) u is upper semi ∗-continuous i.e. for every $\varepsilon \ddot{>} 0$ there exists a ∗-neighborhood U of x_0 such that $u(x) \ddot{\leq} u(x_0) \ddot{+} \varepsilon$ for all $x \in U$.(see Def. 2.6)

(iv) The set $supp\ u = cl^* \left\{x \in \mathbb{R} : u(x) \ddot{>} \ddot{0}\right\}$ is ∗-compact, (see Proposition 2.8), where $cl^*\{x \in \mathbb{R}_\alpha : u(x) \ddot{>} \ddot{0}\}$ denotes the ∗-closure of the set $\{x \in \mathbb{R}_\alpha : u(x) \ddot{>} \ddot{0}\}$ in the usual ∗-topology of $\mathbb{C}^*$. (see Def. 2.7 and Def. 2.10).

We denote the set of all fuzzy numbers on $\mathbb{C}^*$ by E^1_* and called it as *the space of ∗-fuzzy numbers.*

Definition 3.2. Suppose that $u : \mathbb{R}_\alpha \to [\ddot{0}, \ddot{1}]$ is a ∗-fuzzy set and $\ddot{0} \ddot{\leq} \lambda \ddot{\leq} \ddot{1}$, define $[u]_\lambda$ by

$$[u]_\lambda := \begin{cases} \{x \in \mathbb{R}_\alpha : u(x) \ddot{\geq} \lambda\} & , \quad \ddot{0} \ddot{<} \lambda \ddot{\leq} \ddot{1}, \\ supp\ u & , \quad \lambda = \ddot{0}, \end{cases}$$

for $\lambda \in \mathbb{R}_\beta$. Then it is easily established that u is a ∗-fuzzy number if and only if $[u]_\lambda$ is α-closed and α-bounded interval for each λ, $\ddot{0} \ddot{\leq} \lambda \ddot{\leq} \ddot{1}$, and $[u]_\lambda \neq \emptyset$.

From this characterization of ∗-fuzzy numbers, it follows that a ∗-fuzzy number u is completely determined by the end points of the α-intervals $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. This leads to the following characterization of a ∗-fuzzy number in terms of the two *endpoint* functions u^- and u^+. The proof of this theroem is straightforward and is omitted.

Proposition 3.3. *u is ∗-fuzzy convex if and only if $[u]_\lambda$ is a ∗-convex set for all $\lambda \in [\ddot{0}, \ddot{1}]$.*

Proof. It is pointed out that an alternative definition of ∗-convexity is the form of (ii) in Definition 3.1 and this definition does not hold that $u(x)$ must be a ∗-convex function of x.

Suppose that $[u]_\lambda$ is a ∗-convex and $\lambda = u(x) \ddot{\leq} u(y)$, then $y \in [u]_\lambda$ and $\mu_1 \dot{\times} x \dot{+} \mu_2 \dot{\times} y \in [u]_\lambda$ holds by the convexity of $[u]_\lambda$. Hence

$$u(\mu_1 \dot{\times} x \dot{+} \mu_2 \dot{\times} y) \ddot{\geq} \lambda = u(x) = \min\{u(x), u(y)\}, \tag{1}$$

for $\mu_1 \dot{+} \mu_2 = \dot{1}$ for all $\mu_1, \mu_2 \in [\dot{0}, \dot{1}]$. Conversely, if u is ∗-fuzzy convex and $\lambda = u(x)$, then $[u]_\lambda$ may be taken as the set of all points y for which $u(y) \ddot{\geq} u(x)$. By taking into account the inclusion (1), every point of the form $\mu_1 \dot{\times} x \dot{+} \mu_2 \dot{\times} y$ is also in $[u]_\lambda$. Hence $[u]_\lambda$ is a $*$-convex set. □

Theorem 3.4. *Let $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for $u \in E^1_*$ and for each $\lambda \in [\ddot{0}, \ddot{1}]$. Then the following statements hold:*

(i) *$u^- : [\ddot{0}, \ddot{1}] \to \mathbb{R}_\alpha$ is an α- bounded, (see Def. 2.3), and increasing left $*$-continuous function on the interval $(\ddot{0}, \ddot{1}]$.*
(ii) *$u^+ : [\ddot{0}, \ddot{1}] \to \mathbb{R}_\alpha$ is an α-bounded and decreasing left $*$-continuous function on $(\ddot{0}, \ddot{1}]$.*
(iii) *The functions $u^-(\lambda)$ and $u^+(\lambda)$ are right $*$-continuous at the point $\lambda = \ddot{0}$.*
(iv) *$u^-(\ddot{1}) \dot{\leq} u^+(\ddot{1})$.*

Proof. Let $u : \mathbb{R}_\alpha \to [\ddot{0}, \ddot{1}]$ be defined by $u(x) := \sup\{\lambda : u^-(\lambda) \dot{\leq} x \dot{\leq} u^+(\lambda)\}$ for $\lambda \in \mathbb{R}_\beta$ a $*$-fuzzy number corresponding to the pair of functions u^- and u^+. Moreover, if u is a $*$-fuzzy number with respect to the Definition 3.1, then the functions u^- and u^+ satisfy the conditions (i)-(iv). Conversely, if the pair of functions u^- and u^+ satisfies the conditions (i)-(iv), then there exists a unique $u \in E^1_*$ such that $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$ for each $\lambda \in [\ddot{0}, \ddot{1}]$. The rest of proof is straightforward and is omitted. □

Let $u, v, w \in E^1_*$ and $k \in \mathbb{R}_\alpha$. Then the operations α-addition, scalar α-multiplication and α-product defined on

$$\begin{aligned} u \dot{+} v = w &\Leftrightarrow [w]_\lambda = [u]_\lambda \dot{+} [v]_\lambda \text{ for all } \lambda \in [\ddot{0}, \ddot{1}] \\ &\Leftrightarrow w^-(\lambda) = u^-(\lambda) \dot{+} v^-(\lambda) \text{ and } w^+(\lambda) = u^+(\lambda) \dot{+} v^+(\lambda), \end{aligned}$$

$$[k \dot{\times} u]_\lambda = k \dot{\times} [u]_\lambda \text{ and } u \dot{\times} v = w \Leftrightarrow [w]_\lambda = [u]_\lambda \dot{\times} [v]_\lambda,$$

where it is immediate that

$$\begin{aligned} w^-(\lambda) &= \min\{u^-(\lambda) \dot{\times} v^-(\lambda), u^-(\lambda) \dot{\times} v^+(\lambda), u^+(\lambda) \dot{\times} v^-(\lambda), u^+(\lambda) \dot{\times} v^+(\lambda)\}, \\ w^+(\lambda) &= \max\{u^-(\lambda) \dot{\times} v^-(\lambda), u^-(\lambda) \dot{\times} v^+(\lambda), u^+(\lambda) \dot{\times} v^-(\lambda), u^+(\lambda) \dot{\times} v^+(\lambda)\}. \end{aligned}$$

3.1. **Some Types of Non-Newtonian Fuzzy Numbers.** In this section by taking into account Definition 2.11, we generalize some types of well-known fuzzy numbers i.e. trapezoidal and triangular form with respect to the two generating functions α and β.

Definition 3.5. *(Generalized trapezoidal $*$-fuzzy number) We can define generalized trapezoidal $*$-fuzzy number u as $u = (u_1, u_2, u_3, u_4)$, whose membership function will be interpreted as follows:*

$$\mu_{(u)}(x) = \begin{cases} \frac{\iota(x \dot{-} u_1)}{\iota(u_2 \dot{-} u_1)} : & , \quad u_1 \dot{\leq} x \dot{\leq} u_2 \\ \ddot{1} & , \quad u_2 \dot{\leq} x \dot{\leq} u_3 \\ \frac{\iota(u_4 \dot{-} x)}{\iota(u_4 \dot{-} u_3)} : & , \quad u_3 \dot{\leq} x \dot{\leq} u_4 \\ \ddot{0} & , \quad x \dot{<} u_1, \ x \dot{>} u_4 \end{cases}$$

where $u_i \in \mathbb{R}_\alpha$, $i \in \{1,2,3,4\}$ *and, which implies*

$$\mu_{(u)}(x) = \begin{cases} \beta\left\{\frac{\alpha^{-1}(x)-\alpha^{-1}(u_1)}{\alpha^{-1}(u_2)-\alpha^{-1}(u_1)}\right\} & , \quad \alpha^{-1}(u_1) \le \alpha^{-1}(x) \le \alpha^{-1}(u_2) \\ \beta\{1\} & , \quad \alpha^{-1}(u_2) \le \alpha^{-1}(x) \le \alpha^{-1}(u_3) \\ \beta\left\{\frac{\alpha^{-1}(u_4)-\alpha^{-1}(x)}{\alpha^{-1}(u_4)-\alpha^{-1}(u_3)}\right\} & , \quad \alpha^{-1}(u_3) \le \alpha^{-1}(x) \le \alpha^{-1}(u_4) \\ \beta\{0\} & , \quad \alpha^{-1}(x) < \alpha^{-1}(u_1),\ \alpha^{-1}(x) > \alpha^{-1}(u_4) \end{cases}$$

where $u^- : [\ddot{0}, \ddot{1}] \to [u_1, u_2] \subset \mathbb{R}_\alpha$ *and* $u^+ : [\ddot{0}, \ddot{1}] \to [u_3, u_4] \subset \mathbb{R}_\alpha$ *are two strictly monotonical with* α*-order* "$\dot{<}$"*. Thus* λ*-level set for this shape is written below:*

$$\begin{aligned} [u]_\lambda^N &= \big[\alpha\big\{\beta^{-1}(\lambda)(\alpha^{-1}(u_2) - \alpha^{-1}(u_1)) + \alpha^{-1}(u_1)\big\}, \\ &, \quad \alpha\big\{\alpha^{-1}(u_4) - \beta^{-1}(\lambda)(\alpha^{-1}(u_4) - \alpha^{-1}(u_3))\big\}\big] \end{aligned}$$

for each β*-real number* $\lambda \in [\ddot{0}, \ddot{1}]$.

Definition 3.6. (Geometric trapezoidal family) It is a $*$-fuzzy number represented with four points $u_1, u_2, u_3, u_4 \in \mathbb{R}_{exp} = (0, \infty)$. If we choose the generating functions as $\alpha = \exp$ and $\beta = I$ then the membership function in Definition 3.5 turns out

$$\mu_{(G)}(x) = \begin{cases} \frac{\ln(x)-\ln(u_1)}{\ln(u_2)-\ln(u_1)} & , \quad u_1 \le x \le u_2 \\ 1 & , \quad u_2 \le x \le u_3 \\ \frac{\ln(u_4)-\ln(x)}{\ln(u_4)-\ln(u_3)} & , \quad u_3 \le x \le u_4 \\ 0 & , \quad x < u_1,\ x > u_4 \end{cases}$$

and is called trapezoidal geometric fuzzy number. It can be expressed by

$$[u]_\lambda^G := [u^-(\lambda), u^+(\lambda)] = [u_2^\lambda u_1^{1-\lambda}, u_3^\lambda u_4^{1-\lambda}] \text{ for all } \lambda \in [0,1].$$

Similarly if we take $(\alpha = \exp, \beta = \exp)$ and $(\alpha = I, \beta = \exp)$ then the membership functions can be written as

$$\mu_{(BG)}(x) = \begin{cases} \left(\frac{x}{u_1}\right)^{\frac{1}{\ln(u_2/u_1)}} & , \quad u_1 \le x \le u_2 \\ e & , \quad u_2 \le x \le u_3 \\ \left(\frac{u_4}{x}\right)^{\frac{1}{\ln(u_4/u_3)}} & , \quad u_3 \le x \le u_4 \\ 1 & , \quad x < u_1,\ x > u_4 \end{cases},$$

$$\mu_{(AG)}(x) = \begin{cases} \exp\left\{\frac{x-u_1}{u_2-u_1}\right\} & , \quad u_1 \le x \le u_2 \\ e & , \quad u_2 \le x \le u_3 \\ \exp\left\{\frac{u_4-x}{u_4-u_3}\right\} & , \quad u_3 \le x \le u_4 \\ 1 & , \quad x < u_1,\ x > u_4 \end{cases},$$

and, are called trapezoidal bigeometric(anageometric) fuzzy numbers. Hence we can write

$[u]_\lambda^{BG} := [u_2^{\ln\lambda} u_1^{1-\ln\lambda}, u_3^{\ln\lambda} u_4^{1-\ln\lambda}]$ and $[u]_\lambda^{AG} := [(\ln\lambda)(u_2 - u_1) + u_1, u_4 - (\ln\lambda)(u_4 - u_3)]$

for all $\lambda \in [1, e]$ where e is logarithmic number.

Definition 3.7. (q-trapezoidal family) Let $u = (u_1, u_2, u_3, u_4)$ be a $*$-fuzzy number for $u_1, u_2, u_3, u_4 \in \mathbb{R}_q \subseteq \mathbb{R}$. If we choose the generating functions as $\alpha = q$ and $\beta = I$ then the membership function in Definition 3.5 can be expressed by

$$\mu_Q(x) = \begin{cases} \frac{x^p - u_1^p}{u_2^p - u_1^p} & , \quad u_1 \le x \le u_2, \\ 1 & , \quad u_2 \le x \le u_3 \\ \frac{u_4^p - x^p}{u_4^p - u_3^p} & , \quad u_3 \le x \le u_4, \\ 0 & , \quad x < u_1, \ x > u_4 \end{cases}$$

for all $p \in \mathbb{R} \setminus \{0\}$, and is called trapezoidal q-fuzzy number, which imply

$[u]_\lambda^Q := [u^-(\lambda), u^+(\lambda)] = [(\lambda(u_2^p - u_1^p) + u_1^p)^{1/p}, (u_4^p - \lambda(u_4^p - u_3^p))^{1/p}]$ for all $\lambda \in [0, 1]$.

It is clear that all trapezoidal q-fuzzy numbers are reduced to the classic fuzzy numbers for $p = 1$. Besides this if we take $(\alpha = q, \beta = q)$ and $(\alpha = I, \beta = q)$ then the membership functions can be written by

$$\mu_{(BQ)}(x) = \begin{cases} \left(\frac{x^p - u_1^p}{u_2^p - u_1^p}\right)^{1/p} & , \quad u_1 \le x \le u_2, \\ 1 & , \quad u_2 \le x \le u_3, \\ \left(\frac{u_4^p - x^p}{u_4^p - u_3^p}\right)^{1/p} & , \quad u_3 \le x \le u_4, \\ 0 & , \quad x < u_1, \ x > u_4 \end{cases},$$

$$\mu_{(AQ)}(x) = \begin{cases} \left(\frac{x - u_1}{u_2 - u_1}\right)^{1/p} & , \quad u_1 \le x \le u_2, \\ 1 & , \quad u_2 \le x \le u_3, \\ \left(\frac{u_4 - x}{u_4 - u_3}\right)^{1/p} & , \quad u_3 \le x \le u_4, \\ 0 & , \quad x < u_1, \ x > u_4 \end{cases},$$

for all $p \in \mathbb{R} \setminus \{0\}$, and are called trapezoidal BQ(AQ)-fuzzy numbers, which imply

$$[u]_\lambda^{BQ} := [(\lambda^p(u_2^p - u_1^p) + u_1^p)^{1/p}, (u_4^p - \lambda^p(u_4^p - u_3^p))^{1/p}]$$

and

$$[u]_\lambda^{AQ} := [\lambda^p(u_2 - u_1) + u_1, u_4 - \lambda^p(u_4 - u_3)]$$

hold for all $\lambda \in [0, 1]$ and $p \in \mathbb{R} \setminus \{0\}$.

Remark 3.8. The important point to note here that is the α-order "$\dot{<}$" between the numbers u_i, $i \in \{1, 2, 3, 4\}$ implies $u_1 \dot{<} u_2 \Leftrightarrow u_2 > u_1$ in the form of trapezoidal q-fuzzy number for $p \in \mathbb{R}^-$. Especially taking $p = -1$ and $p = 2$ in Definition 3.7 we obtain harmonic and quadratic family of trapezoidal $*$-fuzzy number, respectively.

In particular, when $u_2 \equiv u_3$, the trapezoidal $*$-fuzzy number is reduced to a triangular $*$-fuzzy number denoted by $u = (u_1, u_2, u_3)$. So, triangular forms are special cases of trapezoidal forms.

Example 3.9. Consider the trapezoidal $*$-fuzzy number $u(x)$ defined by $u = (1/4, 1/3, 3/8, 1/2)$. Then the membership functions of the geometric and q-trapezoidal numbers are given by

Case 1: $(\alpha = \exp, \beta = I)$ and $(\alpha = q, \beta = I)$

$$\mu_{(G)}(x) = \begin{cases} \frac{\ln(x)-\ln(1/4)}{\ln(1/3)-\ln(1/4)} & , \quad 1/4 \le x \le 1/3, \\ 1 & , \quad 1/3 \le x \le 3/8, \\ \frac{\ln(1/2)-\ln(x)}{\ln(1/2)-\ln(3/8)} & , \quad 3/8 \le x \le 1/2, \\ 0 & , \quad x > 1/2, \ x < 1/4, \end{cases},$$

$$\mu_{Q}(x) = \begin{cases} \frac{x^p - \frac{1}{4}^p}{\frac{1}{3}^p - \frac{1}{4}^p} & , \quad 1/4 \le x \le 1/3, \\ 1 & , \quad 1/3 \le x \le 3/8, \\ \frac{\frac{1}{2}^p - x^p}{\frac{1}{2}^p - \frac{3}{8}^p} & , \quad 3/8 \le x \le 1/2, \\ 0 & , \quad x > 1/2, \ x < 1/4. \end{cases}.$$

Thus we have $[u]^G_\lambda := \left[\frac{4^{\lambda-1}}{3^\lambda}, \frac{3^\lambda}{2^{2\lambda+1}}\right]$ and

$$[u]^Q_\lambda := \left[\left\{\lambda\left(\frac{1}{3}^p - \frac{1}{4}^p\right) + \frac{1}{4}^p\right\}^{\frac{1}{p}}, \left\{\frac{1}{2}^p - \lambda(\frac{1}{2}^p - \frac{3}{8}^p)\right\}^{\frac{1}{p}}\right]$$

for all $\lambda \in [0,1]$, respectively.

Case 2: $(\alpha = \exp, \beta = \exp)$ and $(\alpha = I, \beta = \exp)$

$$\mu_{(BG)}(x) = \begin{cases} (4x)^{\frac{1}{\ln(4/3)}} & , \quad 1/4 \le x \le 1/3, \\ e & , \quad 1/3 \le x \le 3/8, \\ \left(\frac{1}{2x}\right)^{\frac{1}{\ln(4/3)}} & , \quad 3/8 \le x \le 1/2, \\ 1 & , \quad x > 1/2, \ x < 1/4, \end{cases},$$

$$\mu_{(AG)}(x) = \begin{cases} e^{12x-3} & , \quad 1/4 \le x \le 1/3, \\ e & , \quad 1/3 \le x \le 3/8, \\ e^{4-8x} & , \quad 3/8 \le x \le 1/2, \\ 1 & , \quad x > 1/2, \ x < 1/4. \end{cases}.$$

It is trivial that $[u]^{BG}_\lambda := \left[\frac{4^{\ln\lambda - 1}}{3^{\ln\lambda}}, \frac{3^{\ln\lambda}}{2^{2\ln\lambda+1}}\right]$ and $[u]^{AG}_\lambda := \left[\frac{\ln\lambda+3}{12}, \frac{4-\ln\lambda}{8}\right]$ for all $\lambda \in [1, e]$.

Case 3: $(\alpha = q, \beta = q)$ and $(\alpha = I, \beta = q)$, $p \in \mathbb{R}^+$

$$\mu_{(BQ)}(x) = \begin{cases} \left(\frac{x^p - \frac{1}{4}^p}{\frac{1}{3}^p - \frac{1}{4}^p}\right)^{1/p} & , \quad 1/4 \le x \le 1/3, \\ 1 & , \quad 1/3 \le x \le 3/8, \\ \left(\frac{\frac{1}{2}^p - x^p}{\frac{1}{2}^p - \frac{3}{8}^p}\right)^{1/p} & , \quad 3/8 \le x \le 1/2, \\ 0 & , \quad x > 1/2, \ x < 1/4, \end{cases},$$

$$\mu_{(AQ)}(x) = \begin{cases} (12x-3)^{1/p} & , \quad 1/4 \le x \le 1/3, \\ 1 & , \quad 1/3 \le x \le 3/8, \\ (4-8x)^{1/p} & , \quad 3/8 \le x \le 1/2, \\ 0 & , \quad x > 1/2, \ x < 1/4. \end{cases}.$$

Hence we get $[u]^{BQ}_\lambda := \left[\left\{\lambda^p\left(\frac{1}{3}^p - \frac{1}{4}^p\right) + \frac{1}{4}^p\right\}^{\frac{1}{p}}, \left\{\frac{1}{2}^p - \lambda^p(\frac{1}{2}^p - \frac{3}{8}^p)\right\}^{\frac{1}{p}}\right]$ and $[u]^{AQ}_\lambda := \left[\frac{\lambda^p+3}{12}, \frac{4-\lambda^p}{8}\right]$ for all $\lambda \in [0,1]$, respectively.

Case 4: $(\alpha = \exp, \beta = q, \lambda \in [0,1])$ and $(\alpha = q, \beta = exp, \lambda \in [1,e], p \in \mathbb{R}^+)$

$$\mu_{(GQ)}(x) \quad = \quad \begin{cases} \left(\frac{\ln(x)-\ln(1/4)}{\ln(1/3)-\ln(1/4)}\right)^{1/p} & , \quad 1/4 \leq x \leq 1/3, \\ 1 & , \quad 1/3 \leq x \leq 3/8, \\ \left(\frac{\ln(1/2)-\ln(x)}{\ln(1/2)-\ln(3/8)}\right)^{1/p} & , \quad 3/8 \leq x \leq 1/2, \\ 0 & , \quad x > 1/2, \ x < 1/4 \end{cases},$$

$$\mu_{(QG)}(x) \quad = \quad \begin{cases} \exp\left\{\frac{x^p - \frac{1}{4}^p}{\frac{1}{3}^p - \frac{1}{4}^p}\right\} & , \quad 1/4 \leq x \leq 1/3, \\ e & , \quad 1/3 \leq x \leq 3/8, \\ \exp\left\{\frac{\frac{1}{2}^p - x^p}{\frac{1}{2}^p - \frac{3}{8}^p}\right\} & , \quad 3/8 \leq x \leq 1/2, \\ 1 & , \quad x > 1/2, \ x < 1/4 \end{cases}.$$

That is to say that $[u]_\lambda^{GQ} := \left[\frac{4^{\lambda^p - 1}}{3^{\lambda^p}}, \frac{3^{\lambda^p}}{2^{2(\lambda^p)+1}}\right]$ and

$$[u]_\lambda^{QG} := \left[\left\{\ln\lambda\left(\frac{1}{3}^p - \frac{1}{4}^p\right) + \frac{1}{4}^p\right\}^{\frac{1}{p}}, \left\{\frac{1}{2}^p - \ln\lambda(\frac{1}{2}^p - \frac{3}{8}^p)\right\}^{\frac{1}{p}}\right].$$

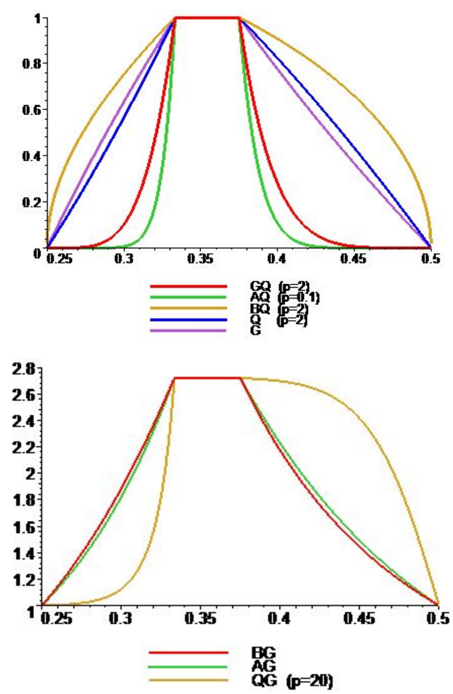

FIGURE 1. This Figure Shows Some Types of $*$-fuzzy Numbers on Cartesian Plane

Figure 1 shows that different types of $*$-fuzzy numbers can be transformed to the real plane based on the choice of generator functions. Moreover, it is pointed out that each of the sets is non-Newtonian fuzzy numbers which are defined over the non-Newtonian Cartesian plane for the x-axis as $\mathbb{R}_\alpha$ and y-axis as $\mathbb{R}_\beta$.

4. **Convergence and Completeness**

Let W be the set of all closed bounded α-intervals A of α-real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e. $A := [\underline{A}, \overline{A}]$. Define the relation d_α on W by

$$d_\alpha(A,B) := \max\{|\underline{A}\dot{-}\underline{B}|_\alpha, |\overline{A}\dot{-}\overline{B}|_\alpha\}$$

with respect to the α-order "$\dot{<}$".

Theorem 4.1. *The space (W, d_α) is a complete α-metric space.*

Proof. One can easily conclude that d_α is an α-metric on W. Since other parts of theorem are basic consequences of Bolzano-Weierstrass theorem in metric space on $\mathbb{R}$, we omit details. □

Now, we may define the metric D^* on E^1_* by means of the Hausdorff metric d_α as

$$\begin{aligned} D^*(u,v) &:= \sup_{\lambda\in[\ddot{0},\ddot{1}]} d_\alpha([u]_\lambda,[v]_\lambda) \\ &:= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|u^-(\lambda)\dot{-}v^-(\lambda)|_\alpha, |u^+(\lambda)\dot{-}v^+(\lambda)|_\alpha\}. \end{aligned}$$

The partial α-ordering relation on E^1_* is defined as follows:

$$u \preceq v \Leftrightarrow [u]_\lambda \preceq [v]_\lambda \Leftrightarrow u^\pm(\lambda)\dot{\leq} v^\pm(\lambda) \Leftrightarrow \alpha^{-1}\{u^\pm(\lambda)\} \leq \alpha^{-1}\{v^\pm(\lambda)\} \quad \text{for all } \lambda \in [\ddot{0},\ddot{1}].$$

One can see that

$$D^*(u,\overline{0}) = \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|u^-(\lambda)|_\alpha, |u^+(\lambda)|_\alpha\} = \max\{|u^-(\ddot{0})|_\alpha, |u^+(\ddot{0})|_\alpha\}.$$

Now, we may give:

Proposition 4.2. *Let $u, v, w, z \in E^1_*$ and $k \in \mathbb{R}_\alpha$. Then,*

(i) $D^*(k\dot{\times}u, k\dot{\times}v) = |k|_\alpha \dot{\times} D^*(u,v)$.
(ii) $D^*(u\dot{+}v, w\dot{+}v) = D^*(u,w)$.
(iii) $D^*(u\dot{+}v, w\dot{+}z) \dot{\leq} D^*(u,w)\dot{+}D^*(v,z)$.
(iv) $|D^*(u,\overline{0})\dot{-}D^*(v,\overline{0})|_\alpha \dot{\leq} D^*(u,v) \dot{\leq} D^*(u,\overline{0})\dot{+}D^*(v,\overline{0})$, $(\overline{0} = [\dot{0},\dot{0}])$.

Proof. We prove only the cases (i)-(iii). The case (iv) can be obtained similarly.

(i) $D^*(k\dot{\times}u, k\dot{\times}v)$

$$\begin{aligned} &= D\big([k\dot{\times}u^-(\lambda), k\dot{\times}u^+(\lambda)], [k\dot{\times}v^-(\lambda), k\dot{\times}v^+(\lambda)]\big) \\ &= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|k\dot{\times}u^-(\lambda)\dot{-}k\dot{\times}v^-(\lambda)|_\alpha, |k\dot{\times}u^+(\lambda)\dot{-}k\dot{\times}v^+(\lambda)|_\alpha\}) \\ &= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\big\{\alpha|\alpha^{-1}(k)(\alpha^{-1}\{u^-(\lambda)\} - \alpha^{-1}\{v^-(\lambda)\}|, \\ &\qquad \alpha|\alpha^{-1}(k)(\alpha^{-1}\{u^+(\lambda)\} - \alpha^{-1}\{v^+(\lambda)\}|\big\} \\ &= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\big\{\alpha^{-1}(k)\dot{\times}\alpha|(\alpha^{-1}\{u^-(\lambda)\} - \alpha^{-1}\{v^-(\lambda)\}|, \\ &\qquad k\dot{\times}\alpha|(\alpha^{-1}\{u^+(\lambda)\} - \alpha^{-1}\{v^+(\lambda)\}|\big\} \\ &= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\big\{|k|_\alpha\dot{\times}|u^-(\lambda)\dot{-}v^-(\lambda)|_\alpha, |k|_\alpha\dot{\times}|u^+(\lambda)\dot{-}v^+(\lambda)|_\alpha\big\} \\ &= |k|_\alpha\dot{\times}D^*(u,v) \end{aligned}$$

(ii) $D^*(u\dot{+}v, w\dot{+}v)$

$$\begin{aligned} &= D\big([u^-(\lambda)\dot{+}v^-(\lambda), u^+(\lambda)\dot{+}v^+(\lambda)], [w^-(\lambda)\dot{+}v^-(\lambda), w^+(\lambda)\dot{+}v^+(\lambda)]\big) \\ &= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\big\{|u^-(\lambda)\dot{-}w^-(\lambda)|_\alpha, |u^+(\lambda)\dot{-}w^+(\lambda)|_\alpha\big\} = D^*(u,w) \end{aligned}$$

(iii) $D^*(u\dot{+}v, w\dot{+}z)$

$$
\begin{aligned}
&= D([u^-(\lambda)\dot{+}v^-(\lambda), u^+(\lambda)\dot{+}v^+(\lambda)], [w^-(\lambda)\dot{+}z^-(\lambda), w^+(\lambda)\dot{+}z^+(\lambda)]) \\
&= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|(u^-(\lambda)\dot{-}w^-(\lambda))\dot{+}(w^-(\lambda)\dot{-}z^-(\lambda))|_\alpha, \\
&\quad |(u^+(\lambda)\dot{-}w^+(\lambda))\dot{+}(v^+(\lambda)\dot{-}z^+(\lambda))|_\alpha\} \\
&\dot{\leq} \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|u^-(\lambda)\dot{-}w^-(\lambda)|_\alpha, |u^+(\lambda)\dot{-}w^+(\lambda)|_\alpha\}\dot{+} \\
&\quad \dot{+} \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|v^-(\lambda)\dot{-}z^-(\lambda)|_\alpha, |v^+(\lambda)\dot{-}v^+(\lambda)|_\alpha\} \\
&= D^*(u,w)\dot{+}D^*(v,z)
\end{aligned}
$$

□

Example 4.3. Consider the trapezoidal $*$-fuzzy numbers u and v defined by q-fuzzy number and BQ-fuzzy number for $p = 2$, respectively in Example 3.9 (see Figure 1). We can calculate the distance $D^*(u, v)$ as follows:

$$
\begin{aligned}
&:= \sup_{\lambda\in[0,1]} d_q([u]^Q_\lambda, [v]^{BQ}_\lambda) := \sup_{\lambda\in[0,1]} \max\{|u^-(\lambda)\dot{-}v^-(\lambda)|_q, |u^+(\lambda)\dot{-}v^+(\lambda)|_q\} \\
&:= \sup_{\lambda\in[0,1]} \max\left\{\left|\left(\frac{7\lambda+9}{144}\right)^{1/2} \dot{-} \left(\frac{7\lambda^2+9}{144}\right)^{1/2}\right|, \left|\left(\frac{16-7\lambda}{64}\right)^{1/2} \dot{-} \left(\frac{16-7\lambda^2}{64}\right)^{1/2}\right|\right\} \\
&:= \sup_{\lambda\in[0,1]} \max\left\{\left(\frac{7\lambda-7\lambda^2}{144}\right)^{1/2}, \left(\frac{7\lambda^2-7\lambda}{64}\right)^{1/2}\right\} \approxeq 0.16
\end{aligned}
$$

As we saw above the distance between two different types of $*$-fuzzy numbers can be calculated by using the same α-generators on the same β-intervals. Depending on the choice of the value of p the distance may vary i.e. for increasing values of p the distance increases and vice versa.

According to Matloka [9], we give some definitions concerning the sequences of $*$-fuzzy numbers below, which are needed in the text.

Definition 4.4.

(i) A sequence $u = (u_k)$ of $*$-fuzzy numbers is a function u from the set $\mathbb{N}$ into the set E^1_*. The $*$-fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the k^{th} term of the sequence. By $\omega^*(F)$ we denote the set of all sequences of $*$-fuzzy numbers.

(ii) A sequence $(u_k) \in w^*(F)$ is called $*$-convergent with the $*$-limit $u \in E^1_*$, if and only if for every $\varepsilon \ddot{>} \ddot{0}$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D^*(u_k, u) \ddot{<} \varepsilon$ for all $k \geq n_0$, $\varepsilon \in \mathbb{R}_\beta$ and, is denoted by ${}^*\lim_{k\to\infty} u_k = u$ or $u_k \xrightarrow{*} u$, as $k \to \infty$.

(iii) (uniformly α-convergent) Suppose a sequence of functions $(u^\pm_n(\lambda))$ with $u^\pm_n(\lambda) : [\ddot{0}, \ddot{1}] \to \mathbb{R}_\alpha$ uniformly α-converges to a function $u : [\ddot{0}, \ddot{1}] \to \mathbb{R}_\alpha$ if for every $\varepsilon \dot{>} 0$ there exists a number N which depends only on $\varepsilon \in \mathbb{R}_\alpha$ such that $d_\alpha(u^\pm_n(\lambda), u^\pm(\lambda)) \dot{<} \varepsilon$ for all $\lambda \in [\ddot{0}, \ddot{1}]$ whenever $n > N$.

(iv) A sequence $(u_k) \in w^*(F)$ is called $*$-bounded if and only if the set of $*$-fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set. That is to say that a sequence $(u_k) \in w^*(F)$ is said to be $*$-bounded if and only if there exist two $*$-fuzzy numbers m and M such that $m \preceq u_k \preceq M$ for all $k \in \mathbb{N}$. This means that $m^-(\lambda) \dot{\leq} u^-_k(\lambda) \dot{\leq} M^-(\lambda)$ and $m^+(\lambda) \dot{\leq} u^+_k(\lambda) \dot{\leq} M^+(\lambda)$ for all $\lambda \in [\ddot{0}, \ddot{1}]$.

Remark 4.5.

(i) If the sequence $(u_k) \in w^*(F)$ $*$-converges to a $*$-fuzzy number u then the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly α-convergent to $u^-(\lambda)$ and $u^+(\lambda)$ in $[\ddot{0}, \ddot{1}]$, respectively.
(ii) If the sequence $(u_k) \in w^*(F)$ $*$-bounded then the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly α-bounded.

Lemma 4.6. *(Nested level sets) Suppose (I_n) is a sequence of $*$-fuzzy numbers, with $I_n = [u_n^-(\lambda), u_n^+(\lambda)]$, such that $I_{n+1} \subset I_n$ for all n and $\ddot{0}\ddot{\leq}\lambda\ddot{\leq}\ddot{1}$.*

(a) *Then the sequences $(u_n^-(\lambda))$ and $(u_n^+(\lambda))$ α-converge uniformly (see Def. 4.4), so $u^-(\lambda) = {}^{\alpha}\lim_n u_n^-(\lambda)$ and $u^+(\lambda) = {}^{\alpha}\lim_n u_n^+(\lambda)$ exist. Also, $\bigcap I_n = [u^-(\lambda), u^+(\lambda)]$.*
(b) *Moreoever, if ${}^{\alpha}\lim_n[u_n^+(\lambda)\dot{-}u_n^-(\lambda)] = \dot{0}$ then $u^-(\lambda) = u^+(\lambda)$ is the only point in common to all the $I_n^{\prime}s$.*

Proof. So the sequence $(u_n^-(\lambda))$ is increasing and α-bounded above, so the sequence $(u_n^-(\lambda))$ α-converges to some α-real number $u^-(\lambda)$. Likewise, the sequence $(u_n^+(\lambda))$ is decreasing and is α-bounded below, so $(u_n^+(\lambda))$ α-converges to some α-real number $u^+(\lambda)$. Since $u_n^-(\lambda)\dot{\leq}u_n^+(\lambda)$ for all n, we have $u^-(\lambda)\dot{\leq}u^+(\lambda)$. It is not hard to see why $\bigcap I_n = [u^-(\lambda), u^+(\lambda)]$. This is because if an $u_n^-(\lambda)\dot{\leq}x\dot{\leq}u_n^+(\lambda)$, then in the α-limit $u^-(\lambda)\dot{\leq}x\dot{\leq}u^+(\lambda)$. Now, if $\lim_n[u_n^+(\lambda)\dot{-}u_n^-(\lambda)] = \dot{0}$ we must have $u^-(\lambda)\dot{-}u^+(\lambda) = \dot{0}$, so $u^-(\lambda) = u^+(\lambda)$ in that case. Also, $u^-(\lambda) = u^+(\lambda)$ is the only number in all of the level sets, in that case. □

Theorem 4.7. *(Bolzano-Weierstrass) Every $*$-bounded infinite sequence of $*$-fuzzy numbers with the level sets contains an $*$-convergent subsequence.*

Proof. By using Remark 4.5, the proof of this theorem can be obtained by a routine verification via defined algebraic operations over $\mathbb{R}_\alpha$, hence is omitted. □

Theorem 4.8. *The space (E_*^1, D^*) is a complete $*$-metric space.*

Proof. Suppose that (u_n) be any Cauchy sequence in E_*^1 of $*$-fuzzy numbers with the level sets, we say that the sequences $(u_n^{\pm}(\lambda))$ are α-bounded for all $\lambda \in [\ddot{0}, \ddot{1}]$. From Theorem 4.7 the sequences $(u_n^{\pm}(\lambda))$ contains α-convergent subsequences $(u_{n_k}^{\pm}(\lambda))$. Let ${}^{\alpha}\lim_n u_{n_k}^{\pm}(\lambda) = u^{\pm}(\lambda)$, by taking into account Lemma 4.6, we have ${}^{\alpha}\lim_n u_n^{\pm}(\lambda) = u^{\pm}(\lambda)$ which implies that $u_n \xrightarrow{*} u$, as $n \to \infty$. Therefore (E_*^1, D^*) is $*$-complete. □

Lemma 4.9. *The following statements hold:*

(i) $D^*(u\dot{\times}v, \overline{0}) \;\dot{\leq}\; D^*(u, \overline{0})\dot{\times}D^*(v, \overline{0})$ *for all* $u, v \in E_*^1, (\overline{0} = [\dot{0}, \dot{0}])$.
(ii) *If* $u_k \xrightarrow{*} u$, *as* $k \to \infty$ *then* $D^*(u_k, \overline{0}) \xrightarrow{*} D^*(u, \overline{0})$, *as* $k \to \infty$*; where* $(u_k) \in w^*(F)$.

Proof. (i) It is trivial that the inequalities $|u^-(\lambda)|_\alpha \dot{\leq} D^*(u,\overline{0})$ and $|u^+(\lambda)|_\alpha \dot{\leq} D^*(u,\overline{0})$ hold for all $\lambda \in [\ddot{0},\ddot{1}]$. By considering these facts, one can see that

$$\begin{aligned} D^*(u\dot{\times}v,\overline{0}) &= \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|(u\dot{\times}v)^-(\lambda)|_\alpha, |(u\dot{\times}v)^+(\lambda)|_\alpha\} \\ &\dot{\leq} \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|u^-(\lambda)|_\alpha\dot{\times}|v^-(\lambda)|_\alpha, |u^-(\lambda)|_\alpha\dot{\times}|v^+(\lambda)|_\alpha, \\ &\qquad |u^+(\lambda)|_\alpha\dot{\times}|v^-(\lambda)|_\alpha, |u^+(\lambda)|_\alpha\dot{\times}|v^+(\lambda)|_\alpha\} \\ &= D^*(u,\overline{0}) \dot{\times} \sup_{\lambda\in[\ddot{0},\ddot{1}]} \max\{|v^-(\lambda)|_\alpha, |v^+(\lambda)|_\alpha\} \\ &= D^*(u,\overline{0}) \dot{\times} D^*(v,\overline{0}), \end{aligned}$$

which completes the proof of part (i).

(ii) This is trivial by using the fact given by (v) of Proposition 4.2. □

Theorem 4.10. *Let* $(u_k),(v_k) \in w^*(F)$ *with* $u_k \xrightarrow{*} x$, $v_k \xrightarrow{*} y$ *as* $k \to \infty$. *Then,*

(i) $u_k\dot{+}v_k \xrightarrow{*} x\dot{+}y$ *as* $k \to \infty$,
(ii) $u_k\dot{-}v_k \xrightarrow{*} x\dot{-}y$ *as* $k \to \infty$,
(iii) $u_k\dot{\times}v_k \xrightarrow{*} x\dot{\times}y$ *as* $k \to \infty$.

Proof. The case (i) of the theorem is immediately obtained from the condition Proposition 4.2(iv), and the rest can be obtained similarly. Really;

$$D^*(u_k\dot{+}v_k, x\dot{+}y)\dot{\leq}D^*(u_k,x)\dot{+}D^*(v_k,y)$$

which leads us by passing to $*$-limit as $k \to \infty$ that

$$\mathop{{}^*\!\lim}_{k\to\infty} D^*(u_k\dot{+}v_k, x\dot{+}y)\dot{\leq} \mathop{{}^*\!\lim}_{k\to\infty} D^*(u_k,x)\dot{+} \mathop{{}^*\!\lim}_{k\to\infty} D^*(v_k,y) \to \dot{0}.$$

□

Example 4.11. Consider the sequence $(u_k) = \{-1/(k+1), 1/(k+3), 1/(k+2), 1/k\}$ of $*$-fuzzy numbers and whose membersip function defined by

$$(\mu_k)_{AQ}(x) := \begin{cases} \left(\frac{(xk+x+1)(k+3)}{2k+4}\right)^{1/p} & , \quad x \in [-1/(k+1), 1/(k+3)], \\ 1 & , \quad x \in]1/(k+3), 1/(k+2)], \\ \left(\frac{(1-xk)(k+2)}{2}\right)^{1/p} & , \quad x \in]1/(k+2), 1/k], \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for all $k \in \mathbb{N}$ and $p \in \mathbb{R}^+$. Then $u_k^-(\lambda) = \left(\frac{1}{k+1} + \frac{1}{k+3}\right)\lambda^p - \frac{1}{k+1}$, $u_k^+(\lambda) = \frac{1}{k} - \lambda^p\left(\frac{1}{k} - \frac{1}{k+2}\right)$. Since $\lim_{k\to\infty} u_k^-(\lambda) = 0$ and $\lim_{k\to\infty} u_k^+(\lambda) = 0$ classically, then the sequence $\{u_k(x)\}$ $*$-converges to $*$-fuzzy number $u = \overline{0}$.

Remark 4.12. As shown in Example 4.11 depending on the choice of α and β, the α-convergence of $\{u_k^\pm(\lambda)\}$ turns out to the classical convergence. Moreover, if we take the forms of membership functions as Example 3.9 then the convergence turn out to the ordinary form.

Definition 4.13. Let $(u_k) \in w^*(F)$. Then the expression ${}_*\sum_{k=0} u_k$ is called a *series of ∗-fuzzy numbers*. Denote $s_k = {}_*\sum_{k=0}^{n} u_k$ for all $n \in \mathbb{N}$, if $s_k \xrightarrow{*} u$ then we say that the series ${}_*\sum_{k=0} u_k \xrightarrow{*} u$ and write ${}_*\sum_{k=0}^{\infty} u_k = u$ which implies as $n \to \infty$ that

$${}_\alpha\sum_{k=0}^{\infty} u_k^-(\lambda) \xrightarrow{\alpha} u^-(\lambda) \text{ and } {}_\alpha\sum_{k=0}^{\infty} u_k^+(\lambda) \xrightarrow{\alpha} u^+(\lambda),$$

α-uniformly in $\lambda \in [\ddot{0}, \ddot{1}]$(see Def. 4.4). Otherwise we say the series of ∗-fuzzy numbers ∗-diverges. Additionally, if the sequence (s_k) is ∗-bounded then we say that the series ${}_*\sum u_k$ is ∗-bounded.

Now following Talo and Basar [13], we provides that uniform α-convergence does not necessary in order for defining a ∗-fuzzy number by the series ${}_\alpha\sum_k u_k^-(\lambda) = u^-(\lambda)$ and ${}_\alpha\sum_k u_k^+(\lambda) = u^+(\lambda)$.

Lemma 4.14. *Let* $u_k = \{(u_k^-(\lambda), u_k^+(\lambda)) : \lambda \in [\ddot{0}, \ddot{1}]\}$ *be ∗-fuzzy numbers and* ${}_\alpha\sum_k u_k^-(\lambda) = u^-(\lambda)$ *and* ${}_\alpha\sum_k u_k^+(\lambda) = u^+(\lambda)$ *uniformly α-converge in λ, then* $u = \{(u^-(\lambda), u^+(\lambda)) : \lambda \in [\ddot{0}, \ddot{1}]\}$ *defines a ∗-fuzzy number such that* $u = {}_*\sum u_k$.

Proof. Firstly, we must show that the functions $u^\pm$ satisfy the conditions of Theorem 3.4. For this, we prove that u^- is an α-bounded, non-decreasing, left ∗-continuous function on $(\ddot{0}, \ddot{1}]$ and right ∗-continuous at $\lambda = \ddot{0}$. u_k^-'s are the α-bounded, non-decreasing, left ∗-continuous functions on $(\ddot{0}, \ddot{1}]$ and right ∗-continuous at $\lambda = \ddot{0}$ for each $k \in \mathbb{N}$.

(i) Let $\lambda_1 \ddot{<} \lambda_2$. Then, $u_k^-(\lambda_1) \dot{\leq} u_k^-(\lambda_2)$ for each $k \in \mathbb{N}$. Thus, we obtain ${}_\alpha\sum_k u_k^-(\lambda_1) \dot{\leq} {}_\alpha\sum_k u_k^-(\lambda_2)$ which implies that $u^-(\lambda_1) \dot{\leq} u^-(\lambda_2)$ and u^- is non-decreasing.

(ii) By using the definition of uniform α-convergence in λ, we have

$${}^\alpha\lim_{\lambda \to \lambda_0^-} u_k^-(\lambda) = u_k^-(\lambda_0), \quad {}_\alpha\sum_k u_k^-(\lambda) = u^-(\lambda) \text{ for each } k \in \mathbb{N} \text{ and } \lambda_0 \in (\ddot{0}, \ddot{1}].$$

Then

$${}^\alpha\lim_{\lambda \to \lambda_0^-} u^-(\lambda) = {}^\alpha\lim_{\lambda \to \lambda_0^-} {}_\alpha\sum_k u_k^-(\lambda) = {}_\alpha\sum_k \lim_{\lambda \to \lambda_0^-} u_k^-(\lambda) = {}_\alpha\sum_k u_k^-(\lambda_0) = u^-(\lambda_0)$$

which yields that u^- is a left ∗-continuous function on $(\ddot{0}, \ddot{1}]$.

(iii) By using the expressions ${}^\alpha\lim_{\lambda \to \ddot{0}^+} u_k^-(\lambda) = u_k^-(\ddot{0})$ for each $k \in \mathbb{N}$ and ${}_\alpha\sum_k u_k^-(\lambda) = u^-(\lambda)$, we get

$${}^\alpha\lim_{\lambda \to \ddot{0}^+} u^-(\lambda) = \lim_{\lambda \to \ddot{0}^+} {}_\alpha\sum_k u_k^-(\lambda) = {}_\alpha\sum_k \lim_{\lambda \to \ddot{0}^+} u_k^-(\lambda) = {}_\alpha\sum_k u_k^-(\ddot{0}) = u^-(\ddot{0}).$$

Hence, u^- is a right ∗-continuous function at the point $\lambda = \ddot{0}$.

(iv) There exists $M_k \dot{>} \dot{0}$ such that $|u_k^-(\lambda)|_\alpha \dot{\leq} M_k$ for all $\lambda \in [\ddot{0}, \ddot{1}]$. Since the series ${}_\alpha\sum_k u_k^-(\lambda) = u^-(\lambda)$ uniformly α-converges there exists $n_0 \in \mathbb{N}$ for all $\varepsilon \dot{>} 0$ such that $\left| {}_\alpha\sum_{k=n+1}^{\infty} u_k^-(\lambda) \right|_\alpha \dot{<} \varepsilon$ for all $n \geq n_0$. Therefore, we obtain

$$\begin{aligned} |u^-(\lambda)|_\alpha = \left| {}_\alpha\sum_k u_k^-(\lambda) \right|_\alpha &= \left| {}_\alpha\sum_{k=0}^{n} u_k^-(\lambda) \dot{+} {}_\alpha\sum_{k=n+1}^{\infty} u_k^-(\lambda) \right|_\alpha \\ &\dot{\leq} {}_\alpha\sum_{k=0}^{n} |u_k^-(\lambda)|_\alpha \dot{+} \left| {}_\alpha\sum_{k=n+1}^{\infty} u_k^-(\lambda) \right|_\alpha \\ &\dot{\leq} {}_\alpha\sum_{k=0}^{n} M_k \dot{+} \varepsilon \dot{\leq} K_\varepsilon. \end{aligned}$$

Therefore, u^- is an α-bounded function.

Similarly u^+ can be obtained in a similar way. Therefore, it is deduced that $[u]_\lambda = [u_k^-(\lambda), u_k^+(\lambda)]$ defines a $*$-fuzzy number. Finally, we show that ${}_*\sum u_k = u$. Since the series of functions ${}_\alpha\sum_k u_k^\pm(\lambda)$ uniformly α-converge to $u^\pm(\lambda)$, respectively, for all $\varepsilon \dot{>} \dot{0}$ there exists $n_0 \in \mathbb{N}$ such that $D^*\left({}_\alpha\sum_{k=0}^{n} u_k, u \right)$ can be interpreted as

$$\begin{aligned} &= \sup_{\lambda \in [\ddot{0}, \ddot{1}]} \max \left\{ \left| {}_\alpha\sum_{k=0}^{n} u_k^-(\lambda) \dot{-} u^-(\lambda) \right|_\alpha, \left| {}_\alpha\sum_{k=0}^{n} u_k^+(\lambda) \dot{-} u^+(\lambda) \right|_\alpha \right\} \\ &\dot{\leq} \max \left\{ \sup_{\lambda \in [\ddot{0}, \ddot{1}]} \left| {}_\alpha\sum_{k=0}^{n} u_k^-(\lambda) \dot{-} u^-(\lambda) \right|_\alpha, \sup_{\lambda \in [\ddot{0}, \ddot{1}]} \left| {}_\alpha\sum_{k=0}^{n} u_k^+(\lambda) \dot{-} u^+(\lambda) \right|_\alpha \right\} \dot{<} \varepsilon \end{aligned}$$

for all $n \geq n_0$. Then the sequence $\left({}_*\sum_{k=0}^{n} u_k \right)$ $*$-converges to the $*$-fuzzy number u, i.e. ${}_*\sum u_k = u$. □

Theorem 4.15. *If* ${}_*\sum u_k$ *and* ${}_*\sum v_k$ *$*$-converge, then*

$$D^*\left({}_*\sum u_k, {}_*\sum v_k \right) \dot{\leq} {}_*\sum D^*(u_k, v_k)$$

.

Proof. Let ${}_*\sum u_k = u$ and ${}_*\sum v_k = v$, i.e. ${}^*\lim_{n\to\infty} D^*\left({}_*\sum_{k=0}^{n} u_k, u \right) = \dot{0}$ and ${}^*\lim_{n\to\infty} D^*\left({}_*\sum_{k=0}^{n} v_k, v \right) = \dot{0}$. It is trivial that there is no problem in the case

${}_{*}\sum D^*(u_k, v_k) = \infty$. Suppose that ${}_{*}\sum D^*(u_k, v_k)\dot{<}\infty$. Then, by using the properties of the metric D^* given by Proposition 4.2, we derive that

$$\begin{aligned} D^*(u,v) &= D^*\left(u\dot{+}\,{}_{*}\sum_{k=0}^{n} u_k\dot{+}\,{}_{*}\sum_{k=0}^{n} v_k, v\dot{+}\,{}_{*}\sum_{k=0}^{n} u_k\dot{+}\,{}_{*}\sum_{k=0}^{n} v_k\right) \\ &\dot{\leq} D^*\left({}_{*}\sum_{k=0}^{n} u_k, u\right)\dot{+}D^*\left({}_{*}\sum_{k=0}^{n} v_k, v\right)\dot{+}D^*\left({}_{*}\sum_{k=0}^{n} u_k,\ {}_{\alpha}\sum_{k=0}^{n} v_k\right) \\ &\dot{\leq} D^*\left({}_{*}\sum_{k=0}^{n} u_k, u\right)\dot{+}D^*\left({}_{*}\sum_{k=0}^{n} v_k, v\right)\dot{+}\,{}_{*}\sum_{k=0}^{n} D^*\left(u_k, v_k\right) \end{aligned}$$

which leads us by passing to $*$-limit as $n \to \infty$ that

$$D^*(u,v) = D^*\left({}_{*}\sum u_k,\ {}_{*}\sum v_k\right)\ \dot{\leq}\ {}_{*}\sum D^*(u_k, v_k).$$

□

Example 4.16. Consider $(u_k) = (2^{-k}, 2^{1-k}, 2^{2-k}, 2^{3-k})$ in which each term belongs to $\mathbb{R}_{exp} = (0,\infty)$. Taking $(\alpha = \exp,\ \beta = I)$ we get the trapezoidal geometric fuzzy number defined by

$$(\mu_k)_G(x) = \begin{cases} \frac{\ln(x)-\ln(2^{-k})}{\ln(2^{1-k})-\ln(2^{-k})} &, \quad 2^{-k} \leq x \leq 2^{1-k}, \\ 1 &, \quad 2^{1-k} < x \leq 2^{2-k}, \\ \frac{\ln(2^{3-k})-\ln(x)}{\ln(2^{3-k})-\ln(2^{2-k})} &, \quad 2^{2-k} < x \leq 2^{3-k}, \\ 0 &, \quad otherwise, \end{cases}$$

for all $k \in \mathbb{N}$. It is obvious that $u_k^-(\lambda) = 2^{\lambda(1-k)}2^{-k(1-\lambda)} = 2^{\lambda-k}$ and $u_k^+(\lambda) = 2^{\lambda(2-k)}2^{(3-k)(1-\lambda)} = 2^{3-\lambda-k}$ for all $\lambda \in [0,1]$. Then, $\sum_{k=0}^{\infty}(u_k)_\lambda^- = 2^{\lambda+1}$ and $\sum_{k=0}^{\infty}(u_k)_\lambda^+ = 2^{4-\lambda}$ in classical mean. Therefore the series ${}_{\alpha=exp}\sum u_k$ converges to a trapezoidal geometric fuzzy number $[u]_\lambda^G = [2^{\lambda+1}, 2^{4-\lambda}]$ for all $\lambda \in [0,1]$.

By taking into account the trapezoidal q-fuzzy number i.e. $(\alpha = q,\ \beta = I)$ we have

$$(\mu_k)_Q(x) = \begin{cases} \frac{x^p-2^{-kp}}{2^{(1-k)p}-2^{-kp}} &, \quad 2^{-k} \leq x \leq 2^{1-k} \\ 1, &, \quad 2^{1-k} < x \leq 2^{2-k} \\ \frac{2^{(3-k)p}-x^p}{2^{(3-k)p}-2^{(2-k)p}} &, \quad 2^{2-k} < x \leq 2^{3-k} \\ 0 &, \quad otherwise \end{cases}$$

for all $k \in \mathbb{N}$ and $p \in \mathbb{R}^+$. It is clear that $u_k^-(\lambda) = [2^{-kp}(\lambda(2^p-1)+1)]^{1/p}$ and $u_k^+(\lambda) = [2^{-kp}(2^{3p}-\lambda(2^{3p}-2^{2p}))]^{1/p}$ hold for all $\lambda \in [0,1]$. Thus $\sum_{k=0}^{\infty}(u_k)_\lambda^- = 2[\lambda(2^p-1)+1]^{1/p}$ and $\sum_{k=0}^{\infty}(u_k)_\lambda^+ = 2[(2^{3p}-\lambda(2^{3p}-2^{2p}))]^{1/p}$ imply ${}_{\alpha=q}\sum u_k$ converges to a trapezoidal q-fuzzy number $[u]_\lambda^Q = [2\{\lambda(2^p-1)+1\}^{1/p}, 2\{(2^{3p}-\lambda(2^{3p}-2^{2p}))\}^{1/p}]$. Moreover the convergence of series of $*$-fuzzy numbers varies depending on the choice of p with respect to the q-arithmetic.

5. Conclusion

In 1914, Carl Friedrich Gauss, as quoted in Gauss, pointed out that "In general the position as regards all such new calculi is this. That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is, that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able without the unconscious inspiration of genius which no one can command to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, and in a more limited region with Lagrange's calculus of variations, with my calculus of congruences, and with Mobius' calculus. Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius" (Werke, Bd. 8, page 298; and as quoted in Memorabilia Mathematica (or The Philomath's Quotation Book) (1914) by Robert Edouard Moritz).

In this paper it is shown that depending on the structure of the generators, some different types of fuzzy numbers can be expressed by proposed method over real plane. Especially in the q-forms, to choose the real values of p, we have infinitely many fuzzy systems. Based on the selection of different generators, the systems within the different method varies under consideration. Besides this the geometric forms of fuzzy numbers may give us a new perspective i.e. the membership function is defined on $[1, e]$. Therefore, this model may be convenient for macro and micro systems of real-life situations. As a future work we will study the solution of fuzzy differential equations with respect to the non-Newtonian calculus and construct certain fuzzy sequence spaces via these arithmetics.

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