A COMPARATIVE STUDY OF FUZZY INNER PRODUCT SPACES

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Abstract. In the present paper, we investigate a connection between two fuzzy inner product one of which arises from Felbin's fuzzy norm and the other is based on Bag and Samanta's fuzzy norm. Also we show that, considering a fuzzy inner product space, how one can construct another kind of fuzzy inner product on this space.

1. **Introduction**

We know that in classical mechanics the algebra of observables is the commutative algebra of functions on some space. Moreover, in quantum mechanics or quantum field theory, the observables are operators on a Hilbert space, and the algebra of operators on a Hilbert space is a non-commutative algebra. On the one hand, there are no solutions to equations like $rs - sr = 1$ in commutative algebra. In fact, it has no solution in operators on a finite dimensional Hilbert space. So in the dynamics of quantum theory, one must study operators on infinite dimensional Hilbert spaces. On the other hand, the usual uncertainty principle of Heisenberg ultimates generalized uncertainty principle, this has been motivated by string theory and non-commutative geometry. In strong quantum gravity regime spacetime points are determined in a fuzzy manner. For this reason, fuzzy structure, fuzzy Hilbert space and operators on a fuzzy Hilbert space are required. Thus, one needs to discuss a new family of fuzzy inner product space.

The notion of fuzzy norm on a linear space was first introduced by Katrasas [10]. Feblin [4] gave an idea of a fuzzy norm on a linear space whose associated metric is Kalva type [8]. Cheng and Menderson [3] considered a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [9]. Felbins definition of a fuzzy norm of a linear operator between two fuzzy normed spaces was generalized by Xiao and Zhu [15]. Bag and Samanta [2] introduced a notion of boundedness of a linear operator between fuzzy normed spaces, and studied the relation between fuzzy continuity and fuzzy boundedness. They also considered fuzzy bounded linear functionals, the concept of fuzzy dual spaces, and established some fundamental theorems in the area of fuzzy functional analysis.

Studies on fuzzy inner product spaces are relatively new and few work have been done in fuzzy inner product spaces. The definition introduced by Mazumder and Samanta [11], Hasankhani, Nazari and Saheli [7], Goudarzia, Vaezpour, Saadati [6],

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Goudarzi and Vaezpour [5], Vijayabalaji [13, 14], Mukherjee and Bag [12] are very recent.
In the present paper, first we modify the concepts of the fuzzy inner product space. And an attempt is made to find such relation by making a comparative study of the fuzzy inner product space defined in this paper and [7].

2. Preliminaries

First, we introduce fuzzy inner product space which was given in [5, 6, 12].

Definition 2.1. [6] A fuzzy inner product space is a triplet $(X, F, *)$, where X is a real vector space, $*$ is a t-norm and F is a fuzzy set on $X^2 \times R$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in R$:
(FI1) $F(x, y, 0) = 0$,
(FI2) $F(x, y, t) = F(y, x, t)$,
(FI3) $F(x, x, t) = H(t)$, for all $t \in R$, if and only if $x = 0$,
(FI4) For any real number $c \in R$,

$$F(cx, y, t) = \begin{cases} F(x, y, t/c) & , \quad c > 0 \\ H(t) & , \quad c = 0 \\ 1 - F(x, y, t/c) & , \quad c < 0, \end{cases}$$

where

$$H(t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t \leq 0, \end{cases}$$

(FI5) $\sup_{r+s=t}\{F(x, z, r) * F(y, z, s)\} = F(x + y, z, t)$,
(FI6) $F(x, y, .) : R \longrightarrow [0, 1]$ is continuous on $R - \{0\}$,
(FI7) $\lim_{t\to\infty} F(x, y, t) = 1$.

Definition 2.2. [5] A fuzzy inner product space is a triplet $(X, F, *)$, where X is a real vector space, $*$ is a t-norm and F is a fuzzy set on $X^2 \times R$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in R$:
(FI1) $F(x, x, 0) = 0$ and $F(x, x, t) > 0$, for each $t > 0$
(FI2) $F(x, y, t) = F(y, x, t)$,
(FI3) $F(x, x, t) = H(t)$, for all $t \in R$, if and only if $x = 0$,
(FI4) For any real number $c \in R$ and $t \neq 0$,

$$F(cx, y, t) = \begin{cases} F(x, y, t/c) & , \quad c > 0 \\ H(t) & , \quad c = 0 \\ 1 - F(x, y, t/(-c)) & , \quad c < 0, \end{cases}$$

where

$$H(t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t \leq 0, \end{cases}$$

(FI5) $F(x, x, t) * F(y, y, s) \leq F(x + y, x + y, t + s)$,
(FI6) $\sup_{r+s=t}\{F(x, z, r) * F(y, z, s)\} = F(x + y, x + y, t)$,
(FI7) $F(x, y, .) : R \longrightarrow [0, 1]$ is continuous on $R - \{0\}$,
(FI8) $\lim_{t\to\infty} F(x, y, t) = 1$.

Definition 2.3. [12] A fuzzy inner product space is a pair (X, F), where X is a real vector space and F is a fuzzy set on $X^2 \times R$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in R$:
(FI1) $F(x, x, t) = 0$, for each $t < 0$
(FI2) $F(x, y, t) = F(y, x, t)$,
(FI3) $F(x, x, t) = 1$, for all $t > 0$, if and only if $x = 0$,
(FI4) For any real number $c \in R$ and $t \neq 0$,

$$F(cx, y, t) = \begin{cases} F(x, y, t/c) & , \quad c > 0 \\ H(t) & , \quad c = 0 \\ 1 - F(x, y, t/c) & , \quad c < 0, \end{cases}$$

where

$$H(t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t \leq 0, \end{cases}$$

(FI5) $\min\{F(x, z, t), F(y, z, s)\} \leq F(x + y, z, t + s)$,
(FI6) $\lim_{t\to\infty} F(x, y, t) = 1$.

Now ,we give below some basic preliminaries required for this paper. Also we modify the definition of a fuzzy inner product space and give some examples.

Definition 2.4. [8] A mapping $\eta : R \longrightarrow [0, 1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:
(N1) there exists $t_0 \in R$ such that $\eta(t_0) = 1$.
(N2) for each $\alpha \in (0, 1]$, there exist real numbers $\eta^-_\alpha \leq \eta^+_\alpha$ such that the $\alpha-$level set $[\eta]_\alpha$ is equal to the closed interval $[\eta^-_\alpha, \eta^+_\alpha]$.
The set of all fuzzy real numbers is denoted by $F(R)$. Since each $r \in R$ can be considered as the fuzzy real number $r \in F(R)$ defined by

$$r(t) = \begin{cases} 1 & , \quad t = r \\ 0 & , \quad t \neq r, \end{cases}$$

it follows that R can be embedded in $F(R)$.

Lemma 2.5. [8] *Let* $[a_\alpha, b_\alpha]$*,* $0 < \alpha \leq 1$*, be a family of non-empty intervals. Assume*
a) $[a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$ *for all* $0 < \alpha_1 \leq \alpha_2$*,*
b) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ *whenever* $\{\alpha_k\}$ *is an increasing sequence in* $(0, 1]$ *converging to* α*,*
c) $-\infty < a_\alpha \leq b_\alpha < +\infty$*, for all* $\alpha \in (0, 1]$*.*
Then the family $[a_\alpha, b_\alpha]$ *represents the* α*-level sets of a fuzzy real number* $\eta \in F(R)$*. Conversely if* $[a_\alpha, b_\alpha]$*,* $0 < \alpha \leq 1$*, are the* α*-level sets of a fuzzy number* $\eta \in F(R)$*, then the conditions (a),(b) and (c) are satisfied.*

Definition 2.6. [5] A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a t-norm if it satisfies the following for every $r, s, t, u \in [0, 1]$:
(i) $1 * t = t$,
(ii) $t * s = s * t$,
(iii) $t * (r * s) = (t * r) * s$,
(iv) If $t \leq s$ and $r \leq u$ then $t * r \leq s * u$.

Theorem 2.7. *If $*$ is a t-norm. Then*

$$t *_0 s \leq t * s \leq t *_1 s, \text{ for all } t, s \in [0, 1],$$

where

$$t *_0 s = \begin{cases} \min\{t, s\} & , \quad \max\{t, s\} = 1 \\ 0 & , \quad \text{otherwise}, \end{cases}$$

and

$$t *_1 s = \min\{t, s\}.$$

Definition 2.8. [1] A fuzzy Normed space is a pair (X, N), where X is a real vector space and N is a fuzzy set on $X \times R$ satisfying the following conditions for every $x, y \in X$ and $t, s \in R$:
(N1) $N(x, t) = 0$, for all $t \leq 0$,
(N2) $N(x, t) = 1$, for all $t > 0$ if and only if $x = 0$,
(N3) $N(cx, t) = N(x, t/|c|)$, for all $(0 \neq)c \in R$,
(N4) $\min\{N(x, t), N(y, s)\} \leq N(x + y, t + s)$,
(N5) $N(x, .) : R \longrightarrow [0, 1]$ is a non-decreasing function,
(N6) $\lim_{t\to\infty} N(x, t) = 1$.

Definition 2.9. A fuzzy inner product space is a triplet $(X, F, *)$, where X is a real vector space, $*$ is a t-norm and F is a fuzzy set on $X^2 \times R$ satisfying the following conditions for every $x, y, z \in X$ and $t, s \in R$:
(FI1) $F(x, y, 0) = 0$,
(FI2) $F(x, y, t) = F(y, x, t)$,
(FI3) $F(x, x, t) = 1$, for all $t > 0$, if and only if $x = 0$,
(FI4) For any real number $c \in R$ and $t \neq 0$,

$$F(cx, y, t) = \begin{cases} F(x, y, t/c) & , \quad c > 0 \\ H(t) & , \quad c = 0 \\ 1 - F(x, y, t/c) & , \quad c < 0, \end{cases}$$

where

$$H(t) = \begin{cases} 1 & , \quad t > 0 \\ 0 & , \quad t \leq 0, \end{cases}$$

(FI5) $F(x, z, t) * F(y, z, s) \leq F(x + y, z, t + s)$, for all $t, s > 0$,
(FI6) $\lim_{t\to\infty} F(x, y, t) = 1$.

In some Theorems, we have considered the following conditions:
(FI7) $F(x + y, z, t + s) \leq 1 - ((1 - F(x, z, t)) * (1 - F(y, z, s)))$, for all $t, s > 0$.
(FI8) $F(x, x, t) = 0$, for all $t < 0$.
(FI9) There exists $0 < t_x$ such that $F(x, x, t_x) = 0$, for all $x \neq 0$.
(FI10) $F(x, y, .)$ is a strictly increasing on the subset $\{t > 0 : 0 < F(x, , y, t) < 1\}$ of R, for all $x, y \in X$.
(FI11) $F(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ $(F(x, y, .) : (-\infty, 0) \longrightarrow [0, 1])$ is continuous, for all $x, y \in X$.
(FI12) $F(x, x, t^2) * F(y, y, s^2) \leq F(x, y, ts)$, for all $t, s > 0$ and all $x, y \in X$.

Example 2.10. Let $(X, <.,.>)$ be an ordinary inner product space. We define a mapping $F : X^2 \times R \longrightarrow [0,1]$ as follows

$$F(x,y,t) = \begin{cases} t^{1/2}/(t^{1/2} + |<x,y>|^{1/2}) & , \quad t > 0 \\ 0 & , \quad t = 0 \\ |<x,y>|^{1/2}/((-t)^{1/2} + |<x,y>|^{1/2}) & , \quad t < 0, \end{cases}$$

let $*$ be arbitrary t-norm. Then $(X, F, *)$ is a fuzzy inner product space.
(FI1) It is clear $F(x,y,0) = 0$.
(FI2) Since $<x,y>=<y,x>$ it follows that $F(x,y,t) = F(y,x,t)$.
(FI3) Let $x = 0$ and $t > 0$. We have $F(x,x,t) = t^{1/2}/(t^{1/2} + |<x,x>|^{1/2}) = 1$. conversely, let $F(x,x,t) = 1$, for all $t > 0$. Hence $t^{1/2}/(t^{1/2} + |<x,x>|^{1/2}) = 1$, for all $t > 0$. Thus $|<x,x>| = 0$, so $x = 0$.
(FI4) Let $c > 0$. Suppose that $t > 0$. We get

$$\begin{aligned} F(cx,y,t) &= t^{1/2}/(t^{1/2} + |<cx,y>|^{1/2}) \\ &= t^{1/2}/(t^{1/2} + c^{1/2}|<x,y>|^{1/2}) \\ &= F(x,y,t/c). \end{aligned}$$

Similarly, $F(cx,y,t) = F(x,y,t/c)$, for all $t < 0$.
Let $c < 0$. Suppose that $t > 0$. We have

$$\begin{aligned} F(cx,y,t) &= t^{1/2}/(t^{1/2} + |<cx,y>|^{1/2}) \\ &= t^{1/2}/(t^{1/2} + (-c)^{1/2}|<x,y>|^{1/2}) \\ &= (t/(-c))^{1/2}/((t/(-c))^{1/2} + |<x,y>|^{1/2}) \\ &= 1 - F(x,y,t/c). \end{aligned}$$

Similarly, $F(cx,y,t) = 1 - F(x,y,t/c)$, for all $t < 0$.
Let $c = 0$. It is clear that $F(cx,y,t) = H(t)$ for all $t \in R$.
(FI5) Let $t, s > 0$. Suppose that $F(x,z,t) = \min\{F(x,z,t), F(y,z,s)\}$. Hence

$$t^{1/2}/(t^{1/2} + |<x,z>|^{1/2}) \leq s^{1/2}/(s^{1/2} + |<y,z>|^{1/2}).$$

Thus $t|<y,z>| \leq s|<x,z>|$. So

$$\begin{aligned} t|<x+y,z>| &\leq t|<x,z>| + t|<y,z>| \\ &\leq t|<x,z>| + s|<x,z>| \\ &\leq (t+s)|<x,z>|. \end{aligned}$$

Then $t^{1/2}/(t^{1/2} + |<x,z>|^{1/2}) \leq (t+s)^{1/2}/((t+s)^{1/2} + |<x+y,z>|^{1/2})$. By Theorem 2.7., we have

$$F(x,z,t) * F(y,z,s) \leq \min\{F(x,z,t), F(y,z,s)\} \leq F(x+y,z,t+s).$$

(FI6) Clearly, $\lim_{t\to\infty} F(x,y,t) = 1$.

Example 2.11. Let $X = R$ be a real number set. We define a mapping $F : X^2 \times R \longrightarrow [0,1]$ as follows

$$F(x,y,t) = \begin{cases} 1 \quad , & t > 0 \text{ and } t > xy \\ (t-(xy/2))/(xy/2) \quad , & t > 0 \text{ and } xy/2 \leq t \leq xy \\ 0 & t > 0 \text{ and } t < xy/2 \\ 0 \quad , & t = 0 \\ 1 \ , & t < 0 \text{ and } t > xy/2 \\ (xy-t)/(xy/2) \ , & t < 0 \text{ and } xy \leq t \leq xy/2 \\ 0 \ , & t < 0 \text{ and } t < xy. \end{cases}$$

We show that $(X, F, \min)$ is a fuzzy inner product space.
(FI1) It is clear that $F(x,y,0) = 0$.
(FI2) Since $xy = yx$ it follows that $F(x,y,t) = F(y,x,t)$.
(FI3) It is clear that $F(x,x,t) = 1$, for all $t > 0$ if and only if $x = 0$.
(FI4) Let $c > 0$ and $t > 0$. Suppose that $xy > 0$.

Case1: Let $t > cxy$. Hence $t/c > xy$. We have $F(cx,y,t) = 1 = F(x,y,t/c)$.

Case2: Let $cxy/2 \leq t \leq cxy$. Thus $xy/2 \leq t/c \leq xy$. So

$$F(cx,y,t) = (t-(cxy/2))/(cxy/2) = ((t/c)-(xy/2))/(xy/2) = F(x,y,t/c)$$

Case3: Let $t < cxy/2$. Therefore $t/c < xy/2$. Hence $F(cx,y,t) = 0 = F(x,y,t/c)$.
Suppose that $xy < 0$. Then $F(cx,y,t) = 1 = F(x,y,t/c)$.
Let $c > 0$ and $t < 0$. Similarly, $F(cx,y,t) = F(x,y,t/c)$.
Let $c < 0$ and $t > 0$. Suppose that $xy > 0$. Thus $cxy < 0$. Hence $F(cx,y,t) = 1 = 1 - F(x,y,t/c)$.
Assume that $xy < 0$.

Case1: Let $t > cxy$. Hence $t/c < xy$. So $F(cx,y,t) = 1 = 1 - F(x,y,t/c)$.

Case2: Let $cxy/2 \leq t \leq cxy$. Thus $xy \leq t/c \leq xy/2$. So

$$F(cx,y,t) = (t-(cxy/2))/(cxy/2) = ((t/c)-(xy/2))/(xy/2) = 1 - F(x,y,t/c)$$

Case3: Let $t < cxy/2$. Therefore $t/c > xy/2$. Hence $F(cx,y,t) = 0 = 1 - F(x,y,t/c)$.
Let $c < 0$ and $t < 0$. Similarly, $F(cx,y,t) = F(x,y,t/c)$.
Let $c = 0$. It is clear that $F(cx,y,t) = H(t)$ for all $t \in R$.
(FI5) Let $t, s > 0$.

Case1: Let $t+s > (x+y)z$. Then $F(x+y,z,t+s) = 1 \geq \min\{F(x,z,t), F(y,z,s)\}$.

Case2: Let $t+s < (x+y)z/2$. Then $t < xz/2$ or $s < yz/2$. Hence $F(x,z,t) = 0$ or $F(y,z,s) = 0$. Thus $F(x+y,z,t+s) = 0 = \min\{F(x,z,t), F(y,z,s)\}$.

Case3: Let $(x+y)z \leq t+s \leq (x+y)z/2$. Suppose that $F(x,z,t) \leq F(y,z,s)$. Therefore $tyz/2 \leq sxz/2$. Hence

$$\begin{aligned} F(x+y,z,t+s) &= ((t+s)-((x+y)z)/2)/((x+y)z/2) \\ &\geq (t-(xz/2))/(xz/2) \\ &= F(x,z,t). \end{aligned}$$

(FI6) Clearly, $\lim_{t\to\infty} F(x,y,t) = 1$.
(FI7) Similar to (FI5),

$$F(x+y,z,t+s) \le \max\{F(x,z,t), F(y,z,s)\} = 1-\min\{1-F(x,z,t), 1-F(y,z,s)\}.$$

(FI8) Since $x^2 > 0$, $F(x,x,t) = 0$, for all $t < 0$.
(FI9) We have $F(x,x,x^2/2) = 0$, for all $x \neq 0$.
(FI10)It is clear that $F(x,y,.)$ is strictly increasing on the subset $\{t > 0 : 0 < F(x,,y,t) < 1\}$ of R, for all $x,y \in X$.
(FI11) It is clear that $F(x,y,.) : (0,\infty) \longrightarrow [0,1]$ $(F(x,y,.) : (-\infty,0) \longrightarrow [0,1])$ is continuous, for all $x,y \in X$.
(FI12) Let $x,y \in X$ and $s,t > 0$.

Case1: Let $ts > xy$. Then $F(x,y,zts) = 1 \ge \min\{F(x,x,t^2), F(y,y,s^2)\}$.

Case2: Let $ts < xy/2$. Then $t^2s^2 < x^2y^2/4$. So $t^2 < x^2/2$ or $s^2 < y^2/2$. Hence $F(x,x,t^2) = 0$ or $F(y,y,s^2) = 0$. Thus $F(x,y,ts) = 0 = \min\{F(x,z,t), F(y,z,s)\}$.

Case3: Let $xy/2 \le ts \le xy$. Assume that $F(x,x,t^2) \le F(y,y,s^2)$. Hence $(t^2 - (x^2/2))/(x^2/2) \le (s^2 - (y^2/2))/(y^2/2)$. So $t^2y^2 \le s^2x^2$. Thus $ty \le sx$. Therefore

$$\begin{aligned} F(x,y,ts) &= ((ts) - ((xy)/2)/((xy/2) \\ &\ge (t^2 - (x^2/2))/(x^2/2) \\ &= F(x,x,t^2) \\ &= \min\{F(x,z,t), F(y,z,s)\}. \end{aligned}$$

Theorem 2.12. *Let $(X,F,*)$ be a fuzzy inner product space and $x,y \in X$. Then $F(x,y,.) : (0,\infty) \longrightarrow [0,1]$ $(F(x,y,.) : (-\infty,0) \longrightarrow [0,1])$ is non-decreasing.*

Definition 2.13. [4] Let X be a vector space over R. Assume the mappings $L,R : [0,1]\times[0,1] \longrightarrow [0,1]$ are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$. Let $\|.\| : X \longrightarrow F^+(R)$. The quadruple $(X,\|.\|,L,R)$ is called a fuzzy normed linear space with the fuzzy norm $\|.\|$, if the following conditions are satisfied:
(F_1) if $x \neq 0$ then $\inf_{0<\alpha\le 1} \|x\|_\alpha^- > 0$,
(F_2) $\|x\| = 0$ if and only if $x = 0$,
(F_3) $\|rx\| = |r| \odot \|x\|$ for $x \in X$ and $r \in R$,
(F_4) for all $x,y \in X$,
$(F_4L)\|x+y\|(s+t) \ge L(\|x\|(s), \|y\|(t))$ whenever $s \le \|x\|_1^-$, $t \le \|y\|_1^-$ and $s+t \le \|x+y\|_1^-$,
$(F_4R)\|x+y\|(s+t) \le R(\|x\|(s), \|y\|(t))$ whenever $s \ge \|x\|_1^-$, $t \ge \|y\|_1^-$ and $s+t \ge \|x+y\|_1^-$.

In the sequel we fix $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$ for all $s,t \in [0,1]$. And we write $(X,\|.\|)$ or simply X when L and R are as indicated just above.
The following result is an analogue of the usual triangle inequality.

Theorem 2.14. *In a fuzzy normed linear space $(X,\|.\|)$, the condition (F_4) is equivalent to*

$$\|x+y\| \le \|x\| \oplus \|y\|.$$

Definition 2.15. [7] A fuzzy inner product space is a pair $(X, <.,.>)$, where X is a real vector space and $<.,.>$ is a function $\langle .,. \rangle : X \times X \longrightarrow F(R)$ such that for all vectors $x, y, z \in X$ and all $c \in R$, we have:
(IP1) $\langle x + y, z \rangle = \langle x, z \rangle \oplus \langle y, z \rangle$,
(IP2) $\langle cx, y \rangle = c \odot \langle x, y \rangle$,
(IP3) $\langle x, y \rangle = \langle y, x \rangle$,
(IP4) $\langle x, x \rangle \geq 0$,
(IP5) $0 < \inf_{\alpha \in (0,1]} \langle x, x \rangle_\alpha^-$ if $x \neq 0$,
(IP6) $\langle x, x \rangle = 0$ if and only if $x = 0$.

In some Theorems, we have considered the following condition:
(IP7) for any decreasing sequence $\{\alpha_k\}$ in $(0, 1]$ such that $\alpha_k \longrightarrow \alpha \in (0, 1]$ implies that $< x, y >_{\alpha_k}^+ \longrightarrow < x, y >_\alpha^+$, for all $x, y \in X$.

Lemma 2.16. [7] *A fuzzy inner product space $(X, <.,.>)$ satisfy the Schwarz inequality*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \odot \sqrt{\langle y, y \rangle}, \text{ for all } x, y \in X.$$

Theorem 2.17. [7] *The function $\|.\| : X \longrightarrow F^+(R)$ defined by $\|x\| = \sqrt{\langle x, x \rangle}$, for all $x \in X$, is a fuzzy norm on X.*

We assume that
(IP7) for any decreasing sequence $\{\alpha_k\}$ in $(0, 1]$ such that $\alpha_k \longrightarrow \alpha \in (0, 1]$ implies that $< x, y >_{\alpha_k}^+ \longrightarrow < x, y >_\alpha^+$, for all $x, y \in X$.

3. Main Results

Proposition 3.1. *Let $(X, F, *)$ be a fuzzy inner product space. Then*

$$F(x+y, z, t+s) \leq 1-((1-F(x, z, t))*(1-F(y, z, s))), \text{ for all } x, y \in X \text{ and all } t, s < 0.$$

Proof. Let $x, y \in X$ and $s, t < 0$. We have

$$\begin{aligned} 1 - F(x + y, z, t + s) &= F(-x - y, z, -t - s) \\ &\geq F(-x, z, -t) * F(-y, z, -s) \\ &= (1 - F(x, z, t)) * (1 - F(y, z, s)). \end{aligned}$$

Hence $F(x + y, z, t + s) \leq 1 - ((1 - F(x, z, t)) * (1 - F(y, z, s)))$. □

Proposition 3.2. *Let $(X, F, *)$ be a fuzzy inner product space satisfying (FI7). Then*

$$F(x + y, z, r) \geq F(x, z, t) * F(y, z, s) \text{ for all } x, y \in X \text{ and all } t, s < 0.$$

Proof. Let $x, y \in X$ and $s, t < 0$. We have

$$\begin{aligned} 1 - F(x + y, z, t + s) &= F(-x - y, z, -t - s) \\ &\leq 1 - ((1 - F(-x, z, -t)) * (1 - F(-y, z, -s))) \\ &= 1 - (F(x, z, t) * F(y, z, s)). \end{aligned}$$

Hence $F(x + y, z, t + s) \geq F(x, z, t) * F(y, z, s)$. □

Theorem 3.3. *Let (X, F, min) be a fuzzy inner product space satisfying (FI7), (FI8) and (FI9). Suppose that*

$$< x, y >_\alpha^- = \inf\{t < 0 : \ \alpha \leq F(x, y, t)\} + \sup\{t > 0 : \ F(x, y, t) = 0\},$$

and

$$< x, y >_\alpha^+ = \sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha\} + \inf\{t < 0 : \ F(x, y, t) = 1\},$$

for all $\alpha \in (0, 1]$. Then there is a fuzzy inner product $< ., . >$ on X such that

$$[< x, y >]_\alpha = [< x, y >_\alpha^-, < x, y >_\alpha^+] \text{ for all } \alpha \in (0, 1] \text{ and } x, y \in X.$$

Proof. Let $x, y \in X$. First we show that $[< x, y >_\alpha^-, < x, y >_\alpha^+]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number.

(a) Assume that $0 < \alpha_1 \leq \alpha_2 \leq 1$. Then

$$\{t < 0 : \ \alpha_2 \leq F(x, y, t)\} \subseteq \{t < 0 : \ \alpha_1 \leq F(x, y, t)\}$$

and

$$\{t > 0 : \ F(x, y, t) \leq 1 - \alpha_2\} \subseteq \{t > 0 : \ F(x, y, t) \leq 1 - \alpha_1\}.$$

Hence $< x, y >_{\alpha_1}^- \leq < x, y >_{\alpha_2}^-$ and $< x, y >_{\alpha_2}^- \leq < x, y >_{\alpha_1}^-$. This implies that

$$[< x, y >_{\alpha_2}^-, < x, y >_{\alpha_2}^+] \subseteq [< x, y >_{\alpha_1}^-, < x, y >_{\alpha_1}^+].$$

(b) Let $\{\alpha_k\}$ be an increasing sequence in $(0, 1]$ converging to α. Since $\alpha_k \leq \alpha_{k+1} \leq \alpha$,

$$< x, y >_{\alpha_k}^- \leq < x, y >_{\alpha_{k+1}}^- \leq < x, y >_\alpha^-, \text{ for all } k > 0.$$

Then $\lim_{k\to\infty} < x, y >_{\alpha_k}^- = \sup_{k>0} < x, y >_{\alpha_k}^- \leq < x, y >_\alpha^-$. If

$$\lim_{k\to\infty} < x, y >_{\alpha_k}^- < t_0 << x, y >_\alpha^-,$$

then $\alpha_k \leq F(x, y, t_0) < \alpha$, for all $k > 0$. As $k \longrightarrow \infty$, we obtain that $\alpha \leq F(x, y, t_0) < \alpha$, which is a contradiction.

Since $\alpha_k \leq \alpha_{k+1} \leq \alpha$, $< x, y >_{\alpha_k}^+ \geq < x, y >_{\alpha_{k+1}}^+ \geq < x, y >_\alpha^+$, for all $k > 0$. Hence $\lim_{k\to\infty} < x, y >_{\alpha_k}^+ = \inf_{k>0} < x, y >_{\alpha_k}^+ \geq < x, y >_\alpha^+$. If

$$\lim_{k\to\infty} < x, y >_{\alpha_k}^- > t_1 >< x, y >_\alpha^-,$$

then $1 - \alpha < F(x, y, t_1) \leq 1 - \alpha_k$, for all $k > 0$. As $k \longrightarrow \infty$, we obtain that $1 - \alpha < F(x, y, t_0) \leq 1 - \alpha$, which is a contradiction. Thus

$$[\lim_{k\to\infty} < x, y >_{\alpha_k}^-, \lim_{k\to\infty} < x, y >_{\alpha_k}^+] = [< x, y >_\alpha^-, < x, y >_\alpha^+].$$

(c) Since $\lim_{t\to\infty} F(x, y, t) = 1$ and $\lim_{t\to-\infty} F(x, y, t) = 0$ it follows that

$$\begin{aligned}
-\infty &< \inf\{t < 0 : \ \alpha \leq F(x, y, t)\} \\
&\leq \inf\{t < 0 : \ \alpha \leq F(x, y, t)\} + \sup\{t > 0 : \ F(x, y, t) = 0\} \\
&\leq \sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha\} + \inf\{t < 0 : \ F(x, y, t) = 1\} \\
&\leq \sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha\} \\
&< +\infty.
\end{aligned}$$

Hence $-\infty << x,y >_\alpha^- \leq < x,y >_\alpha^+ < +\infty$.
By Lemma 2.5, $< x,y > \in F(R)$, for all $x,y \in X$.
Now we show that $< .,. >$ is a fuzzy inner product. Let $x,y,z \in X$ and $c \in R$.

(IP1) If $\inf\{t < 0 : \alpha \leq F(x+y,z,t)\} < r < \inf\{t < 0 : \alpha \leq F(x,z,t)\} + \inf\{s < 0 : \alpha \leq F(y,z,s)\} \leq 0$, then $\alpha \leq F(x+y,z,r)$. Suppose that $t+s=r$ and $t < \inf\{t < 0 : \alpha \leq F(x,z,t)\} \leq 0$ and $s < \inf\{s < 0 : \alpha \leq F(y,z,s)\} \leq 0$. Then $F(x,z,t) < \alpha$ and $F(y,z,s) < \alpha$. By proposition 3.1, we have

$$F(x+y,z,r) \leq 1 - \min\{1-F(x,z,t), 1-F(y,z,s)\} < \alpha$$

which is a contradiction. Thus $\inf\{t < 0 : \alpha \leq F(x,z,t)\} + \inf\{s < 0 : \alpha \leq F(y,z,s)\} \leq \inf\{t < 0 : \alpha \leq F(x+y,z,t)\}$.
If $\sup\{t > 0 : F(x+y,z,t) = 0\} < r < \sup\{t > 0 : F(x,z,t) = 0\} + \sup\{s > 0 : F(y,z,s) = 0\}$, then $0 < F(x+y,z,r)$. Let $t,s > 0$, $t+s=r$, $t < \sup\{t > 0 : F(x,z,t) = 0\}$ and $s < \sup\{s > 0 : F(y,z,s) = 0\}$. Hence $F(x,z,t) = 0$ and $F(y,z,s) = 0$. By (FI7), we have

$$F(x+y,z,r) \leq 1 - \min\{1-F(x,z,t), 1-F(y,z,s)\} = 0,$$

which is a contradiction. So $\sup\{t > 0 : F(x,z,t) = 0\} + \sup\{s > 0 : F(y,z,s) = 0\} \leq \sup\{t > 0 : F(x+y,z,t) = 0\}$.
Hence $< x,z >_\alpha^- + < y,z >_\alpha^- \leq < x+y,z >_\alpha^-$.
If $0 \geq \inf\{t < 0 : \alpha \leq F(x+y,z,t)\} > r > \inf\{t < 0 : \alpha \leq F(x,z,t)\} + \inf\{s < 0 : \alpha \leq F(y,z,s)\}$, then $F(x+y,z,r) < \alpha$. Let $t,s < 0$, $t+s=r$, $t > \inf\{t < 0 : \alpha \leq F(x,z,t)\}$ and $s > \inf\{s < 0 : \alpha \leq F(y,z,s)\}$. Hence $\alpha \leq F(x,z,t)$ and $\alpha \leq F(y,z,s)$. By proposition 3.2, we have

$$F(x+y,z,r) \geq \min\{F(x,z,t), F(y,z,s)\} \geq \alpha$$

which is a contradiction. So $\inf\{t < 0 : \alpha \leq F(x+y,z,t)\} \leq \inf\{t < 0 : \alpha \leq F(x,z,t)\} + \inf\{s < 0 : \alpha \leq F(y,z,s)\}$.
If $\sup\{t > 0 : F(x+y,z,t) = 0\} > r > \sup\{t > 0 : F(x,z,t) = 0\} + \sup\{s > 0 : F(y,z,s) = 0\}$, then $F(x+y,z,r) = 0$. Let $t,s > 0$, $t+s=r$, $t > \sup\{t > 0 : F(x,z,t) = 0\}$ and $s > \sup\{s > 0 : F(y,z,s) = 0\}$. Hence $0 < F(x,z,t)$ and $0 < F(y,z,s)$. Thus $0 < \min\{F(x,z,t), F(y,z,s)\}$. Therefore

$$F(x+y,z,r) \geq \min\{F(x,z,t), F(y,z,s)\} > 0,$$

which is a contradiction. So $\sup\{t > 0 : F(x+y,z,t) = 0\} \leq \sup\{t > 0 : F(x,z,t) = 0\} + \sup\{s > 0 : F(y,z,s) = 0\}$.
Hence $< x+y,z >_\alpha^- \leq < x,z >_\alpha^- + < y,z >_\alpha^-$. Thus

$$< x+y,z >_\alpha^- = < x,z >_\alpha^- + < y,z >_\alpha^- .$$

Similarly, we obtain that

$$< x+y,z >_\alpha^+ = < x,z >_\alpha^+ + < y,z >_\alpha^+ .$$

Therefore, $< x+y,z > = < x,z > \oplus < y,z >$.

(IP2) If $c = 0$. It is clear that $< cx,y > = < 0,y > = 0 = 0 < x,y > = c < x,y >$.
Let $0 < c$. We have

$$< cx,y >^-_\alpha = \inf\{t<0:\ \alpha \le F(cx,y,t)\} + \sup\{t>0:\ F(cx,y,t)=0\}$$
$$= \inf\{t<0:\ \alpha \le F(x,y,t/c)\} + \sup\{t>0:\ F(x,y,t/c)=0\}$$
$$= \inf\{ct<0:\ \alpha \le F(x,y,t)\} + \sup\{ct>0:\ F(x,y,t)=0\}$$
$$= c\inf\{t<0:\ \alpha \le F(x,y,t)\} + c\sup\{t>0:\ F(x,y,t)=0\}$$
$$= c < x,y >^-_\alpha,$$

and

$$< cx,y >^+_\alpha = \sup\{t>0:\ F(cx,y,t) \le 1-\alpha\} + \inf\{t<0:\ F(cx,y,t)=1\}$$
$$= \sup\{t>0:\ F(x,y,t/c) \le 1-\alpha\} + \inf\{t<0:\ F(x,y,t/c)=1\}$$
$$= \sup\{ct>0:\ F(x,y,t) \le 1-\alpha\} + \inf\{ct<0:\ F(x,y,t)=1\}$$
$$= c\sup\{t>0:\ F(x,y,t) \le 1-\alpha\} + c\inf\{t<0:\ F(x,y,t)=1\}$$
$$= c < x,y >^+_\alpha .$$

Let $c<0$. We get

$$< cx,y >^-_\alpha = \inf\{t<0:\ \alpha \le F(cx,y,t)\} + \sup\{t>0:\ F(cx,y,t)=0\}$$
$$= \inf\{t<0:\ \alpha \le 1-F(x,y,t/c)\} + \sup\{t>0:\ 1-F(x,y,t/c)=0\}$$
$$= \inf\{ct<0:\ \alpha \le 1-F(x,y,t)\} + \sup\{ct>0:\ 1-F(x,y,t)=0\}$$
$$= c\sup\{t>0:\ F(x,y,t) \le 1-\alpha\} + c\inf\{t<0:\ F(x,y,t)=1\}$$
$$= c < x,y >^+_\alpha,$$

and

$$< cx,y >^+_\alpha = \sup\{t>0:\ F(cx,y,t) \le 1-\alpha\} + \inf\{t<0:\ F(cx,y,t)=1\}$$
$$= \sup\{t>0:\ 1-F(x,y,t/c) \le 1-\alpha\} + \inf\{t<0:\ 1-F(x,y,t/c)=1\}$$
$$= \sup\{ct>0:\ 1-F(x,y,t) \le 1-\alpha\} + \inf\{ct<0:\ 1-F(x,y,t)=1\}$$
$$= c\inf\{t<0:\ \alpha \le F(x,y,t) \le 1-\alpha\} + c\sup\{t>0:\ F(x,y,t)=0\}$$
$$= c < x,y >^-_\alpha .$$

Hence $< cx,y >= c\odot < x,y >$.

(IP3) Since $F(x,y,t)=F(y,x,t)$, for all $t\in R$ it follows that $< x,y >=< y,x >$.

(IP4) By (FI8), we have $\inf\{t<0:\ \alpha \le F(x,x,t)\}=0$, for all $\alpha\in(0,1]$. Hence $0 \le < x,x >^-_\alpha$, for all $\alpha\in(0,1]$. Thus $0 \le < x,x >$, for all $x\in X$.

(IP5) Let $x\neq 0$. By (FI8), we have $\inf\{t<0:\ \alpha \le F(x,x,t)\}=0$, for all $\alpha\in(0,1]$. By Theorem 2.12, $F(x,y,.):(0,\infty)\longrightarrow[0,1]$ is increasing. So by (FI9), $F(x,x,t)=0$, for all $0<t\le t_x$. Which implies that $0<t_x\le \sup\{t>0:\ F(x,x,t)=0\}$. Hence $0<t_x \le < x,x >^-_\alpha = \inf\{t<0:\ \alpha\le F(x,x,t)\} + \sup\{t>0:\ F(x,x,t)=0\}$, for all $\alpha\in(0,1]$. Thus $0<t_x\le \inf_{\alpha\in(0,1]}\langle x,x\rangle^-_\alpha$.

(IP6) Let $x=0$. By (FI4), $F(x,x,t)=H(t)$, for all $t\in R$. It is clear that $< x,x >^-_\alpha = 0 = < x,x >^-_\alpha$. So $< x,x >= 0$.

Conversely, let $< x,x >= 0$. Assume that $0<t$. Then $< x,x >^+_\alpha = 0$, for all $\alpha\in(0,1]$. Since $F(x,x,t)=0$, for all $t\le 0$, it follows that $\inf\{t<0:\ F(x,x,t)=$

$1\} = 0$. Thus $\sup\{t > 0 : F(x,x,t) \leq 1-\alpha\} = 0$, for all $\alpha \in (0,1]$. Hence $1-\alpha < F(x,x,t)$, for all $\alpha \in (0,1]$. As $\alpha \longrightarrow 1$, we have $F(x,x,t) = 1$. So

$$F(x,x,t) = 1, \text{ for all } 0 < t.$$

By (FI3), $x = 0$. □

Note. Let (X, F, min) be a fuzzy inner product space.
i) If $\{t < 0 : \alpha \leq F(x,y,t)\} = \emptyset$ then $\inf\{t < 0 : \alpha \leq F(x,y,t)\} = 0$,
ii) If $\{t > 0 : F(x,y,t) = 0\} = \emptyset$ then $\sup\{t > 0 : F(x,y,t) = 0\} = 0$,
iii) If $\{t > 0 : F(x,y,t) \leq 1-\alpha\} = \emptyset$ then $\sup\{t > 0 : F(x,y,t) \leq 1-\alpha\} = 0$,
iv) If $\{t < 0 : F(x,y,t) = 1\} = \emptyset$ then $\inf\{t < 0 : F(x,y,t) = 1\} = 0$.

Theorem 3.4. *Let $(X, <.,.>)$ be a fuzzy inner product space satisfying (IP7) and $[<x,y>]_\alpha = [<x,y>^-_\alpha, <x,y>^+_\alpha]$, for all $\alpha \in (0,1]$. Moreover, let F be a functions on $X \times X \times R$ defined by*

$$F(x,y,t) = \begin{cases} 1 - \inf\{\alpha \in (0,1] : <x,y>^+_\alpha \leq t\} & , \quad t > 0 \\ 0 & , \quad t = 0 \\ \inf\{\alpha \in (0,1] : <x,y>^-_\alpha \geq t\} & , \quad t < 0, \end{cases}$$

Then $(X, F, *)$ is a fuzzy inner product space for any t-norm $*$.

Proof. Let $x, y, z \in X$ and $t, s \in R$.
(FI1) It is clear that $F(x,y,0) = 0$.
(FI2) Since $<x,y>=<y,x>$ it follows that $F(x,y,t) = F(x,y,t)$.
(FI3) Let $x = 0$. By (IP6), we have $<x,x>= 0$. Hence

$$\inf\{\alpha \in (0,1] : <x,x>^+_\alpha \leq t\} = 0, \text{ for all } t > 0.$$

Thus $F(x,x,t) = 1$, for all $t > 0$.
Conversely, let $F(x,x,t) = 1$, for all $t > 0$. So $\inf\{\alpha \in (0,1] : <x,x>^+_\alpha \leq t\} = 0$, for all $t > 0$. Then $<x,x>^+_\alpha \leq t$, for all $t > 0$. Hence $<x,x>^+_\alpha = 0$. Therefore $<x,x>= 0$. By (IP6), we obtain that $x = 0$.
(FI4) Let $c \in R$ and $t \neq 0$. Assume that $t, c > 0$. We get

$$\begin{aligned} F(cx,y,t) &= 1 - \inf\{\alpha \in (0,1] : <cx,y>^+_\alpha \leq t\} \\ &= 1 - \inf\{\alpha \in (0,1] : c<x,y>^+_\alpha \leq t\} \\ &= 1 - \inf\{\alpha \in (0,1] : <x,y>^+_\alpha \leq t/c\} \\ &= F(x,y,t/c). \end{aligned}$$

Suppose that $c > 0$ and $t < 0$. we have

$$\begin{aligned} F(cx,y,t) &= \inf\{\alpha \in (0,1] : <cx,y>^-_\alpha \geq t\} \\ &= \inf\{\alpha \in (0,1] : c<x,y>^-_\alpha \geq t\} \\ &= \inf\{\alpha \in (0,1] : <x,y>^-_\alpha \geq t/c\} \\ &= F(x,y,t/c). \end{aligned}$$

Assume that $c < 0$ and $t > 0$. We obtain that

$$\begin{aligned} F(cx, y, t) &= 1 - \inf\{\alpha \in (0,1] :< cx, y >^+_\alpha \le t\} \\ &= 1 - \inf\{\alpha \in (0,1] : c < x, y >^-_\alpha \le t\} \\ &= 1 - \inf\{\alpha \in (0,1] :< x, y >^-_\alpha \ge t/c\} \\ &= 1 - F(x, y, t/c). \end{aligned}$$

Let $t, c < 0$. We have

$$\begin{aligned} F(cx, y, t) &= \inf\{\alpha \in (0,1] :< cx, y >^-_\alpha \ge t\} \\ &= \inf\{\alpha \in (0,1] : c < x, y >^+_\alpha \ge t\} \\ &= \inf\{\alpha \in (0,1] :< x, y >^+_\alpha \le t/c\} \\ &= 1 - F(x, y, t/c). \end{aligned}$$

(FI5) Let $s, t > 0$. Suppose that $F(x, z, t) = \min\{F(x, z, t), F(y, z, s)\}$. Therefore $F(x, z, t) \le F(y, z, s)$. So

$$\inf\{\alpha \in (0,1] :< y, z >^+_\alpha \le s\} \le \inf\{\alpha \in (0,1] :< x, z >^+_\alpha \le t\}.$$

Now Assume that $< x, z >^+_\alpha \le t$. Then

$$\inf\{\alpha \in (0,1] :< y, z >^+_\alpha \le s\} \le \inf\{\alpha \in (0,1] :< x, z >^+_\alpha \le t\} \le \alpha < \alpha + 1/n,$$

for all $n \in N$. Hence there exist $\beta_n, \beta'_n \in [\alpha, \alpha + 1/n)$ such that $< x, z >^+_{\beta_n} \le t$ and $< y, z >^+_{\beta'_n} \le s$, for all $n \in N$. Thus

$$\begin{aligned} < x + y, z >^+_{\beta_n} &\le < x, z >^+_{\beta_n} + < y, z >^+_{\beta_n} \\ &= < x, z >^+_{\beta_n} + < y, z >^+_{\beta'_n} + < y, z >^+_{\beta_n} - < y, z >^+_{\beta'_n} \\ &\le t + s + < y, z >^+_{\beta_n} - < y, z >^+_{\beta'_n}, \text{ for all } n \in N. \end{aligned}$$

As $n \longrightarrow \infty$. By (IP7), we have $< x + y, z >^+_\alpha \le t + s$. Hence

$$\inf\{\alpha \in (0,1] :< x + y, z >^+_\alpha \le t + s\} \le \alpha.$$

So $\inf\{\alpha \in (0,1] :< x + y, z >^+_\alpha \le t + s\} \le \inf\{\alpha \in (0,1] :< x, z >^+_\alpha \le t\}$. Which implies that $\min\{F(x, z, t), F(y, z, s)\} = F(x, z, t) \le F(x+y, z, t+s)$. By Theorem 2.7., we obtain that

$$F(x, z, t) * F(y, z, s) \le \min\{F(x, z, t), F(y, z, s)\} \le F(x + y, z, t + s).$$

(FI6) Let $\epsilon \in (0,1)$ be given. Assume that $\alpha = \epsilon$ and $M =< x, y >_\alpha$. If $M \le t$ then $\inf\{\alpha \in (0,1] :< x, y >^+_\alpha \le t\} \le \alpha$. Hence $1 - \alpha \le F(x, y, t) \le 1 \le 1 + \alpha$. So $|F(x, y, t) - 1| \le \epsilon$. Thus $\lim_{t\to\infty} F(x, y, t) = 1$. □

Note. Let $(X, < ., . >)$ be a fuzzy inner product space.
i) If $\{\alpha \in (0,1] :< x, y >^+_\alpha \le t\} = \emptyset$ then $\inf\{\alpha \in (0,1] :< x, y >^+_\alpha \le t\} = 1$,
ii) If $\{\alpha \in (0,1] :< x, y >^-_\alpha \ge t\} = \emptyset$ then $\inf\{\alpha \in (0,1] :< x, y >^-_\alpha \ge t\} = 1$.

Theorem 3.5. *Let $(X, <.,.>)$ be a fuzzy inner product space satisfying (IP7) and $[<x,y>]_\alpha = [<x,y>_\alpha^-, <x,y>_\alpha^+]$, for all $\alpha \in (0,1]$. Moreover, let F be a functions on $X \times X \times R$ defined by*

$$F(x,y,t) = \begin{cases} 1 - \inf\{\alpha \in (0,1] : <x,y>_\alpha^+ \leq t\} & , \quad t > 0 \\ 0 & , \quad t = 0 \\ \inf\{\alpha \in (0,1] : <x,y>_\alpha^- \geq t\} & , \quad t < 0, \end{cases}$$

Then $(X, F, \min)$ is a fuzzy inner product space satisfying (FI7), (FI8), (FI9) and (FI12).

Proof. By Theorem 3.4, $(X, F, \min)$ is a fuzzy inner product space.
(FI7) Let $x, y, z \in X$ and $t, s > 0$. Assume that $<x+y, z>_\alpha^+ \leq t+s$. Hence $<x,z>_\alpha^+ + <y,z>_\alpha^+ \leq t+s$. Thus $<x,z>_\alpha^+ \leq t$ or $<y,z>_\alpha^+ \leq s$. So $\inf\{\alpha \in (0,1] : <x,y>_\alpha^+ \leq t\} \leq \alpha$ or $\inf\{\alpha \in (0,1] : <y,z>_\alpha^+ \leq s\} \leq \alpha$. Therefore $1-\alpha \leq F(x,z,t)$ or $1-\alpha \leq F(y,z,s)$. This implies that $1-\alpha \leq \max\{F(x,z,t), F(y,z,s)\}$. Hence $1 - \max\{F(x,z,t), F(y,z,s)\} \leq \alpha$. Thus $1 - \max\{F(x,z,t), F(y,z,s)\} \leq \inf\{\alpha \in (0,1] : <x+y,z>_\alpha^+ \leq t+s\}$. Therefore $F(x+y, z, t+s) \leq \max\{F(x,z,t), F(y,z,s)\} = 1 - \min\{1-F(x,z,t), 1-F(y,z,s)\}$.
(FI8) Let $x \in X$ and $t < 0$. By (IP4), we have $t < 0 \leq <x,x>_\alpha^-$, for all $\alpha \in (0,1]$. Then $F(x,x,t) = 0$.
(FI9) Let $0 \neq x \in X$. By (IP5), we have $0 < \inf_{\alpha\in(0,1]}\langle x,x\rangle_\alpha^-$. Suppose that $0 < t_x < \inf_{\alpha\in(0,1]}\langle x,x\rangle_\alpha^- \leq \langle x,x\rangle_\alpha^+$, for all $\alpha \in (0,1]$. So $F(x,x,t_x) = 0$.
(FI12) Let $x, y \in X$ and $t, s > 0$. Assume that $F(x,x,t^2) \leq F(y,y,s^2)$. So $\inf\{\alpha \in (0,1] : <y,y>_\alpha^+ \leq s^2\} \leq \inf\{\alpha \in (0,1] : <x,x>_\alpha^+ \leq t^2\}$. Now suppose that $<x,x>_\alpha^+ \leq t^2$. Then

$$\inf\{\alpha \in (0,1] : <y,y>_\alpha^+ \leq s^2\} \leq \inf\{\alpha \in (0,1] : <x,x>_\alpha^+ \leq t^2\} \leq \alpha < \alpha + 1/n,$$

for all $n \in N$. Hence there exist $\beta_n, \beta'_n \in [\alpha, \alpha+1/n)$ such that $<x,x>_{\beta_n}^+ \leq t^2$ and $<y,y>_{\beta'_n}^+ \leq s^2$, for all $n \in N$. As $n \longrightarrow \infty$. By (IP7), we have $<x,x>_\alpha^+ \leq t^2$ and $<y,y>_\alpha^+ \leq s^2$. By Lemma 3.2. in [7], we obtain that

$$<x,y>_\alpha^+ \leq | <x,y>_\alpha^+ | \leq (<x,x>_\alpha^+ <y,y>_\alpha^+)^{1/2} \leq ts.$$

Hence $\inf\{\alpha \in (0,1] : <x,y>_\alpha^+ \leq ts\} \leq \alpha$. So $1-\alpha \leq F(x,y,ts)$. Therefore $1 - F(x,y,ts) \leq \alpha$. This implies that $1 - F(x,y,ts) \leq \inf\{\alpha \in (0,1] : <x,x>_\alpha^+ \leq t^2\}$. Thus $\min\{F(x,x,t^2), F(y,y,s^2)\} = F(x,x,t^2) \leq F(x,y,ts)$. □

Theorem 3.6. *Let (X, F, min) be a fuzzy inner product space satisfying (FI7), (FI8), (FI9) and (FI10). Moreover, let $(X, <.,.>)$ be a fuzzy inner product space defined in Theorem 3.3. Then $(X, <.,.>)$ satisfies in (IP7).*

Proof. Let $\{\alpha_k\}$ be a decreasing sequence in $(0,1]$ such that $\alpha_k \longrightarrow \alpha \in (0,1]$. Suppose that $x, y \in X$. We have $\alpha \leq \alpha_{n+1} \leq \alpha_n$, for all $n \in N$. So

$$<x,y>_{\alpha_n}^+ \leq <x,y>_{\alpha_{n+1}}^+ \leq <x,y>_\alpha^+, \text{ for all } n \in N.$$

Hence $\lim_{n\to\infty} < x, y >^+_{\alpha_n} = \sup_{n>0} < x, y >^+_{\alpha_n} \leq < x, y >^+_\alpha$.
If $\lim_{n\to\infty} < x, y >^+_{\alpha_n} < t_0 < s_0 << x, y >^+_\alpha$. Then

$$\sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha_n\} < t_0 < s_0 < \sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha\},$$

for all $n \in N$. By (FI10), we have $1 - \alpha_n < F(x, y, t_0) < F(x, y, s_0) \leq 1 - \alpha$, for all $n \in N$. As $n \longrightarrow \infty$, we obtain that $1 - \alpha \leq F(x, y, t_0) < F(x, y, s_0) \leq 1 - \alpha$, which is a contradiction. □

Theorem 3.7. *Let $(X, < ., . >)$ be a fuzzy inner product space satisfying (IP7) and $[< x, y >]_\alpha = [< x, y >^-_\alpha, < x, y >^+_\alpha]$, for all $\alpha \in (0, 1]$. Moreover, let (X, F, min) be a fuzzy inner product space defined in Theorem 3.4. Then $F(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ ($F(x, y, .) : (-\infty, 0) \longrightarrow [0, 1]$) is continuous from the right (left), for all $x, y \in X$.*

Proof. Let $x, y \in X$ and $s > 0$. Suppose that $s < t$. By Theorem 2.12, $F(x, y, s) \leq F(x, y, t)$. Thus $F(x, y, s) \leq \inf_{s<t} F(x, y, t) = \lim_{t\to s^+} F(x, y, t)$.
If $F(x, y, s) < 1 - \alpha < \inf_{s<t} F(x, y, t) = \lim_{t\to s^+} F(x, y, t)$, then $F(x, y, s) < 1 - \alpha < F(x, y, t)$, for all $s < t$. So

$$\inf\{\alpha \in (0, 1] :< x, y >^+_\alpha \leq t\} < \alpha < \inf\{\alpha \in (0, 1] :< x, y >^+_\alpha \leq s\}, \text{ for all } s < t.$$

Thus $s << x, y >^+_\alpha \leq t$, for all $s < t$. As $t \longrightarrow s$, we obtain that $s << x, y >^+_\alpha \leq s$, which is a contradiction. Hence $F(x, y, s) = \lim_{t\to s^+} F(x, y, t)$. □

Theorem 3.8. *Let (X, F, min) be a fuzzy inner product space satisfying (FI12). Then the function $N : X \times R \longrightarrow [0, 1]$ defined by*

$$N(x, t) = \begin{cases} F(x, x, t^2) & , \quad t > 0 \\ 0 & , \quad t \leq 0, \end{cases}$$

is a fuzzy norm on X.

Proof. Let $x, y \in X$ and $t, s \in R$.
(N1) It is clear that $N(x, t) = 0$, for all $t \leq 0$.
(N2) Let $x = 0$. Then $F(x, x, t) = 1$, for all $t > 0$. Hence $N(x, t) = 1$, for all $t > 0$. Conversely, let $N(x, t) = 1$, for all $t > 0$. Thus $F(x, x, t) = 1$, for all $t > 0$. So $x = 0$.
(N3) Let $t > 0$. We have

$$N(cx, t) = F(cx, cx, t^2) = F(x, x, t^2/c^2) = N(x, t/|c|).$$

(N4) Let $t, s > 0$. By (FI12), we have

$$\begin{aligned} N(x + y, t + s) &= F(x + y, x + y, (t + s)^2) \\ &\geq \min\{F(x + y, x, t^2 + ts), F(x + y, y, s^2 + ts)\} \\ &\geq \min\{F(x, x, t^2), F(y, y, s^2), F(x, y, ts)\} \\ &\geq \min\{F(x, x, t^2), F(y, y, s^2)\} \\ &= \min\{N(x, t), N(y, s)\}. \end{aligned}$$

(N5) By Theorem 3.7., $N(x, .) : R \longrightarrow [0, 1]$ is a non-decreasing function.
(N6) By (FI6), $\lim_{t\to\infty} N(x, t) = 1$. □

Theorem 3.9. *Let $(X,<.,.>)$ be a fuzzy inner product space satisfying (IP7). And, let (X,F,min) be a fuzzy inner product space defined in Theorem 3.4. Suppose that $\{x_n\} \subseteq X$ and $x \in X$. Then $x_n \longrightarrow x$ in fuzzy normed linear space (X,N) defined in Theorem 3.8. if and only if $x_n \longrightarrow x$ in fuzzy normed linear space $(X,\|.\|)$ defined in Theorem 2.17.*

Proof. Let $\{x_n\} \subseteq X$ and $x \in X$. Assume that $x_n \longrightarrow x$ in fuzzy normed linear space (X,N) defined in Theorem 3.8. Then $\lim_{n\to\infty} F(x_n - x, x_n - x, t^2) = 1$, for all $t > 0$.
Let $\epsilon > 0$ be given. Suppose that $\alpha \in (0,1]$. We have $\lim_{n\to\infty} F(x_n - x, x_n - x, \epsilon^2) = 1$. Hence there exists $N > 0$ such that $|F(x_n - x, x_n - x, \epsilon^2) - 1| < \alpha$, for all $n > N$. Thus $\inf\{\alpha \in (0,1] :< x_n - x, x_n - x >_\alpha^+ \leq \epsilon^2\} < \alpha$, for all $n > N$. So $< x_n - x, x_n - x >_\alpha^+ \leq \epsilon^2$, for all $n > N$. Then $\|x_n - x\|_\alpha^+ \leq \epsilon^2$, for all $n > N$. This implies that $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$. Therefore $x_n \longrightarrow x$ in fuzzy normed linear space $(X,\|.\|)$ defined in Theorem 2.17.
Conversely, let $x_n \longrightarrow x$ in fuzzy normed linear space $(X,\|.\|)$ defined in Theorem 2.17. Then $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$. Assume that $t > 0$ and $\epsilon \in (0,1)$. Hence there exists $N > 0$ such that $\|x_n - x\|_\epsilon^+ < t$, for all $n > N$. So $\inf\{\alpha \in (0,1] :< x_n - x, x_n - x >_\alpha^+ \leq t^2\} < \epsilon$, for all $n > N$. Hence $|F(x_n - x, x_n - x, t^2) - 1| < \epsilon$, for all $n > N$. Therefore $\lim_{n\to\infty} F(x_n - x, x_n - x, t^2) = 1$. Thus $x_n \longrightarrow x$ in fuzzy normed linear space (X,N) defined in Theorem 3.8. □

Corollary 3.10. *Let $(X,<.,.>)$ be a fuzzy Hilbert space satisfying (IP7). And, let (X,F,min) be a fuzzy inner product space defined in Theorem 3.4. Then (X,F,min) is a Fuzzy Hilbert space.*

Theorem 3.11. *Let (X,F,min) be a fuzzy inner product space satisfying (FI7), (FI8), (FI9) and (FI12). And, let $(X,<.,.>)$ be a fuzzy inner product space defined in Theorem 3.3. Suppose that $\{x_n\} \subseteq X$ and $x \in X$. Then $x_n \longrightarrow x$ in fuzzy normed linear space (X,N) defined in Theorem 3.8. if and only if $x_n \longrightarrow x$ in fuzzy normed linear space $(X,\|.\|)$ defined in Theorem 2.17.*

Proof. Let $\{x_n\} \subseteq X$ and $x \in X$. Assume that $x_n \longrightarrow x$ in fuzzy normed linear space (X,N) defined in Theorem 3.8. Then $\lim_{n\to\infty} F(x_n - x, x_n - x, t^2) = 1$, for all $t > 0$.
Let $\epsilon > 0$ be given. Suppose that $\alpha \in (0,1]$. We have $\lim_{n\to\infty} F(x_n - x, x_n - x, \epsilon^2) = 1$. Hence there exists $N > 0$ such that $|F(x_n - x, x_n - x, \epsilon^2) - 1| < \alpha$, for all $n > N$. Thus $1 - \alpha < F(x_n - x, x_n - x, \epsilon^2)$, for all $n > N$. So

$$\sup\{t > 0 : F(x_n - x, x_n - x, t) \leq 1 - \alpha\} \leq \epsilon^2, \text{ for all } n > N.$$

Then $\|x_n - x\|_\alpha^+ \leq \epsilon$, for all $n > N$. This implies that $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$. Therefore $x_n \longrightarrow x$ in fuzzy normed linear space $(X,\|.\|)$ defined in Theorem 2.17.
Conversely, let $x_n \longrightarrow x$ in fuzzy normed linear space $(X,\|.\|)$ defined in Theorem 2.17. Then $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$. Assume that $t > 0$ and $\epsilon \in (0,1)$. Hence there exists $N > 0$ such that $\|x_n - x\|_\epsilon^+ < t$, for all $n > N$. So $\sup\{t > 0 : F(x_n - x, x_n - x, t) \leq 1 - \epsilon\} < t^2$, for all $n > N$. Hence $1 - \epsilon < F(x_n - x, x_n - x, t^2) \leq 1 < 1 + \epsilon$, for all $n > N$. Thus $|F(x_n - x, x_n - x, t^2) - 1| < \epsilon$,

for all $n > N$. Therefore $\lim_{n\to\infty} F(x_n - x, x_n - x, t^2) = 1$. Thus $x_n \longrightarrow x$ in fuzzy normed linear space (X, N) defined in Theorem 3.8. □

Corollary 3.12. *Let (X, F, min) be a fuzzy Hilbert space satisfying (FI7), (FI8), (FI9) and (FI12). And, let $(X, <.,.>)$ be a fuzzy inner product space defined in Theorem 3.3. Then $(X, <.,.>)$ is a Fuzzy Hilbert space.*

Definition 3.13. Let (X, F, min) be a fuzzy inner product space. And, let $\alpha \in (0, 1)$.
i) A sequence $\{x_n\} \subseteq X$ is said to be α-converge to $x \in X$, if for each $t > 0$ there exists $N > 0$ such that $1 - \alpha < F(x_n - x, x_n - x, t)$, for all $n > N$.
ii) A sequence $\{x_n\} \subseteq X$ is called α-Cauchy, if for each $t > 0$ there exists $N > 0$ such that $1 - \alpha < F(x_n - x_m, x_n - x_m, t)$, for all $m, n > N$.
iii) A subset Y of X is said to be α-complete, if every α-Cauchy sequence in Y α-converges in Y.

Lemma 3.14. *Let (X, F, min) be a fuzzy Hilbert space satisfying (FI7), (FI8), (FI9) and (FI11). And, let $(X, <.,.>)$ be a fuzzy inner product space defined in Theorem 3.3. Then $x\perp y$ in fuzzy inner product space (X, F, min) if and only if $x\perp y$ in fuzzy inner product space $(X, <.,.>)$.*

Proof. Let $x\perp y$ in fuzzy inner product space (X, F, min). We have $F(x, y, t) = H(t)$, for each $t \in R$. It is clear that $< x, y >= 0$. Hence $x\perp y$ in fuzzy inner product space $(X, <.,.>)$.
Conversely, let $x\perp y$ in fuzzy inner product space $(X, <.,.>)$. We have $< x, y >= 0$. So $< x, y >^-_\alpha = 0$ and $< x, y >^+_\alpha = 0$, for all $\alpha \in (0, 1]$. Hence

$$\inf\{t < 0 : \ \alpha \leq F(x, y, t)\} + \sup\{t > 0 : \ F(x, y, t) = 0\} = 0$$

and

$$\sup\{t > 0 : \ F(x, y, t) \leq 1 - \alpha\} + \inf\{t < 0 : \ F(x, y, t) = 1\} = 0,$$

for all $\alpha \in (0, 1]$. If $t_0 = \sup\{t > 0 : \ F(x, y, t) = 0\} > 0$ then

$$\inf\{t < 0 : \ \alpha \leq F(x, y, t)\} = -t_0, \text{ for all } \alpha \in (0, 1].$$

Thus $\alpha \leq F(x, y, -t_0+\epsilon)$, for all $\alpha \in (0, 1]$ and all $\epsilon > 0$. As $\alpha \longrightarrow 1$, we obtain that $F(x, y, -t_0+\epsilon) = 1$, for all $\epsilon > 0$. By (FI11), we have $F(x, y, -t_0) = 1$. Let $t < -t_0$. So $t < \inf\{t < 0 : \ \alpha \leq F(x, y, t)\}$, for all $\alpha \in (0, 1]$. Therefore $F(x, y, t) < \alpha$, for all $\alpha \in (0, 1]$. As $\alpha \longrightarrow 0$, we get $F(x, y, t) = 0$. Hence $F(x, y, t) = 0$, for all $t < t_0$. By (FI11), $0 = \lim_{t\to t_0^-} F(x, y, t) = F(x, y, t_0)$, which is a contradiction. So $\sup\{t > 0 : \ F(x, y, t) = 0\} = 0$. Hence $\inf\{t < 0 : \ \alpha \leq F(x, y, t)\} = 0$, for all $\alpha \in (0, 1]$. Therefore $F(x, y, t) < \alpha$, for all $\alpha \in (0, 1]$ and all $t < 0$. As $\alpha \longrightarrow 0$, we have $F(x, y, t) = 0$, for all $t < 0$. Similarly, we obtain that $F(x, y, t) = 1$, for all $t > 0$. Hence $F(x, y, t) = H(t)$, for all $t \in R$. Thus $x\perp y$ in fuzzy inner product space (X, F, min). □

Corollary 3.15. *Let (X, F, min) be a fuzzy Hilbert space satisfying (FI7), (FI8), (FI9) and (FI11). And, let $(X, <.,.>)$ be a fuzzy inner product space defined in Theorem 3.3. Suppose that $Y \subseteq X$. Then*

$$\{x \in X : x\perp Y \ \textit{in fuzzy inner product space } (X, F, min)\} =$$

$$\{x \in X : x \perp Y \text{ in fuzzy inner product space } (X, <.,.>)\}$$

Lemma 3.16. *Let $(X, <.,.>)$ be a fuzzy inner product space satisfying (IP7). And, let (X, F, min) be a fuzzy inner product space defined in Theorem 3.4. Then $x \perp y$ in fuzzy inner product space $(X, <.,.>)$ if and only if $x \perp y$ in fuzzy inner product space (X, F, min).*

Proof. Let $x \perp y$ in fuzzy inner product space $(X, <.,.>)$. We have $< x, y >= 0$. So $< x, y >_\alpha^- = 0$ and $< x, y >_\alpha^+ = 0$, for all $\alpha \in (0, 1]$. It is clear that $F(x, y, t) = H(t)$, for all $t \in R$.
Conversely, let $x \perp y$ in fuzzy inner product space (X, F, min). We have $F(x, y, t) = H(t)$, for each $t \in R$. Hence $\inf\{\alpha \in (0, 1] :< x, y >_\alpha^+ \leq t\} = 0$, for all $t > 0$, and $\inf\{\alpha \in (0, 1] :< x, y >_\alpha^- \geq t\} = 0$, for all $t < 0$. Let $\alpha \in (0, 1]$. Thus $< x, y >_\alpha^+ \leq t$, for all $t > 0$, and $< x, y >_\alpha^- \geq t$, for all $t < 0$. Since $< x, y >_\alpha^- \leq < x, y >_\alpha^+$ it follows that $< x, y >_\alpha^- = 0 =< x, y >_\alpha^+$. Therefore $< x, y >= 0$. Thus $x \perp y$ in fuzzy inner product space $(X, <.,.>)$. □

Theorem 3.17. *Let (X, F, min) be a fuzzy Hilbert space satisfying (FI7), (FI8), (FI9) and (FI11). And, let Y be any subspace of X such that Y is α-complete, for all $\alpha \in (0, 1)$. Then $X = Y \oplus Z$ where $Z = Y^\perp$.*

Proof. Let $(X, <.,.>)$ be a fuzzy inner product space defined in Theorem 3.3. First, we show that fuzzy normed linear spaces $(Y, \|.\|_\alpha)$ are complete for all $\alpha \in (0, 1)$. Assume that $\alpha \in (0, 1)$ and $\{x_n\}$ is a Cauchy sequence in $(Y, \|.\|_\alpha)$. Let $t > 0$. Hence for $\epsilon = \sqrt{t} > 0$ there exists $N > 0$ such that $\|x_n - x_m\|_\alpha^+ < \sqrt{t}$, for all $m, n > N$. Thus $\sup\{t > 0 : \; F(x_n - x_m, x_n - x_m, t) \leq 1 - \alpha\} < t$, for all $m, n > N$. So $1 - \alpha < F(x_n - x_m, x_n - x_m, t)$, for all $m, n > N$. Therefore $\{x_n\}$ is a α-Cauchy sequence in Y. Since Y is α-complete, there is $x \in Y$ such that $\{x_n\}$ α-converges to x. Suppose that $\epsilon > 0$. Hence for $\epsilon^2 = t > 0$ there exists $N > 0$ such that $1 - \alpha < F(x_n - x, x_n - x, \epsilon^2)$, for all $, n > N$. So

$$\sup\{t > 0 : \; F(x_n - x, x_n - x, t) \leq 1 - \alpha\} < \epsilon^2, \text{ for all } n > N.$$

Thus $\|x_n - x\|_\alpha^+ < \epsilon$, for all $n > N$. Hence fuzzy normed linear space $(Y, \|.\|_\alpha)$ is complete. By Theorem 4.7. in [7] and Corollary 3.15, we obtain that $X = Y \oplus Z$ where $Z = Y^\perp$. □

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