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THE URYSOHN, COMPLETELY HAUSDORFF AND COMPLETELY REGULAR AXIOMS IN L-FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, the Urysohn, completely Hausdorff and completely regular axioms in L-topological spaces are generalized to L-fuzzy topological spaces. Each L-fuzzy topological space can be regarded to be Urysohn, completely Hausdorff and completely regular to some degree. Some properties of them are investigated. The relations among them and T_2 in L-fuzzy topological spaces are discussed.

1. Introduction

Since Chang [1] introduced fuzzy set theory to topology, many aspects of fuzzy topology were introduced and discussed. In a Chang I-topology, the open sets are fuzzy, but the topology comprising those open sets is a crisp subset of the I-powerset I^X. In 1980, Höhle [9] presented a new topological notion from a logical point of view and called it fuzzifying topology. Kubiak [11] and Šostak [21] independently extended Höhle's fuzzy topology to M-subset of L^X and I-subset of I^X, respectively, which is called (L, M)-fuzzy topology. In 1991, from a logical point of view, Ying [23] studied Höhle's topology and called it fuzzifying topology.

As it is well known, the Urysohn, completely Hausdorff and completely regular axioms are important separation axioms in general topology. Chen and Wu generalized the Urysohn axiom to topological molecular lattice and called this generalized version the U_2 axiom [2]. Fang also generalized these axioms to L-topological space and topological molecular lattice in [4, 5]. However, none of the L-extended real line, the L-real line, and the L-unit interval satisfy these axioms.

In 2006, Shi [17] introduced L-Urysohn and L-completely Hausdorff axioms in L-topological spaces in such a way as to be compatible with the canonical topologies on the L-extended real line, L-real line, L-unit interval and the topology of pointwise pseudo-metric. They have been generalized to I-fuzzy topological spaces in [12]. Moreover, Shi [18] introduced the Urysohn and completely Hausdorff axioms to L-topological space such that they are compatible with pointwise metrics in L-set theory. In [15, 16], the completely regular axiom had been generalized to L-topological spaces in order to characterize pointwise uniformity.

With the development of fuzzy topology, separation axioms in L-fuzzy topological spaces have endowed with some degrees. In [22], Šostak firstly considered the

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graded topological (in particular, separation) properties in the context of L-fuzzy topological spaces. Afterwards, many researchers proposed the degrees of separation axioms. In the framework of I-fuzzy topologies, Yue and Fang [25] studied some low-level separation axioms defined by conditions on fuzzy points with different supports. In [19, 20], Shi introduced some separation axioms in (L, M)-fuzzy topology in such a way as to be compatible with (L, M)-fuzzy metrics.

The aim of this paper is to define the degree to which an L-fuzzy topological space is Urysohn, completely Hausdorff and completely regular. The relations among them are discussed.

2. Preliminaries

Throughout this paper, $I = [0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. $(L, \bigvee, \bigwedge, ')$ is a completely distributive De Morgan algebra, i.e., a completely distributive lattice with an order-reversing involution $'$. The smallest element and the largest element in L are denoted by $\bot$ and $\top$, respectively. For $a, b \in L$, we say that a is wedge below b, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$, there exists $d \in D$, $d \geq a$ [3]. The wedge below relation in a completely distributive lattice has the interpolation property, this means $\lambda \prec \mu \Rightarrow \exists\, \gamma \in L$ such that $\lambda \prec \gamma \prec \mu$. Moreover we can see that $\lambda \prec \bigvee_{i \in \Omega} \mu_i \;\Rightarrow\; \exists\, \mu_i$ such that $\lambda \prec \mu_i$. A complete lattice L is completely distributive if and only if $b = \bigvee \{a \in L |\ a \prec b\}$ for each $b \in L$. An element a in L is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ [7]. The set of non-zero co-prime elements in L is denoted by $M(L)$.

In a completely distributive De Morgan algebra L, there exists an implication operation $\to$: $L \times L \to L$ as the right adjoint for the meet operation $\wedge$ by

$$a \to b = \bigvee \{\, c \in L \mid a \wedge\ c \leqslant\ b\}.$$

We list some properties of implication operation.

Lemma 2.1. [10] *Suppose that $(L, \bigvee, \bigwedge)$ is a completely distributive lattice and $\to$ is the implication operation corresponding to $\wedge$. Then for all $a, b, c \in L$, $\{a_j\}_{j \in J}$, $\{b_j\}_{j \in J} \subseteq L$, the following conditions hold:*

(1) $(a \to b) \geq c \;\Leftrightarrow\; a \wedge c \leq b$;
(2) $a \leqslant b \;\Leftrightarrow\; a \to b = \top$;
(3) $a \to (b \to c) = (a \wedge b) \to c$;
(4) $(c \to a) \wedge (a \to b) \leq c \to b$;
(5) $c \to a \leq (a \to b) \to (c \to b)$;
(6) $a \to \bigwedge_{j \in J} a_j = \bigwedge_{j \in J} (a \to a_j)$, *hence* $a \to b \leqslant a \to c$ *whenever* $b \leqslant c$;
(7) $\bigvee_{j \in J} a_j \to b = \bigwedge_{j \in J} (a_j \to b)$, *hence* $a \to c \geqslant b \to c$ *whenever* $a \leqslant b$.

For a completely distributive lattice L and a nonempty set X, L^X denotes the set of all L-fuzzy sets on X. L^X is also a completely distributive lattice when it inherits the structure of lattice L in a natural way, by defining $\vee, \wedge, \leq$ pointwisely. The set of non-zero co-prime elements in L^X is denoted by $M(L^X)$. Each member in $M(L^X)$ is also called a point. It is easy to see that $M(L^X)$ is exactly the set of all fuzzy points $x_\lambda\,(\lambda \in M(L))$.

An L-topological space is a pair (X,δ), where $\delta(\subseteq L^X)$ contains $\underline{\perp}, \underline{\top}$ and is closed under arbitrary supremums and finite infimums. Elements in δ are called open L-sets and their quasi-complements are called closed L-sets. A closed L-set P is called a remote-neighborhood (or R-neighborhood for short) of $e \in M(L^X)$ if $e \not\leq P$. The set of all closed R-neighborhoods of e is denoted by $\eta^-(e)$.

Definition 2.2. [11, 21] An L-fuzzy topology on a set X is a map $\mathcal{T} : L^X \to L$ such that

(1) $\mathcal{T}(\underline{\top}) = \mathcal{T}(\underline{\perp}) = \top$;

(2) $\forall\, U, V \in L^X, \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$;

(3) $\forall\, \{U_j |\ j \in J\} \subseteq L^X, \mathcal{T}(\bigvee\limits_{j\in J} U_j) \geq \bigwedge\limits_{j\in J} \mathcal{T}(U_j)$.

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an open set. $\mathcal{T}^*(U) = \mathcal{T}(U')$ will be called the degree of closedness of U. The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

A map $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called continuous if $\mathcal{T}(f^{\leftarrow}(B)) \geq \mathcal{U}(B)$ holds for all $B \in L^Y$, where $f^{\leftarrow}$ is defined by $f^{\leftarrow}(B)(x) = B(f(x))$. A map $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called open if $\mathcal{T}(B) \leq \mathcal{U}(f^{\rightarrow}(B))$ holds for all $B \in L^X$, where $f^{\rightarrow}(B)(y) = \bigvee\limits_{f(x)=y} B(x)$. A map $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ is called a homeomorphism if and only if f is bijective and f is both continuous and open.

Definition 2.3. [6, 24] (1) Let $\mathcal{T}$ be an L-fuzzy topology on X. A map $\mathcal{B} : L^X \to L$ is called a base of $\mathcal{T}$ if $\mathcal{B}$ satisfies the following condition:

$$\mathcal{T}(A) = \bigvee_{\vee_{j\in J} B_j = A} \bigwedge_{j\in J} \mathcal{B}(B_j).$$

(2) Let $\phi : L^X \to L$ be a map. Then ϕ is called a subbase of $\mathcal{T}$ if and only if $\phi^{(\sqcap)} : L^X \to L$ is a base of $\mathcal{T}$, where

$$\phi^{(\sqcap)}(A) = \bigvee_{\sqcap_{j\in J} B_j = A} \bigwedge_{j\in J} \phi(B_j)$$

with $\sqcap$ standing for "finite intersection".

(3) Let $\{(X_j,\mathcal{T}_j)\}_{j\in J}$ be a collection of L-fuzzy topological spaces and $P_i : \prod_{j\in J} X_j \to X_i$ be the projection. Then the L-fuzzy topology on $\prod_{j\in J} X_j$ determined by the subbase

$$\forall\ A \in L^{\prod_{j\in J} X_j},\ \phi(A) = \bigvee_{j\in J} \bigvee_{P_j^{\leftarrow}(U)=A} \mathcal{T}_j(U)$$

is called the product L-fuzzy topology of $\{\mathcal{T}_j | j \in J\}$, denoted by $\prod_{j\in J} \mathcal{T}_j$, and $\left(\prod_{j\in J} X_j, \prod_{j\in J} \mathcal{T}_j\right)$ is called the product space of $\{(X_j,\mathcal{T}_j)\}_{j\in J}$.

Definition 2.4. [13] Given an L-fuzzy topological space $(X,\mathcal{T})$ and a subset $Y \subseteq X$, we call $(Y, \mathcal{T}|_Y)$ (where $\mathcal{T}|_Y(U) = \bigvee_{V\in L^X, V|_Y = U} \mathcal{T}(V)$) a subspace of $(X,\mathcal{T})$.

Definition 2.5. [19] For an (L, M)-fuzzy topological space $(X, \mathcal{T})$, define the degree $T_1(X, \mathcal{T})$ to which $(X, \mathcal{T})$ is T_1 as follows:

$$T_1(X, \mathcal{T}) = \bigwedge_{a \not\leq b} \bigvee_{a \not\leq P \geq b} \mathcal{T}(P^{'}).$$

Definition 2.6. [19] For an (L, M)-fuzzy topological space $(X, \mathcal{T})$, define the degree $T_2(X, \mathcal{T})$ to which $(X, \mathcal{T})$ is T_2 as follows:

$$T_2(X, \mathcal{T}) = \bigwedge_{a \not\leq b} \bigvee \left\{ \mathcal{T}(P^{'}) \wedge \mathcal{T}(Q) \mid a \not\leq P \text{ and } P \geq Q \geq b \right\}.$$

Definition 2.7. [8] The L-fuzzy unit interval $I(L)$ is defined as the set of all equivalence classes $[\lambda]$ of antitone maps $\lambda : \mathbb{R} \to L$, for any $t < 0$, $\lambda(t) = \top$ and for any $t > 1$, $\lambda(t) = \bot$. The equivalence identifies two such maps λ, μ iff $\lambda(t+) = \mu(t+)$, $\forall\, t \in \mathbb{R}$. The canonical L-topology is generated from the subbase $\{\mathcal{L}_t, \mathcal{R}_t |\ t \in \mathbb{R}\}$, where

$$\mathcal{L}_t : I(L) \to L \text{ by } \mathcal{L}_t(\lambda) = \lambda(t-)^{'}$$
$$\mathcal{R}_t : I(L) \to L \text{ by } \mathcal{R}_t(\lambda) = \lambda(t+).$$

Definition 2.8. [15] An L-topological space (X, δ) is said to be pointwise completely regular if for each $e \in M(L^X)$ and each $B \in \eta^-(e)$, there exists a family of sets $\{A(t) : \ t \in (0, 1)\} \subseteq L^X$ such that for all $e \not\leq A(s) \geq A(s)^\circ \geq A(r)^- \geq A(r) \geq B$ for all $r, s \in (0, 1)$ with $s < r$.

Lemma 2.9. [14] *An L-topological space (X, δ) is pointwise completely regular if and only if for each $e \in M(L^X)$ and each $B \in \eta^-(e)$, there exists $\{A(t) : \ t \in (0, 1)\} \subseteq L^X$ such that for all $s < r$, $e \not\leq \bigvee_{t>0} A(t)$, $B \leq \bigwedge_{t<1} A(t)$, and $A(s)^\circ \geq A(r)^-$.*

The following definitions and two theorems were presented for an I-fuzzy topology. They can be easily transformed to an L-fuzzy topology as follows:

Definition 2.10. [12] Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $A \in L^X$. The degree to which $D \in L^X$ is the interior of A is defined by

$$Int(A, D) = \begin{cases} \bot, & \exists\, B \not\leq D \text{ with } B \leq A \text{ such that } \mathcal{T}(B) \geq \mathcal{T}(D); \\ \bigwedge\limits_{x_\lambda q D} \bigvee\limits_{x_\lambda q B \leq A} \mathcal{T}(B), & \text{otherwise.} \end{cases}$$

The degree to which $D \in L^X$ is the closure of A is defined by

$$Cl(A, D) = \begin{cases} \bot, & \exists\, B \not\geq D \text{ with } B \geq A \\ & \text{such that } \mathcal{T}^*(B) \geq \mathcal{T}^*(D); \\ \bigwedge\limits_{x_\lambda q D'} \bigvee\limits_{x_\lambda q B' \leq A'} \mathcal{T}^*(B), & \text{otherwise.} \end{cases}$$

Theorem 2.11. [12] *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $A, P, Q \in L^X$. Then the following facts are valid.*

(1) *If $Int(A, P) \neq \bot$, then $Int(A, P) \geq \mathcal{T}(P)$, $P \leq A$ and for all $B \not\leq P$ with $B \leq A$, $\mathcal{T}(B) \not\geq \mathcal{T}(P)$;*

(2) *$Int(A,A)=\mathcal{T}(A)$;*
(3) *For all $P\leq A$, there exists a $Q\in L^X$ such that $P\leq Q\leq A$, $Int(A,Q)\geq \mathcal{T}(P)$;*
(4) *If $Int(A,P)=a\neq\bot$, then for all $b\prec a$, there exists a $Q\in L^X$ such that $P\leq Q\leq A$, $\mathcal{T}(Q)\geq b$ and $Int(A,Q)\geq b$.*

Theorem 2.12. [12] *Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $A,P,Q\in L^X$. Then the following statements are true:*

(1) *If $Cl(A,P)\neq\bot$, then $Cl(A,P)\geq\mathcal{T}^*(P)$, $P\geq A$ and for all $P\not\leq B$ with $A\leq B$, $\mathcal{T}^*(B)\not\geq\mathcal{T}^*(P)$;*
(2) *$Cl(A,A)=\mathcal{T}^*(A)$;*
(3) *For all $P\geq A$, there exists a $Q\in L^X$ such that $P\geq Q\geq A$, $Cl(A,Q)\geq \mathcal{T}^*(P)$;*
(4) *If $Cl(A,P)=a\neq\bot$, then for all $b\prec a$, there exists a $Q\in L^X$ such that $P\geq Q\geq A$, $\mathcal{T}^*(Q)\geq b$ and $Cl(A,Q)\geq b$.*

3. The Urysohn Axiom

In this section, we define the degree to which an L-fuzzy topological space is Urysohn separated, and discuss its relation with T_2 in L-fuzzy topological space.

Definition 3.1. For an L-fuzzy topological space $(X,\mathcal{T})$, we define the degree $U(a,b)$ to which two points a,b with $a\not\leq b$ are Urysohn separated as follows:

$$U(a,b)=\bigvee_{P_2\leq Q_2,b\leq Q_1,a\not\leq P_1}\Big(\mathcal{T}(P_1^{'})\wedge\mathcal{T}(Q_1)\wedge Cl(Q_1,P_2)\wedge Int(P_1,Q_2)\Big).$$

The degree to which $(X,\mathcal{T})$ is Urysohn separated is defined by

$$U(X,\mathcal{T})=\bigwedge_{a\not\leq b}U(a,b).$$

Theorem 3.2. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Then*

$$U(X,\mathcal{T})=\bigwedge_{a\not\leq b}\ \bigvee_{b\leq Q_1\leq P_2\leq Q_2\leq P_1\not\geq a}\Big(\mathcal{T}(P_1^{'})\wedge\mathcal{T}(Q_1)\wedge Cl(Q_1,P_2)\wedge Int(P_1,Q_2)\Big).$$

Proof. It is obvious that

$$U(X,\mathcal{T})\geq\bigwedge_{a\not\leq b}\ \bigvee_{b\leq Q_1\leq P_2\leq Q_2\leq P_1\not\geq a}\Big(\mathcal{T}(P_1^{'})\wedge\mathcal{T}(Q_1)\wedge Cl(Q_1,P_2)\wedge Int(P_1,Q_2)\Big).$$

Now we prove

$$U(X,\mathcal{T})\leq\bigwedge_{a\not\leq b}\ \bigvee_{b\leq Q_1\leq P_2\leq Q_2\leq P_1\not\geq a}\Big(\mathcal{T}(P_1^{'})\wedge\mathcal{T}(Q_1)\wedge Cl(Q_1,P_2)\wedge Int(P_1,Q_2)\Big).$$

Let $a,b\in M(L^X),P_1,P_2,Q_1,Q_2\in L^X$, satisfying $a\not\leq P_1,b\leq Q_1$ and $P_2\leq Q_2$. If $Q_1\not\leq P_2$, then $Cl(Q_1,P_2)=\bot$. If $Q_2\not\leq P_1$, then $Int(P_1,Q_2)=\bot$. This implies that

$$\begin{aligned} & \mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_2) \wedge Int(P_1, Q_2) \\ \leq & \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_2) \wedge Int(P_1, Q_2) \Big). \end{aligned}$$

Therefore, we can obtain that

$$U(X, \mathcal{T}) \leq \bigwedge_{a \not\leq b} \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_2) \wedge Int(P_1, Q_2) \Big).$$

The proof is completed. □

Theorem 3.3. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. Then for any points a, b with $a \not\leq b$, it follows that*

$$U(a, b) = \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2) \Big).$$

Proof. First we check

$$U(a, b) \geq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2) \Big).$$

Suppose that $\gamma \in L$ and $\gamma \prec \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2) \Big)$. Then there exist $P_1, P_2, Q_1, Q_2 \in L^X$ such that $b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a$ and $\gamma \prec \mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2)$.

By Theorem 2.11 (3), we know that there exists a $Q_0 \in L^X$ such that $Q_2 \leq Q_0 \leq P_1$ and $Int(P_1, Q_0) \geq \mathcal{T}(Q_2)$. By Theorem 2.12 (3), we know that there exists a $P_0 \in L^X$ such that $Q_1 \leq P_0 \leq P_2$ and $Cl(Q_1, P_0) \geq \mathcal{T}(P_2^{'})$.

Therefore we can obtain $b \leq Q_1 \leq P_0 \leq P_2 \leq Q_2 \leq Q_0 \leq P_1 \not\geq a$ and $\gamma \prec \mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_0) \wedge Int(P_1, Q_0)$. Then we have

$$\gamma \prec \bigvee_{P_0 \leq Q_0, b \leq Q_1, a \not\leq P_1} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_0) \wedge Int(P_1, Q_0) \Big) = U(a, b).$$

Since γ is arbitrary, we know

$$U(a, b) \geq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2) \Big).$$

Secondly, we check

$$U(a, b) \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2) \Big).$$

Let $P_1, P_2, Q_1, Q_2 \in L^X$ with $a \not\leq P_1, b \leq Q_1, P_2 \leq Q_2$. Now we prove that

$$\begin{aligned}&\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_2) \wedge Int(P_1, Q_2) \\ \leq & \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2)\Big).\end{aligned}$$

Let $Int(P_1, Q_2) = g \neq \perp, Cl(Q_1, P_2) = h \neq \perp$. By Theorem 2.11 (4), for all $c \prec g$, there exists a $Q_c \in L^X$ such that $Q_2 \leq Q_c \leq P_1,\ \mathcal{T}(Q_c) \geq c$ and $Int(P_1, Q_c) \geq c$. By Theorem 2.12 (4), for all $d \prec h$, there exists a $P_d \in L^X$ such that $Q_1 \leq P_d \leq P_2,\ \mathcal{T}(P_d^{'}) \geq d$ and $Cl(Q_1, P_d) \geq d$. Therefore $b \leq Q_1 \leq P_d \leq P_2 \leq Q_2 \leq Q_c \leq P_1 \not\geq a$, then we have

$$\begin{aligned}&\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_2) \wedge Int(P_1, Q_2) \\ \leq\ & \mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \left(\bigvee_{d \prec h} \mathcal{T}(P_d^{'})\right) \wedge \left(\bigvee_{c \prec g} \mathcal{T}(Q_c)\right) \\ =\ & \bigvee_{d \prec h} \bigvee_{c \prec g} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_d^{'}) \wedge \mathcal{T}(Q_c)\Big) \\ \leq\ & \bigvee_{b \leq Q_1 \leq P_d \leq Q_c \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_d^{'}) \wedge \mathcal{T}(Q_c)\Big).\end{aligned}$$

This implies

$$\begin{aligned}&\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge Cl(Q_1, P_2) \wedge Int(P_1, Q_2) \\ \leq & \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1^{'}) \wedge \mathcal{T}(Q_1) \wedge \mathcal{T}(P_2^{'}) \wedge \mathcal{T}(Q_2)\Big).\end{aligned}$$

The proof is completed. □

By Theorem 3.3, we have the following theorems.

Theorem 3.4. *If $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a homeomorphism, then for all $a \not\leq b$, $U_{\mathcal{T}}^X(a, b) = U_{\mathcal{U}}^Y(f^{\to}(a), f^{\to}(b))$.*

Theorem 3.5. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $(Y, \mathcal{T}|_Y)$ be a subspace of $(X, \mathcal{T})$. Then $U(X, \mathcal{T}) \leq U(Y, \mathcal{T}|_Y)$.*

Theorem 3.6. *Let $(X, \mathcal{T})$ be the product of L-fuzzy topological spaces $\{(X_j, \mathcal{T}_j)\}_{j \in J}$. Then $\bigwedge_{j \in J} U(X_j, \mathcal{T}_j) \leq U(X, \mathcal{T})$. In addition, if $(X_j, \mathcal{T}_j)$ is stratified for all $j \in J$, then $\bigwedge_{j \in J} U(X_j, \mathcal{T}_j) = U(X, \mathcal{T})$.*

Proof. Suppose that $\gamma \prec \bigwedge_{j \in J} U(X_j, \mathcal{T}_j)$. Then for any $j \in J,\ \gamma \prec U(X_j, \mathcal{T}_j)$. Let $a, b \in M(L^X)$ with $a \not\leq b$. Then there exists a $k \in J$ such that $P_k^{\to}(a) \not\leq P_k^{\to}(b)$.

By Theorem 3.3 we have

$$\begin{aligned}\gamma &\prec U(X_k,\mathcal{T}_k)\\ &= \bigwedge_{a_k\not\leq b_k}\ \bigvee_{b_k\leq A_k\leq B_k\leq C_k\leq D_k\not\geq a_k}\Big(\mathcal{T}_k(D_k^{'})\wedge\mathcal{T}_k(A_k)\wedge\mathcal{T}_k(B_k^{'})\wedge\mathcal{T}_k(C_k)\Big)\\ &\leq \bigvee_{P_k^{\rightarrow}(b)\leq A_k\leq B_k\leq C_k\leq D_k\not\geq P_k^{\rightarrow}(a)}\Big(\mathcal{T}_k(D_k^{'})\wedge\mathcal{T}_k(A_k)\wedge\mathcal{T}_k(B_k^{'})\wedge\mathcal{T}_k(C_k)\Big).\end{aligned}$$

This implies that there exist $A_k, B_k, C_k, D_k \in L^{X_k}$, such that $P_k^{\rightarrow}(b) \leq A_k \leq B_k \leq C_k \leq D_k \not\geq P_k^{\rightarrow}(a)$ and

$$\begin{aligned}\gamma &\leq \mathcal{T}_k(D_k^{'})\wedge\mathcal{T}_k(A_k)\wedge\mathcal{T}_k(B_k^{'})\wedge\mathcal{T}_k(C_k)\\ &\leq \mathcal{T}(P_k^{\leftarrow}(D_k^{'}))\wedge\mathcal{T}(P_k^{\leftarrow}(A_k))\wedge\mathcal{T}(P_k^{\leftarrow}(B_k^{'}))\wedge\mathcal{T}(P_k^{\leftarrow}(C_k)).\end{aligned}$$

By $b \leq P_k^{\leftarrow}(A_k) \leq P_k^{\leftarrow}(B_k) \leq P_k^{\leftarrow}(C_k) \leq P_k^{\leftarrow}(D_k) \not\geq a$, we obtain

$$\gamma \leq \bigvee_{b\leq A\leq B\leq C\leq D\not\geq a}\Big(\mathcal{T}(D^{'})\wedge\mathcal{T}(A)\wedge\mathcal{T}(B^{'})\wedge\mathcal{T}(C)\Big).$$

Therefore,

$$\gamma \leq \bigwedge_{a\not\leq b}\ \bigvee_{b\leq A\leq B\leq C\leq D\not\geq a}\Big(\mathcal{T}(D^{'})\wedge\mathcal{T}(A)\wedge\mathcal{T}(B^{'})\wedge\mathcal{T}(C)\Big).$$

Since γ is arbitrary, we know $\bigwedge_{j\in J} U(X_j,\mathcal{T}_j) \leq U(X,\mathcal{T})$.

In addition, assume that $(X_j,\mathcal{T}_j)$ is stratified for all $j \in J$, then $(X_j,\mathcal{T}_j)$ is homeomorphic to a subspace $\left(\tilde{X}_j, \mathcal{T}|_{\tilde{X}_j}\right)$ of $(X,\mathcal{T})$, where $\tilde{X}_j$ is a subset of X parallel to X_j through $x = (x_j)_{j\in J}$. From Theorem 3.4 and 3.5, we have

$$\bigwedge_{j\in J} U(X_j,\mathcal{T}_j) = \bigwedge_{j\in J} U\left(\tilde{X}_j, \mathcal{T}|_{\tilde{X}_j}\right) \geq U(X,\mathcal{T}).$$

□

By Definition 2.6 and Theorem 3.3, we can easily obtain the relation between the Urysohn axiom and T_2 axiom.

Theorem 3.7. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Then $U(X,\mathcal{T}) \leq T_2(X,\mathcal{T})$.*

4. **The Completely Hausdorff Axiom**

In this section, the degree to which an L-fuzzy topological space is completely Hausdorff is introduced and its relation with the degree of Urysohn separation is discussed.

Definition 4.1. For an L-fuzzy topological space $(X,\mathcal{T})$, we define the degree $H(a,b)$ to which two points a, b with $a \not\leq b$ are completely Hausdorff separated as follows:

$$H(a,b) = \bigvee_{F\in\mathcal{F}(a,b)}\ \bigwedge_{s<r}\ \bigvee_{P\geq Q}\left(Cl(F(r),Q)\wedge Int(F(s),P)\right),$$

where $\mathcal{F}(a,b) = \left\{F : (0,1) \to L^X \mid \forall\ s < r, a \not\leq F(s) \geq F(r) \geq b\right\}$. The degree to which $(X,\mathcal{T})$ is completely Hausdorff separated is defined by:

$$H\left(X,\mathcal{T}\right) = \bigwedge_{a \not\leq b} H\left(a,b\right).$$

Theorem 4.2. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Then for all points $a,b \in M(L^X)$ with $a \not\leq b$,*

$$H\left(a,b\right) = \bigvee_{G \in \mathcal{G}(a,b)} \left(\bigwedge_{t \in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t \in I_1} \mathcal{T}\left(G(t+)\right)\right),$$

where

$\mathcal{G}(a,b) = \left\{G : (0,1) \to L^X \mid G \text{ is an antitone map with } a \not\leq G(0+),\ b \leq G(1-)\right\}$.

Proof. Suppose that $\lambda \prec H\left(a,b\right)$. Then there exists an $F \in \mathcal{F}(a,b)$ such that for any $s,r \in (0,1)$ with $s < r$, there exist P,Q with $P \geq Q$ such that $\lambda \prec Cl\left(F(r),Q\right) \wedge Int\left(F(s),P\right)$. It is easy to check that $F \in \mathcal{G}(a,b)$.

Since $\lambda \prec Cl\left(F(r),Q\right) \wedge Int\left(F(s),P\right)$, it follows that $\lambda \prec Cl\left(F(r),Q\right)$ and $\lambda \prec Int\left(F(s),P\right)$. By Theorems 2.11 and 2.12, we know that there exist $Q_{(r,s)}, P_{(r,s)}$ such that $F(r) \leq Q_{(r,s)} \leq Q \leq P \leq P_{(r,s)} \leq F(s)$ and $\lambda \leq \mathcal{T}\left(P_{(r,s)}\right), \lambda \leq \mathcal{T}^*\left(Q_{(r,s)}\right)$. From

$$F(t-) \leq \bigwedge_{l<t}\bigwedge_{\beta<l} F(\beta) \leq \bigwedge_{l<t}\bigwedge_{\beta<l}\bigwedge_{\alpha<\beta} Q_{(\beta,\alpha)} \leq \bigwedge_{l<t}\bigwedge_{\beta<l}\bigwedge_{\alpha<\beta} F(\alpha) = F(t-), \quad (t \in I_0)$$

and

$$F(t+) \geq \bigvee_{l>t}\bigvee_{\alpha>l} F(\alpha) \geq \bigvee_{l>t}\bigvee_{\alpha>l}\bigvee_{\beta>\alpha} P_{(\beta,\alpha)} \geq \bigvee_{l>t}\bigvee_{\alpha>l}\bigvee_{\beta>\alpha} F(\beta) = F(t+), \quad (t \in I_1)$$

we know that

$$\mathcal{T}^*\left(F(t-)\right) = \mathcal{T}^*\left(\bigwedge_{l<t}\bigwedge_{\beta<l}\bigwedge_{\alpha<\beta} Q_{(\beta,\alpha)}\right) \geq \bigwedge_{l<t}\bigwedge_{\beta<l}\bigwedge_{\alpha<\beta} \mathcal{T}^*\left(Q_{(\beta,\alpha)}\right) \geq \lambda, \quad (t \in I_0)$$

and

$$\mathcal{T}\left(F(t+)\right) = \mathcal{T}\left(\bigvee_{l>t}\bigvee_{\alpha>l}\bigvee_{\beta>\alpha} P_{(\beta,\alpha)}\right) \geq \bigwedge_{l>t}\bigwedge_{\alpha>l}\bigwedge_{\beta>\alpha} \mathcal{T}\left(P_{(\beta,\alpha)}\right) \geq \lambda. \quad (t \in I_1)$$

Hence

$$\lambda \leq \bigvee_{G \in \mathcal{G}(a,b)} \left(\bigwedge_{t \in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t \in I_1} \mathcal{T}\left(G(t+)\right)\right).$$

This means that

$$H\left(a,b\right) \leq \bigvee_{G \in \mathcal{G}(a,b)} \left(\bigwedge_{t \in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t \in I_1} \mathcal{T}\left(G(t+)\right)\right).$$

On the other hand, let $\mu \prec \bigvee_{G\in\mathcal{G}(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right)$. Then there exists a $G\in\mathcal{G}(a,b)$ such that $\mu\prec\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))$. Then for all $t\in I_0,\ \mu\prec\mathcal{T}^*(G(t-))$ and for all $t\in I_1,\ \mu\prec\mathcal{T}(G(t+))$. It is easy to check that $G\in\mathcal{F}(a,b)$.

For all $s<r$, from $G(s+)\leq G(s)$ and $G(r)\leq G(r-)$, we know that there exist $Q_{(r,s)}, P_{(r,s)}$ such that $G(r)\leq Q_{(r,s)}\leq G(r-)\leq G(s+)\leq P_{(r,s)}\leq G(s)$ and $Int\left(G(s),P_{(r,s)}\right)\geq\mathcal{T}(G(s+))\geq\mu, Cl\left(G(r),Q_{(r,s)}\right)\geq\mathcal{T}^*(G(r-))\geq\mu$. Therefore,

$$\mu\leq\bigvee_{F\in\mathcal{F}(a,b)}\bigwedge_{s<r}\bigvee_{P\geq Q}\left(Cl\left(F(r),Q\right)\wedge Int\left(F(s),P\right)\right).$$

This means that

$$H(a,b)\geq\bigvee_{G\in\mathcal{G}(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right).$$

The proof is completed. □

By Theorem 4.2, we can obtain the following theorem.

Theorem 4.3. *If $f:(X,\mathcal{T})\to(Y,\mathcal{U})$ is a homeomorphism, then for all $a\not\leq b$, $H^X_{\mathcal{T}}(a,b)=H^Y_{\mathcal{U}}\left(f^{\to}(a),f^{\to}(b)\right)$.*

Theorem 4.4. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $(Y,\mathcal{T}|_Y)$ be a subspace of $(X,\mathcal{T})$. Then $H(X,\mathcal{T})\leq H(Y,\mathcal{T}|_Y)$.*

Proof. By Theorem 4.2, we need to prove

$$\begin{aligned}&\bigwedge_{e\not\leq d}\bigvee_{G\in\mathcal{G}(e,d)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right)\\ \leq&\bigwedge_{a\not\leq b}\bigvee_{G_Y\in\mathcal{G}_Y(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}|_Y^*(G_Y(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}|_Y(G_Y(t+))\right).\end{aligned}$$

Suppose that $\lambda\prec\bigwedge_{e\not\leq d}\bigvee_{G\in\mathcal{G}(e,d)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right)$. Then for any $e,d\in M(L^X)$ with $e\not\leq d$, there exists $G\in\mathcal{G}(e,d)$ such that $\lambda\prec\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))$.

For any $a,b\in M(L^Y)$ with $a\not\leq b$, we have $a,b\in M(L^X)$. Then there exists $G\in\mathcal{G}(a,b)$ such that $\lambda\prec\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))$.

Define $\mathcal{G}|_Y(a,b) = \{F|_Y : (0,1) \to L^Y \mid F \in \mathcal{G}(a,b)\}$, where $F|_Y(t)(y) = F(t)(y)$ for any $t \in (0,1)$, and $y \in Y$. Then we have

$$\begin{aligned} \lambda &\prec \bigwedge_{t\in I_0} \mathcal{T}^*(G(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}(G(t+)) \\ &\leq \bigwedge_{t\in I_0} \mathcal{T}|_Y^*(G|_Y(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y(G|_Y(t+)) \\ &\leq \bigvee_{G|_Y\in\mathcal{G}|_Y(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*(G|_Y(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y(G|_Y(t+)) \right) \\ &\leq \bigvee_{G_Y\in\mathcal{G}_Y(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*(G_Y(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y(G_Y(t+)) \right). \end{aligned}$$

Therefore,

$$\lambda \leq \bigwedge_{a\not\leq b} \bigvee_{G_Y\in\mathcal{G}_Y(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*(G_Y(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y(G_Y(t+)) \right).$$

Since λ is arbitrary, it follows that

$$\begin{aligned} &\bigwedge_{e\not\leq d} \bigvee_{G\in\mathcal{G}(e,d)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*(G(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}(G(t+)) \right) \\ \leq\ &\bigwedge_{a\not\leq b} \bigvee_{G_Y\in\mathcal{G}_Y(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*(G_Y(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y(G_Y(t+)) \right). \end{aligned}$$

□

Theorem 4.5. *Let $(X,\mathcal{T})$ be the product of L-fuzzy topological spaces $\{(X_j,\mathcal{T}_j)\}_{j\in J}$. Then $\bigwedge_{j\in J} H(X_j,\mathcal{T}_j) \leq H(X,\mathcal{T})$. In addition, if $(X_j,\mathcal{T}_j)$ is stratified for all $j\in J$, then $\bigwedge_{j\in J} H(X_j,\mathcal{T}_j) = H(X,\mathcal{T})$.*

Proof. Suppose that $\gamma \prec \bigwedge_{j\in J} H(X_j,\mathcal{T}_j)$. Then for any $j\in J$, $\gamma \prec H(X_j,\mathcal{T}_j)$. Let $a,b\in M(L^X)$ with $a\not\leq b$. Then there exists a $k\in J$ such that $P_k^{\rightarrow}(a) \not\leq P_k^{\rightarrow}(b)$. Then

$$\begin{aligned} \gamma &\prec H(X_k,\mathcal{T}_k) \\ &= \bigwedge_{a_k\not\leq b_k} \bigvee_{G_k\in\mathcal{G}_k(a_k,b_k)} \left(\bigwedge_{t\in I_0} \mathcal{T}_k^*(G_k(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}_k(G_k(t+)) \right) \\ &\leq \bigvee_{G_k\in\mathcal{G}_k(P_k^{\rightarrow}(a),P_k^{\rightarrow}(b))} \left(\bigwedge_{t\in I_0} \mathcal{T}_k^*(G_k(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}_k(G_k(t+)) \right). \end{aligned}$$

This implies that there exists a $G_k \in \mathcal{G}_k(P_k^{\rightarrow}(a), P_k^{\rightarrow}(b))$ such that

$$\gamma \prec \bigwedge_{t\in I_0} \mathcal{T}_k^*(G_k(t-)) \wedge \bigwedge_{t\in I_1} \mathcal{T}_k(G_k(t+)).$$

For any $t \in (0,1)$, let $G(t) = P_k^{\leftarrow}(G_k(t))$. Then $G \in \mathcal{G}(a,b)$.
From

$$\begin{aligned}\mathcal{T}^*(G(t-)) &= \mathcal{T}^*\left(\bigwedge_{s<t} P_k^{\leftarrow}(G_k(s))\right) \\ &= \mathcal{T}^*\left(P_k^{\leftarrow}\left(\bigwedge_{s<t} G_k(s)\right)\right) \geq \mathcal{T}_k^*\left(G_k(t-)\right),\ (t \in I_1)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}(G(t+)) &= \mathcal{T}\left(\bigvee_{s>t} P_k^{\leftarrow}(G_k(s))\right) \\ &= \mathcal{T}\left(P_k^{\leftarrow}\left(\bigvee_{s>t} G_k(s)\right)\right) \geq \mathcal{T}_k\left(G_k(t+)\right),\ (t \in I_0)\end{aligned}$$

we know

$$\begin{aligned}\gamma &\prec \bigwedge_{t\in I_0} \mathcal{T}_k^*\left(G_k(t-)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}_k\left(G_k(t+)\right) \\ &\leq \bigwedge_{t\in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(G(t+)\right) \\ &\leq \bigvee_{G\in\mathcal{G}(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(G(t+)\right)\right).\end{aligned}$$

Hence

$$\gamma \leq \bigwedge_{a\not\leq b} \bigvee_{G\in\mathcal{G}(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(G(t+)\right)\right) = H\left(X,\mathcal{T}\right).$$

Since γ is arbitrary, we know $\bigwedge_{j\in J} H\left(X_j,\mathcal{T}_j\right) \leq H\left(X,\mathcal{T}\right)$.

In addition, assume that $(X_j,\mathcal{T}_j)$ is stratified for all $j \in J$, then $(X_j,\mathcal{T}_j)$ is homeomorphic to a subspace $\left(\tilde{X}_j, \mathcal{T}|_{\tilde{X}_j}\right)$ of $(X,\mathcal{T})$, where $\tilde{X}_j$ is a subset of X parallel to X_j through $x = (x_j)_{j\in J}$. From Theorems 4.3 and 4.4, we have

$$\bigwedge_{j\in J} H\left(X_j,\mathcal{T}_j\right) = \bigwedge_{j\in J} H\left(\tilde{X}_j, \mathcal{T}|_{\tilde{X}_j}\right) \geq H\left(X,\mathcal{T}\right).$$

□

Now we study the relation between the Urysohn axiom and the completely Hausdorff axiom.

Theorem 4.6. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Then $H\left(X,\mathcal{T}\right) \leq U\left(X,\mathcal{T}\right)$.*

Proof. By Theorems 3.3 and 4.2, we only need to prove that for all $a, b \in M(L^X)$,

$$\begin{aligned}&\bigvee_{G\in\mathcal{G}(a,b)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(G(t-)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(G(t+)\right)\right) \\ \leq &\bigvee_{b\leq Q_1\leq P_2\leq Q_2\leq P_1\not\geq a} \left(\mathcal{T}(P_1') \wedge \mathcal{T}\left(Q_1\right) \wedge \mathcal{T}(P_2') \wedge \mathcal{T}\left(Q_2\right)\right).\end{aligned}$$

For each $\lambda \in L$ and $\lambda \prec \bigvee_{G \in \mathcal{G}(a,b)} \left(\bigwedge_{t \in I_0} \mathcal{T}^* \left(G(t-) \right) \wedge \bigwedge_{t \in I_1} \mathcal{T} \left(G(t+) \right) \right)$, there exists a $G \in \mathcal{G}(a,b)$ such that $\lambda \prec \bigwedge_{t \in I_0} \mathcal{T}^* \left(G(t-) \right) \wedge \bigwedge_{t \in I_1} \mathcal{T} \left(G(t+) \right)$. Since $G \in \mathcal{G}(a,b)$, we have $a \not\leq G(0+)$, and $b \leq G(1-)$. Then there exist m, s with $0 < m < s < 1$ such that $a \not\leq G(m+) \geq G(s-) \geq G(1-) \geq b$. Let $P_1 = Q_2 = G(m+)$, $Q_1 = P_2 = G(s-)$. Then $b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a$ and

$$
\begin{aligned}
\lambda &\leq \mathcal{T}^*(G(m+)) \wedge \mathcal{T}(G(m+)) \wedge \mathcal{T}^*(G(s-)) \wedge \mathcal{T}(G(s-)) \\
&= \mathcal{T}(P_1') \wedge \mathcal{T}\left(Q_1\right) \wedge \mathcal{T}(P_2') \wedge \mathcal{T}\left(Q_2\right).
\end{aligned}
$$

Then

$$
\lambda \leq \bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1') \wedge \mathcal{T}\left(Q_1\right) \wedge \mathcal{T}(P_2') \wedge \mathcal{T}\left(Q_2\right) \Big).
$$

Since λ is arbitrary, we have

$$
\begin{aligned}
&\bigvee_{G \in \mathcal{G}(a,b)} \left(\bigwedge_{t \in I_0} \mathcal{T}^* \left(G(t-) \right) \wedge \bigwedge_{t \in I_1} \mathcal{T} \left(G(t+) \right) \right) \\
\leq &\bigvee_{b \leq Q_1 \leq P_2 \leq Q_2 \leq P_1 \not\geq a} \Big(\mathcal{T}(P_1') \wedge \mathcal{T}\left(Q_1\right) \wedge \mathcal{T}(P_2') \wedge \mathcal{T}\left(Q_2\right) \Big).
\end{aligned}
$$

Therefore $H\left(X, \mathcal{T}\right) \leq U\left(X, \mathcal{T}\right)$. □

5. **The Completely Regular Axiom**

In this section, the completely regular axiom in L-topological space is generalized to L-fuzzy topological space. Each L-fuzzy topological space can be regarded to be completely regular to some degree. Moreover, the relation between $T_{3\frac{1}{2}}$ and the degree of completely Hausdorff separation is investigated.

Definition 5.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space. The degree to which $(X, \mathcal{T})$ is completely regular is defined by:

$$
\begin{aligned}
Creg\left(X, \mathcal{T}\right) = &\bigwedge_{e \in M(L^X)} \bigwedge_{e \not\leq B \in L^X} \Big\{ \mathcal{T}^*(B) \to \\
&\bigvee_{A \in \mathcal{K}(e,B)} \bigwedge_{s<r} \bigvee_{P \geq Q} \left(Cl(A(r), Q) \wedge Int(A(s), P) \right) \Big\},
\end{aligned}
$$

where $\mathcal{K}(e,B) = \left\{ A : (0,1) \to L^X \mid e \not\leq \bigvee_{t>0} A(t),\ B \leq \bigwedge_{t<1} A(t) \right\}$.

Moreover define

$$
T_{3\frac{1}{2}}\left(X, \mathcal{T}\right) = Creg\left(X, \mathcal{T}\right) \wedge T_1\left(X, \mathcal{T}\right).
$$

Theorem 5.2. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. Then*

$$Creg\,(X,\mathcal{T}) = \bigwedge_{e\in M(L^X)} \bigwedge_{e\not\leq B\in L^X} \Big\{\mathcal{T}^*(B) \to \bigvee_{H\in\mathcal{H}(e,B)} \Big(\bigwedge_{t\in I_0} \mathcal{T}^*\left(H(t-)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(H(t+)\right)\Big)\Big\},$$

where

$$\mathcal{H}(e,B) = \Big\{H:(0,1)\to L^X \mid H \text{ is an antitone map with } e\not\leq H(0+),\ B\leq H(1-)\Big\}.$$

Proof. The proof is similar to that of Theorem 4.2. □

Theorem 5.3. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $(Y,\mathcal{T}|_Y)$ be a subspace of $(X,\mathcal{T})$. Then $Creg\,(X,\mathcal{T}) \leq Creg\,(Y,\mathcal{T}|_Y)$.*

Proof. By Theorem 5.2, we only need to prove

$$\bigwedge_{e\in M\left(L^X\right)} \bigwedge_{e\not\leq B\in L^X} \left\{\mathcal{T}^*\left(B\right) \to \bigvee_{H\in\mathcal{H}(e,B)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(H\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(H\left(t+\right)\right)\right)\right\}$$
$$\leq \bigwedge_{a\in M\left(L^Y\right)} \bigwedge_{a\not\leq A\in L^Y} \Big\{\mathcal{T}|_Y^*\left(A\right) \to \bigvee_{H_Y\in\mathcal{H}_Y(a,A)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*\left(H_Y\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y\left(H_Y\left(t+\right)\right)\right)\Big\}.$$

Let $\lambda \prec \bigwedge_{e\in M\left(L^X\right)} \bigwedge_{e\not\leq B\in L^X} \left\{\mathcal{T}^*\left(B\right) \to \bigvee_{H\in\mathcal{H}(e,B)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(H\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(H\left(t+\right)\right)\right)\right\}$.
Then for any $e\in M\left(L^X\right), B\in L^X$ with $e\not\leq B$ we have

$$\lambda\wedge\mathcal{T}^*\left(B\right) \leq \bigvee_{H\in\mathcal{H}(e,B)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(H\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(H\left(t+\right)\right)\right).$$

Let $a\in M\left(L^Y\right), A\in L^Y$ with $a\not\leq A$ and $V\in L^X$ with $V|_Y = A$. For any $H\in\mathcal{H}\left(a,V\right)$, define $\mathcal{H}|_Y\left(a,V|_Y\right) = \{H|_Y:(0,1)\to L^Y \mid H\in\mathcal{H}\left(a,V\right)\}$, where $H|_Y\left(t\right)\left(y\right) = H\left(t\right)\left(y\right)$ for any $t\in(0,1)$ and $y\in Y$. Then we have

$$\begin{aligned}\lambda\wedge\mathcal{T}^*\left(V\right) &\leq \bigvee_{H\in\mathcal{H}(a,V)} \left(\bigwedge_{t\in I_0} \mathcal{T}^*\left(H\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}\left(H\left(t+\right)\right)\right)\\ &\leq \bigvee_{H|_Y\in\mathcal{H}|_Y(a,V|_Y)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*\left(H|_Y\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y\left(H|_Y\left(t+\right)\right)\right)\\ &\leq \bigvee_{H_Y\in\mathcal{H}_Y(a,A)} \left(\bigwedge_{t\in I_0} \mathcal{T}|_Y^*\left(H_Y\left(t-\right)\right) \wedge \bigwedge_{t\in I_1} \mathcal{T}|_Y\left(H_Y\left(t+\right)\right)\right).\end{aligned}$$

Hence

$$\begin{aligned}\lambda\wedge\mathcal{T}|_Y^*(A) &= \lambda\wedge\left(\bigvee_{V\in L^X, V|_Y=A}\mathcal{T}^*(V)\right)\\ &\le \bigvee_{H_Y\in\mathcal{H}_Y(a,A)}\left(\bigwedge_{t\in I_0}\mathcal{T}|_Y^*(H_Y(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}|_Y(H_Y(t+))\right).\end{aligned}$$

This implies

$$\lambda\le\mathcal{T}|_Y^*(A)\to\bigvee_{H_Y\in\mathcal{H}_Y(a,A)}\left(\bigwedge_{t\in I_0}\mathcal{T}|_Y^*(H_Y(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}|_Y(H_Y(t+))\right).$$

Therefore,

$$\begin{aligned}\lambda \le & \bigwedge_{a\in M(L^Y)}\bigwedge_{a\not\le A\in L^Y}\Big\{\mathcal{T}|_Y^*(A)\to\\ & \bigvee_{H_Y\in\mathcal{H}_Y(a,A)}\Big(\bigwedge_{t\in I_0}\mathcal{T}|_Y^*(H_Y(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}|_Y(H_Y(t+))\Big)\Big\}.\end{aligned}$$

Since λ is arbitrary, we know that

$$\begin{aligned}&\bigwedge_{e\in M(L^X)}\bigwedge_{e\not\le B\in L^X}\Big\{\mathcal{T}^*(B)\to\bigvee_{H\in\mathcal{H}(e,B)}\Big(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(H(t+))\Big)\Big\}\\ \le\ &\bigwedge_{a\in M(L^Y)}\bigwedge_{a\not\le A\in L^Y}\Big\{\mathcal{T}|_Y^*(A)\to\\ &\bigvee_{H_Y\in\mathcal{H}_Y(a,A)}\Big(\bigwedge_{t\in I_0}\mathcal{T}|_Y^*(H_Y(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}|_Y(H_Y(t+))\Big)\Big\}.\end{aligned}$$

□

Theorem 5.4. *If $f:(X,\mathcal{T})\to(Y,\mathcal{U})$ is a homeomorphism, then $Creg(X,\mathcal{T})=Creg(Y,\mathcal{U})$.*

Proof. By Theorem 5.2, we can easily obtain this theorem. □

Theorem 5.5. *Let $(X,\mathcal{T})$ be the product of L-fuzzy topological spaces $\{(X_j,\mathcal{T}_j)\}_{j\in J}$. Then $\bigwedge_{j\in J}Creg(X_j,\mathcal{T}_j)\le Creg(X,\mathcal{T})$. In addition, if $(X_j,\mathcal{T}_j)$ is stratified for all $j\in J$, then $\bigwedge_{j\in J}Creg(X_j,\mathcal{T}_j)=Creg(X,\mathcal{T})$.*

Proof. Let $\gamma\prec\bigwedge_{j\in J}Creg(X_j,\mathcal{T}_j)$. Then for any $j\in J,\gamma\prec Creg(X_j,\mathcal{T}_j)$. This implies for any $e_j\in M\left(L^{X_j}\right), A_j\in L^{X_j}$ with $e_j\not\le A_j$, that

$$\gamma\wedge\mathcal{T}_j^*(A_j)\le\bigvee_{H_j\in\mathcal{H}_j(e_j,A_j)}\left(\bigwedge_{t\in I_0}\mathcal{T}_j^*(H_j(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}_j(H_j(t+))\right).$$

Now we prove for any $e \in M\left(L^X\right), A \in L^X$ with $e \not\leq A$, that

$$\gamma \wedge \mathcal{T}^*(A) \leq \bigvee_{H \in \mathcal{H}(e,A)} \left(\bigwedge_{t \in I_0} \mathcal{T}^*(H(t-)) \wedge \bigwedge_{t \in I_1} \mathcal{T}(H(t+)) \right).$$

Let $\mu \prec \gamma \wedge \mathcal{T}^*(A)$. Then $\mu \prec \gamma$ and $\mu \prec \mathcal{T}^*(A)$. Therefore, there exists a finite set $\Omega \subseteq J$ and $A_k \in L^{X_k} (k \in \Omega)$ such that $e \not\leq \bigvee_{k \in \Omega} P_k^{\leftarrow}(A_k) \geq A$ and $\mu \prec \mathcal{T}_k(A_k^{'})$. Obviously, for any $k \in \Omega, P_k^{\rightarrow}(e) \not\leq A_k$, we have

$$\gamma \wedge \mathcal{T}_k^*(A_k) \leq \bigvee_{H_k \in \mathcal{H}_k\left(P_k^{\rightarrow}(e), A_k\right)} \left(\bigwedge_{t \in I_0} \mathcal{T}_k^*(H_k(t-)) \wedge \bigwedge_{t \in I_1} \mathcal{T}_k(H_k(t+)) \right).$$

By $\mu \prec \gamma$ and $\mu \prec \mathcal{T}_k^*(A_k)$, we know for any $k \in \Omega$ that

$$\mu \leq \gamma \wedge \mathcal{T}_k^*(A_k) \leq \bigvee_{H_k \in \mathcal{H}_k\left(P_k^{\rightarrow}(e), A_k\right)} \left(\bigwedge_{t \in I_0} \mathcal{T}_k^*(H_k(t-)) \wedge \bigwedge_{t \in I_1} \mathcal{T}_k(H_k(t+)) \right).$$

Let $\alpha \prec \mu$. Then for any $k \in \Omega$, there exists an $H_k \in \mathcal{H}_k(P_k^{\rightarrow}(e), A_k)$ such that $\alpha \prec \bigwedge_{t \in I_0} \mathcal{T}_k^*(H_k(t-)) \wedge \bigwedge_{t \in I_1} \mathcal{T}_k(H_k(t+))$. Then

$$\alpha \leq \bigwedge_{k \in \Omega} \left(\bigwedge_{t \in I_0} \mathcal{T}_k^*(H_k(t-)) \wedge \bigwedge_{t \in I_1} \mathcal{T}_k(H_k(t+)) \right).$$

We define a map $H : (0,1) \to L^X$ as follows:

$$H(t) = \bigvee_{k \in \Omega} P_k^{\leftarrow}(H_k(t)), \quad \forall\, t \in (0,1).$$

Now we prove $H \in \mathcal{H}(e,A)$. It is obvious that H is an antitone map. From $P_k^{\rightarrow}(e) \not\leq \bigvee_{t>0} H_k(t)$ $(\forall\, k \in \Omega)$, we know that $e \not\leq \bigvee_{t>0} H(t)$. In fact, if $e \leq \bigvee_{t>0} H(t) = \bigvee_{t>0} \bigvee_{k \in \Omega} P_k^{\leftarrow}(H_k(t))$, then there exists $k_0 \in \Omega$ such that $e \leq \bigvee_{t>0} P_{k_0}^{\leftarrow}(H_{k_0}(t))$. Therefore, $P_{k_0}^{\rightarrow}(e) \leq \bigvee_{t>0} H_{k_0}(t)$, which is a contradiction.

From $A_k \leq \bigwedge_{s<1} H_k(s)$, we know that

$$\begin{aligned} \bigwedge_{s<1} H(s) &= \bigwedge_{s<1} \bigvee_{k \in \Omega} P_k^{\leftarrow}(H_k(s)) = \bigvee_{k \in \Omega} \bigwedge_{s<1} P_k^{\leftarrow}(H_k(s)) \\ &= \bigvee_{k \in \Omega} P_k^{\leftarrow}\left(\bigwedge_{s<1} H_k(s) \right) \geq \bigvee_{k \in \Omega} P_k^{\leftarrow}(A_k) \geq A. \end{aligned}$$

Therefore, $H \in \mathcal{H}(e,A)$.

For any $t \in I_0$, we have

$$\begin{aligned}
\mathcal{T}^*(H(t-)) &= \mathcal{T}^*\left(\bigwedge_{s<t} H(s)\right) &&= \mathcal{T}^*\left(\bigwedge_{s<t}\bigvee_{k\in\Omega} P_k^{\leftarrow}(H_k(s))\right)\\
&\geq \bigwedge_{k\in\Omega}\mathcal{T}^*\left(\bigwedge_{s<t} P_k^{\leftarrow}(H_k(s))\right) &&= \bigwedge_{k\in\Omega}\mathcal{T}^*\left(P_k^{\leftarrow}\left(\bigwedge_{s<t} H_k(s)\right)\right)\\
&\geq \bigwedge_{k\in\Omega}\mathcal{T}_k^*\left(\bigwedge_{s<t} H_k(s)\right) &&= \bigwedge_{k\in\Omega}\mathcal{T}_k^*(H_k(t-)).
\end{aligned}$$

For any $t \in I_1$, we have

$$\begin{aligned}
\mathcal{T}(H(t+)) &= \mathcal{T}\left(\bigvee_{s>t} H(s)\right) &&= \mathcal{T}\left(\bigvee_{s>t}\bigvee_{k\in\Omega} P_k^{\leftarrow}(H_k(s))\right)\\
&\geq \bigwedge_{k\in\Omega}\mathcal{T}\left(\bigvee_{s>t} P_k^{\leftarrow}(H_k(s))\right) &&= \bigwedge_{k\in\Omega}\mathcal{T}\left(P_k^{\leftarrow}\left(\bigvee_{s>t} H_k(s)\right)\right)\\
&\geq \bigwedge_{k\in\Omega}\mathcal{T}_k\left(\bigvee_{s>t} H_k(s)\right) &&= \bigwedge_{k\in\Omega}\mathcal{T}_k(H_k(t+)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha &\leq \bigwedge_{k\in\Omega}\left(\bigwedge_{t\in I_0}\mathcal{T}_k^*(H_k(t-)) \wedge \bigwedge_{t\in I_1}\mathcal{T}_k(H_k(t+))\right)\\
&\leq \bigwedge_{t\in I_0}\mathcal{T}^*(H(t-)) \wedge \bigwedge_{t\in I_1}\mathcal{T}(H(t+))\\
&\leq \bigvee_{H\in\mathcal{H}(e,A)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-)) \wedge \bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right).
\end{aligned}$$

Since α is arbitrary, we know that

$$\mu \leq \bigvee_{H\in\mathcal{H}(e,A)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-)) \wedge \bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right).$$

Since μ is arbitrary, it follows that

$$\gamma \wedge \mathcal{T}^*(A) \leq \bigvee_{H\in\mathcal{H}(e,A)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-)) \wedge \bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right).$$

Therefore, $\gamma \leq Creg(X,\mathcal{T})$. Since γ is arbitrary, we have $\bigwedge_{j\in J} Creg(X_j,\mathcal{T}_j) \leq Creg(X,\mathcal{T})$.

In addition, assume that $(X_j,\mathcal{T}_j)$ is stratified for all $j \in J$, then $(X_j,\mathcal{T}_j)$ is homeomorphic to a subspace $\left(\tilde{X}_j, \mathcal{T}|_{\tilde{X}_j}\right)$ of $(X,\mathcal{T})$, where $\tilde{X}_j$ is a subset of X parallel to X_j through $x = (x_j)_{j\in J}$. From Theorems 5.3 and 5.4, we have

$$\bigwedge_{j\in J} Creg(X_j,\mathcal{T}_j) = \bigwedge_{j\in J} Creg\left(\tilde{X}_j, \mathcal{T}|_{\tilde{X}_j}\right) \geq Creg(X,\mathcal{T}).$$

□

Theorem 5.6. *Let* $(X,\mathcal{T})$ *be an L-fuzzy topological space. Then* $T_{3\frac{1}{2}}(X,\mathcal{T}) \leq H(X,\mathcal{T})$.

Proof. By Theorems 4.2 and Theorem 5.2, we need to prove

$$\bigwedge_{e\in M(L^X)}\bigwedge_{e\nleq B\in L^X}\left\{\mathcal{T}^*(B)\to\bigvee_{H\in\mathcal{H}(e,B)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right)\right\}$$
$$\wedge\bigwedge_{a\nleq b}\bigvee_{a\nleq P\geq b}\mathcal{T}(P')$$
$$\leq\bigwedge_{a\nleq b}\bigvee_{G\in\mathcal{G}(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right).$$

Let $\lambda\prec\bigwedge_{e\in M(L^X)}\bigwedge_{e\nleq B\in L^X}\left\{\mathcal{T}^*(B)\to\bigvee_{H\in\mathcal{H}(e,B)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right)\right\}$ $\wedge\bigwedge_{a\nleq b}\bigvee_{a\nleq P\geq b}\mathcal{T}(P')$. Then for any $e\in M(L^X), B\in L^X$ with $e\nleq B$ we have

$$\lambda\wedge\mathcal{T}^*(B)\leq\bigvee_{H\in\mathcal{H}(e,B)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right).$$

For any $a,b\in M(L^X)$ with $a\nleq b$, there exists $P\in L^X$ such that $a\nleq P\geq b$ and $\lambda\prec\mathcal{T}^*(P)$. From $\mathcal{H}(a,P)\subseteq\mathcal{G}(a,b)$, we know that

$$\begin{aligned}\lambda &= \lambda\wedge\mathcal{T}^*(P)\\ &\leq \bigvee_{H\in\mathcal{H}(a,P)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right)\\ &\leq \bigvee_{G\in\mathcal{G}(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right).\end{aligned}$$

Hence

$$\lambda\leq\bigwedge_{a\nleq b}\bigvee_{G\in\mathcal{G}(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right).$$

Since λ is arbitrary, we know that

$$\bigwedge_{e\in M(L^X)}\bigwedge_{e\nleq B\in L^X}\left\{\mathcal{T}^*(B)\to\right.$$
$$\left.\bigvee_{H\in\mathcal{H}(e,B)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(H(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(H(t+))\right)\right\}\wedge\bigwedge_{a\nleq b}\bigvee_{a\nleq P\geq b}\mathcal{T}(P')$$
$$\leq\bigwedge_{a\nleq b}\bigvee_{G\in\mathcal{G}(a,b)}\left(\bigwedge_{t\in I_0}\mathcal{T}^*(G(t-))\wedge\bigwedge_{t\in I_1}\mathcal{T}(G(t+))\right).$$

Therefore, $T_{3\frac{1}{2}}(X,\mathcal{T})\leq H(X,\mathcal{T})$. □

By Theorems 3.7, 4.6 and 5.6, we can obtain the following theorem.

Theorem 5.7. *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space. Then*

$$T_{3\frac{1}{2}}(X, \mathcal{T}) \leq H(X, \mathcal{T}) \leq U(X, \mathcal{T}) \leq T_2(X, \mathcal{T}).$$

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