DECISION MAKING IN MEDICAL INVESTIGATIONS USING NEW DIVERGENCE MEASURES FOR INTUITIONISTIC FUZZY SETS

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Abstract. In recent times, intuitionistic fuzzy sets introduced by Atanassov has been one of the most powerful and flexible approaches for dealing with complex and uncertain situations of real world. In particular, the concept of divergence between intuitionistic fuzzy sets is important since it has applications in various areas such as image segmentation, decision making, medical diagnosis, pattern recognition and many more. The aim of this paper is to introduce a new divergence measure for Atanassov's intuitionistic fuzzy sets (*AIFS)*. The properties of the proposed divergence measure have been studied and the findings are applied in medical diagnosis of some diseases with a common set of symptoms.

1. Introduction

The concept of fuzzy set proposed by Zadeh [24] has attracted widespread attention from various researchers, due to its effectiveness in dealing with uncertain situations of real world. Since then, various generalizations of fuzzy sets have been proposed. Atanassov [1] extended the concept of fuzzy sets to intuitionistic fuzzy sets, characterized by a membership function and non membership function, which provides a better and more effective mathematical framework in handling situations where a decision maker is not sure whether to favour a given decision or not and he abstains from voting. The hesitation parameter defined in Atanassov's intuitionistic fuzzy sets ($AIFS$) gives us an additional possibility in quantifying lacking knowledge and illustrate various real life problems in a more satisfactory way. The theory of Atanassov's intuitionistic fuzzy sets ($AIFS$) has been extensively studied from different aspects and one application in which it has been extremely successful is in quantifying the similarity/dissimilarity between two objects by means of some real valued functions. These functions are termed as similarity measures, distance measures, divergence measures etc depending upon the properties they satisfy [7, 16]. However authors [14, 15] have shown that the definition of divergence is more restrictive when compared with other measures of comparison, which is absolutely necessary for avoiding counter intuitive situations. Motivated by the notion of divergence between two probability distributions, Bhandari & Pal [2] , Shang & Jiang [17], Fan & Xie [6] and Montes, Couso, Gil & Bertoluzza [13] have proposed

Received: November 2014; Revised: April 2015; Accepted: October 2015

Key words and phrases: Fuzzy Sets, Intuitionistic Fuzzy sets, Divergence measure, Medical diagnosis.

various divergence measures between fuzzy sets. The fuzzy divergence measures proposed by Bhandari & Pal [2] and Shang & Jiang [17] are based on logarithmic information gain functions and that of Fan & Xie [6] are based on exponential information gain functions with the same approach being followed in Chaira & Ray [4]. Like for fuzzy sets, divergence measures between Atanassov's intuitionistic fuzzy sets ($AIFS$) have been proposed by Vlachos & Sergiadis [21], Hung & Yang [8],Zhang & Jiang [25], Wei & Ye [22], Xia & Xu [23], Junjun, Dengbao & Cuicui [10]and the findings have been applied in medical diagnosis, pattern recognition and image segmentation problems. Recently, Montes, Pal, Janis & Montes [15] proposed an axiomatic definition of the notion of divergence for Atanassov's intuitionistic fuzzy sets ($AIFS$) and suggested some new approaches for building divergence measures between Atanassov's intuitionistic fuzzy sets ($AIFS$). In the present work, motivated by the well known inequality

$$min(x, y) \leq \left(\frac{x+y}{2}\right) \tag{1}$$

between two real numbers x and y, we will define a new divergence measure for Atanassov's intuitionistic fuzzy sets ($AIFS$). The properties of the proposed divergence measure will be studied and the findings are applied in medical diagnosis of some diseases with a common set of symptoms. In the next section, we introduce some basic definitions related to fuzzy divergence measures and intuitionistic fuzzy divergence measures.

2. Preliminaries

As we noted above Atanassov's intuitionistic fuzzy sets ($AIFS$) are the extensions of the standard fuzzy sets.For basic definitions and operations related to fuzzy sets and intuitionistic fuzzy sets, the readers can refer Zadeh [24] and Atanassov [1]. We however start by listing some basic axioms which a fuzzy divergence measure and a intuitionistic fuzzy divergence measure should satisfy.

Definition 2.1. Fuzzy Divergence Measures - Let $\hat{A} = \left\{ \left\langle x_i, \mu_{\hat{A}}\left(x_i\right) \right\rangle;\ x_i \in X \right\}$ and $\hat{B} = \left\{ \left\langle x_i, \mu_{\hat{B}}\left(x_i\right) \right\rangle;\ x_i \in X \right\}$ be two fuzzy sets in the finite universe of discourse $X = \{x_1,\ x_2,\ \ldots,\ x_n\ \}$. Let $F\ (X)$ denote the set of all fuzzy sets on X. A mapping $J : F\,(X) \times F(X) \to \mathbb{R}$ is said to be a divergence measure for fuzzy sets in the sense of Montes [13] if it satisfies the following axioms:

A1. $J\left(\hat{A}||\hat{B}\right) = J\left(\hat{B}||\hat{A}\right)$.

A2. $J\left(\hat{A}||\hat{B}\right) = 0$ if and only if $\hat{A} = \hat{B}$.

A3. $J\left(\hat{A}\bigcap\hat{C}||\hat{B}\bigcap\hat{C}\right) \leq J\left(\hat{A}||\hat{B}\right)$for every $\hat{C} \in F(X)$.

A4. $max\left(J\left(\hat{A}\bigcap\hat{C}||\hat{B}\bigcap\hat{C}\right), J\left(\hat{A}\bigcup\hat{C}||\hat{B}\bigcup\hat{C}\right)\right) \leq J\left(\hat{A}||\hat{B}\right)$for every $\hat{C} \in F(X)$.

The non negativity of the divergence is not required in the above axioms. However it is trivial to deduce it from A2 and A3 (or A2 and A4). Several fuzzy divergence measures between fuzzy sets are reviewed next.

Bhandari & Pal [2]

$$J_1(\hat{A},\hat{B}) = \sum_{i=1}^{n}\left(\mu_{\hat{A}}(x_i)ln\left(\frac{\mu_{\hat{A}}(x_i)}{\mu_{\hat{B}}(x_i)}\right) + \left(1-\mu_{\widehat{A}}(x_i)\right)ln\left(\frac{(1-\mu_{\hat{A}}(x_i))}{(1-\mu_{\hat{B}}(x_i))}\right)\right). \quad (2)$$

The measure (2) is based on the well known Kullback - Leibler's divergence [10].
Shang & Jiang[17]

$$J_2(\hat{A},\hat{B}) = \sum_{i=1}^{n}\left(\begin{array}{c}\mu_{\hat{A}}(x_i)ln\left(\frac{\mu_{\hat{A}}(x_i)}{((1/2)(\mu_{\hat{A}}(x_i)+\mu_{\hat{B}}(x_i)))}\right)\\ +\left(1-\mu_{\widehat{A}}(x_i)\right)ln\left(\frac{(1-\mu_{\hat{A}}(x_i))}{(1-(1/2(\mu_{\hat{A}}(x_i)+\mu_{\widehat{B}}(x_i))))}\right)\end{array}\right). \quad (3)$$

Shang & Jiang [17] also proposed a symmetric discrimination information measure based on $J\left(\hat{A},\hat{B}\right)$ is given by

$$J_{sym}(\hat{A},\hat{B}) = J_2(\hat{A},\hat{B}) + J_2(\hat{B},\hat{A}). \quad (4)$$

Fan & Xie[6]

$$J_3(\hat{A},\hat{B}) = \sum_{i=1}^{n}\left(1-(1-\mu_{\hat{A}}(x_i))\,e^{\left(\mu_{\hat{A}}(x_i)-\mu_{\hat{B}}(x_i)\right)} - \mu_{\hat{A}}(x_i)e^{\left(\mu_{\hat{B}}(x_i)-\mu_{\hat{A}}(x_i)\right)}\right). \quad (5)$$

Montes, Couso, Gil & Bertoluzza [13]

$$J_4(\hat{A}||\hat{B}) = \underset{x\in X}{\mathrm{T}}\left(\left|\hat{A}(x)-\hat{B}(x)\right|\right). \quad (6)$$

where T is a t - conorm.

Definition 2.2. Intuistionistic Fuzzy Divergence Measures - Let

$$A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle;\ x_i \in X\}$$

and

$$B = \{\langle x_i, \mu_B(x_i),\ \nu_B(x_i)\rangle;\ x_i \in X\}$$

be two Atanassov's intuitionistic fuzzy sets in X. A mapping $D: AIFS(X) \times AIFS(X) \to \mathbb{R}$ is a divergence measure for Atanassov's intuitionistic fuzzy sets if it satisfies the following axioms [15].
M1. $D(A||B) = D(B||A)$.
M2. $D(A||B) = 0$ if and only if $A = B$.
M3. $D(A\bigcap C||B\bigcap C) \leq D(A||B)$for every $C \in AIFS(X)$.
M4. $D(A\bigcup C||B\bigcup C) \leq D(A||B)$for every $C \in AIFS(X)$.

Again, the non negativity of the divergence measure is not required in the above axioms. However, it is trivial to deduce it from M2 and M3 (or M2 and M4). The existing divergence measures between $AIFS(X)$ are reviewed next.

Vlachos & Sergiadis[21]

$$D_1(A||B) = \sum_{i=1}^{n} \left(\begin{array}{l} \mu_A(x_i) ln \left(\frac{\mu_A(x_i)}{(1/2)(\mu_A(x_i)+\mu_B(x_i))} \right) \\ +\nu_A(x_i) ln \left(\frac{\nu_A(x_i)}{(1/2)(\nu_A(x_i)+\nu_B(x_i))} \right) \end{array} \right). \quad (7)$$

Vlachos & Sergiadis[21] also defined the symmetric version of measure (7) given by

$$D_{sym}(A||B) = D_1(A||B) + D_1(B||A). \quad (8)$$

Hung & Yang [8]

$$D_2(A||B) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \left(\begin{array}{l} \left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2} \right)^{\alpha} - \left(\frac{(\mu_A(x_i))^{\alpha}+(\mu_B(x_i))^{\alpha}}{2} \right) \\ + \left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2} \right)^{\alpha} - \left(\frac{(\nu_A(x_i))^{\alpha}+(\nu_B(x_i))^{\alpha}}{2} \right) \\ + \left(\frac{\pi_A(x_i)+\pi_B(x_i)}{2} \right)^{\alpha} - \left(\frac{(\pi_A(x_i))^{\alpha}+(\pi_B(x_i))^{\alpha}}{2} \right) \end{array} \right). \quad (9)$$

where $\alpha \neq 1 (\alpha > 0)$.

Zhang & Yiang [25]

$$D_3(A||B) = \sum_{i=1}^{n} \left(\begin{array}{l} \left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2} \right) ln \frac{(\mu_A(x_i)+1-\nu_A(x_i))}{(1/2)((\mu_A(x_i)+1-\nu_A(x_i))+(\mu_B(x_i)+1-\nu_B(x_i)))} \\ + \left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2} \right) ln \frac{(\nu_A(x_i)+1-\mu_A(x_i))}{(1/2)((\nu_A(x_i)+1-\mu_A(x_i))+(\nu_B(x_i)+1-\mu_B(x_i)))} \end{array} \right). \quad (10)$$

Wei & Yei [22] & K. C. Hung [9]

$$D_4(A||B) = \sum_{i=1}^{n} \left(\begin{array}{l} \mu_A(x_i) ln \left(\frac{\mu_A(x_i)}{(1/2)(\mu_A(x_i)+\mu_B(x_i))} \right) \\ +\nu_A(x_i) ln \left(\frac{\nu_A(x_i)}{(1/2)(\nu_A(x_i)+\nu_B(x_i))} \right) \\ +\pi_A(x_i) ln \left(\frac{\pi_A(x_i)}{(1/2)(\pi_A(x_i)+\pi_B(x_i))} \right) \end{array} \right). \quad (11)$$

Xia & Xu [23]

$$D_5(A||B) = \left(\frac{1}{t} \right) \sum_{i=1}^{n} \left(\begin{array}{l} \left(\frac{(1+q\mu_A(x_i)) ln(1+q\mu_A(x_i))+(1+q\mu_B(x_i)) ln(1+q\mu_B(x_i))}{2} \right) \\ - \left(\frac{1+q\mu_A(x_i)+1+q\mu_B(x_i)}{2} \right) ln \left(\frac{1+q\mu_A(x_i)+1+q\mu_B(x_i)}{2} \right) \\ + \left(\frac{(1+q\nu_A(x_i)) ln(1+q\nu_A(x_i))+(1+q\nu_B(x_i)) ln(1+q\nu_B(x_i))}{2} \right) \\ - \left(\frac{1+q\nu_A(x_i)+1+q\nu_B(x_i)}{2} \right) ln \left(\frac{1+q\nu_A(x_i)+1+q\nu_B(x_i)}{2} \right) \\ + \left(\frac{(1+q\pi_A(x_i)) ln(1+q\pi_A(x_i))+(1+q\pi_B(x_i)) ln(1+q\pi_B(x_i))}{2} \right) \\ - \left(\frac{1+q\pi_A(x_i)+1+q\pi_B(x_i)}{2} \right) ln \left(\frac{1+q\pi_A(x_i)+1+q\pi_B(x_i)}{2} \right) \end{array} \right). \quad (12)$$

where $t = (1+q)\ln(1+q) - (2+q)(\ln(2+q) - \ln 2)$and $q > 0$.

Junjun, Dengbao & Cuicui [10]

$$D_6(A||B) = \sum_{i=1}^{n} \left(\begin{array}{l} \pi_A(x_i) ln \left(\frac{\pi_A(x_i)}{(1/2)(\pi_A(x_i)+\pi_B(x_i))} \right) \\ +\Delta_A(x_i) ln \left(\frac{\Delta_A(x_i)}{(1/2)(\Delta_A(x_i)+\Delta_B(x_i))} \right) \end{array} \right). \quad (13)$$

where $\Delta_A(x_i) = \left|\mu_{\hat{A}}(x) - \nu_A(x_i)\right|$, denotes that how close the membership and non membership degrees are.
Junjun, Dengbao & Cuicui [10] also defined the symmetric version of measure (13) given by

$$D_{sym}(A||B) = D_6(A||B) + D_6(B||A). \tag{14}$$

However, we will now show that the measures (7) and (13) does not satisfy the basic requirement of non − negativity.

Example 2.3. Consider two $AIFS(X)$ given by

$$A = \{\langle x_1, 0.44, 0.385\rangle, \langle x_2, 0.43, 0.39\rangle, \langle x_3, 0.42, 0.38\rangle\}$$

and

$$B = \{\langle x_1, 0.34, 0.48\rangle, \langle x_2, 0.37, 0.46\rangle, \langle x_3, 0.38, 0.45\rangle\}.$$

For the above two $AIFS(X)$, the measures (7) and (10) assume the values $D_1(A||B) = -0.00712$ and $D_6(A||B) = -0.04547$.
Therefore, the measure (7) and (10) do not satisfy the basic condition of non negativity.

This is due to the fact that the measures (7) and (10) are based on the information theoretic measure proposed by Lin [12] given by

$$K(P,Q) = \sum_{i=1}^{n} p_i ln\left(\frac{2p_i}{(p_i+q_i)}\right); \quad 0 \le p_i, q_i \le 1 \; ; \; \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1. \tag{15}$$

Here $P = (p_1, p_2, \ ..., \ p_n)$ and $Q = (q_1, q_2, \ ..., \ q_n)$ are two finite discrete probability distributions and as a result, the above measure by Lin [12] is always non-negative by virtue of Shannon inequality. But in case of measures (7) and (13), neither the pair $(\mu_A(x_i), \ \nu_A(x_i))$ nor $(\Delta_A(x_i), \ \pi_A(x_i))$ forms a probability distribution because $0 \le \mu_A(x_i) + \nu_A(x_i) \le 1$and $0 \le \Delta_A(x_i) + \pi_A(x_i) \le 1$ for all $x_i \in X$. As a result, both measures (7) and (13) does not satisfy the Shannon inequality forcing them to assume some negative values. However the measure (11) proposed by Wei and Yei [22] is always positive since the pair $(\mu_A(x_i), \nu_A(x_i), \pi_A(x_i))$ forms a probability distribution because $0 \le \mu_A(x_i) + \nu_A(x_i) + \pi_A(x_i) \le 1$.

Example 2.4. Consider the two $AIFS(X)$ A and B, given by

$$A = \{\langle x_1, 0, 0.5\rangle, \langle x_2, 0.5, 0\rangle, \langle x_3, 0, 0\rangle\}$$

and

$$B = \{\langle x_1, 0.5, 0.5\rangle, \langle x_2, 0.5, 0.5\rangle, \langle x_3, 0.5, 0\rangle\}.$$

For these two $AIFS$(X), we have $D_1(A||B) = 0$. But here the two sets A and B are not equal. Therefore axiom M2 is violated.

Example 2.5. Consider the two $AIFS(X)$ A and B, given by

$$A = \{\langle x_1, 0, 0.5\rangle, \langle x_2, 0, 0.5\rangle\}$$

and

$$B = \{\langle x_1, 0.5, 0\rangle, \langle x_2, 0.5, 0\rangle\}.$$

we have $D_3(A||B) = 0$. Again axiom M2 is violated.

Example 2.6. Let $A, B \in AIFS(X)$, given by $A = \{\langle x_1, 0, 0.5\rangle\}$ and $B = \{\langle x_1, 0.5, 0\rangle\}$, for which we have $D_6(A||B) = 0$. This is again a case of violation of axiom M2.
Therefore, it is clear that measures (7), (10) and (13) do not satisfy the axiom M2 *i.e.* the divergence of an intuitionistic fuzzy set with itself is not zero. As a result, the measure(8) [symmetric version of measure (7)] and measure (14)[symmetric version of measure(13)] also do not satisfy axiom M2. However the measures (9), (11) and (12) satisfy axiom M2.

In the next section, we will define a new divergence measure for Atanassov's intuitionistic fuzzy sets ($AIFS$) which does not suffer from the weaknesses mentioned above. Further we have shown that the proposed divergence measure satisfies a number of additional properties apart from the basic axioms which a divergence measure should satisfy.

3. New Intuitionistic Fuzzy Divergence Measure

Definition 3.1. Let $A, B \in AIFS(X)$ then the intuitionistic fuzzy cross-entropy of A against B is defined as

$$D\,(A||B) = 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} min\,(\mu_A(x_i), \mu_B(x_i)) \\ +min\,(\nu_A(x_i), \nu_B(x_i)) \\ +min\,(\pi_A(x_i), \pi_B(x_i)) \end{pmatrix}\right). \quad (16)$$

The above divergence measure satisfies all the properties which a divergence measure between intuitionistic fuzzy sets should satisfy. It should be noted that the logarithms are taken to base 2 in measure (16). Since for any $x, y \in [0,1]$, we have

$$min\,(x, y) = \frac{x + y - |x - y|}{2} = 1 - max\,(1 - x, 1 - y)\,. \quad (17)$$

Therefore, the measure (16) can be written in the form

$$D\,(A||B) = 1 - log_2\left(1 + \tfrac{1}{2n}\textstyle\sum_{i=1}^{n}\begin{pmatrix} 2 - |\mu_A(x_i) - \mu_B(x_i)| \\ - |\nu_A(x_i) - \nu_B(x_i)| \\ - |\pi_A(x_i) - \pi_B(x_i)| \end{pmatrix}\right)$$

$$= 1 - log_2\left(1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\begin{pmatrix} 3 - max\,(1 - \mu_A(x_i), 1 - \mu_B(x_i)) \\ - max\,(1 - \nu_A(x_i), 1 - \nu_B(x_i)) \\ - max\,(1 - \pi_A(x_i), 1 - \pi_B(x_i)) \end{pmatrix}\right)$$

$$= 1 - log_2\left(1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n}\begin{pmatrix} 3 - max\,(\nu_A(x_i) + \pi_A(x_i), \nu_B(x_i) + \pi_B(x_i)) \\ - max\,(\mu_A(x_i) + \pi_A(x_i), \mu_B(x_i) + \pi_B(x_i)) \\ - max\,(\nu_A(x_i) + \mu_A(x_i), \nu_B(x_i) + \mu_B(x_i)) \end{pmatrix}\right).$$

The properties of the proposed divergence measure $D\,(A||B)$given by (16) are listed next.

Theorem 3.2. ***Properties of the Proposed Intuitionistic Fuzzy Divergence Measure***-*Let $A, B, C \in AIFS(X)$,then the measure $D\,(A||B)$ given by (16) satisfies the following properties:*

D1. $D(A||B) = D(B||A)$ *and* $0 \leq D(A||B) \leq 1$.
D2. $D(A||B) = 0$ *if and only if* $A = B$.
D3. $D(A\bigcap C||B\bigcap C) \leq D(A||B)$*for every* $C \in IFS(X)$.
D4. $D(A\bigcup C||B\bigcup C) \leq D(A||B)$*for every* $C \in IFS(X)$.
D5. $D(A||B) = D\left(A^C||B^C\right)$.
D6. $D\left(A||B^C\right) = D\left(A^C||B\right)$.
D7. $D\left(A||A^C\right) = 1$ *if and only if* A *is a crisp set.*
D8. $D\left(A||A^C\right) = 0$ *if and only if* $\mu_A(x_i) = \nu_A(x_i)$*for all* $x_i \in X$.
D9. $D(A\bigcap B||B) = D(A||A\bigcup B) \leq D(A||B)$ *for* $A \subseteq B and B \subseteq A$.
D10. $D(A\bigcap B||A\bigcup B) = D(A||B)$.
D11. $D(A||B) \leq D(A||C)$ *for* $A \subseteq B \subseteq C$.
D12. $D(B||C) \leq D(A||C)$ *for* $A \subseteq B \subseteq C$.

Proof. For proof of the properties (D1 - D12), kindly refer Appendix . □

4. **Applications in Medical Diagnosis**

Preliminary investigation is essential in any illness for proper medical diagnosis, however at the same time it is very difficult due to the limited information available to the physician about the patient's medical history. Uncertainty in medical investigations is an indicator of correspondence between the symptoms and diseases. In fact, a symptom is an uncertain indicator of a disease since its occurrence with the disease is not guaranteed. Therefore, coping with uncertainty is entirely essential for proper diagnosis of a patient with a given set of symptoms. One approach which exhibits immense potential in handling uncertainty in medical diagnostic investigations is the fuzzy set theory. Fuzzy set theory makes it possible to define the inexact medical information as fuzzy sets; therefore researchers (De, Biswas & Roy [5]; Szmidt, & Kacprzyk [18, 19, 20]; Vlachos & Sergiadis [21]; Hung [9]; Boran [3]) in different streams have studied various fuzzy approaches and their generalizations for modeling the medical diagnostic processes. We will illustrate an application of the proposed intutionistic fuzzy divergence measure given by (16) in medical diagnosis of some diseases with a common set of symptoms. We consider the same example as discussed by De, Biswas & Roy [5]. The data comprised of a set of patients given by $P = \{\text{Al, Bob, Joe, Ted}\}$, a set of diagnosis represented as $D = \{\text{viral fever, malaria, stomach problem, chest pain}\}$ and finally a set of symptoms defined as $S = \{\text{temperature, headache, stomach pain, chest pain}\}$. Table 1 gives the characteristic symptoms for the diagnosis with row indicating symptoms and column indicating diseases. Table 2 indicates the symptoms for each patient with row indicating patients and column indicating symptoms. Each element of the both tables is given in the form of a triplet representing the membership function, non-membership function and hesitation values respectively, for example, headache for viral fever is described by $(\mu, \nu, \pi) = (0.3, 0.5, 0.2)$ in Table 1. In order to accomplish a proper diagnosis for each patient, we evaluate the proposed intuitionistic fuzzy divergence measure given by (16) between a diagnosis and all patients in the context of symptoms. This process is repeated for all diagnosis. Finally we assign

to the i^{th} patient the diagnosis whose symptoms have the lowest intuitionistic fuzzy divergence measure from patient's symptoms.
Since the patient's condition and variation of symptoms will change $w\ .r.\ t$ time, for a physician, it might not be feasible to make a decision on an examination of the patient. One approach which may be effective in most situations is to inspect the patient at regular time intervals and suggest diagnosis accordingly. But whatever is the approach, the symptoms for the diagnosis and the patients are bound to change with passage of time. In particular, the hesitation margin will decrease resulting in an increase of both membership and non membership functions since a particular symptom for diagnosis (or, for a patient) will become less or more prevalent with passage of time. As a result, representation of medical data by intuitionistic fuzzy sets, given by the triplet (μ, ν, π) will also undergo changes. However we intend to reflect this change in the divergence measure which we are using for obtaining a proper diagnosis and not in the medical data as such. We now replace the triplet (μ, ν, π) by $(\mu + \alpha\pi, \nu + \alpha\pi, \pi - 2\alpha\pi)$ in the measure (16) thereby obtaining a modified biparametric divergence measure given by

$$D_{(\alpha_1,\alpha_2)}(A||B)$$

$$= 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\left(\mu_A(x_i) + \alpha_1\pi_A(x_i)\right), \left(\mu_B(x_i) + \alpha_1\pi_B(x_i)\right)\right) \\ + min\left(\left(\nu_A(x_i) + \alpha_2\pi_A(x_i)\right), \left(\nu_B(x_i) + \alpha_2\pi_B(x_i)\right)\right) \\ + \min((\pi_A(x_i) - (\alpha_1 + \alpha_2)\pi_A(x_i)), \\ (\pi_B(x_i) - (\alpha_1 + \alpha_2)\pi_B(x_i))) \end{array}\right)\right). \quad (18)$$

Similarly the parametric version of the measure (11) can be written as

$$D_{4(\alpha)}(A||B) = \sum_{i=1}^{n}\left(\begin{array}{c} (\mu_A(x_i) + \alpha\pi_A(x_i))\left(ln\frac{(\mu_A(x_i)+\alpha\pi_A(x_i))}{(1/2)(\mu_A(x_i)+\alpha\pi_A(x_i))+\mu_B(x_i)+\alpha\pi_B(x_i)}\right) \\ + (\nu_A(x_i) + \alpha\pi_A(x_i))\, ln\left(\frac{(\nu_A(x_i)+\alpha\pi_A(x_i))}{(1/2)((\nu_A(x_i)+\alpha\pi_A(x_i))+(\nu_B(x_i)+\alpha\pi_B(x_i)))}\right) \\ + (\pi_A(x_i) - 2\alpha\pi_A(x_i))\, ln\left(\frac{(\pi_A(x_i)-2\alpha\pi_A(x_i))}{(1/2)((\pi_A(x_i)-2\alpha\pi_A(x_i))+(\pi_B(x_i)-2\alpha\pi_B(x_i)))}\right) \end{array}\right). \quad (19)$$

However, the same process can't be repeated for the measure given by (7) because the measure (7) proposed by Vlachos & Sergiadis [21]assumes negative values (as shown in section 2) and replacing the triplet (μ, ν, π) by $(\mu + \alpha\pi, \nu + \alpha\pi, \pi - 2\alpha\pi)$ in (7) will only worsen the situation. However if we replace (μ, ν, π) by $\{\mu + \alpha\pi, \nu + (1-\alpha)\pi, 0\}$ in (7), the resulting biparametric measure is given by $D_{1(\alpha_1,\alpha_2)}(A||B)$

$$= \sum_{i=1}^{n}\left(\begin{array}{l} (\mu_A(x_i) + \alpha_1\pi_A(x_i))\, ln\left(\frac{(\mu_A(x_i)+\alpha_1\pi_A(x_i))}{(1/2)((\mu_A(x_i)+\alpha_1\pi_A(x_i))+(\mu_B(x_i)+\alpha_2\pi_B(x_i)))}\right) \\ + (\nu_A(x_i) + (1-\alpha_1)\pi_A(x_i)) \\ ; \qquad \times ln\left(\frac{(\nu_A(x_i)+(1-\alpha_1)\pi_A(x_i))}{(1/2)((\nu_A(x_i)+(1-\alpha_1)\pi_A(x_i))+(\nu_B(x_i)+(1-\alpha_2)\pi_B(x_i)))}\right) \end{array}\right) \quad (20)$$

will always be positive because the pair $\{\mu + \alpha\pi, \nu + (1-\alpha)\pi, 0\}$ forms a probability distribution and the measure given by (20) will satisfy the Shannon inequality. Equations (18) and (20) have two parameters pair α_1 and α_2 which basically represents the amount of variations which the membership and non membership functions(and as a result the hesitation parameter) can possibly have. However the value of parameters $\alpha_1$and α_2 should be such that

$$0 \leq (\mu_A(x_i) + \alpha_1\pi_A(x_i)) \leq 1; 0 \leq (\mu_B(x_i) + \alpha_1\pi_B(x_i)) \leq 1;$$
$$0 \leq (\nu_A(x_i) + \alpha_2\pi_A(x_i)) \leq 1; 0 \leq (\nu_B(x_i) + \alpha_2\pi_B(x_i)) \leq 1;$$
$$0 \leq (\pi_A(x_i) - (\alpha_1 + \alpha_2)\pi_A(x_i)) \leq 1; 0 \leq (\pi_B(x_i) - (\alpha_1 + \alpha_2)\pi_B(x_i)) \leq 1.$$

We have evaluated the measures (18) and (20) for different values of $\alpha_1$and α_2 in tables 4 - 20 which shows that such $\alpha_1$and α_2 exists satisfying the above inequalities. The same approach could be applied for other measures also; however we restrict ourselves to the three parametric divergence measures given by (18), (19) and (20) and utilized these measures for obtaining a proper diagnosis for the data given in Tables 1 and 2. We have not considered the measures (7), (8), (10), (13) and (14) since they do not satisfy axiom M2. Further the measures (9) and (12) already contain parameters and as such further parametrization will worsen the situations as the dependence between the original parameters and the introduced parameters can't be established. The evaluated results for measures (16), (18), (19) and (20) are given in Tables 3-19. The same problem has been considered in (De, Biswas & Roy [5]; Szmidt & Kacprzyk [18, 19, 20]; Vlachos & Sergiadis [21]; Hung [9]; Boran [3]). The results obtained in these studies and the evaluated results for measures(16), (18), (19) and (20) are shown in Table 20.

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Temperature	(0.4, 0.0,0.6)	(0.7, 0.0,0.3)	(0.3, 0.3,0.4)	(0.1, 0.7,0.2)	(0.1, 0.8, 0.1)
Headache	(0.3, 0.5,0.2)	(0.2, 0.6,0.2)	(0.6, 0.1,0.3)	(0.2, 0.4,0.4)	(0.0, 0.8,0.2)
Stomach pain	(0.1, 0.7,0.2)	(0.0, 0.9,0.1)	(0.2, 0.7,0.1)	(0.8, 0.0,0.2)	(0.2, 0.8, 0.0)
Cough	(0.4, 0.3,0.3)	(0.7, 0.0,0.3)	(0.2, 0.6,0.2)	(0.2, 0.7,0.1)	(0.2, 0.8, 0.0)
Chest Pain	(0.1, 0.7,0.2)	(0.1, 0.8,0.1)	(0.1, 0.9,0.0)	(0.2, 0.7,0.1)	(0.8, 0.1,0.1)

Table 1. Symptoms Characteristics for the Diagnosis Considered by De et al. [5]

	Temperature	Headache	Stomach pain pain	Cough	Chest Pain
Al	(0.8, 0.1, 0.1)	(0.6, 0.1, 0.3)	(0.2, 0.8, 0.0)	(0.6, 0.1, 0.3)	(0.1, 0.6, 0.3)
Bob	(0.0, 0.8, 0.2)	(0.4, 0.4, 0.2)	(0.6, 0.1, 0.3)	(0.1, 0.7, 0.2)	(0.1, 0.8, 0.1)
Joe	(0.8, 0.1, 0.1)	(0.8, 0.1, 0.1)	(0.0, 0.6, 0.4)	(0.2, 0.7, 0.1)	(0.0, 0.5, 0.5)
Ted	(0.6, 0.1, 0.3)	(0.5, 0.4, 0.1)	(0.3, 0.4, 0.3)	(0.7, 0.2, 0.1)	(0.3, 0.4, 0.3)

Table 2. Symptoms Characteristics for the Patients Considered by De et al. [5]

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.488633**	0.497873	0.516739	0.648146	0.680382
Bob	0.589985	0.659267	0.551522	**0.460767**	0.592593
Joe	0.510971	0.566079	**0.482976**	0.598151	0.633188
Ted	**0.501239**	0.534914	0.56672	0.604389	0.652901
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 3. Diagnosis Results for the Proposed Biparametric Divergence Measure (18); α_1=58/100; α_2=41/100

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.512308**	0.515604	0.536378	0.672543	0.697593
Bob	0.615287	0.674383	0.572608	**0.463947**	0.611535
Joe	0.546245	0.606254	**0.517461**	0.617941	0.656308
Ted	**0.520769**	0.556606	0.593354	0.618827	0.678303
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 4. Diagnosis Results for the Proposed Biparametric Divergence Measure(18) ;α_1=25/50; α_2=4/50

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.524915	**0.520769**	0.541618	0.680382	0.701342
Bob	0.613741	0.673463	0.571322	**0.471929**	0.617056
Joe	0.561707	0.610433	**0.532199**	0.639355	0.675765
Ted	**0.532199**	0.557455	0.596187	0.627048	0.683854
Optimal – Al(Malaria);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 5. Diagnosis Results for the Proposed Biparametric Divergence Measure(18); α_1=3/10; α_2=1/10

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.500918	**0.491903**	0.507293	0.643235	0.651569
Bob	0.564843	0.645091	0.530589	**0.46853**	0.567731
Joe	0.521682	0.543001	**0.489014**	0.622051	0.632842
Ted	**0.503434**	0.506643	0.537439	0.599871	0.627125
Optimal – Al(Malaria);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 6. Diagnosis Results for the Proposed Biparametric Divergence Measure(18); α_1=23/100; α_2=69/100

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.500218**	0.50226	0.522841	0.655172	0.685014
Bob	0.590745	0.659723	0.552156	**0.467932**	0.597278
Joe	0.524915	0.571322	**0.498179**	0.614845	0.648372
Ted	**0.512514**	0.537424	0.571322	0.612637	0.659723
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 7. Diagnosis Results for the Proposed Biparametric Divergence Measure (18); α_1=2/5; α_2=2/5

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.4935**	0.497364	0.517875	0.648372	0.680845
Bob	0.586189	0.65699	0.548354	**0.46554**	0.592049
Joe	0.515604	0.562346	**0.489646**	0.605596	0.638905
Ted	**0.507583**	0.533034	0.565545	0.608012	0.653582
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 8. Diagnosis Results for the Proposed Biparametric Divergence Measure (18); α_1=23/50; α_2=23/50

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.508403**	0.512514	0.533242	0.668868	0.694321
Bob	0.606035	0.668868	0.564905	**0.467932**	0.608232
Joe	0.53952	0.595097	**0.514573**	0.62371	0.659723
Ted	**0.520769**	0.550043	0.586406	0.619271	0.673463
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 9. Diagnosis Results for the Proposed Biparametric Divergence Measure (18); α_1=2/5; α_2=1/5

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.506353	**0.504305**	0.526992	0.664288	0.68966
Bob	0.586406	0.650635	0.550043	**0.459973**	0.606035
Joe	0.545824	0.592919	**0.522841**	0.632629	0.673463
Ted	**0.514573**	0.537424	0.577767	0.612637	0.673463
Optimal – Al(Malaria);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 10. Diagnosis Results for the Proposed Biparametric Divergence Measure (18); α_1=1/5; α_2=1/5

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.177016**	0.227676	0.312961	1.031289	1.255492
Bob	0.666913	1.247152	0.420522	**0.069276**	0.787391
Joe	0.28676	0.590486	**0.180747**	0.921666	1.15896
Ted	**0.161949**	0.315276	0.394506	0.652821	0.863239
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 11. Diagnosis Results for the Parametric Divergence Measure (19);α = 23/50

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.055784**	0.168215	0.174799	0.855219	1.110792
Bob	0.798154	1.3707	0.510177	**0.174128**	0.729255
Joe	0.286258	0.546736	**0.124977**	1.027321	1.222081
Ted	**0.065272**	0.240187	0.258661	0.56799	0.74152
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 12. Diagnosis Results for the Biparametric Divergence Measure (20); α_1=1/5; α_2= 3/5

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.360805	**0.322654**	0.464794	1.161537	1.361289
Bob	0.534151	1.100336	0.329363	**0.052015**	0.812097
Joe	0.397358	0.619977	**0.279363**	0.96693	1.202559
Ted	**0.354572**	0.402452	0.514533	0.750427	0.970804
Optimal – Al(Malaria);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

Table 13. Diagnosis Results for the Biparametric Divergence Measure (20); α_1=3/5; α_2= 1/5

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.117421**	0.174262	0.228721	0.86654	1.162256
Bob	0.652285	1.260407	0.415404	**0.145314**	0.737283
Joe	0.329741	0.506986	**0.211793**	1.167457	1.39844
Ted	**0.110392**	0.227214	0.270552	0.591795	0.777143
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

TABLE 14. Diagnosis Results for the Biparametric Divergence Measure (20); α_1=5/50; α_2= 15/50

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.280841	**0.257648**	0.37038	1.015481	1.270626
Bob	0.51058	1.108767	0.301251	**0.073779**	0.759974
Joe	0.356082	0.529442	**0.244808**	1.070323	1.312468
Ted	**0.276479**	0.32286	0.400472	0.686173	0.887152
Optimal – Al(Malaria);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

TABLE 15. Diagnosis Results for the Biparametric Divergence Measure (20); α_1=15/50; α_2= 5/50

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.235099	**0.228613**	0.313754	0.918091	1.25778
Bob	**0.144185**	1.227575	0.358342	0.15742	0.802338
Joe	0.475717	0.539414	**0.413363**	1.442464	1.722656
Ted	**0.213692**	0.265239	0.313684	0.656491	0.854581
Optimal – Al(Malaria);Bob(Viral Fever);Joe(Typhoid);Ted(Viral Fever)					

TABLE 16. Diagnosis Results for the Biparametric Divergence Measure (20); α_1= 0; α_2= 0

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.208148**	0.212968	0.300096	0.921133	1.223396
Bob	0.556283	1.181196	0.339663	**0.116246**	0.752464
Joe	0.358839	0.497613	**0.265784**	1.214807	1.466645
Ted	**0.195826**	0.262139	0.319113	0.640677	0.83245
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

TABLE 17. Diagnosis Results for the Biparametric Divergence Measure (20); α_1= 0.1; α_2= 0.9

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.320863**	0.396169	0.29821	1.164064	1.470964
Bob	0.387038	0.888615	0.050633	**0.089844**	0.887247
Joe	0.548721	**0.783782**	**0.345765**	1.266508	1.540685
Ted	**0.216316**	0.361034	0.241797	0.765748	1.077892
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

TABLE 18. Diagnosis Results for the Biparametric Divergence Measure (20); α_1= 0.4; α_2= 0.6

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	**0.269747**	0.339439	0.381499	1.168964	1.337748
Bob	0.82411	1.373496	0.515239	**0.125549**	0.856779
Joe	0.355791	0.726914	**0.208327**	0.900051	1.111062
Ted	**0.282256**	0.455587	0.494222	0.750974	0.955336
Optimal – Al(Viral Fever);Bob(Stomach Problem);Joe(Typhoid);Ted(Viral Fever)					

TABLE 19. Diagnosis Results for the Biparametric Divergence Measure (20); $\alpha_1 = 0.7$; $\alpha_2 = 0.3$

5. Discussion

It is clear that Bob suffers from a stomach problem and this diagnosis does not change even after variation in parameter α. Joe suffers from typhoid in eleven out of the thirteen approaches considered. The remaining two approaches are for non parametric measures given by (11) and (16) suggesting viral fever as the possible diagnosis. However this converts into typhoid after the parameter α is introduced in these measures. Therefore we can say that typhoid is the correct diagnosis for Joe. Again Ted suffers from Viral Fever in twelve out of thirteen approaches, so we can say viral fever is the correct diagnosis for Joe because this diagnosis does not change with change in parameter α. Finally four out of the thirteen methods indicate that Al suffers from malaria, and other methods indicate that Al suffers from viral fever. Also we observe that parametric measures given by (18) and (19) suggests viral fever as the possible diagnosis however the parametric measure (20) suggests malaria as the possible diagnosis for smaller values of α and viral fever for higher values of α. Therefore, we can say that it is slightly difficult to predict that Al suffers from viral fever or malaria although odds are tilted slightly in favour of viral fever. On the basis of above analysis, it is clear that determining a medical diagnosis on the basis of symptoms is both difficult and challenging. Many signs and symptoms are non specific with many symptoms being attributed to multiple conditions Thus performing differential diagnosis, in which several possible explanations are compared and contrasted, is necessary for correct diagnosis. Information collected from patient's medical history and proper physical examination is also necessary for proper diagnosis. However the advice of a medical professional is always advisable before determining a medical diagnosis based on symptoms.

6. Conclusion

In the present work, we have defined a new symmetric divergence measure for Atanassov's intuitionistic fuzzy sets ($AIFS$). Its properties were studied and finally the efficiency of the proposed divergence measure in the context of medical diagnosis has been demonstrated. Work on one and two parametric generalizations of the proposed measure is in progress and will be reported elsewhere.

7. Appendix : Proof of the Properties

D1.

Proof. The symmetry of measure (16) $w.r.t$ their arguments is obvious. Further for all $x_i \in X$, we have

$$min\left(\mu_A(x_i), \mu_B(x_i)\right) \leq \left(\frac{\mu_A(x_i) + \mu_B(x_i)}{2}\right)$$

$$min\left(\nu_A(x_i), \nu_B(x_i)\right) \le \left(\frac{\nu_A(x_i) + \nu_B(x_i)}{2}\right)$$

$$min\left(\pi_A(x_i), \pi_B(x_i)\right) \le \left(\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right)$$

On adding the above three equations and take summations on both sides, we get

$$\begin{aligned}\textstyle\sum_{i=1}^{n} &\left(min\left(\mu_A(x_i), \mu_B(x_i)\right) + min\left(\nu_A(x_i), \nu_B(x_i)\right) + min\left(\pi_A(x_i), \pi_B(x_i)\right)\right) \\ &\le \textstyle\sum_{i=1}^{n} \left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2} + \frac{\nu_A(x_i)+\nu_B(x_i)}{2} + \frac{\pi_A(x_i)+\pi_B(x_i)}{2}\right)\end{aligned}$$

$$\Rightarrow 0 \le \frac{1}{n}\sum_{i=1}^{n}\left(min\left(\mu_A(x_i), \mu_B(x_i)\right) + min\left(\nu_A(x_i), \nu_B(x_i)\right) + min\left(\pi_A(x_i), \pi_B(x_i)\right)\right) \le 1$$

$$\Rightarrow \frac{1}{2} \le \frac{1}{2}\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l} min\left(\mu_A(x_i), \mu_B(x_i)\right) \\ +min\left(\nu_A(x_i), \nu_B(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_B(x_i)\right)\end{array}\right)\right) \le 1$$

$$\Rightarrow 0 \le 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l} min\left(\mu_A(x_i), \mu_B(x_i)\right) \\ +min\left(\nu_A(x_i), \nu_B(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_B(x_i)\right)\end{array}\right)\right) \le 1$$

$$\Rightarrow 0 \le D\left(A||B\right) \le 1.$$

□

D2.

Proof. Let $A = B$, then it is obvious that$D\left(A||B\right) = 0$. Consider $D\left(A||B\right) = 0$

$$\Rightarrow 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l} min\left(\mu_A(x_i), \mu_B(x_i)\right) \\ +min\left(\nu_A(x_i), \nu_B(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_B(x_i)\right)\end{array}\right)\right) = 0$$

$$\Rightarrow min\left(\mu_A(x_i), \mu_B(x_i)\right) + min\left(\nu_A(x_i), \nu_B(x_i)\right) + min\left(\pi_A(x_i), \pi_B(x_i)\right) = 1$$

$$\Rightarrow \frac{2 - |\mu_A(x_i) - \mu_B(x_i)| - |\nu_A(x_i) - \nu_B(x_i)| - |\pi_A(x_i) - \pi_B(x_i)|}{2} = 1$$

$$\Rightarrow |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)| = 0$$

$\Rightarrow \mu_A(x_i) = \mu_B(x_i)$, $\nu_A(x_i) = \nu_B(x_i)$, $\pi_A(x_i) = \pi_B(x_i)$for all $x_i \in X$.

Therefore, two sets coincide, *i.e.* $A = B$. □

D3.

Proof. For $A, B \in AIFS(X)$,we have

$$|min\left(\left(\mu_A(x_i), \mu_C(x_i)\right) - min\left(\left(\mu_B(x_i), \mu_C(x_i)\right)\right| \le |\mu_A(x_i) - \mu_B(x_i)| \tag{21}$$

$$|max\left(\left(\nu_A(x_i), \nu_C(x_i)\right) - max\left(\left(\nu_B(x_i), \nu_C(x_i)\right)\right| \le |\nu_A(x_i) - \nu_B(x_i)| \tag{22}$$

and

$$\begin{aligned}&\left|\begin{array}{l}\left(1 - min\left(\left(\mu_A(x_i), \mu_C(x_i)\right) - max\left(\left(\nu_A(x_i), \nu_C(x_i)\right)\right)\right) \\ -\left(1 - min\left(\left(\mu_B(x_i), \mu_C(x_i)\right) - max\left(\left(\nu_B(x_i), \nu_C(x_i)\right)\right)\right)\end{array}\right| \\ &\qquad \le |(1 - \mu_A(x_i) - \nu_A(x_i)) - (1 - \mu_B(x_i) - \nu_B(x_i))|\end{aligned} \tag{23}$$

Adding (21), (22) and (23) yields

$$\begin{aligned}&|min\left((\mu_A(x_i),\mu_C(x_i)\right)-min\left((\mu_B(x_i),\mu_C(x_i)\right)|\\&+|max\left((\nu_A(x_i),\nu_C(x_i)\right)-max\left((\nu_B(x_i),\nu_C(x_i)\right)|\\&+\left|\begin{array}{l}(1-min\left((\mu_A(x_i),\mu_C(x_i)\right)-max\left((\nu_A(x_i),\nu_C(x_i)\right))\\-(1-min\left((\mu_B(x_i),\mu_C(x_i)\right)-max\left((\nu_B(x_i),\nu_C(x_i)\right))\end{array}\right|\\&\quad\le|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|\\&\quad+|(1-\mu_A(x_i)-\nu_A(x_i))-(1-\mu_B(x_i)-\nu_B(x_i))|\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &2-|min\left((\mu_A(x_i),\mu_C(x_i)\right)-min\left((\mu_B(x_i),\mu_C(x_i)\right)|\\&-|max\left((\nu_A(x_i),\nu_C(x_i)\right)-max\left((\nu_B(x_i),\nu_C(x_i)\right)|\\&-\left|\begin{array}{l}(1-min\left((\mu_A(x_i),\mu_C(x_i)\right)-max\left((\nu_A(x_i),\nu_C(x_i)\right))\\-(1-min\left((\mu_B(x_i),\mu_C(x_i)\right)-max\left((\nu_B(x_i),\nu_C(x_i)\right))\end{array}\right|\\&\quad\ge 2-|\mu_A(x_i)-\mu_B(x_i)|-|\nu_A(x_i)-\nu_B(x_i)|\\&\quad-|(1-\mu_A(x_i)-\nu_A(x_i))-(1-\mu_B(x_i)-\nu_B(x_i))|\end{aligned}$$

$$\begin{aligned}&\Rightarrow 1-log_2\left(1+\tfrac{1}{2n}\sum_{i=1}^{n}\left(\begin{array}{l}2-|min\left((\mu_A(x_i),\mu_C(x_i)\right)-min\left((\mu_B(x_i),\mu_C(x_i)\right)|\\-|max\left((\nu_A(x_i),\nu_C(x_i)\right)-max\left((\nu_B(x_i),\nu_C(x_i)\right)|\\-\left|\begin{array}{l}(1-min\left((\mu_A(x_i),\mu_C(x_i)\right)-max\left((\nu_A(x_i),\nu_C(x_i)\right))\\-(1-min\left((\mu_B(x_i),\mu_C(x_i)\right)-max\left((\nu_B(x_i),\nu_C(x_i)\right))\end{array}\right|\end{array}\right)\right)\\&\le 1-log_2\left(1+\tfrac{1}{2n}\sum_{i=1}^{n}\left(2-|\mu_A(x_i)-\mu_B(x_i)|-|\nu_A(x_i)-\nu_B(x_i)|-|\pi_A(x_i)-\pi_B(x_i)|\right)\right)\end{aligned}$$

$\Rightarrow D\left(A\bigcap C||B\bigcap C\right)\le D\left(A||B\right)$for every $C\in IFS(X)$, ☐

D4.

Proof. The proof is on similar lines as in D3. ☐

D5.

Proof. We have

$$\begin{aligned}D\left(A||B\right)&=1-log_2\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l}min\left(\mu_A(x_i),\mu_B(x_i)\right)\\+min\left(\nu_A(x_i),\nu_B(x_i)\right)\\+min\left(\pi_A(x_i),\pi_B(x_i)\right)\end{array}\right)\right)\\&=1-log_2\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l}min\left(\nu_A(x_i),\nu_B(x_i)\right)\\+min\left(\mu_A(x_i),\mu_B(x_i)\right)\\+min\left(\pi_A(x_i),\pi_B(x_i)\right)\end{array}\right)\right)\end{aligned}$$

$=D\left(A^C||B^c\right).$ ☐

D6.

Proof. We have

$$\begin{aligned}D\left(A||B^C\right)&=1-log_2\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l}min\left(\mu_A(x_i),\nu_B(x_i)\right)\\+min\left(\nu_A(x_i),\mu_B(x_i)\right)\\+min\left(\pi_A(x_i),\pi_B(x_i)\right)\end{array}\right)\right)\\&=1-log_2\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l}min\left(\nu_A(x_i),\mu_B(x_i)\right)\\+min\left(\mu_A(x_i),\nu_B(x_i)\right)\\+min\left(\pi_A(x_i),\pi_B(x_i)\right)\end{array}\right)\right)\end{aligned}$$

$=D\left(A^C||B\right).$ ☐

D7.

Proof. Let A be a crisp set, *i.e.* $\mu_A(x_i)=0,\nu_A(x_i)=1$, then it is obvious that $D\left(A||A^C\right)=1$.

Now, consider

$$D\left(A||A^C\right) = 1$$

$$\Rightarrow 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\mu_A(x_i), \nu_A(x_i)\right) \\ +min\left(\nu_A(x_i), \mu_A(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_A(x_i)\right)\end{array}\right)\right) = 1$$

$$\Rightarrow log_2\left(\frac{2}{1 + \frac{1}{n}\sum_{i=1}^{n}\left(2\,min\left(\mu_A(x_i), \nu_A(x_i)\right) + \pi_A(x_i)\right)}\right) = log2$$

$\Rightarrow 2\,min\left(\mu_A(x_i), \nu_A(x_i)\right) + \pi_A(x_i) = 0$ for all $x_i \in X$.
$\Rightarrow \mu_A(x_i) + \nu_A(x_i) - 2\left(\frac{\mu_A(x_i)+\nu_A(x_i)-|\mu_A(x_i)-\nu_A(x_i)|}{2}\right) = 1$ for all $x_i \in X$.
$\Rightarrow |\mu_A(x_i) - \nu_A(x_i)| = 1$ for all $x_i \in X$.
$\Rightarrow \mu_A(x_i) + \nu_A(x_i) - 2\left(\frac{\mu_A(x_i)+\nu_A(x_i)-|\mu_A(x_i)-\nu_A(x_i)|}{2}\right) = 1$ for all $x_i \in X$.
$\Rightarrow |\mu_A(x_i) - \nu_A(x_i)| = 1$ for all $x_i \in X$. □

D8.

Proof. Let $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$,then from (16), it is obvious

$$D\left(A||A^C\right) = 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\mu_A(x_i), \nu_A(x_i)\right) \\ +min\left(\nu_A(x_i), \mu_A(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_A(x_i)\right)\end{array}\right)\right)$$

$$= 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\mu_A(x_i) + \nu_A(x_i) + \pi_A(x_i)\right)\right)$$

$$= 0.$$

Now, if we consider

$$D\left(A||A^C\right) = 0$$

$$\Rightarrow 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\mu_A(x_i), \nu_A(x_i)\right) \\ +min\left(\nu_A(x_i), \mu_A(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_A(x_i)\right)\end{array}\right)\right) = 0$$

$$\Rightarrow -log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\mu_A(x_i), \nu_A(x_i)\right) \\ +min\left(\nu_A(x_i), \mu_A(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_A(x_i)\right)\end{array}\right)\right) = -log_2 2$$

$$\Rightarrow min\left(\mu_A(x_i), \nu_A(x_i)\right) + min\left(\nu_A(x_i), \mu_A(x_i)\right) + min\left(\pi_A(x_i), \pi_A(x_i)\right) = 1.$$

$$\Rightarrow 2min\left(\mu_A(x_i), \nu_A(x_i)\right) + \pi_A(x_i) = 1.$$

$$\Rightarrow \mu_A(x_i) + \nu_A(x_i) - 2\left(\frac{\mu_A(x_i) + \nu_A(x_i) - |\mu_A(x_i) - \nu_A(x_i)|}{2}\right) = 0.$$

$\Rightarrow |\mu_A(x_i) - \nu_A(x_i)| = 0$ for all $x_i \in X$.
$\Rightarrow \mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.
Therefore, $D\left(A||A^C\right) = 0$ if and only if $\mu_A(x_i) = \nu_A(x_i)$for all $x_i \in X$. □

D9.

Proof. From (16), we have

$$D\left(A||A\bigcup B\right) = 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\mu_A(x_i), max\left(\mu_A(x_i), \mu_B(x_i)\right)\right) \\ + min\left(\nu_A(x_i), min\left(\nu_A(x_i), \nu_B(x_i)\right)\right) \\ + min\left(\begin{array}{c} \left(1 - \mu_A(x_i) - \nu_A(x_i)\right), \\ \left(\begin{array}{c} 1 - max\left(\mu_A(x_i), \mu_B(x_i)\right) \\ -min\left(\nu_A(x_i), \nu_B(x_i)\right)\end{array}\right)\end{array}\right)\end{array}\right)\right)$$

$$D\left(A\bigcap B||B\right) = 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(min\left(\mu_A(x_i), \mu_B(x_i)\right), \mu_B(x_i)\right) \\ + min\left(max\left(\nu_A(x_i), \nu_B(x_i)\right), \nu_B(x_i)\right) \\ + min\left(\begin{array}{c} \left(\begin{array}{c} 1 - min\left(\mu_A(x_i), \mu_B(x_i)\right) \\ - max\left(\nu_A(x_i), \nu_B(x_i)\right)\end{array}\right), \\ \left(1 - \mu_B(x_i) - \nu_B(x_i)\right)\end{array}\right)\end{array}\right)\right)$$

For $A \subseteq B$

$$D\left(A||A\bigcup B\right) = D\left(A\bigcap B||B\right) = D\left(A||B\right). \tag{24}$$

Again for$B \subseteq A$

$$D\left(A||A\bigcup B\right) = D\left(A\bigcap B||B\right) = 0 \leq D\left(A||B\right). \tag{25}$$

The proof follows from (24) and (25). □

D10.

Proof. We have
$D\left(A\bigcap B||A\bigcup B\right)$

$$= 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(min\left(\mu_A(x_i), \mu_B(x_i)\right), max\left(\mu_A(x_i), \mu_B(x_i)\right)\right) \\ +min\left(max\left(\nu_A(x_i), \nu_B(x_i)\right), min\left(\nu_A(x_i), \nu_B(x_i)\right)\right) \\ + min\left(\begin{array}{c} \left(1 - min\left(\mu_A(x_i), \mu_B(x_i)\right) - max\left(\nu_A(x_i), \nu_B(x_i)\right)\right), \\ \left(1 - max\left(\mu_A(x_i), \mu_B(x_i)\right) - min\left(\nu_A(x_i), \nu_B(x_i)\right)\right)\end{array}\right)\end{array}\right)\right)$$
$$= 1 - log_2\left(1 + \frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{c} min\left(\mu_A(x_i), \mu_B(x_i)\right) \\ +min\left(\nu_A(x_i), \nu_B(x_i)\right) \\ +min\left(\pi_A(x_i), \pi_B(x_i)\right)\end{array}\right)\right)$$
$$= D\left(A||B\right).$$ □

D11.

Proof. For $A \subseteq B \subseteq C$ we have

$$\begin{array}{l} \left|\mu_A(x_i) - \mu_B(x_i)\right| \leq \left|\mu_A(x_i) - \mu_C(x_i)\right|; \left|\nu_A(x_i) - \nu_B(x_i)\right| \\ \leq \left|\nu_A(x_i) - \nu_c(x_i)\right|; \left|\pi_A(x_i) - \pi_B(x_i)\right| \leq \left|\pi_A(x_i) - \pi_c(x_i)\right| \end{array}$$

The above three inequalities taken together gives

$$\begin{array}{l} 2\left(1 - \left|\mu_A(x_i) - \mu_B(x_i)\right| - \left|\nu_A(x_i) - \nu_B(x_i)\right| - \left|\pi_A(x_i) - \pi_B(x_i)\right|\right) \\ \geq 2\left(1 - \left|\mu_A(x_i) - \mu_C(x_i)\right| - \left|\nu_A(x_i) - \nu_c(x_i)\right| - \left|\pi_A(x_i) - \pi_c(x_i)\right|\right) \end{array}$$

$$\Rightarrow 1 - log_2\left(1 + \frac{1}{2n}\sum_{i=1}^{n}\left(2\left(1 - \left|\mu_A(x_i) - \mu_B(x_i)\right| - \left|\nu_A(x_i) - \nu_B(x_i)\right| - \left|\pi_A(x_i) - \pi_B(x_i)\right|\right)\right)\right)$$
$$\leq 1 - log_2\left(1 + \frac{1}{2n}\sum_{i=1}^{n}\left(2\left(1 - \left|\mu_A(x_i) - \mu_C(x_i)\right| - \left|\nu_A(x_i) - \nu_c(x_i)\right| - \left|\pi_A(x_i) - \pi_c(x_i)\right|\right)\right)\right)$$
$$\Rightarrow D\left(A||B\right) \leq D\left(A||C\right),$$ □

D12.

Proof. The proof is on similar lines as in **D11.** □

	Szmidt & Kacprzcyk [16]	Szmidt & Kacprzyk [17]	Wei & Yei [20]	Boran & Akay [3]	Vlachos & Sergiadis [19]	PBPM (26) for α_1=58/100, α_2=41/100	PBPM (26) for α_1=25/50, α_2=4/50	PBPM (26) for α_1=3/10, α_2=1/10
Al	Malaria	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Malaria
Bob	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem
Joe	Typhoid	Typhoid	Viral Fever	Typhoid	Typhoid	Typhoid	Typhoid	Typhoid
Ted	Viral Fever	Malaria	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Viral Fever

	PBPM (26) for α_1=23/100, α_2=69/100	PBPM (26) for α_1=2/5, α_2=2/5	PBPM (26) for α_1=23/50, α_2=23/50	PBPM (26) for α_1=2/5, α_2=1/5,	PBPM (26) for α_1=1/5, α_2=1/5,	Parametric measure (27) for α=23/50	BPM (28) for α_1=1/5, α_2=3/5	BPM (28) for α_1=3/5 α_2=1/5
Al	Malaria	Viral Fever	Viral Fever	Viral Fever	Malaria	Viral Fever	Viral Fever	Malaria
Bob	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem
Joe	Typhoid	Typhoid	Typhoid Fever	Typhoid	Typhoid	Typhoid	Typhoid	Typhoid
Ted	Viral Fever	Malaria	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Viral Fever	Viral Fever

	BPM (28) for α_1=5/50, α_2=15/50	BPM (28) for α_1=15/50, α_2=5/50	BPM (28) for α_1=0, α_2=0	BPM (28) for α_1=0.1, α_2=0.9,	BPM (28) for α_1=0.4, α_2=0.6,	BPM (28) for α=0.7, α_2=0.3
Al	Malaria	Viral Fever	Viral Fever	Viral Fever	Malaria	Viral Fever
Bob	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem	Stomach Problem
Joe	Typhoid	Typhoid	Typhoid	Typhoid	Typhoid	Typhoid
Ted	Viral Fever	Malaria	Viral Fever	Viral Fever	Viral Fever	Viral Fever

TABLE 20. Comparison of Results

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