

ARITHMETIC AGGREGATION OPERATORS FOR INTERVAL-VALUED INTUITIONISTIC LINGUISTIC VARIABLES AND APPLICATION TO MULTI-ATTRIBUTE GROUP DECISION MAKING

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ABSTRACT. The intuitionistic linguistic set (ILS) is an extension of linguistic variable. To overcome the drawback of using single real number to represent membership degree and non-membership degree for ILS, the concept of interval-valued intuitionistic linguistic set (IVILS) is introduced through representing the membership degree and non-membership degree with intervals for ILS in this paper. The operation law, score function, accuracy function, and certainty function for interval-valued intuitionistic linguistic variables (IVILVs) are defined. Hereby a lexicographic method is proposed to rank the IVILVs. Then, three kinds of interval-valued intuitionistic linguistic arithmetic average operators are defined, including the interval-valued intuitionistic linguistic weighted arithmetic average (IVILWAA) operator, interval-valued intuitionistic linguistic ordered weighted arithmetic (IVILOWA) operator, and interval-valued intuitionistic linguistic hybrid arithmetic (IVILHA) operator, and their desirable properties are also discussed. Based on the IVILWAA and IVILHA operators, two methods are proposed for solving multi-attribute group decision making problems with IVILVs. Finally, an investment selection example is illustrated to demonstrate the applicability and validity of the methods proposed in this paper.

1. Introduction

Since fuzzy set was introduced by Zadeh [36], it has been receiving much attention from many fields, such as decision making, supply chain management science, investment selection and artificial intelligence. Fuzzy set has great superiority for representing the uncertainty and fuzziness. However, it is common that decision makers (DMs) or experts feel more comfortable providing their knowledge by using terms close to human beings cognitive model (e.g. when evaluating the comfort or design of a car, terms like “bad”, “poor”, “tolerable”, “average”, or “good” can be used). It is more suitable to provide their preferences by means of linguistic variables [37] than fuzzy sets or real numbers, enhancing the flexibility and reliability of the decision models.

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Multiple attribute group decision making (MAGDM) problems with linguistic information arise from a wide range of real-world situations. Many methods for MAGDM with linguistic variables and their extensions have been proposed [7, 9, 16, 17]. Currently, there are several extensions of linguistic variables, such as uncertain linguistic variables, 2-tuple linguistic information, duplex linguistic variables, intuitionistic linguistic sets, intuitionistic two-semantics, and intuitionistic uncertain linguistic variables, which are respectively reviewed as follows.

Because of time pressure, lack of knowledge, and limited expertise related with problem domain, Xu [26] introduced the uncertain linguistic variables and developed two uncertain linguistic aggregation operators called uncertain linguistic ordered weighted averaging (OWA). (ULOWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator. Based on the ULOWA and the ULHA operators, he proposed an approach to MAGDM with uncertain linguistic information. Xu [28] developed some induced ULOWA operators and applied to MAGDM with uncertain linguistic information. Xu [29] proposed an approach based on the uncertain linguistic ordered weighted geometric (LOWG) and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations. Xu [30] developed an interactive approach to MAGDM with multigranular uncertain linguistic information. Xu et al.[31] investigated a consensus based method for MAGDM under uncertain linguistic setting. It should be pointed out that there may be resulting in the information distortion and losing during the computation process of uncertain linguistic variables.

To avoid information distortion and losing which occur formerly in the linguistic information processing, Herrera and Martnez [4] and Herrera et al. [5] proposed 2-tuple linguistic representation model, which is composed of a linguistic term and a real number. The 2-tuple linguistic model has exact characteristic in linguistic information processing. The talent of 2-tuple linguistic model in handling linguistic information makes it popular with researchers quickly, and quite a lot of work has already been done in MAGDM problems. For example,Wei [23] developed three new aggregation operators: generalized 2-tuple weighted average (G-2TWA) operator, generalized 2-tuple ordered weighted average (G-2TOWA) operator and induced generalized 2-tuple ordered weighted average (IG-2TOWA) operator. Pei et al. [12] analyzed three kinds of weight information, i.e., belief degrees of linguistic evaluation values, weights of experts about indicators and strengths of indicators and proposed a weighted linguistic aggregation operator. Wei [22] proposed the extended 2-tuple weighted geometric (ET-WG) operator and the extended 2-tuple ordered weighted geometric (ET-OWG) operator and analyzed the properties of these operators. Then, a MAGDM method was presented based on the ET-WG and ET-OWG operators. Wei and Zhao [25] developed some dependent aggregation operators with 2-tuple linguistic information and applied to MAGDM. Wei [24] proposed the grey relational analysis method for 2-tuple linguistic MAGDM with incomplete weight information. Zeng et al. [38] developed a method based on OWA operator and distance measures for multiple attribute decision making with 2-Tuple linguistic information. Wan [14, 15] introduced some hybrid arithmetic and

geometric aggregation operators with 2-tuple linguistic information and applied to MAGDM.

Although uncertain linguistic variables and 2-tuple linguistic model have some dominance to deal with linguistic information, there still exist some disadvantages. When using linguistic labels, there exists an assumption that the membership degree of an element to a linguistic term equals 1. This cannot describe the confidence of the judgment of a DM. In practice, the membership function has been widely adopted to describe the relevance degree of a fuzzy label to an element and then reflect the DM's confidence. However, it is not easy to elicit the membership degree in using linguistic decision information. It can be hard to describe the relevance degree of a linguistic label to a real number and a more realistic technique is to sketch this degree with a linguistic variable as well. Therefore, Yang et al. [35] introduced the concept of a duplex linguistic (DL) set. An element in a DL set includes a pair of linguistic terms, one of which describes the evaluation with respect to a criterion and the other reflects the confidence about this evaluation. They proposed an outranking method for multi-criteria decision making with duplex linguistic information.

Besides the aforementioned three extensions of linguistic variables, recently Wang and Li [19] combined the intuitionistic fuzzy set (IFSs) [1] and the linguistic assessment set to propose some new extensions of linguistic variables, such as the intuitionistic linguistic set (ILS), the intuitionistic linguistic fuzzy number (ILFN), and the intuitionistic two-semantic. They argued that for a linguistic assessment value it is usually implied that the membership degree is one, and the non-membership degree and hesitation degree of DMs cannot be expressed. Wang and Li [19] defined the Hamming distance between two intuitionistic two-semantics and ranked the alternatives by calculating the comprehensive membership degree to the ideal solution for each alternative. Subsequently, Wang and Li [20] gave the operational laws, expected value, score function, and accuracy function of ILFNs. The intuitionistic linguistic weighted arithmetic average (ILWAA) operator and intuitionistic linguistic weighted geometric average (ILWGA) operator are developed. They proposed two methods based ILWAA and ILWGA operators respectively for solving multi-criteria decision making problems with ILFNs.

Since uncertain linguistic variables more easily express fuzzy information than linguistic variables, Liu and Jin [6] further proposed the concept of intuitionistic uncertain linguistic set (IULS) based on the ILS. They defined the operational laws, expected values and accuracy functions of intuitionistic uncertain linguistic variables (IULVs). Then, some intuitionistic uncertain linguistic geometric operators are developed, including the intuitionistic uncertain linguistic weighted geometric average (IULWGA) operator, intuitionistic uncertain linguistic ordered weighted geometric (IULOWG) operator, and intuitionistic uncertain linguistic hybrid geometric (IULHG) operator. Based on these operators, they proposed two approaches to solving MAGDM problems with IULVs. Compared with the IFSs, the interval-valued intuitionistic fuzzy sets (IVIFSs) [2] are characterized by a membership function and a non-membership function with values that are intervals rather than crisp numbers. IVIFSs may express more abundant and flexible information than

IFSs[18]. Therefore, based on the ILS and IVIFS, the concepts of interval-valued intuitionistic linguistic set (IVILS) and interval-valued intuitionistic linguistic variable (IVILV) are proposed. For example, an investment company wants to invest a sum of money to a car company, the growth index of the car company may be assessed using linguistic variable: “very good” with membership degree between interval $[0.3, 0.5]$, non-membership degree between interval $[0.1, 0.2]$ and hesitancy degree between interval $[0.3, 0.6]$. Namely, we can express this assessment as an IVILV $(s_5, [0.3, 0.5], [0.1, 0.2])$. This example shows that IVILV can not only express the linguistic evaluation value, but also reflect the information of membership and non-membership degrees with intervals, which greatly enhance the ability of linguistic variable for representing uncertain and fuzzy information.

Therefore, it is necessary to pay attention to theoretical analysis and real application of IVILSs and IVILVs. However, there exists some challenges and difficulties. On the one hand, since the IVILS is remarkably different to the IULS and ILS, the existing operational laws of IULSs and ILSs cannot suit for the IVILSs. Meanwhile, the ranking methods for IULSs and ILSs are not appropriate for the IVILSs. On the other hand, the aggregation operators of IULSs and ILSs cannot be directly applied to the IVILSs. Consequently, this paper firstly gives the operational laws of IVILVs and defines the score function, accuracy function, and certainty function of IVILVs. Thereby, a lexicographic method is developed to rank the IVILVs. As the aggregation operators of IVILVs are important tools of information fusion in MAGDM problems with interval-valued intuitionistic linguistic information, some arithmetic aggregation operators of IVILVs are introduced in this paper. These operators involve the interval-valued intuitionistic linguistic weighted arithmetic average (IVILWAA) operator, interval-valued intuitionistic linguistic ordered weighted arithmetic (IVILOWA) operator, and interval-valued intuitionistic linguistic hybrid arithmetic (IVILHA) operator. Their desirable properties are also discussed in detail. Based on the IVILWAA and IVILHA operators, two methods are proposed for solving MAGDM problems in which both the attribute weights and the expert weights take the form of real numbers, and the attribute values take the form of IVILVs.

The rest of this paper is organized as follows. In Section 2, we present the definition and operation laws of IVILSs. The score function, accuracy function, and certainty function for IVILV are also defined and thereby a lexicographic method is developed to rank the IVILVs. In Section 3, three kinds of interval-valued intuitionistic linguistic arithmetic aggregation operators are defined and their desirable properties are discussed in detail. Two decision methods for MAGDM problems with IVILVs are proposed in Section 4. An example of investment selection and comparison analysis are given in Section 5. Short conclusions are made in Section 6.

2. The Interval-valued Intuitionistic Linguistic Set

In this section, the definition and operation laws for the IVILS are introduced.

Let $S = \{s_0, s_1, s_2, \dots, s_{t-1}\}$ be a finite and totally ordered discrete linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable, t is an odd number.

To preserve all the given information, the discrete term set S is extended to a continuous term set $\bar{S} = \{s_l | s_0 \leq s_l \leq s_q, l \in [0, q]\}$, where $q \geq t$ and q is a sufficiently large positive integer, whose elements also meet all the characteristics of S . If $s_l \in S$, then we call s_l the original term, otherwise, we call s_l the virtual term. In general, the DM uses the original linguistic term to evaluate attributes and alternatives, and the virtual linguistic terms can only appear in calculation [26].

2.1. Definition of the Interval-valued Intuitionistic Linguistic Set.

Definition 2.1. [19] Let X be a finite universe of discourse. An intuitionistic linguistic set (ILS) in X is defined as follows:

$$A = \{ \langle x, [s_{\theta(x)}, \mu_A(x), \nu_A(x)] \rangle | x \in X \},$$

where $s_{\theta(x)} \in \bar{S}$, $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$, satisfying that $\mu_A(x) + \nu_A(x) \leq 1$. $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree, respectively, of the element x to the linguistic variable $s_{\theta(x)}$.

For each ILS $A = \{ \langle x, [s_{\theta(x)}, \mu_A(x), \nu_A(x)] \rangle | x \in X \}$, let $\Pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, which is called an intuitionistic fuzzy index of the element x to the linguistic variable $s_{\theta(x)}$.

For ILS $A = \{ \langle x, [s_{\theta(x)}, \mu_A(x), \nu_A(x)] \rangle | x \in X \}$, the triple $\langle s_{\theta(x)}, (\mu_A(x), \nu_A(x)) \rangle$ is called an intuitionistic linguistic fuzzy number (ILFN) [19].

Definition 2.2. Let X be a finite universal set. An interval-valued intuitionistic linguistic set (IVILS) in X is defined as follows:

$$A = \{ \langle x, [s_{\theta(x)}, \mu_A(x), \nu_A(x)] \rangle | x \in X \},$$

where $s_{\theta(x)} \in \bar{S}$, $\mu_A(x) \subseteq [0, 1]$ and $\nu_A(x) \subseteq [0, 1]$ denote respectively the membership degree interval and the non-membership degree interval of the element x to the linguistic variable $s_{\theta(x)}$, with the condition: $\sup \mu_A(x) + \sup \nu_A(x) \leq 1, \forall x \in X$.

For simplicity, denote $\mu_A(x) = [\underline{\mu}_A(x), \bar{\mu}_A(x)]$ and $\nu_A(x) = [\underline{\nu}_A(x), \bar{\nu}_A(x)]$. Then, IVILS A can be represented as $A = \{ \langle x, (s_{\theta(x)}, [\underline{\mu}_A(x), \bar{\mu}_A(x)], [\underline{\nu}_A(x), \bar{\nu}_A(x)]) \rangle | x \in X \}$.

For each IVILS A , let $\Pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = [1 - \bar{\mu}_A(x) - \bar{\nu}_A(x), 1 - \underline{\mu}_A(x) - \underline{\nu}_A(x)]$, which is an interval called an intuitionistic fuzzy index of the element x to the linguistic variable $s_{\theta(x)}$.

Remark 2.3. Since $\mu_A(x)$ represents the degree to which the element x belongs to the linguistic evaluation value $s_{\theta(x)}$ and $\nu_A(x)$ represents the degree to which the element x does not belong to the linguistic evaluation value $s_{\theta(x)}$, the intuitionistic fuzzy index $\Pi_A(x)$ may be viewed as the hesitancy degree to which the element x belongs or does not belong to the linguistic evaluation value $s_{\theta(x)}$. The intuitionistic fuzzy index $s_{\theta(x)}$ expresses the lack of knowledge on the membership of the element x to the linguistic evaluation value $s_{\theta(x)}$ other than the IVILS A . For an IFS $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle | x \in X \}$, its intuitionistic fuzzy index $\Pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ expresses the lack of knowledge on the membership of the element x to the IFS A .

IFS can only roughly describe the degree to which the element x belongs or does not belong to some fuzzy concept, whereas the IVILS can clearly represent the degree to which the element belongs or does not belong to the linguistic variable. Therefore, the IVILS integrates the advantages of IFS and linguistic term set.

Definition 2.4. Let $A = \{ \langle x, (s_{\theta(x)}, [\underline{\mu}_A(x), \bar{\mu}_A(x)], [\underline{\nu}_A(x), \bar{\nu}_A(x)]) \rangle | x \in X \}$ be an IVILS. The quintuple $(s_{\theta(x)}, [\underline{\mu}_A(x), \bar{\mu}_A(x)], [\underline{\nu}_A(x), \bar{\nu}_A(x)])$ is called an interval-valued intuitionistic linguistic variable (IVILV).

The IVILS A can also be viewed as the collection of the IVILVs. Thus, the IVILS A can also be represented as $A = \{ (s_{\theta(x)}, [\underline{\mu}_A(x), \bar{\mu}_A(x)], [\underline{\nu}_A(x), \bar{\nu}_A(x)]) | x \in X \}$.

2.2. Operation Laws of the Interval-valued Intuitionistic Linguistic Variables.

Definition 2.5. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2)$ be any two IVILVs and $\lambda \geq 0$ be any real number. Then, the operation laws for IVILVs are defined as follows:

- 1) $\tilde{a}_1 + \tilde{a}_2 = (s_{\theta(\tilde{a}_1) + \theta(\tilde{a}_2)}, [\underline{\mu}(\tilde{a}_1) + \underline{\mu}(\tilde{a}_2) - \underline{\mu}(\tilde{a}_1)\underline{\mu}(\tilde{a}_2), \bar{\mu}(\tilde{a}_1) + \bar{\mu}(\tilde{a}_2) - \bar{\mu}(\tilde{a}_1)\bar{\mu}(\tilde{a}_2)], [\underline{\nu}(\tilde{a}_1)\underline{\nu}(\tilde{a}_2), \bar{\nu}(\tilde{a}_1)\bar{\nu}(\tilde{a}_2)]);$
- 2) $\tilde{a}_1 \tilde{a}_2 = (s_{\theta(\tilde{a}_1)\theta(\tilde{a}_2)}, [\underline{\mu}(\tilde{a}_1)\underline{\mu}(\tilde{a}_2), \bar{\mu}(\tilde{a}_1)\bar{\mu}(\tilde{a}_2)], [\underline{\nu}(\tilde{a}_1) + \underline{\nu}(\tilde{a}_2) - \underline{\nu}(\tilde{a}_1)\underline{\nu}(\tilde{a}_2), \bar{\nu}(\tilde{a}_1) + \bar{\nu}(\tilde{a}_2) - \bar{\nu}(\tilde{a}_1)\bar{\nu}(\tilde{a}_2)]);$
- 3) $\lambda \tilde{a}_1 = (s_{\lambda\theta(\tilde{a}_1)}, [1 - (1 - \underline{\mu}(\tilde{a}_1))^\lambda, 1 - ((1 - \mu(a_1))^\lambda), [\nu(a_1)^\lambda, \bar{\nu}(\tilde{a}_1)^\lambda]);$
- 4) $\tilde{a}_1^\lambda = (s_{\theta(\tilde{a}_1)^\lambda}, [\underline{\mu}(\tilde{a}_1)^\lambda, \bar{\mu}(\tilde{a}_1)^\lambda], [\underline{\nu}(\tilde{a}_1)^\lambda, 1 - (1 - \bar{\nu}(\tilde{a}_1)^\lambda)]).$

Remark 2.6. If $\underline{\mu}(\tilde{a}_i) = \bar{\mu}(\tilde{a}_i) = 1, \underline{\nu}(\tilde{a}_i) = \bar{\nu}(\tilde{a}_i) = 0 (i = 1, 2)$, then the IVILVs $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2)$ are reduced to linguistic variables $s_{\theta(\tilde{a}_i)} (i = 1, 2)$, and thus the above operation laws are degenerated to the operation laws of linguistic variables [26]. If ignoring the linguistic parts of IVILVs, then the IVILVs $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ are reduced to IVIFSs $\tilde{a}_i = ([\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2)$, and thus the above operation laws are degenerated to the operation laws of IVIFSs [28]. These observations show the reasonability of Definition 2.5.

Theorem 2.7. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2, 3)$ be any three IVILVs and $\lambda, \lambda_1, \lambda_2, k, k_1, k_2 \geq 0$ be any real numbers. Then, the following properties hold.

- 1) *Multiplication commutativity:* $\tilde{a}_1 \tilde{a}_2 = \tilde{a}_2 \tilde{a}_1$;
- 2) *Addition commutativity:* $\tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1$;
- 3) *Scalar multiplication distribution:* $\lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda \tilde{a}_1 + \lambda \tilde{a}_2$;
- 4) $\lambda_1 \tilde{a}_1 + \lambda_2 \tilde{a}_1 = (\lambda_1 + \lambda_2) \tilde{a}_1$;
- 5) $(\tilde{a}_1 \tilde{a}_2)^k = \tilde{a}_1^k \tilde{a}_2^k$;
- 6) $\tilde{a}_1^{k_1 + k_2} = \tilde{a}_1^{k_1} \cdot \tilde{a}_1^{k_2}$;
- 7) *Puissance:* $(\tilde{a}_1^{k_1})^{k_2} = \tilde{a}_1^{k_1 k_2}$;
- 8) $\tilde{a}_1^{k_1} \tilde{a}_1^{k_2} = \tilde{a}_1^{k_1 + k_2}$;

- 9) *Multiplication associativity*: $\tilde{a}_1(\tilde{a}_2\tilde{a}_3) = (\tilde{a}_1\tilde{a}_2)\tilde{a}_3$;
 10) *Multiplication distribution*: $\tilde{a}_1(\tilde{a}_2 + \tilde{a}_3) = \tilde{a}_1\tilde{a}_2 + \tilde{a}_1\tilde{a}_3$.

2.3. Ranking Method of the Interval-valued Intuitionistic Linguistic Variables.

The IVILV \tilde{a} is composed of two parts: linguistic part and interval-valued intuitionistic part. To compare two IVILVs, it is necessary to take these two parts into account simultaneously. For IVILV $\tilde{a} = (s_{\theta(\tilde{a})}, [\underline{\mu}(\tilde{a}), \bar{\mu}(\tilde{a})], [\underline{\nu}(\tilde{a}), \bar{\nu}(\tilde{a})])$, the interval-valued intuitionistic part is an interval-valued intuitionistic value (IV-IFV) $([\underline{\mu}(\tilde{a}), \bar{\mu}(\tilde{a})], [\underline{\nu}(\tilde{a}), \bar{\nu}(\tilde{a})])$, we can define its score and accuracy functions as $\frac{1}{4}[2 + \underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) - \underline{\nu}(\tilde{a}) - \bar{\nu}(\tilde{a})]$ and $\frac{1}{2}[\underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) + \underline{\nu}(\tilde{a}) + \bar{\nu}(\tilde{a})]$, respectively; the linguistic part is a linguistic variable $s_{\theta(\tilde{a})}$. It is apparent that the larger the linguistic variable $s_{\theta(\tilde{a})}$ and IVIFV $([\underline{\mu}(\tilde{a}), \bar{\mu}(\tilde{a})], [\underline{\nu}(\tilde{a}), \bar{\nu}(\tilde{a})])$, the bigger the IVILV. Integrating the linguistic variable and IVIFV, we give the following score, accuracy and certainty functions to rank the IVIFVs.

Definition 2.8. Let $\tilde{a} = (s_{\theta(\tilde{a})}, [\underline{\mu}(\tilde{a}), \bar{\mu}(\tilde{a})], [\underline{\nu}(\tilde{a}), \bar{\nu}(\tilde{a})])$ be an IVILV. The score function, accuracy function and certainty function for IVILV are respectively defined as follows:

$$E(\tilde{a}) = \frac{1}{4}[2 + \underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) - \underline{\nu}(\tilde{a}) - \bar{\nu}(\tilde{a})]s_{\theta(\tilde{a})} = s_{\frac{1}{4}[2 + \underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) - \underline{\nu}(\tilde{a}) - \bar{\nu}(\tilde{a})]\theta(\tilde{a})} \quad (1)$$

$$H(\tilde{a}) = \frac{1}{2}[\underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) + \underline{\nu}(\tilde{a}) + \bar{\nu}(\tilde{a})]s_{\theta(\tilde{a})} = s_{\frac{1}{2}[\underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) + \underline{\nu}(\tilde{a}) + \bar{\nu}(\tilde{a})]\theta(\tilde{a})} \quad (2)$$

$$C(\tilde{a}) = \frac{1}{2}[\underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a})]s_{\theta(\tilde{a})} = s_{\frac{1}{2}[\underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a})]\theta(\tilde{a})} \quad (3)$$

Owing to that the core of IVILV \tilde{a} is the linguistic variable $s_{\theta(\tilde{a})}$, the score, accuracy and certainty functions for IVILV \tilde{a} are also defined as the linguistic variable in Definition 2.8. The score function and accuracy function for IVILV are similar to the mean and variance of random variables. They can be used to quantitatively characterize the values of IVILV. As to the certainty function, the certainty of IVILV \tilde{a} positively depends on the value of the membership degree interval $[\underline{\mu}(\tilde{a}), \bar{\mu}(\tilde{a})]$.

Obviously, the greater the score, accuracy and certainty functions, the bigger the corresponding IVILV.

Let $E(\tilde{a}_i)$, $H(\tilde{a}_i)$ and $C(\tilde{a}_i)$ be the score, accuracy and certainty functions for IVILVs \tilde{a}_i , respectively. Thereby, a lexicographic ranking method between two IVILVs can be summarized as follows:

- (1) If $E(\tilde{a}_1) > E(\tilde{a}_2)$, then $E(\tilde{a}_1)$ is bigger than $E(\tilde{a}_2)$, denoted by $\tilde{a}_1 > \tilde{a}_2$;
- (2) If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $H(\tilde{a}_1) > H(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;
- (3) If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $H(\tilde{a}_1) = H(\tilde{a}_2)$ and $C(\tilde{a}_1) > C(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;
- (4) If $E(\tilde{a}_1) = E(\tilde{a}_2)$, $H(\tilde{a}_1) = H(\tilde{a}_2)$ and $C(\tilde{a}_1) = C(\tilde{a}_2)$, then \tilde{a}_1 and \tilde{a}_2 represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$.

Example 2.9. (1) Let $\tilde{a}_1 = (s_1, [0.2, 0.3], [0.1, 0.2])$ and $\tilde{a}_2 = (s_1, [0.1, 0.2], [0.3, 0.4])$. We have $E(\tilde{a}_1) = s_{0.55} > E(\tilde{a}_2) = s_{0.4}$, so $\tilde{a}_1 > \tilde{a}_2$.

(2) Let $\tilde{a}_1 = (s_1, [0.2, 0.3], [0.3, 0.6])$ and $\tilde{a}_2 = (s_1, [0.1, 0.2], [0.3, 0.4])$. We have $E(\tilde{a}_1) = E(\tilde{a}_2) = s_{0.4}$, $H(\tilde{a}_1) = s_{0.7} > H(\tilde{a}_2) = s_{0.5}$, so $\tilde{a}_1 > \tilde{a}_2$.

(3) Let $\tilde{a}_1 = (s_1, [0.2, 0.3], [0.2, 0.3])$ and $\tilde{a}_2 = (s_1, [0.1, 0.4], [0.1, 0.4])$. We have $E(\tilde{a}_1) = E(\tilde{a}_2) = s_{0.5}$, $H(\tilde{a}_1) = H(\tilde{a}_2) = s_{0.5}$, and $C(\tilde{a}_1) = C(\tilde{a}_2) = s_{0.25}$, so $\tilde{a}_1 = \tilde{a}_2$.

3. Some Arithmetic Aggregation Operators of the Interval-valued Intuitionistic Linguistic Variables

This section is devoted to developing some arithmetic aggregation operators of the IVILVs.

3.1. The Interval-valued Intuitionistic Linguistic Weighted Arithmetic Average Operator.

Definition 3.1. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. Let IVILWAA: $\Omega^n \rightarrow \Omega$, if

$$\text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{j=1}^n w_j \tilde{a}_j,$$

where Ω is the set of all IVILVs, $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of \tilde{a}_i ($i = 1, 2, \dots, n$), satisfying that $0 \leq w_i \leq 1$ ($i = 1, 2, \dots, n$) and $\sum_{i=1}^n w_i = 1$, then the function IVILWAA is called the interval-valued intuitionistic linguistic weighted arithmetic average operator.

Theorem 3.2. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. Then, the aggregated value by using IVILWAA operator is also an IVILV, and

$$\begin{aligned} \text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = & (s_{\sum_{j=1}^n w_j \theta(\tilde{a}_j)}, [1 - \prod_{j=1}^n (1 - \underline{\mu}(\tilde{a}_j))^{w_j}, \\ & 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}_j))^{w_j}], [\prod_{j=1}^n \underline{\nu}(\tilde{a}_j)^{w_j}, \prod_{j=1}^n \bar{\nu}(\tilde{a}_j)^{w_j}]) \end{aligned} \quad (4)$$

Theorem 3.2 can be easily proven by mathematical induction on n according to Definition 2.5.

The IVILWAA operator has some desirable properties, such as idempotency, boundedness, and so on.

Theorem 3.3. (Idempotency). Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. If all IVILVs \tilde{a}_i ($i = 1, 2, \dots, n$) are equal, i.e., $\tilde{a}_1 = \tilde{a}_2 = \dots = \tilde{a}_n = \tilde{a}$, then $\text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \tilde{a}$.

Proof. Since $0 \leq w_i \leq 1 (i = 1, 2, \dots, n)$ and $\sum_{i=1}^n w_i = 1$, by Theorem 2.7 we have

$$\text{IVILWAA}_{\mathbf{w}}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{j=1}^n w_j \tilde{a}_j = \sum_{j=1}^n w_j \tilde{a} = \tilde{a} \sum_{j=1}^n w_j = \tilde{a},$$

which completes the proof of Theorem 3.3. \square

Theorem 3.4. (*Boundedness*). Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2, \dots, n)$ be a collection of the IVILVs and let

$$\tilde{a}^+ = (\max\{s_{\theta(\tilde{a}_i)}\}, [\max\{\underline{\mu}(\tilde{a}_i)\}, \max\{\bar{\mu}(\tilde{a}_i)\}], [\min\{\underline{\nu}(\tilde{a}_i)\}, \min\{\bar{\nu}(\tilde{a}_i)\}])$$

$$\tilde{a}^- = (\min\{s_{\theta(\tilde{a}_i)}\}, [\min\{\underline{\mu}(\tilde{a}_i)\}, \min\{\bar{\mu}(\tilde{a}_i)\}], [\max\{\underline{\nu}(\tilde{a}_i)\}, \max\{\bar{\nu}(\tilde{a}_i)\}])$$

Then,

$$\tilde{a}^- \leq \text{IVILWAA}_{\mathbf{w}}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \leq \tilde{a}^+$$

Proof. Since $0 \leq w_i \leq 1 (i = 1, 2, \dots, n)$ and $\sum_{i=1}^n w_i = 1$, by Theorems 2.7. and 3.2. we have

$$\begin{aligned} \min\{s_{\theta(\tilde{a}_j)}\} &= s_{\min\{\theta(\tilde{a}_j)\}} = s_{\sum_{j=1}^n w_j \min\{\theta(\tilde{a}_j)\}} \leq s_{\sum_{j=1}^n w_j \theta(\tilde{a}_j)}, \\ s_{\sum_{j=1}^n w_j \theta(\tilde{a}_j)} &\leq s_{\sum_{j=1}^n w_j \max\{\theta(\tilde{a}_j)\}} = s_{\max\{\theta(\tilde{a}_j)\} \sum_{j=1}^n w_j} = s_{\max\{\theta(\tilde{a}_j)\}} = \max\{s_{\theta(\tilde{a}_j)}\}, \\ 1 - \prod_{j=1}^n (1 - \underline{\mu}(\tilde{a}_j))^{w_j} &\leq 1 - \prod_{j=1}^n (1 - \max\{\underline{\mu}(\tilde{a}_j)\})^{w_j} \\ &\leq 1 - (1 - \max\{\underline{\mu}(\tilde{a}_j)\})^{\sum_{j=1}^n w_j} = \max\{\underline{\mu}(\tilde{a}_j)\}, \\ \min\{\bar{\mu}(\tilde{a}_j)\} &= 1 - (1 - \min\{\bar{\mu}(\tilde{a}_j)\})^{\sum_{j=1}^n w_j} \\ &= 1 - \prod_{j=1}^n (1 - \min\{\bar{\mu}(\tilde{a}_j)\})^{w_j} \leq 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}_j))^{w_j}, \\ 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}_j))^{w_j} &\leq 1 - \prod_{j=1}^n (1 - \max\{\bar{\mu}(\tilde{a}_j)\})^{w_j} \\ &\leq 1 - (1 - \max\{\bar{\mu}(\tilde{a}_j)\})^{\sum_{j=1}^n w_j} = \max\{\bar{\mu}(\tilde{a}_j)\}, \\ \min\{\underline{\nu}(\tilde{a}_j)\} &= \min\{\underline{\nu}(\tilde{a}_j)\}^{\sum_{j=1}^n w_j} = \prod_{j=1}^n \min\{\underline{\nu}(\tilde{a}_j)\}^{w_j} \leq \prod_{j=1}^n \underline{\nu}(\tilde{a}_j)^{w_j}, \\ \prod_{j=1}^n \underline{\nu}(\tilde{a}_j)^{w_j} &\leq \prod_{j=1}^n \max\{\underline{\nu}(\tilde{a}_j)\}^{w_j} = \max\{\underline{\nu}(\tilde{a}_j)\}^{\sum_{j=1}^n w_j} = \max\{\underline{\nu}(\tilde{a}_j)\}, \end{aligned}$$

$$\min\{\bar{\nu}(\tilde{a}_j)\} = \min\{\bar{\nu}(\tilde{a}_j)\}^{\sum_{j=1}^n w_j} = \prod_{j=1}^n \min\{\bar{\nu}(\tilde{a}_j)\}^{w_j} \leq \prod_{j=1}^n \bar{\nu}(\tilde{a}_j)^{w_j},$$

$$\prod_{j=1}^n \bar{\nu}(\tilde{a}_j)^{w_j} \leq \prod_{j=1}^n \max\{\bar{\nu}(\tilde{a}_j)\}^{w_j} = \max\{\bar{\nu}(\tilde{a}_j)\}^{\sum_{j=1}^n w_j} = \max\{\bar{\nu}(\tilde{a}_j)\},$$

Let $IVILWAA_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \tilde{a} = (s_{\theta(\tilde{a})}, [\underline{\mu}(\tilde{a}), \bar{\mu}(\tilde{a})], [\underline{\nu}(\tilde{a}), \bar{\nu}(\tilde{a})])$. Then,

$$\min\{s_{\theta(\tilde{a}_j)}\} \leq s_{\theta(\tilde{a})} \leq \max\{s_{\theta(\tilde{a}_j)}\},$$

$$\min\{\bar{\mu}(\tilde{a}_j)\} \leq \bar{\mu}(\tilde{a}) \leq \max\{\bar{\mu}(\tilde{a}_j)\}, \min\{\underline{\mu}(\tilde{a}_j)\} \leq \underline{\mu}(\tilde{a}) \leq \max\{\underline{\mu}(\tilde{a}_j)\},$$

$$\min\{\underline{\nu}(\tilde{a}_j)\} \leq \underline{\nu}(\tilde{a}) \leq \max\{\underline{\nu}(\tilde{a}_j)\}, \min\{\bar{\nu}(\tilde{a}_j)\} \leq \bar{\nu}(\tilde{a}) \leq \max\{\bar{\nu}(\tilde{a}_j)\}$$

Thus,

$$\frac{1}{4}[2 + \underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) - \underline{\nu}(\tilde{a}) - \bar{\nu}(\tilde{a})]s_{\theta(\tilde{a})} \leq \frac{1}{4}[2 + \max\{\underline{\mu}(\tilde{a})\} + \max\{\bar{\mu}(\tilde{a})\}$$

$$- \min\{\underline{\nu}(\tilde{a})\} - \min\{\bar{\nu}(\tilde{a})\}] \max\{s_{\theta(\tilde{a}_j)}\},$$

$$\frac{1}{4}[2 + \underline{\mu}(\tilde{a}) + \bar{\mu}(\tilde{a}) - \underline{\nu}(\tilde{a}) - \bar{\nu}(\tilde{a})]s_{\theta(\tilde{a})} \geq \frac{1}{4}[2 + \min\{\underline{\mu}(\tilde{a})\} + \min\{\bar{\mu}(\tilde{a})\} - \max\{\underline{\nu}(\tilde{a})\}$$

$$- \max\{\bar{\nu}(\tilde{a})\}] \min\{s_{\theta(\tilde{a}_j)}\},$$

It yields from Theorem 3.2 and Definition 2.8 that

$$E(\tilde{a}^-) \leq E(\tilde{a}) \leq E(\tilde{a}^+),$$

i.e.,

$$\tilde{a}^- \leq IVILWAA_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \leq \tilde{a}^+.$$

Hence, the proof of Theorem 3.4 is completed. \square

Theorem 3.5. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ and $\tilde{b}_i = (s_{\theta(\tilde{b}_i)}, [\underline{\mu}(\tilde{b}_i), \bar{\mu}(\tilde{b}_i)], [\underline{\nu}(\tilde{b}_i), \bar{\nu}(\tilde{b}_i)])$ be two collections of the IVILVs, $\tilde{b} = (s_{\theta(\tilde{b})}, [\underline{\mu}(\tilde{b}), \bar{\mu}(\tilde{b})], [\underline{\nu}(\tilde{b}), \bar{\nu}(\tilde{b})])$ be another IVILV. If $\lambda \geq 0$, then,

$$IVILWAA_w(\lambda\tilde{a}_1, \lambda\tilde{a}_2, \dots, \lambda\tilde{a}_n) = \lambda \cdot IVILWAA_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n),$$

$$IVILWAA_w(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \dots, \tilde{a}_n\tilde{b}) = IVILWAA_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \cdot \tilde{b},$$

$$IVILWAA_w(\lambda\tilde{a}_1\tilde{b}, \lambda\tilde{a}_2\tilde{b}, \dots, \lambda\tilde{a}_n\tilde{b}) = \lambda \cdot IVILWAA_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \cdot \tilde{b},$$

$$IVILWAA_w(\lambda(\tilde{a}_1 + \tilde{b}_1), \lambda(\tilde{a}_2 + \tilde{b}_2), \dots, \lambda(\tilde{a}_n + \tilde{b}_n)) = \lambda \cdot IVILWAA_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$$

$$+ \lambda \cdot IVILWAA_w(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n).$$

Proof. By Theorem 2.7 and Definitions 2.5 and 3.1, we have

$$\begin{aligned}
& \text{IVILWAA}_w(\lambda\tilde{a}_1, \lambda\tilde{a}_2, \dots, \lambda\tilde{a}_n) = \sum_{j=1}^n w_j \lambda \tilde{a}_j \\
& = \lambda \sum_{j=1}^n w_j \tilde{a}_j = \lambda \cdot \text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n), \\
& \text{IVILWAA}_w(\tilde{a}_1 \tilde{b}, \tilde{a}_2 \tilde{b}, \dots, \tilde{a}_n \tilde{b}) = \sum_{j=1}^n w_j \tilde{a}_j \tilde{b} \\
& = \tilde{b} \sum_{j=1}^n w_j \tilde{a}_j = \text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \cdot \tilde{b}, \\
& \text{IVILWAA}_w(\lambda\tilde{a}_1 \tilde{b}, \lambda\tilde{a}_2 \tilde{b}, \dots, \lambda\tilde{a}_n \tilde{b}) = \sum_{j=1}^n w_j \lambda \tilde{a}_j \tilde{b} \\
& = \lambda \tilde{b} \sum_{j=1}^n w_j \tilde{a}_j = \lambda \cdot \text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \cdot \tilde{b}, \\
& \text{IVILWAA}_w(\lambda(\tilde{a}_1 + \tilde{b}_1), \lambda(\tilde{a}_2 + \tilde{b}_2), \dots, \lambda(\tilde{a}_n + \tilde{b}_n)) \\
& = \sum_{j=1}^n w_j \lambda(\tilde{a}_j + \tilde{b}_j) = \lambda \sum_{j=1}^n w_j \tilde{a}_j + \lambda \sum_{j=1}^n w_j \tilde{b}_j \\
& = \lambda \cdot \text{IVILWAA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \lambda \cdot \text{IVILWAA}_w(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n).
\end{aligned}$$

Thus, the proof of Theorem 3.5 is completed. \square

3.2. The Interval-valued Intuitionistic Linguistic Ordered Weighted Arithmetic Average Operator.

Definition 3.6. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. Let IVILOWA: $\Omega^n \rightarrow \Omega$, if

$$\text{IVILOWA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{j=1}^n w_j \tilde{a}_{\sigma(j)},$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector with IVILOWA, satisfying that $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$, $\sigma(j)$ is the index of the j th largest of the IVILV \tilde{a}_i ($i = 1, 2, \dots, n$), then the function IVILOWA is called the interval-valued intuitionistic linguistic ordered weighted arithmetic operator.

It should be noted that the IVILOWA operator is developed based on the idea of the OWA operator [32]. The main characterization of the OWA operator is its reordering step. The associated weight ω_j is decided only by the j th position in the aggregation process. Therefore, the associated weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ can also be called the position-weighted vector.

The position-weighted vector w can be determined according to actual needs. Moreover, there are also many methods to obtain the position-weighted vector, such as fuzzy linguistic quantifiers [32] orness measure and dispersion measure [11], exponential smoothing [3], normal distribution method [27], and combination number method [21].

Theorem 3.7. *Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. Then, the aggregated value by using IVILOWA operator is also an IVILV, and*

$$\begin{aligned} \text{IVILOWA}_\omega(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = & (s_{\sum_{j=1}^n \omega_j \theta(\tilde{a}_{\sigma(j)})}, [1 - \prod_{j=1}^n (1 - \underline{\mu}(\tilde{a}_{\sigma(j)}))^{\omega_j}, \\ & 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}_{\sigma(j)}))^{\omega_j}], [\prod_{j=1}^n \underline{\nu}(\tilde{a}_{\sigma(j)})^{\omega_j}, \prod_{j=1}^n \bar{\nu}(\tilde{a}_{\sigma(j)})^{\omega_j}]). \end{aligned} \quad (5)$$

Theorem 3.7 can be easily proven by using mathematical induction on n according to Definition 2.5.

Theorem 3.8. *Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. Then, the following properties hold:*

- 1) *If $\omega = (1, 0, \dots, 0)^T$, then $\text{IVILOWA}_\omega(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \max\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$;*
- 2) *If $\omega = (0, 0, \dots, 1)^T$, then $\text{IVILOWA}_\omega(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \min\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$;*
- 3) *If $\omega_j = 1$ and $\omega_i = 0$ ($i \neq j$), then $\text{IVILOWA}_\omega(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \tilde{a}_{\sigma(j)}$, $\tilde{a}_{\sigma(j)}$ is the j th largest of the IVILVs \tilde{a}_i ($i = 1, 2, \dots, n$).*

Similar to the IVILVWA operator, the IVILOWA operator has some desirable properties, such as idempotency, boundedness, commutativity, and so on.

Theorem 3.9. *(Idempotency). Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. If all IVILVs \tilde{a}_i ($i = 1, 2, \dots, n$) are equal, i.e., $\tilde{a}_1 = \tilde{a}_2 = \dots = \tilde{a}_n = \tilde{a}$, then*

$$\text{IVILOWA}_w(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \tilde{a}.$$

Theorem 3.10. *(Boundedness). Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of the IVILVs. Then,*

$$\tilde{a}^- \leq \text{IVILOWA}_\omega(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \leq \tilde{a}^+,$$

where \tilde{a}^- and \tilde{a}^+ are defined in Theorem 3.4.

Theorem 3.11. *(Commutativity). Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)])$ ($i = 1, 2, \dots, n$) be a collection of IVILVs. If $(\check{a}_1, \check{a}_2, \dots, \check{a}_n)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$, then*

$$\text{IVILOWA}_\omega(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \text{IVILOWA}_\omega(\check{a}_1, \check{a}_2, \dots, \check{a}_n).$$

The proofs of Theorems 3.9 and 3.10 are similar to those of Theorems 3.3 and 3.4, respectively. The proof of Theorem 3.11 is obvious.

3.3. The Interval-valued Intuitionistic Linguistic Hybrid Arithmetic Average Operator.

It can be seen from Definitions 3.1 and 3.6 that the IVILWAA operator weights the IVILV arguments, while the IVILOWA operator weights the ordered positions of the IVILV arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the IVILWAA and IVILOWA operators. However, the two operators consider only one of them. To overcome this drawback, based on Definitions 3.1 and 3.6, a new interval-valued intuitionistic linguistic hybrid arithmetic (IVILHA) operator is developed in the following. The IVILHA operator uses the hybrid aggregation in order to integrate the OWA and the WA in the same formulation. Certainly, there are other methods that could be used for integrating the OWA with the WA including the weighted OWA (WOWA) [13], the importance OWA [33], the immediate weights [9, 34] and the OWA weighted average [8].

Definition 3.12. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2, \dots, n)$ be a collection of the IVILVs. Let IVILHA: $\Omega^n \rightarrow \Omega$, if

$$IVILHA_{\omega, w}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{j=1}^n \omega_j \tilde{a}'_{\sigma(j)},$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector with IVILVHA, satisfying that $0 \leq \omega_i \leq 1$ and $\sum_{i=1}^n \omega_i = 1$, $\tilde{a}'_{\sigma(k)}$ is the k -th largest of the weighted IVILVs $\tilde{a}'_i (i = 1, 2, \dots, n)$, $\tilde{a}'_i = nw_i \tilde{a}_i$, $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $\tilde{a}_i (i = 1, 2, \dots, n)$, satisfying that $0 \leq w_i \leq 1$ and $\sum_{i=1}^n w_i = 1$, n is the balancing coefficient, then the function IVILHA is called the interval-valued intuitionistic linguistic hybrid arithmetic operator.

Especially, if $w = (1/n, 1/n, \dots, 1/n)^T$, then the IVILHA operator reduces to the IVILOWA operator; if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the IVILHA operator reduces to the IVILWAA operator.

Theorem 3.13. Let $\tilde{a}_i = (s_{\theta(\tilde{a}_i)}, [\underline{\mu}(\tilde{a}_i), \bar{\mu}(\tilde{a}_i)], [\underline{\nu}(\tilde{a}_i), \bar{\nu}(\tilde{a}_i)]) (i = 1, 2, \dots, n)$ be a collection of the IVILVs. If $\tilde{a}'_{\sigma(k)} = (s_{\theta(\tilde{a}'_{\sigma(k)})}, [\underline{\mu}(\tilde{a}'_{\sigma(k)}), \bar{\mu}(\tilde{a}'_{\sigma(k)})], [\underline{\nu}(\tilde{a}'_{\sigma(k)}), \bar{\nu}(\tilde{a}'_{\sigma(k)})])$ is the k -th largest of the weighted IVILVs $\tilde{a}'_i (i = 1, 2, \dots, n)$, then the aggregated value by using IVILHA operator is also an IVILV, and

$$IVILHA_{w, \omega}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = (s_{\sum_{j=1}^n w_j \theta(\tilde{a}'_{\sigma(j)})}, [1 - \prod_{j=1}^n (1 - \underline{\mu}(\tilde{a}'_{\sigma(j)}))^{w_j}, 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}'_{\sigma(j)}))^{w_j}], [\prod_{j=1}^n \underline{\nu}(\tilde{a}'_{\sigma(j)})^{w_j}, \prod_{j=1}^n \bar{\nu}(\tilde{a}'_{\sigma(j)})^{w_j}]) \quad (6)$$

Theorem 3.13 can be easily proven by using mathematical induction on n according to Definition 2.5.

The above three kinds of arithmetic average operators of IVILVs are very useful tools to aggregate the interval-valued intuitionistic linguistic information. It is often that the different DMs and different attributes play different roles in real-life MAGDM problems. The IVILWAA operator can be used to integrate the different attribute values of each alternative to obtain the individual overall attribute value of each alternative. Then, the collective overall attribute value of each alternative can be obtained by using the IVILHA operator. Hence, applying these operators to MAGDM field is of great importance for real applications.

4. Application to MAGDM Problems with IVILS Information

In this section, by the IVILWAA and IVILHA operators, two decision making methods are proposed to solve the MAGDM problems with IVILVs information.

4.1. MAGDM Problems Using IVILVs.

MAGDM problems are widespread in real-life decision situations. The motivation of a fuzzy MAGDM problem is to find the best compromise solution from all feasible alternatives assessed on multiple attributes. For MAGDM problems, denote an alternative set by $A = \{A_1, A_2, \dots, A_m\}$ and an attribute set by $C = \{c_1, c_2, \dots, c_n\}$. The weight vector of attributes is $w = (w_1, w_2, \dots, w_n)^T$, satisfying that $0 \leq w_j \leq 1 (j = 1, 2, \dots, n)$ and $\sum_{j=1}^n w_j = 1$. Assume that there are p DMs participating in decision making, denote the set of DMs by $E = \{e_1, e_2, \dots, e_p\}$. The weight vector of DMs is $\mathbf{V} = (v_1, v_2, \dots, v_p)^T$ satisfying that $0 \leq v_k \leq 1 (k = 1, 2, \dots, p)$ and $\sum_{k=1}^p v_k = 1$. The rating of an alternative A_i on an attribute c_j given by the DM e_k is an IVILV $\tilde{a}_{ij}^k = (s_{\theta(\tilde{a}_{ij}^k)}, [\underline{\mu}(\tilde{a}_{ij}^k), \bar{\mu}(\tilde{a}_{ij}^k)], [\underline{\nu}(\tilde{a}_{ij}^k), \bar{\nu}(\tilde{a}_{ij}^k)])$, where the linguistic variable $s_{\theta(\tilde{a}_{ij}^k)} \in S$, $[\underline{\mu}(\tilde{a}_{ij}^k), \bar{\mu}(\tilde{a}_{ij}^k)]$ and $[\underline{\nu}(\tilde{a}_{ij}^k), \bar{\nu}(\tilde{a}_{ij}^k)]$ represent respectively the membership degree interval and non-membership degree interval for the attribute values of A_i on c_j to the linguistic variable $s_{\theta(\tilde{a}_{ij}^k)}$, satisfying $0 \leq \underline{\mu}(\tilde{a}_{ij}^k) \leq \bar{\mu}(\tilde{a}_{ij}^k) \leq 1$, $0 \leq \underline{\nu}(\tilde{a}_{ij}^k) \leq \bar{\nu}(\tilde{a}_{ij}^k) \leq 1$ and $\bar{\mu}(\tilde{a}_{ij}^k) + \bar{\nu}(\tilde{a}_{ij}^k) \leq 1$.

Hence, a MAGDM problem can be concisely expressed in matrix format as $\tilde{A}^k = (\tilde{a}_{ij}^k)_{m \times n} (k = 1, 2, \dots, P)$, which are referred to as IVILV decision matrices usually used to represent the MAGDM problem.

4.2. Two Methods for MAGDM with IVILVs Information.

In the sequel, we will apply the IVILWAA and IVILHA operators to solve MAGDM problems with IVILV information. Two methods are developed as follows.

(1) Method I: Aggregating the Attribute Values First

In this method, the attribute values for each DM with respect to each alternative are aggregated into an individual overall attribute value, and then the individual overall attribute values for different DMs are aggregated into a collective one for each alternative. This method involves the following steps:

Step 1: Utilized the IVILWAA operator (i.e., equation (4)), the individual overall attribute value of alternative can be obtained as follows:

$$\begin{aligned} \tilde{a}_i^k = \text{IVILWAA}_w(\tilde{a}_{i1}^k, \tilde{a}_{i2}^k, \dots, \tilde{a}_{in}^k) = & (s \sum_{j=1}^n w_j \theta(\tilde{a}_{ij}^k), [1 - \prod_{j=1}^n (1 - \underline{\mu}(\tilde{a}_{ij}^k))^{w_j}, \\ & 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}_{ij}^k))^{w_j}], [\prod_{j=1}^n \underline{\nu}(\tilde{a}_{ij}^k)^{w_j}, \prod_{j=1}^n \bar{\nu}(\tilde{a}_{ij}^k)^{w_j}]) \end{aligned} \quad (7)$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of attributes.

Step 2: Combining the weight vector of DMs $\mathbf{V} = (v_1, v_2, \dots, v_p)^T$ with the IVILHA operator, we can calculate the collective overall attribute \tilde{a}_i of alternative A_i as follows:

$$\begin{aligned} \tilde{a}_i = \text{IVILHA}_{\omega, V}(\tilde{a}_i^1, \tilde{a}_i^2, \dots, \tilde{a}_i^p) = & (s \sum_{k=1}^p \omega_k \theta(\tilde{a}_i^{\sigma(k)}), [1 - \prod_{k=1}^p (1 - \underline{\mu}(\tilde{a}_i^{\sigma(k)}))^{\omega_k}, \\ & 1 - \prod_{k=1}^p (1 - \bar{\mu}(\tilde{a}_i^{\sigma(k)}))^{\omega_k}], [\prod_{k=1}^p \underline{\nu}(\tilde{a}_i^{\sigma(k)})^{\omega_k}, \prod_{k=1}^p \bar{\nu}(\tilde{a}_i^{\sigma(k)})^{\omega_k}]) \end{aligned} \quad (8)$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_p)^T$ is the associated weight vector with IVILHA, $\tilde{a}_i^{\sigma(k)}$ is the k -th largest of the weighted IVILVs $\tilde{a}_i^k = v_k \tilde{a}_i^k$ ($k = 1, 2, \dots, p$).

Step 3: Compute the score function $E(\tilde{a}_i)$, accuracy function $H(\tilde{a}_i)$, and certainty function $C(\tilde{a}_i)$ for IVILVs \tilde{a}_i using equations. (1)-(3);

Step 4: The ranking orders of alternatives are generated by $E(\tilde{a}_i)$, $H(\tilde{a}_i)$, and $C(\tilde{a}_i)$ ($i = 1, 2, \dots, m$) according to the ranking method developed in Subsection 2.3.

(2) Method II: First Aggregating the Information from Different DMs

In this method, we first aggregate the decision matrixes from different DMs into the collective decision matrix and then aggregate the attribute values of each alternative into a comprehensive attribute value. The method involves the following steps:

Step 1: Utilized the IVILWAA operator (i.e., equation (4)), the decision matrix $\tilde{A}^k = (\tilde{a}_{ij}^k)_{m \times n}$ ($k = 1, 2, \dots, P$) can be aggregated into a collective decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$, where \tilde{a}_{ij} is calculated as follows:

$$\begin{aligned} \tilde{a}_{ij} = \text{IVILVWAA}_V(\tilde{a}_{ij}^1, \tilde{a}_{ij}^2, \dots, \tilde{a}_{ij}^p) = & (s \sum_{k=1}^p v_k \theta(\tilde{a}_{ij}^k), [1 - \prod_{k=1}^p (1 - \underline{\mu}(\tilde{a}_{ij}^k))^{v_k}, \\ & 1 - \prod_{k=1}^p (1 - \bar{\mu}(\tilde{a}_{ij}^k))^{v_k}], [\prod_{k=1}^p \underline{\nu}(\tilde{a}_{ij}^k)^{v_k}, \prod_{k=1}^p \bar{\nu}(\tilde{a}_{ij}^k)^{v_k}]), \end{aligned} \quad (9)$$

where $\mathbf{v} = (v_1, v_2, \dots, v_p)^T$ is the weight vector of DMs.

Step 2: Combining the weight vector of attributes $w = (w_1, w_2, \dots, w_n)^T$ with the IVILHA operator, we can calculate the collective overall attribute \tilde{a}_i of alternative A_i as follows:

$$\begin{aligned} \tilde{a}_i = \text{IVILVHA}_{\omega, w}(\tilde{a}_{i1}, \tilde{a}_{i2}, \dots, \tilde{a}_{in}) = & (s \sum_{j=1}^n \omega_j \theta(\tilde{a}'_{i\sigma(j)}), [1 - \prod_{j=1}^n (1 - \underline{\mu}(\tilde{a}'_{i\sigma(j)}))^{\omega_j}, \\ & 1 - \prod_{j=1}^n (1 - \bar{\mu}(\tilde{a}'_{i\sigma(j)}))^{\omega_j}], [\prod_{j=1}^n \underline{\nu}(\tilde{a}'_{i\sigma(j)})^{\omega_j}, \prod_{j=1}^n \bar{\nu}(\tilde{a}'_{i\sigma(j)})^{\omega_j}]), \end{aligned} \quad (10)$$

	A_1	A_2	A_3	A_4
c_1	$(s_5, [0.1, 0.2], [0.6, 0.7])$	$(s_5, [0.2, 0.4], [0.4, 0.6])$	$(s_6, [0.0, 0.2], [0.5, 0.7])$	$(s_4, [0.3, 0.5], [0.2, 0.4])$
c_2	$(s_3, [0.2, 0.4], [0.4, 0.6])$	$(s_5, [0.2, 0.4], [0.3, 0.5])$	$(s_4, [0.0, 0.2], [0.5, 0.7])$	$(s_4, [0.0, 0.2], [0.6, 0.8])$
c_3	$(s_6, [0.3, 0.5], [0.3, 0.5])$	$(s_4, [0.0, 0.1], [0.6, 0.8])$	$(s_5, [0.1, 0.3], [0.5, 0.7])$	$(s_5, [0.0, 0.2], [0.4, 0.6])$
c_4	$(s_4, [0.0, 0.2], [0.4, 0.6])$	$(s_4, [0.3, 0.5], [0.3, 0.5])$	$(s_4, [0.0, 0.2], [0.5, 0.7])$	$(s_3, [0.1, 0.3], [0.4, 0.6])$

TABLE 1. The IVILV Decision Matrix Given by DM e_1

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the associated weight vector with IVILHA, $\tilde{a}'_{i\sigma(j)}$ is the j -th largest of the weighted IVILVs $\tilde{a}'_{ij} = w_j \tilde{a}_{ij} (j = 1, 2, \dots, n)$.

Step 3-Step 4: These steps are the same as the steps in Method I.

Remark 4.1. The difference between Method I and Method II is the aggregation order. Method I emphasizes the importance of individual DMs while Method II emphasizes the importance of decision group. In real application, the decision group should select one of methods after discussion and negotiation.

5. An Investment Selection Problem and Comparison Analysis of Computational Results

In this section, an investment selection problem is illustrated to demonstrate the applicability and implementation process of the MAGDM method proposed in this paper. In order to show the superiority of the proposed method, the comparison analyses with the MAGDM method based on intuitionistic uncertain linguistic variables and the MADM method based on intuitionistic linguistic fuzzy numbers are also conducted.

5.1. An Investment Selection Problem and the Analysis Process.

The proposed method is illustrated with an investment selection problem adapted from Liu and Jin [6]. Suppose that an investment company wants to invest a sum of money in the best option. There is a panel with four possible alternatives in which to invest the money: a car company A_1 , a computer company A_2 , a TV company A_3 and a food company A_4 . The decision making committee consists of three DMs e_1 , e_2 and e_3 . They must make a decision according to four attributes, namely risk index c_1 , growth index c_2 , social-political impact index c_3 , and environmental impact index c_4 . Suppose that the weight vector of the attributes is $w = (0.32, 0.26, 0.18, 0.24)^T$ and the weight vector of DMs is $v = (0.40, 0.32, 0.28)^T$. The DMs evaluate these alternatives using the linguistic term set $S = \{s_0 = \text{extremely poor (EP)}; s_1 = \text{very poor (VP)}; s_2 = \text{poor (P)}; s_3 = \text{medium (M)}; s_4 = \text{good (G)}; s_5 = \text{very good (VG)}; s_6 = \text{extremely good (EG)}\}$. After the data acquisition and statistical treatment, the ratings of the alternatives with respect to attributes can be represented by IVILVs shown in Tables 1-3. For example, $(s_5, [0.1, 0.2], [0.6, 0.7])$ in Table 1 is an IVILV which indicates that the mark of the candidate A_1 with respect to the attribute c_1 is about s_5 with the satisfaction degree interval $[0.1, 0.2]$ and the dissatisfaction degree interval $[0.6, 0.7]$. Other IVILVs in Tables 1-3 are explained similarly.

	A_1	A_2	A_3	A_4
c_1	$(s_4, [0.0,0.1], [0.5,0.7])$	$(s_6, [0.3,0.4], [0.3,0.5])$	$(s_5, [0.0,0.2], [0.4,0.6])$	$(s_5, [0.2,0.3], [0.4,0.6])$
c_2	$(s_4, [0.0,0.2], [0.4,0.7])$	$(s_4, [0.2,0.3], [0.4,0.6])$	$(s_4, [0.0,0.2], [0.5,0.7])$	$(s_5, [0.2,0.4], [0.3,0.5])$
c_3	$(s_4, [0.1,0.2], [0.4,0.8])$	$(s_5, [0.0,0.2], [0.4,0.6])$	$(s_3, [0.2,0.4], [0.4,0.6])$	$(s_3, [0.1,0.3], [0.5,0.6])$
c_4	$(s_6, [0.2,0.4], [0.4,0.5])$	$(s_4, [0.1,0.2], [0.5,0.7])$	$(s_4, [0.1,0.3], [0.5,0.7])$	$(s_4, [0.1,0.2], [0.4,0.6])$

TABLE 2. The IVILV Decision Matrix Given by DM e_2

	A_1	A_2	A_3	A_4
c_1	$(s_5, [0.1,0.2], [0.4,0.6])$	$(s_5, [0.2,0.3], [0.5,0.7])$	$(s_4, [0.0,0.2], [0.5,0.7])$	$(s_4, [0.1,0.2], [0.5,0.7])$
c_2	$(s_4, [0.2,0.3], [0.4,0.7])$	$(s_5, [0.2,0.3], [0.4,0.6])$	$(s_4, [0.1,0.3], [0.5,0.6])$	$(s_4, [0.0,0.1], [0.6,0.7])$
c_3	$(s_5, [0.3,0.4], [0.3,0.5])$	$(s_3, [0.0,0.1], [0.6,0.8])$	$(s_3, [0.0,0.1], [0.6,0.8])$	$(s_5, [0.1,0.3], [0.4,0.6])$
c_4	$(s_4, [0.0,0.2], [0.5,0.7])$	$(s_4, [0.3,0.4], [0.4,0.6])$	$(s_4, [0.0,0.2], [0.5,0.7])$	$(s_5, [0.2,0.4], [0.4,0.5])$

TABLE 3. The IVILV Decision Matrix Given by DM e_3

	e_1	e_2	e_3
A_1	$(s_{4.42}, [0.144,0.318], [0.458,0.620])$	$(s_{4.48}, [0.070,0.225], [0.434,0.685])$	$(s_{4.50}, [0.144,0.266], [0.450,0.639])$
A_2	$(s_{4.58}, [0.194,0.382], [0.398,0.605])$	$(s_{4.82}, [0.179,0.295], [0.426,0.599])$	$(s_{4.40}, [0.194,0.294], [0.474,0.678])$
A_3	$(s_{4.82}, [0.019,0.219], [0.500,0.700])$	$(s_{4.14}, [0.06,0.264], [0.452,0.654])$	$(s_{3.82}, [0.027,0.211], [0.520,0.700])$
A_4	$(s_{3.94}, [0.130,0.333], [0.408,0.620])$	$(s_{4.40}, [0.159,0.306], [0.396,0.576])$	$(s_{4.42}, [0.101,0.248], [0.491,0.643])$

TABLE 4. The Individual Overall Attributes

(1) Method I: Aggregating the Attribute Values First

Step 1: Combined the weight vector of the attributes $w = (0.32, 0.26, 0.18, 0.24)^T$ and the IVILWAA operator, the individual overall attributes of alternatives can be obtained by Equation (7) shown in Table 4.

Step 2: Suppose that the position weight vector is $\omega = (0.25, 0.50, 0.25)^T$. Combining the weight vector of DMs $V = (0.40, 0.32, 0.28)^T$ and the IVILHA operator, we can compute the collective overall attribute values of alternatives by Equation (8) as follows:

$$\begin{aligned} \tilde{a}_1 &= \text{IVILHA}_{\omega,V}(\tilde{a}_1^1, \tilde{a}_1^2, \tilde{a}_1^3) = (s_{4.47}, [0.1079, 0.2594], [0.4336, 0.6567]) \\ \tilde{a}_2 &= \text{IVILHA}_{\omega,V}(\tilde{a}_2^1, \tilde{a}_2^2, \tilde{a}_2^3) = (s_{4.6550}, [0.1864, 0.3178], [0.4298, 0.6195]) \\ \tilde{a}_3 &= \text{IVILHA}_{\omega,V}(\tilde{a}_3^1, \tilde{a}_3^2, \tilde{a}_3^3) = (s_{4.23}, [0.0433, 0.2400], [0.4802, 0.6763]) \\ \tilde{a}_4 &= \text{IVILHA}_{\omega,V}(\tilde{a}_4^1, \tilde{a}_4^2, \tilde{a}_4^3) = (s_{4.29}, [0.1378, 0.2989], [0.4207, 0.6030]) \end{aligned}$$

Step 3: Using equations (1) and (2), the score function and accuracy function for IVILVs $\tilde{a}_i (i = 1, 2, 3, 4)$ are respectively calculated as follows:

$$\begin{aligned} E(\tilde{a}_1) &= 1.4272, E(\tilde{a}_2) = 1.6931, E(\tilde{a}_3) = 1.1917, E(\tilde{a}_4) = 1.5154, \\ H(\tilde{a}_1) &= 7.7277, H(\tilde{a}_2) = 8.2707, H(\tilde{a}_3) = 7.2751, H(\tilde{a}_4) = 7.4226. \end{aligned}$$

Step 4: Since $E(\tilde{a}_2) > E(\tilde{a}_4) > E(\tilde{a}_1) > E(\tilde{a}_3)$, according to the ranking method developed in Subsection 2.3, the ranking order of alternatives is generated as $A_2 \succ A_4 \succ A_1 \succ A_3$ and the best alternative is A_2 .

As far as this investment selection example is concerned, to illustrate the influence of the position weight ω on the decision-making, we use a different weight vector ω in Step 2 to rank the alternatives. Generally speaking, we need to consider some special cases [6]. One is an average weight (not considering the importance of the position weight). From this, we can set $\omega = (1/3, 1/3, 1/3)^T$. Secondly, the most optimistic DMs can select the best evaluation value, i.e., the value in the first

Position weight	Weight description	Score function	Ranking order
$\omega = (1/3, 1/3, 1/3)^T$	Not considered	$E(\tilde{a}_1) = 1.46, E(\tilde{a}_2) = 1.67,$ $E(\tilde{a}_3) = 1.17, E(\tilde{a}_4) = 1.47$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\omega = (1/15, 10/15, 4/15)^T$	By fuzzy semantic quantitative operator	$E(\tilde{a}_1) = 1.39, E(\tilde{a}_2) = 1.68,$ $E(\tilde{a}_3) = 1.19, E(\tilde{a}_4) = 1.55$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\omega = (1, 0, 0)^T$	Max	$E(\tilde{a}_1) = 1.53, E(\tilde{a}_2) = 1.80,$ $E(\tilde{a}_3) = 1.25, E(\tilde{a}_4) = 1.41$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\omega = (0, 1, 0)^T$	Middle	$E(\tilde{a}_1) = 1.32, E(\tilde{a}_2) = 1.75,$ $E(\tilde{a}_3) = 1.27, E(\tilde{a}_4) = 1.64$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\omega = (0, 0, 1)^T$	Min	$E(\tilde{a}_1) = 1.53, E(\tilde{a}_2) = 1.47,$ $E(\tilde{a}_3) = 0.97, E(\tilde{a}_4) = 1.34$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$\omega = (0.4, 0.2, 0.4)^T$	Increased at both ends and reduced in middle	$E(\tilde{a}_1) = 1.49, E(\tilde{a}_2) = 1.66,$ $E(\tilde{a}_3) = 1.14, E(\tilde{a}_4) = 1.44$	$A_2 \succ A_1 \succ A_4 \succ A_3$

TABLE 5. Ordering of the Alternatives by Utilizing a Different ω in the $IVILHA_{\omega, V}$ Operator

	A_1	A_2
c_1	$(s_{4.68}, [0.07, 0.17], [0.57, 0.70])$	$(s_{5.32}, [0.23, 0.37], [0.40, 0.57])$
c_2	$(s_{3.60}, [0.14, 0.31], [0.40, 0.63])$	$(s_{4.68}, [0.20, 0.34], [0.33, 0.53])$
c_3	$(s_{5.08}, [0.24, 0.39], [0.33, 0.58])$	$(s_{4.04}, [0.00, 0.13], [0.53, 0.73])$
c_4	$(s_{4.64}, [0.07, 0.27], [0.40, 0.57])$	$(s_{4.}, [0.24, 0.39], [0.35, 0.56])$
	A_3	A_4
c_1	$(s_{5.12}, [0.00, 0.20], [0.47, 0.67])$	$(s_{4.32}, [0.22, 0.37], [0.25, 0.46])$
c_2	$(s_{4.}, [0.03, 0.23], [0.50, 0.70])$	$(s_{4.32}, [0.07, 0.25], [0.48, 0.69])$
c_3	$(s_{3.8}, [0.11, 0.29], [0.47, 0.67])$	$(s_{4.36}, [0.06, 0.26], [0.43, 0.60])$
c_4	$(s_{4.}, [0.03, 0.23], [0.50, 0.70])$	$(s_{3.88}, [0.13, 0.30], [0.40, 0.60])$

TABLE 6. The Collective Decision Matrix

position, and then we can set $\omega = (1, 0, 0)^T$. Similarly, for the most pessimistic DMs, we can set $\omega = (0, 0, 1)^T$. Thirdly, similar to the Olympic games, which discard a maximum point and a minimum point, we can set $\omega = (0, 1, 0)^T$. In addition, we can set the other value in ω , for example, $\omega = (1/15, 10/15, 4/15)^T$, which is calculated by the fuzzy semantic quantitative operator. The position weight $\omega = (0.4, 0.2, 0.4)^T$ emphasizes both ends and reduces the middle. The ranking results are shown in Table 5.

It is easy to see from Table 5 that for the position weights $\omega = (1/3, 1/3, 1/3)^T$, $\omega = (1/15, 10/15, 4/15)^T$ and $\omega = (0, 1, 0)^T$ the ranking orders for four alternatives are the same, i.e., $A_2 \succ A_4 \succ A_1 \succ A_3$; for the position weights $\omega = (1, 0, 0)^T$ and $\omega = (0.4, 0.2, 0.4)^T$ the ranking orders for four alternatives are the same, i.e., $A_2 \succ A_1 \succ A_4 \succ A_3$; for the position weight $\omega = (0, 0, 1)^T$ the ranking order for four alternatives is $A_1 \succ A_2 \succ A_4 \succ A_3$. This analysis shows that the position weight may affect the ranking of alternatives.

(2) Method II: First Aggregating the Information from Different DMs

Step 1: Combined the weight vector of DMs $V = (0.40, 0.32, 0.28)^T$ and the $IVILVWAA$ operator, the collective decision matrix can be obtained by Equation (9) shown in Table 6.

Step 2: Suppose that the position weight vector is $\omega = (0.2, 0.3, 0.3, 0.2)^T$. Combined the weight vector of the attributes $w = (0.32, 0.26, 0.18, 0.24)^T$ and the $IVILHA$ operator, the collective overall attribute values of alternatives can be computed by Equation (10) as follows:

$$\begin{aligned}\tilde{a}_1 &= IVILHA_{\omega, w}(\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{13}, \tilde{a}_{14}) = (s_{4.572}, [0.1384, 0.2983], [0.4043, 0.6082]) \\ \tilde{a}_2 &= IVILHA_{\omega, w}(\tilde{a}_{21}, \tilde{a}_{22}, \tilde{a}_{23}, \tilde{a}_{24}) = (s_{4.476}, [0.1837, 0.3264], [0.3840, 0.5810])\end{aligned}$$

Position weight	Weight description	Score function	Ranking order
$\omega = (0.25, 0.25, 0.25, 0.25)^T$	Not considered	$E(\tilde{a}_1) = 1.56, E(\tilde{a}_2) = 1.7,$ $E(\tilde{a}_3) = 1.18, E(\tilde{a}_4) = 1.54$	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\omega = (0, 0.4, 0.5, 0.1)^T$	By fuzzy semantic quantitative operator	$E(\tilde{a}_1) = 1.85, E(\tilde{a}_2) = 1.75,$ $E(\tilde{a}_3) = 1.08, E(\tilde{a}_4) = 1.34$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$\omega = (1, 0, 0, 0)^T$	Max	$E(\tilde{a}_1) = 1.14, E(\tilde{a}_2) = 2.18,$ $E(\tilde{a}_3) = 1.37, E(\tilde{a}_4) = 2.03$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\omega = (0, 1, 0, 0)^T$	Middle	$E(\tilde{a}_1) = 1.59, E(\tilde{a}_2) = 1.97,$ $E(\tilde{a}_3) = 1.06, E(\tilde{a}_4) = 1.24$	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\omega = (0, 0, 1, 0)^T$	Middle	$E(\tilde{a}_1) = 2.18, E(\tilde{a}_2) = 1.72,$ $E(\tilde{a}_3) = 1.07, E(\tilde{a}_4) = 1.39$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$\omega = (0, 0, 0, 1)^T$	Min	$E(\tilde{a}_1) = 1.28, E(\tilde{a}_2) = 0.89,$ $E(\tilde{a}_3) = 1.20, E(\tilde{a}_4) = 1.41$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$\omega = (0.4, 0.1, 0.1, 0.4)^T$	Increased at both ends and reduced in middle	$E(\tilde{a}_1) = 1.38, E(\tilde{a}_2) = 1.59,$ $E(\tilde{a}_3) = 1.25, E(\tilde{a}_4) = 1.66$	$A_4 \succ A_2 \succ A_1 \succ A_3$

TABLE 7. Ordering of the Alternatives by Utilizing a Different ω in the $IVILHA_{\omega,w}$ Operator

$$\tilde{a}_3 = IVILHA_{\omega,w}(\tilde{a}_{31}, \tilde{a}_{32}, \tilde{a}_{33}, \tilde{a}_{34}) = (s_{4.184}, [0.0408, 0.2364], [0.4859, 0.6863])$$

$$\tilde{a}_4 = IVILHA_{\omega,w}(\tilde{a}_{41}, \tilde{a}_{42}, \tilde{a}_{43}, \tilde{a}_{44}) = (s_{4.196}, [0.1169, 0.2905], [0.3902, 0.5917])$$

Step 3: By equations (1) and (2), the score function and accuracy function for $IVILVs \tilde{a}_i (i = 1, 2, 3, 4)$ are respectively calculated as follows:

$$E(\tilde{a}_1) = 1.628, E(\tilde{a}_2) = 1.7290, E(\tilde{a}_3) = 1.1558, E(\tilde{a}_4) = 1.4954,$$

$$H(\tilde{a}_1) = 7.8850, H(\tilde{a}_2) = 7.7774, H(\tilde{a}_3) = 7.2162, H(\tilde{a}_4) = 7.1106.$$

Step 4: Since $E(\tilde{a}_2) > E(\tilde{a}_1) > E(\tilde{a}_4) > E(\tilde{a}_3)$, according to the ranking method developed in Subsection 2.3, the ranking order of alternatives is generated as $A_2 \succ A_1 \succ A_4 \succ A_3$ and the best alternative is A_2 .

Similar to Method I, we also illustrate the influence of the position weight ω on decision making as far as this example is concerned. For reference on how to obtain the value of ω , please refer to the comments in Method I. The ranking results for different weight values ω are shown in Table 7.

From Table 7, we know that the ordering of the alternatives may be different for a different position weight value ω in the $IVILHA_{\omega,w}$ operator. In general, we must observe two special cases. First, we do not consider the position weight, i.e., the weight $\omega = (1/n, 1/n, \dots, 1/n)^T$. Secondly, we consider increasing the weight value in the middle order and reducing the weight value in both ends, i.e., the weight values of the middle attribute values are bigger, and the weight values of the maximum and minimum attribute values are smaller. This position weight can be determined by the method of the combination number introduced by [21]. In short, the organization can properly select the position weight according to his interest and actual needs.

5.2. Comparison Analysis with the MAGDM Based on Intuitionistic Uncertain Linguistic Variables.

In this subsection, we compare the methods [6] with the proposed method in this paper. Liu and Jin [6] proposed two methods to solve MAGDM with IULVs based on the IULHG, IULWGA and IULOWG operators. The first method is aggregating the attribute values first, the second method is first aggregating the information from different DMs.

If we rewrite IVILV $\tilde{a}_{ij}^k = (s_{\theta(\tilde{a}_{ij}^k)}, [\underline{\mu}(\tilde{a}_{ij}^k), \bar{\mu}(\tilde{a}_{ij}^k)], [\underline{\nu}(\tilde{a}_{ij}^k), \bar{\nu}(\tilde{a}_{ij}^k)])$ as IULV $\tilde{a}_{ij}^k = \langle [s_{\theta(\tilde{a}_{ij}^k)-1}, s_{\theta(\tilde{a}_{ij}^k)}], (\frac{1}{2}[\underline{\mu}(\tilde{a}_{ij}^k) + \bar{\mu}(\tilde{a}_{ij}^k)], \frac{1}{2}[\underline{\nu}(\tilde{a}_{ij}^k) + \bar{\nu}(\tilde{a}_{ij}^k)]) \rangle$ or $\tilde{a}_{ij}^k = \langle [s_{\theta(\tilde{a}_{ij}^k)}, s_{\theta(\tilde{a}_{ij}^k)}], (\bar{\mu}(\tilde{a}_{ij}^k), \bar{\nu}(\tilde{a}_{ij}^k)) \rangle$ or $\tilde{a}_{ij}^k = \langle [s_{\theta(\tilde{a}_{ij}^k)-1}, s_{\theta(\tilde{a}_{ij}^k)}], (\bar{\mu}(\tilde{a}_{ij}^k), \bar{\nu}(\tilde{a}_{ij}^k)) \rangle$ or $\tilde{a}_{ij}^k = \langle [s_{\theta(\tilde{a}_{ij}^k)}, s_{\theta(\tilde{a}_{ij}^k)}], (\frac{1}{2}[\underline{\mu}(\tilde{a}_{ij}^k) + \bar{\mu}(\tilde{a}_{ij}^k)], \frac{1}{2}[\underline{\nu}(\tilde{a}_{ij}^k) + \bar{\nu}(\tilde{a}_{ij}^k)]) \rangle$, then all the IVILVs in Tables 1-3 can be rewritten as IULVs. Using the two methods [6] to solve this investment selection problem with IULVs, the ranking orders of alternatives are the same, i.e., $A_2 \succ A_4 \succ A_1 \succ A_3$. Furthermore, comparing Table 5 and Table 7 in this paper with Table 6 and Table 8 in [6], we find that the ranking results obtained by Liu and Jin[6] and this paper are not completely the same. The main reasons are as follows:

(i) Both the IULV in [6] and the IVILV in this paper are the extensions of ILFN. The IULV extends the linguistic part of ILFN as uncertain linguistic variable, whereas the IVILV extends the intuitionistic part of ILFN as interval-valued intuitionistic fuzzy value. The superiority of IULV is in linguistic part while the superiority of IVILV is in intuitionistic part. Therefore, the motivations of the IULV and the IVILV are different.

(ii) This paper develops some arithmetic aggregation operators of IVILVs, while Liu and Jin [6] developed some geometric aggregation operators of IULVs. The arithmetic aggregation operators aggregate the variables in the form of addition, while the geometric aggregation operators aggregate the variables in the form of multiplication. The decision methods proposed in this paper are based on the arithmetic aggregation operators of IVILVs, while the decision methods proposed in [6] are based on the geometric aggregation operators of IULVs. Therefore, the decision principles of Liu and Jin [6] and this paper are remarkably different.

5.3. Comparison Analysis with MADM Based on Intuitionistic Linguistic Fuzzy Numbers.

If we rewrite IVILV $\tilde{a}_{ij}^k = (s_{\theta(\tilde{a}_{ij}^k)}, [\underline{\mu}(\tilde{a}_{ij}^k), \bar{\mu}(\tilde{a}_{ij}^k)], [\underline{\nu}(\tilde{a}_{ij}^k), \bar{\nu}(\tilde{a}_{ij}^k)])$ as ILFN $\tilde{a}_{ij}^k = \langle s_{\theta(\tilde{a}_{ij}^k)}, \frac{1}{2}[\underline{\mu}(\tilde{a}_{ij}^k) + \bar{\mu}(\tilde{a}_{ij}^k)], \frac{1}{2}[\underline{\nu}(\tilde{a}_{ij}^k) + \bar{\nu}(\tilde{a}_{ij}^k)] \rangle$, then all the IVILVs in Tables 1-3 can be rewritten as ILFNs. Wang and Li [20] proposed two methods for multi-criteria decision making with ILFNs. The first method is based on the intuitionistic linguistic weighted arithmetic averaging operator, the second method is based on the intuitionistic linguistic weighted geometric averaging operators. These two methods only considered single DM.

If single DM e_1 is considered, then the above investment selection problem is reduced to multi-criteria decision making problem with ILFNs. We use the two methods [20] to solve this problem. The ranking orders of alternatives obtained by the two methods [20] are $A_2 \succ A_3 \succ A_4 \succ A_1$ and $A_2 \succ A_3 \succ A_1 \succ A_4$, respectively, which are not accordance with the ranking orders by Method I and Method II in this paper. This analysis shows that the membership degree interval and non-membership degree interval of IVILV play the important roles for the decision results indeed. Since ILFN utilizes the single real number to represent the membership degree and non-membership degree, it has less representation ability

and flexibility than IVILV.

Furthermore, the methods [20] can only be used to solve MADM and are not appropriate for MAGDM. If single DM is considered, Method I and Method II proposed in this paper can also be used to solve MADM only if removing Step 2 from Method I and Method II. In addition, the methods [20] did not consider the position weight.

6. Conclusions

To overcome the drawback of using single real number to represent membership degree and non-membership degree for ILS, the concept of IVILS was introduced through representing the membership degree and non-membership degree with intervals for ILS in this paper. The operation law, score function, accuracy function, and certainty function for IVILVs were defined. A lexicographic method was proposed to rank the IVILVs on the basis of the score, accuracy, and certainty functions. Some arithmetic aggregation operators of IVILVs were developed, including the IVILWAA, IVILOWA, and IVILHA operators. Based on the IVILWAA and IVILHA operators, two methods were proposed for solving MAGDM problems with IVILVs.

In general, many real-world MAGDM problems occur in a complex environment and usually adhere to imprecise data and uncertainty. It is quite common that the DM uses linguistic terms to represent the assessment information and meanwhile there exist some hesitancy degrees when he makes judgments. An IVILS is adequate for dealing with the vagueness of a DM's judgment of alternatives on attributes since it contains the linguistic part and intuitionistic part simultaneously. The IVILS adopts the intervals to characterize the membership and non-membership degrees of the linguistic evaluations and combines the ILS with the IVIFS, whereas the ILS utilizes real numbers to express the membership and non-membership degrees. Thus, IVILS has stronger representation ability and flexibility than ILS. The IVILWAA, IVILOWA, and IVILHA operators are the helpful extensions of the traditional aggregation operators for real numbers.

The investment selection example and comparison analysis show the applicability and effectiveness of the methods proposed in this paper. Obviously, the models and methods proposed in this paper are applicable to lots of similar decision problems although they are illustrated with the example of the investment selection problem, such as the supplier management, water environment assessment, threat evaluation and missile weapon system selection, warship combat plan evaluation. As mentioned in section 3.3, we will study the WOWA, the importance OWA, the immediate weights and the OWA weighted average operators under interval-valued intuitionistic linguistic environment in the near future.

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