

CORRESPONDENCE BETWEEN PROBABILISTIC NORMS AND FUZZY NORMS

H. P. ZHANG

ABSTRACT. In this paper, the connection between Menger probabilistic norms and Höhle probabilistic norms is discussed. In addition, the correspondence between probabilistic norms and Wu-Fang fuzzy (semi-) norms is established. It is shown that a probabilistic norm (with triangular norm \min) can generate a Wu-Fang fuzzy semi-norm and conversely, a Wu-Fang fuzzy norm can generate a probabilistic norm.

1. Introduction

In many cases, the length of a vector is uncertain because of the existence of randomness. This motivates the introduction of probabilistic (also known as random) norms [7, 15, 19]. One may refer to [20] for the history of probabilistic norms. The notion of a probabilistic norm (the values of the norms in a vector space are probability distribution functions rather than numbers) is natural generalization of that of an ordinary norm. The principal class of probabilistic norms are Menger probabilistic norms. From a different perspective, Höhle [8] proposed an alternative definition of a probabilistic norm. Katsaras [11] applied Höhle probabilistic norms to the study of linear fuzzy neighborhood spaces.

However, when the uncertainty is due to fuzziness rather than randomness, it is more appropriate to consider fuzzy norms. There are several definitions of a fuzzy norm by various authors from different points of view in the literature. In 1984, Katsaras [10] initiated the concept of fuzzy norm when studying fuzzy topological vector spaces. Almost at the same time, Wu and Fang [21] introduced their definition of fuzzy norm in order to extend Kolmogoroff's theorem from classical topological vector spaces to fuzzy topological vector spaces. Assigning a non-negative fuzzy number to each element of a vector space, Felbin [6] proposed another notion of fuzzy norm with the associated fuzzy metric of Kaleva-Seikkala type [9]. Cheng and Mordeson [5] introduced an idea of fuzzy norm with the corresponding fuzzy metric of Kramosil-Michálek type [14]. Bag and Samanta [3] redefined fuzzy norms slightly differently from that of Cheng and Mordeson [5] and obtained a useful decomposition theorem for fuzzy norms. Up to now, there have been many papers on these kinds of fuzzy norms and their applications such as [1, 12, 17, 18].

It goes without saying that it is quite necessary to investigate the interrelationship among the various notions of fuzzy norms. In 1985, Ma [16] compared Katsaras

Received: July 2015; Revised: September 2015; Accepted: November 2015

Key words and phrases: Probabilistic norm, Fuzzy norm.

fuzzy norms with Wu-Fang fuzzy norms, and found that a Katsaras fuzzy semi-norm is equivalent to a Wu-Fang fuzzy semi-norm and the only difference between Katsaras fuzzy norms and Wu-Fang fuzzy norms lies in separability. Concretely, the separability of a Wu-Fang fuzzy norm is stronger than that of a Katsaras fuzzy norm. Wu and Ma [22] discussed the connection between Katsaras fuzzy norms and Kaleva-Seikkala fuzzy metrics, and that between Höhle probabilistic norms and Kaleva-Seikkala fuzzy metrics (Note that the correspondence between fuzzy norms and probabilistic norms was not considered in [22]). In 2008, Bag and Samanta [4] made a comparative study among Katsaras fuzzy norms, Felbin fuzzy norms and their own fuzzy norms, and drew the conclusion that the different kinds of fuzzy norms can be broadly classified into two classes (One is of Katsaras type and the other of Felbin type). Note that Wu-Fang fuzzy norms were not included in [4].

The aim of this paper is to discuss the connection between Menger probabilistic norms and Höhle probabilistic norms, and to establish the correspondence between probabilistic norms and Wu-Fang fuzzy (semi-) norms.

2. Preliminaries

Throughout this paper, X always denotes a vector space (with θ as its zero element) over the field \mathbb{K} of real or complex numbers. For each $x \in X$ and $\lambda \in (0, 1]$, x_λ is the fuzzy set on X which takes value λ at x and 0 elsewhere. Such a fuzzy set is called a fuzzy point. X^* denotes the set of all fuzzy points of X . For $x_\lambda, y_\lambda \in X^*$, $k \in \mathbb{K}$, $x_\lambda + y_\lambda$, $k \cdot x_\lambda \in X^*$ are defined as follows:

$$x_\lambda + y_\lambda = (x + y)_\lambda, \quad k \cdot x_\lambda = (kx)_\lambda.$$

Let $\mathbb{R} = (-\infty, +\infty)$. We denote by $\mathcal{D}(\mathbb{R})$ the set of all distance distribution functions $f : \mathbb{R} \rightarrow [0, 1]$, i.e., f satisfies (a) f is non-decreasing, (b) f is left-continuous, (c) $\lim_{t \rightarrow +\infty} f(t) = 1$, and (d) $f(t) = 0$ for all $t \leq 0$.

Let $\mathbb{R}^+ = [0, +\infty)$. $\mathcal{D}(\mathbb{R}^+)$ denotes the set of all functions $f : \mathbb{R}^+ \rightarrow [0, 1]$ satisfying (a), (b), (c) and (e) $f(0) = 0$.

$\varepsilon_0 \in \mathcal{D}(\mathbb{R}^+)$ is a specific function defined by

$$\varepsilon_0(t) = \begin{cases} 0 & t = 0 \\ 1 & t > 0 \end{cases}$$

Let $f, g \in \mathcal{D}(\mathbb{R}^+)$. Then $f \oplus g \in \mathcal{D}(\mathbb{R}^+)$ is defined by

$$(f \oplus g)(t) = \sup_{x, y \in \mathbb{R}^+, x+y=t} \min\{f(x), g(y)\}.$$

A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (t-norm) [2, 13, 19] if the following conditions are satisfied: $T(a, 1) = a$ ($\forall a \in [0, 1]$); $T(a, b) = T(b, a)$ ($\forall a, b \in [0, 1]$); $a \geq b, c \geq d \implies T(a, c) \geq T(b, d)$; $T(a, T(b, c)) = T(T(a, b), c)$ ($\forall a, b, c \in [0, 1]$). Obviously, $T = \min$ is a t-norm. Moreover, it is the largest one among all t-norms (with respect to the pointwise order).

Definition 2.1. [8, 11] A Höhle probabilistic semi-norm on X is a mapping $P : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ with the following properties:

(HPN-1) $P(\theta) = \varepsilon_0$;

(HPN-2) $P(kx)(t) = P(x)(t/|k|)$ for all $t \in \mathbb{R}^+$, $x \in X$ and all $k \in \mathbb{K}$ with $k \neq 0$;

(HPN-3) $P(x) \oplus P(y) \leq P(x + y)$ for all $x, y \in X$.

If, in addition, $P(x)(0+) = 0$ for each $x \neq \theta$, then P is called a Höhle probabilistic norm.

Remark 2.2. It is not difficult to verify that condition (HPN-3) is equivalent to the following condition:

$$\min\{P(x)(s), P(y)(t)\} \leq P(x + y)(s + t) \text{ for all } s > 0, t > 0 \text{ and all } x, y \in X.$$

Definition 2.3. [7, 15, 19] A Menger probabilistic norm on X with respect to a t -norm T is a mapping $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ with the following properties:

(MPN-1) $F(x) = \varepsilon_0 \iff x = \theta$;

(MPN-2) $F(kx)(t) = F(x)(t/|k|)$ for all $t \in \mathbb{R}^+$, $x \in X$ and all $k \in \mathbb{K}$ with $k \neq 0$;

(MPN-3) $F(x + y)(s + t) \geq T(F(x)(s), F(y)(t))$ for all $s > 0, t > 0$ and all $x, y \in X$.

Remark 2.4. Originally, a Menger probabilistic norm is a mapping from X to $\mathcal{D}(\mathbb{R})$. Since $f \in \mathcal{D}(\mathbb{R})$ implies $f(t) = 0$ for each $t < 0$, there is an obvious one-to-one correspondence between $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R}^+)$. Hence, for the convenience of comparison between Menger probabilistic norms and Höhle probabilistic norms, we always assume that a Menger probabilistic norm is a mapping from X to $\mathcal{D}(\mathbb{R}^+)$.

Definition 2.5. [21, 22] A mapping $\|\cdot\| : X^* \rightarrow \mathbb{R}^+$ is called a fuzzy norm if it satisfies the following conditions:

(FN-1) $\|x_1\| = 0$ implies $x = \theta$;

(FN-2) $\|k \cdot x_\lambda\| = |k| \|x_\lambda\|$ for all $k \in \mathbb{K}$ and all $x_\lambda \in X^*$;

(FN-3) $\|x_\lambda + y_\lambda\| \leq \|x_\lambda\| + \|y_\lambda\|$ for all $x_\lambda, y_\lambda \in X^*$;

(FN-4) $\|x_\lambda\| = \inf_{0 < \mu < \lambda} \|x_\mu\|$ for all $x_\lambda \in X^*$.

If $\|\cdot\|$ satisfies (FN-2), (FN-3) and (FN-4), then we call it a fuzzy semi-norm.

Remark 2.6. The above definition of fuzzy norm is the special case of L -fuzzy norm due to Yan and Fang [23] with $L = [0, 1]$. Although, in form, the definition is a little different from that in [21, 22], it is not difficult to check that they are equivalent.

3. Relationship Between Höhle Probabilistic Norms and Menger Probabilistic Norms

Theorem 3.1. Let $P : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ be a Höhle probabilistic norm. Then P is a Menger probabilistic norm with respect to any t -norm T .

Proof. Since $T = \min$ is the largest t -norm, it suffices to show that P is a Menger probabilistic norm with respect to t -norm \min .

By the definition of Höhle probabilistic norm and that of Menger probabilistic norm and Remark 2.2, it remains to check that $P(x) = \varepsilon_0 \implies x = \theta$.

In fact, if $x \neq \theta$, since P is a Höhle probabilistic norm, we have $P(x)(0+) = 0$, which contradicts with $P(x) = \varepsilon_0$. \square

The following example shows that the converse of Theorem 3.1 does not hold in general.

Example 3.2. Let $(X, \|\cdot\|)$ be an ordinary normed space. Define a mapping F from X into $\mathcal{D}(\mathbb{R}^+)$ as follows:

If $x = \theta$, let $F(x) = \varepsilon_0$;

If $x \neq \theta$, let

$$F(x)(t) = \begin{cases} 0 & t = 0 \\ \frac{t}{\|x\|} + \frac{1}{2} & 0 < t < \|x\|/2 \\ 1 & t \geq \|x\|/2 \end{cases}$$

We shall show that F is a Menger probabilistic norm with respect to t-norm min, but it is not a Höhle probabilistic norm.

Since, for each $x \neq \theta$, $F(x)(0+) = 1/2$, F is not a Höhle probabilistic norm. But, we will check that F is a Menger probabilistic norm with respect to t-norm min.

In fact, it is easy to check that $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ is well defined. (MPN-1) and (MPN-2) are also satisfied. It remains to check (MPN-3).

Let $x, y \in X$, $s, t > 0$. Without loss of generality, we can assume that $x + y \neq \theta$, $x \neq \theta$, $y \neq \theta$ and $0 < s + t < \|x + y\|/2$. We will consider the following four possible cases:

Case 1: if $0 < s < \|x\|/2$ and $0 < t < \|y\|/2$, then

$$\begin{aligned} \frac{s+t}{\|x+y\|} + \frac{1}{2} &\geq \frac{s+t}{\|x\| + \|y\|} + \frac{1}{2} \geq \min\left\{\frac{s}{\|x\|}, \frac{t}{\|y\|}\right\} + \frac{1}{2} \\ &= \min\left\{\frac{s}{\|x\|} + \frac{1}{2}, \frac{t}{\|y\|} + \frac{1}{2}\right\}. \end{aligned}$$

So $F(x+y)(s+t) \geq \min\{F(x)(s), F(y)(t)\}$.

Case 2: if $0 < s < \|x\|/2$ and $t \geq \|y\|/2$, then $F(y)(t) = 1$ and

$$\frac{s+t}{\|x+y\|} + \frac{1}{2} \geq \frac{s+t}{\|x\| + \|y\|} + \frac{1}{2} \geq \frac{s}{\|x\|} + \frac{1}{2}.$$

So $F(x+y)(s+t) \geq F(x)(s) = \min\{F(x)(s), F(y)(t)\}$.

Case 3: if $0 < t < \|y\|/2$ and $s \geq \|x\|/2$, it is completely analogous to **Case 2**.

Case 4: if $s \geq \|x\|/2$ and $t \geq \|y\|/2$, then $s+t \geq (\|x\| + \|y\|)/2 \geq \|x+y\|/2$, which contradicts our assumption.

Therefore, (MPN-3) is also satisfied. Hence F is a Menger probabilistic norm with respect to t-norm min.

Remark 3.3. Theorem 3.1 and Example 3.2 indicate that the class of all Höhle probabilistic norms is a proper subclass of that of all Menger probabilistic norms.

4. Wu-Fang Fuzzy Semi-norms Generated by Probabilistic Norms

Theorem 4.1. Let $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ be a Menger probabilistic norm with respect to t-norm min. Define $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ as follows:

$$\|x_\lambda\|_F = \inf\{t : F(x)(t) > 1 - \lambda\} \text{ for all } x_\lambda \in X^*.$$

Then $\|\cdot\|_F$ is a fuzzy semi-norm satisfying the following condition **(A)**:

(A) $\|x_\lambda\|_F = 0$ for all $\lambda \in (0, 1)$ implies $x = \theta$.

Further, $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ is a fuzzy norm iff $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ also satisfies the following condition **(B)**:

(B) $F(x)(t) > 0$ for each $t > 0$ implies $x = \theta$.

Proof. First, we prove that $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ is a fuzzy semi-norm. It is routine to check (FN-2), (FN-3) and (FN-4).

(FN-2). Since $F(\theta)(t) > 1 - \lambda$ for each $t > 0$, $\lambda \in (0, 1]$, $\|\theta_\lambda\|_F = 0$ for each $\lambda \in (0, 1]$. So $\|0 \cdot x_\lambda\|_F = \|\theta_\lambda\|_F = 0\|x_\lambda\|_F$. If $k \in \mathbb{K}$ with $k \neq 0$, then $\|k \cdot x_\lambda\|_F = \|(kx)_\lambda\|_F = \inf\{t : F(kx)(t) > 1 - \lambda\} = \inf\{t : F(x)(t/|k|) > 1 - \lambda\} = |k| \inf\{t : F(x)(t) > 1 - \lambda\} = |k|\|x_\lambda\|_F$. Thus (FN-2) is satisfied.

(FN-3). For each $x_\lambda, y_\lambda \in X^*$, let $\|x_\lambda\|_F = a$ and $\|y_\lambda\|_F = b$. Then, by the definition of $\|\cdot\|_F$, we have $F(x)(a + \varepsilon) > 1 - \lambda$ and $F(y)(b + \varepsilon) > 1 - \lambda$ for each $\varepsilon > 0$. Hence $F(x + y)(a + b + 2\varepsilon) \geq \min\{F(x)(a + \varepsilon), F(y)(b + \varepsilon)\} > 1 - \lambda$. So $\|x_\lambda + y_\lambda\|_F \leq a + b + 2\varepsilon$. By the arbitrariness of ε , we get (FN-3).

(FN-4). It is not difficult to see that $\|x_\lambda\|_F \leq \|x_\mu\|_F$ if $\mu < \lambda$. Thus $\|x_\lambda\|_F \leq \inf_{0 < \mu < \lambda} \|x_\mu\|_F$. Conversely, let $\|x_\lambda\|_F = a$. Then $F(x)(a + \varepsilon) > 1 - \lambda$ for each $\varepsilon > 0$. Hence there exists μ_0 with $0 < \mu_0 < \lambda$ such that $F(x)(a + \varepsilon) > 1 - \mu_0$, which implies $\|x_{\mu_0}\|_F \leq a + \varepsilon$. Therefore $\inf_{0 < \mu < \lambda} \|x_\mu\|_F \leq a + \varepsilon$, which implies $\inf_{0 < \mu < \lambda} \|x_\mu\|_F \leq \|x_\lambda\|_F$ by the arbitrariness of ε . This completes the proof of (FN-4).

Second, we prove that $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ satisfies condition **(A)**.

In fact, since $\|x_\lambda\|_F = 0$ for all $\lambda \in (0, 1)$, by the definition of $\|\cdot\|_F$, $F(x)(t) > 1 - \lambda$ for each $t > 0$, $\lambda \in (0, 1)$, which implies that $F(x) = \varepsilon_0$. Hence $x = \theta$.

Finally, we prove that $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ is a fuzzy norm iff $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ satisfies condition **(B)**.

In fact, since $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ is already a fuzzy semi-norm, $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ is a fuzzy norm iff $\|x_1\|_F = 0$ implies $x = \theta$ iff $F(x)(t) > 0$ for each $t > 0$ implies $x = \theta$ by the definition of $\|\cdot\|_F$. \square

Remark 4.2. It is easy to see that the Menger probabilistic norm F in Example 3.2 does not satisfy condition **(B)**, so $\|\cdot\|_F$ is not necessarily a fuzzy norm in general.

Theorem 4.3. Let $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ be a Höhle probabilistic norm. Define $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ as in Theorem 4.1. Then $\|\cdot\|_F$ is a fuzzy semi-norm satisfying the following condition **(C)**:

(C) $\|x_\lambda\|_F = 0$ for some $\lambda \in (0, 1)$ implies $x = \theta$.

Proof. Since F is also a Menger probabilistic norm, by Theorem 4.1, we need only to prove that $\|\cdot\|_F$ satisfies condition **(C)**.

Suppose that there exists $\lambda \in (0, 1)$ such that $\|x_\lambda\|_F = 0$. Then, by the definition of $\|x_\lambda\|_F$, $F(x)(t) > 1 - \lambda$ for each $t > 0$, which implies $F(x)(0+) \geq 1 - \lambda > 0$. Hence, $x = \theta$. \square

Remark 4.4. (1) It is not difficult to show that, for a fuzzy semi-norm, condition (FN-1) \implies condition **(C)** \implies condition **(A)**.

(2) Does $\|\cdot\|_F$ in Theorem 4.3 also satisfy condition (FN-1) (i.e., is $\|\cdot\|_F$ a fuzzy norm)?

By Theorem 4.1, the question is equivalent to that whether each Höhle probabilistic norm necessarily satisfies condition **(B)**.

The answer is negative as the following example, which is a slight modification of Example 3.2, shows.

Example 4.5. Let $(X, \|\cdot\|)$ be an ordinary normed space. Define a mapping F from X into $\mathcal{D}(\mathbb{R}^+)$ as follows:

If $x = \theta$, let $F(x) = \varepsilon_0$;

If $x \neq \theta$, let

$$F(x)(t) = \begin{cases} 0 & t = 0 \\ \frac{t}{\|x\|} & 0 < t < \|x\| \\ 1 & t \geq \|x\| \end{cases}$$

Similar to Example 3.2, we can prove that F is a Höhle probabilistic norm, but it does not satisfy condition **(B)**.

Theorem 4.6. Let $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ be a Menger probabilistic norm with respect to t -norm min. Define $\|\cdot\|_F : X^* \rightarrow \mathbb{R}^+$ as in Theorem 4.1. Then $\|x_\lambda\|_F < t$ iff $F(x)(t) > 1 - \lambda$.

Proof. Let $\|x_\lambda\|_F < t$. Then it is not difficult to see that $F(x)(t) > 1 - \lambda$ by the definition of $\|x_\lambda\|_F$. Conversely, let $F(x)(t) > 1 - \lambda$. Then, by the left-continuity of $F(x)(\cdot) : \mathbb{R}^+ \rightarrow [0, 1]$, there exists $s < t$ such that $F(x)(s) > 1 - \lambda$. Hence $\|x_\lambda\|_F \leq s < t$. This completes the proof. \square

5. Probabilistic Norms Generated by Wu-Fang Fuzzy Semi-norms

Theorem 5.1. Let $\|\cdot\| : X^* \rightarrow \mathbb{R}^+$ be a fuzzy semi-norm satisfying condition **(A)**. Define $F_{\|\cdot\|} : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ as follows: for each $x \in X$, $t \in \mathbb{R}^+$,

$$F_{\|\cdot\|}(x)(t) = \sup\{\lambda : \lambda \in [0, 1), \|x_{1-\lambda}\| < t\},$$

where we assume that $\sup \emptyset = 0$. Then $F_{\|\cdot\|}$ is a Menger probabilistic norm with respect to t -norm min.

Proof. First, we prove that $F_{\|\cdot\|}(x) \in \mathcal{D}(\mathbb{R}^+)$ for all $x \in X$.

It is obvious that $F_{\|\cdot\|}(x)(\cdot) : \mathbb{R}^+ \rightarrow [0, 1]$ is non-decreasing and satisfies $F_{\|\cdot\|}(x)(0) = 0$. Since, for each $\lambda \in [0, 1)$, there exists $t_0 \in \mathbb{R}^+$ such that $\|x_{1-\lambda}\| < t_0$, we have $\lim_{t \rightarrow +\infty} F_{\|\cdot\|}(x)(t) = 1$. It remains to check that $F_{\|\cdot\|}(x)(\cdot) : \mathbb{R}^+ \rightarrow [0, 1]$ is left-continuous.

In fact, we need only to verify that $\sup_{0 < s < t} F_{\|\cdot\|}(x)(s) = F_{\|\cdot\|}(x)(t)$ for each $t > 0$ since $F_{\|\cdot\|}(x)(\cdot) : \mathbb{R}^+ \rightarrow [0, 1]$ is non-decreasing. $\sup_{0 < s < t} F_{\|\cdot\|}(x)(s) \leq F_{\|\cdot\|}(x)(t)$ is obvious. On the other hand, we assume that $F_{\|\cdot\|}(x)(t) > 0$. For each $a \in [0, 1)$

with $a < F_{\|\cdot\|}(x)(t)$, by the definition of $F_{\|\cdot\|}(x)(t)$, there exists $\lambda_0 \in [0, 1)$ such that $a < \lambda_0$ and $\|x_{1-\lambda_0}\| < t$. Hence there exists $s_0 \in (0, t)$ such that $\|x_{1-\lambda_0}\| < s_0$. So $a < \lambda_0 \leq F_{\|\cdot\|}(x)(s_0) \leq \sup_{0 < s < t} F_{\|\cdot\|}(x)(s)$. Therefore $F_{\|\cdot\|}(x)(t) \leq \sup_{0 < s < t} F_{\|\cdot\|}(x)(s)$ by the arbitrariness of a ,

Finally, we prove that $F_{\|\cdot\|}$ is a Menger probabilistic norm with respect to t -norm \min , i.e., (MPN-1), (MPN-2) and (MPN-3) are satisfied.

(MPN-1). It is easy to see that $F_{\|\cdot\|}(\theta) = \varepsilon_0$. Conversely, let $F_{\|\cdot\|}(x) = \varepsilon_0$. We assume that $x \neq \theta$. Since $\|\cdot\|$ satisfies condition **(A)**, there exists $\lambda_0 \in (0, 1)$ such that $\|x_{1-\lambda_0}\| > 0$. Let $t_0 = \|x_{1-\lambda_0}\|/2$, then $F_{\|\cdot\|}(x)(t_0) \leq \lambda_0 < 1$ by the definition of $F_{\|\cdot\|}(x)(t_0)$, which contradicts $F_{\|\cdot\|}(x) = \varepsilon_0$. Thus (MPN-1) is satisfied.

(MPN-2). For all $k \in \mathbb{K}$ with $k \neq 0$,

$$\begin{aligned} F_{\|\cdot\|}(kx)(t) &= \sup\{\lambda : \lambda \in [0, 1), \|kx_{1-\lambda}\| < t\} \\ &= \sup\{\lambda : \lambda \in [0, 1), |k|\|x_{1-\lambda}\| < t\} \\ &= \sup\{\lambda : \lambda \in [0, 1), \|x_{1-\lambda}\| < t/|k|\} \\ &= F_{\|\cdot\|}(x)(t/|k|). \end{aligned}$$

(MPN-3). We can assume that

$$\min\{F_{\|\cdot\|}(x)(s), F_{\|\cdot\|}(y)(t)\} > 0.$$

For each $a \in [0, 1)$ with $a < \min\{F_{\|\cdot\|}(x)(s), F_{\|\cdot\|}(y)(t)\}$, we have $a < F_{\|\cdot\|}(x)(s)$. By the definition of $F_{\|\cdot\|}(x)(s)$, there exists $\lambda_1 \in [0, 1)$ such that $a < \lambda_1$ and $\|x_{1-\lambda_1}\| < s$. Similarly, there exists $\lambda_2 \in [0, 1)$ such that $a < \lambda_2$ and $\|y_{1-\lambda_2}\| < t$. Let $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. Then $a < \lambda_0$ and $\|(x+y)_{1-\lambda_0}\| \leq \|x_{1-\lambda_0}\| + \|y_{1-\lambda_0}\| \leq \|x_{1-\lambda_1}\| + \|y_{1-\lambda_2}\| < s+t$. Hence $a < \lambda_0 \leq F_{\|\cdot\|}(x+y)(s+t)$. Therefore $F_{\|\cdot\|}(x+y)(s+t) \geq \min\{F_{\|\cdot\|}(x)(s), F_{\|\cdot\|}(y)(t)\}$ by the arbitrariness of a . This completes the proof. \square

Theorem 5.2. Let $\|\cdot\| : X^* \rightarrow \mathbb{R}^+$ be a fuzzy semi-norm satisfying condition **(C)**. Define $F_{\|\cdot\|} : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ as in Theorem 5.1. Then $F_{\|\cdot\|}$ is a Höhle probabilistic norm.

Proof. By Theorem 5.1 and Remark 4.4, it remains to show that $F_{\|\cdot\|}(x)(0+) = 0$ for each $x \neq \theta$.

In fact, let $x \neq \theta$. Since $\|\cdot\|$ satisfies condition **(C)**, $\|x_{1-\lambda}\| > 0$ for all $\lambda \in (0, 1)$. Hence, for each $\lambda \in (0, 1)$, there exists $t > 0$ such that $\|x_{1-\lambda}\| \geq t$, which implies $F_{\|\cdot\|}(x)(0+) \leq F_{\|\cdot\|}(x)(t) \leq \lambda$. Therefore, $F_{\|\cdot\|}(x)(0+) = 0$ by the arbitrariness of λ . \square

Theorem 5.3. Let $\|\cdot\| : X^* \rightarrow \mathbb{R}^+$ be a fuzzy norm and $F_{\|\cdot\|} : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ be defined as in Theorem 5.1, then $F_{\|\cdot\|}$ is a Menger probabilistic norm with respect to t -norm \min . In addition, $F_{\|\cdot\|}$ satisfies condition **(B)**.

Proof. By Theorem 5.1 and Remark 4.4, we need only to prove that $F_{\|\cdot\|}$ satisfies condition **(B)**.

Suppose that $F_{\|\cdot\|}(x)(t) > 0$ for each $t > 0$. Then, for each $t > 0$, there exists $\lambda \in [0, 1)$ such that $\|x_{1-\lambda}\| < t$. Hence, $\|x_1\| \leq \|x_{1-\lambda}\| < t$, which implies $\|x_1\| = 0$ by the arbitrariness of $t > 0$. Therefore, $x = \theta$ by condition (FN-1). \square

Theorem 5.4. *Let $\|\cdot\| : X^* \rightarrow \mathbb{R}^+$ be a fuzzy semi-norm satisfying condition (A). Define $F_{\|\cdot\|} : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ as in Theorem 5.1. Then $F_{\|\cdot\|}(x)(t) > 1 - \lambda$ iff $\|x_\lambda\| < t$.*

Proof. Let $F_{\|\cdot\|}(x)(t) > 1 - \lambda$. Then $\|x_\lambda\| < t$ by the definition of $F_{\|\cdot\|}(x)(t)$. Conversely, let $\|x_\lambda\| < t$. Then, by (FN-4), there exists μ with $0 < \mu < \lambda$ such that $\|x_\mu\| < t$. Hence $F_{\|\cdot\|}(x)(t) \geq 1 - \mu > 1 - \lambda$. This completes the proof. \square

6. Correspondence Between Probabilistic Norms and Wu-Fang Fuzzy Semi-norms

Theorem 6.1. *Let $F : X \rightarrow \mathcal{D}(\mathbb{R}^+)$ be a Menger probabilistic norm with respect to t -norm min. $\|\cdot\|_F$ and $F_{(\|\cdot\|_F)}$ are defined as in Theorem 4.1 and Theorem 5.1, respectively. Then $F_{(\|\cdot\|_F)} = F$.*

Proof. First, we prove that $F(x)(t) \leq F_{(\|\cdot\|_F)}(x)(t)$ for all $x \in X$, $t > 0$.

We can assume that $F(x)(t) > 0$. Let $\lambda \in [0, 1)$ with $\lambda < F(x)(t)$. Then $\|x_{1-\lambda}\|_F < t$ by Theorem 4.6. Hence $\lambda \leq F_{(\|\cdot\|_F)}(x)(t)$. By the arbitrariness of λ , we have $F(x)(t) \leq F_{(\|\cdot\|_F)}(x)(t)$.

Finally, we prove that $F_{(\|\cdot\|_F)}(x)(t) \leq F(x)(t)$ for all $x \in X$, $t > 0$.

We can assume that $F_{(\|\cdot\|_F)}(x)(t) > 0$. Let $\lambda \in [0, 1)$ with $\lambda < F_{(\|\cdot\|_F)}(x)(t)$. Then $\|x_{1-\lambda}\|_F < t$. Hence $\lambda < F(x)(t)$. By the arbitrariness of λ , we have $F_{(\|\cdot\|_F)}(x)(t) \leq F(x)(t)$. This completes the proof. \square

Theorem 6.2. *Let $\|\cdot\| : X^* \rightarrow \mathbb{R}^+$ be a fuzzy semi-norm satisfying condition (A). $F_{\|\cdot\|}$ and $\|\cdot\|_{(F_{\|\cdot\|})}$ are defined as in Theorem 5.1 and Theorem 4.1, respectively. Then $\|\cdot\|_{(F_{\|\cdot\|})} = \|\cdot\|$.*

Proof. First, we prove that $\|x_\lambda\|_{(F_{\|\cdot\|})} \leq \|x_\lambda\|$ for each $x_\lambda \in X^*$.

Let $\|x_\lambda\| < t$. Then $F_{\|\cdot\|}(x)(t) > 1 - \lambda$ by Theorem 5.4. Hence $\|x_\lambda\|_{(F_{\|\cdot\|})} \leq t$. By the arbitrariness of t , we have $\|x_\lambda\|_{(F_{\|\cdot\|})} \leq \|x_\lambda\|$.

Finally, we prove that $\|x_\lambda\| \leq \|x_\lambda\|_{(F_{\|\cdot\|})}$ for each $x_\lambda \in X^*$.

Let $\|x_\lambda\|_{(F_{\|\cdot\|})} < t$. Then $F_{\|\cdot\|}(x)(t) > 1 - \lambda$. Hence $\|x_\lambda\| < t$. By the arbitrariness of t , we have $\|x_\lambda\| \leq \|x_\lambda\|_{(F_{\|\cdot\|})}$. This completes the proof. \square

Theorem 6.3. *There exists a one-to-one correspondence between the class of all Menger probabilistic norms (with t -norm min) and that of all fuzzy semi-norms satisfying condition (A).*

Proof. It follows from Theorems 4.1, 5.1, 6.1 and 6.2. \square

Theorem 6.4. *There exists a one-to-one correspondence between the class of all Höhle probabilistic norms and that of all fuzzy semi-norms satisfying condition (C).*

Proof. It follows from Theorems 4.3, 5.2, 6.1 and 6.2, and Remark 4.4. \square

Theorem 6.5. *There exists a one-to-one correspondence between the class of all fuzzy norms and that of all Menger probabilistic norms (with t -norm \min) satisfying condition (B).*

Proof. It follows from Theorems 4.1, 5.3, 6.1 and 6.2, and Remark 4.4. \square

7. Conclusion

We have discussed the connection between Menger probabilistic norms and Höhle probabilistic norms, and obtained the correspondence between probabilistic norms and Wu-Fang fuzzy (semi-) norms. According to the correspondence, we can study probabilistic normed spaces via Wu-Fang fuzzy (semi-) normed spaces, and vice versa. This is an interesting direction worthy of further research.

Acknowledgements. This work is supported by the National Natural Science Foundation of China (No. 11301281 and No. 11571006).

REFERENCES

- [1] C. Alegre and S. Romaguera, *Characterizations of metrizable topological vector spaces and their asymmetric generalizations in terms of fuzzy (quasi-)norms*, Fuzzy Sets and Systems, **161** (2010), 2181–2192.
- [2] C. Alsina, M. J. Frank and B. Schweizer, *Associative Functions: Triangular Norms and Copulas*, World Scientific Publishing, Singapore, 2006.
- [3] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., **11(3)** (2003), 687–705.
- [4] T. Bag and S. K. Samanta, *A comparative study of fuzzy norms on a linear space*, Fuzzy Sets and Systems, **159** (2008), 670–684.
- [5] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., **86** (1994), 429–436.
- [6] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48** (1992), 239–248.
- [7] O. Hadžić and E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
- [8] U. Höhle, *Minkowski functionals of L -fuzzy sets*, in: P.P. Wang, S.K. Chang (Eds.), *Fuzzy sets: theory and applications to policy analysis and information systems*, Plenum Press, New York, (1980), 13–24.
- [9] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems, **12** (1984), 215–229.
- [10] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems, **12** (1984), 143–154.
- [11] A. K. Katsaras, *Linear fuzzy neighborhood spaces*, Fuzzy Sets and Systems, **16** (1985), 25–40.
- [12] A. K. Katsaras, *Locally convex topologies induced by fuzzy norms*, Global Journal of Mathematical Analysis, **1(3)** (2013), 83–96.
- [13] E. P. Klement, R. Mesiar and E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.
- [14] I. Kramosil and J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11** (1975), 336–344.
- [15] B. Lafuerza-Guillén and P. K. Harikrishnan, *Probabilistic normed spaces*, World Scientific Publishing, Singapore, 2014.
- [16] M. Ma, *A comparison between two definitions of fuzzy normed spaces*, J. Harbin Inst. Technology Suppl. Math., (in Chinese), (1985), 47–49.

- [17] S. Nădăban and I. Dzitac, *Atomic decompositions of fuzzy normed linear spaces for wavelet applications*, Informatica, **25(4)** (2014), 643–662.
- [18] R. Saadati and S. M. Vaezpour, *Some results on fuzzy Banach spaces*, J. Appl. Math. & Computing, **17(1-2)** (2005), 475–484.
- [19] B. Schweizer and A. Sklar, *Probabilistic metric spaces*, North-Holland series in Probability and Applied Mathematics, North-Holland, New York, 1983.
- [20] C. Sempi, *A short and partial history of probabilistic normed spaces*, Mediterr. J. Math., **3** (2006), 283–300.
- [21] C. X. Wu and J. X. Fang, *Fuzzy generalization of Kolmogoroff's theorem*, J. Harbin Inst. Technology, (in Chinese), **1** (1984), 1–7.
- [22] C. X. Wu and M. Ma, *Fuzzy norms, probabilistic norms and fuzzy metrics*, Fuzzy Sets and Systems, **36** (1990), 137–144.
- [23] C. H. Yan and J. X. Fang, *Generalization of Kolmogoroff's theorem to L -topological vector spaces*, Fuzzy Sets and Systems, **125** (2002), 177–183.

HUA-PENG ZHANG, SCHOOL OF SCIENCE, NANJING UNIVERSITY OF POSTS AND TELECOMMUNICATIONS, NANJING 210023, CHINA

E-mail address: huapengzhang@163.com