L-FUZZY APPROXIMATION SPACES AND L-FUZZY TOPOLOGICAL SPACES

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Abstract. The L-fuzzy approximation operator associated with an L-fuzzy approximation space (X, R) turns out to be a saturated L-fuzzy closure (interior) operator on a set X precisely when the relation R is reflexive and transitive. We investigate the relations between L-fuzzy approximation spaces and L-(fuzzy) topological spaces.

1. Introduction

The theory of rough set, proposed by Pawlak [14] is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. As a new method of soft computing, rough set theory has become an important mathematical framework for pattern recognition, image processing, feature selection, signal analysis and decision analysis.

Equivalence relation is an important concept in classical rough set theory which are considered as the building blocks for the lower and upper approximation operators. The notion of an approximation space consisting of a universe of discourse and an indiscernible relation imposed on it is one of the fundamental concept of rough set theory. Based on the approximation space, the primitive notion of lower and upper approximation operators can be induced. From both theoretic and practical needs, many authors have generalized the concept of approximation operators by using nonequivalent binary relations [5, 13, 26, 27], or by using axiomatic approaches [26]. Different kinds of generalization of Pawlak's rough set has been obtained by replacing equivalence relation with an arbitrary binary relation. On the other hand, the relationships between rough sets and topological spaces were studied by many authors [16, 28]. It can be proved that the lower and upper approximation operators derived by a reflexive and transitive relation are exactly a pair of interior and closure operators in a topology. In [16], it was pointed out that there exists a one-to-one corresponding between the set of all reflexive, transitive relations (preorder) and the set of all Alexandroff topologies on an arbitrary universe.

Various fuzzy generalizations of rough approximation have been proposed in the literature [4, 6, 10, 15, 23, 25, 26]. Many papers investigated the relationship between fuzzy rough set models and fuzzy topologies [3, 24]. On the other hand

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Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic. In [17, 23], L-fuzzy rough set based on residuated lattice was studied. Furthermore, it was shown that the lower and upper L-fuzzy approximation operators derived by a reflexive and transitive L-relation are exactly a pair of interior and closure operators of an L-topology.

In the present paper, some basic properties of L-upper (L-lower) quasi-approximation spaces are studied. We investigate the relationship between L-fuzzy rough sets based on residuated lattice and L-(fuzzy) topological spaces. Also, we discuss the continuity of maps between L-fuzzy approximation spaces and L-(fuzzy) topological spaces.

2. **Preliminaries**

Throughout this paper, X is a nonempty set and $L = (L, \vee, \wedge, 0, 1)$ a completely distributive lattice with the least element 0 and the greatest element 1 in L. For each $\alpha \in L$, let $\overline{\alpha}$ denote the constant fuzzy subset with value α. We denote the characteristic function of a subset A of X by 1_A.

A residuated lattice L is a structure $(L, *, \rightarrow, \vee, \wedge, 0, 1)$ where $(L, \vee, \wedge, 0, 1)$ is bounded lattice with the greatest element 1 and the smallest element 0; $(L, *, 1)$ is a commutative monoid and $*$ is isotonic at both arguments; and $(x * z) \leq y$ if and only if $z \leq (x \rightarrow y)$ for all $x, y, z \in L$. A residuated lattice is said to be complete if the underline lattice is complete.

In what follows, $*$ is sometimes called a generalized triangular norm and the implicator $\rightarrow$ is called the residuum of $*$. An implicator I is called left monotonic (resp. right monotonic) if $I(\alpha)$ is decreasing (resp. increasing) for every $\alpha \in L$. If I is both left monotonic and right monotonic, then it is called hybrid monotonic.

An operator $' : L \rightarrow L$ defined by $x' = x \rightarrow 0$, for every $x \in L$, is called a strong negation if $x'' = x$.

In this paper, we always assume that $L = (L, *, \rightarrow, \vee, \wedge, ', 0, 1)$ is a complete residuated lattice with a strong negation. Some basic properties of a complete residuated lattices are as follows.

Lemma 2.1. [1, 2, 8, 9, 22] *Suppose that L is a complete residuated lattice, then $\forall x, y, z \in L$, $\{x_j : j \in J\} \subseteq L$, the following conditions hold*

(I1) $1 \rightarrow x = x,\ x \rightarrow (y \rightarrow x) = 1$,
(I2) $x \leq y$ *iff* $x \rightarrow y = 1$,
(I3) $(x \rightarrow y) * (y \rightarrow z) \leq (x \rightarrow z),\ x \rightarrow y \leq (x * z) \rightarrow (y * z)$,
(I4) $x \rightarrow \bigwedge_{j \in J} x_j = \bigwedge_{j \in J}(x \rightarrow x_j)$, *and hence if* $y \leq z$, *then* $x \rightarrow y \leq x \rightarrow z$,
(I5) $(\bigvee_{j \in J} x_j) \rightarrow y = \bigwedge_{j \in J}(x_j \rightarrow y)$, *and hence if* $x \leq y$, *then* $x \rightarrow z \geq y \rightarrow z$,
(I6) $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I7) $x * (x \rightarrow y) \leq y$ *and* $x \leq (x \rightarrow y) \rightarrow y$,
(I8) $y \leq x \rightarrow y$ *and* $y \leq x \rightarrow (x * y)$,
(I9) $(x \rightarrow y) * (z \rightarrow w) \leq x * z \rightarrow y * w$, *where* $w \in L$,
(I10) $(x \rightarrow y) \wedge (z \rightarrow w) \leq x \wedge z \rightarrow y \wedge w$,
(I11) $x \rightarrow y' = (x * y)'$,
(I12) $(\bigvee_{j \in J} x_j) * y = \bigvee_{j \in J}(x_j * y)$,

(I13) $(\bigwedge_{j\in J} x_j)' = \bigvee_{j\in J} x_j'$,
(I14) $(\bigvee_{j\in J} x_j)' = \bigwedge_{i\in I} x_j'$,
(I15) $x \to y = y' \to x'$,
(I16) $x * y = (x \to y')', \ x \to y = (x * y')'$.

Let X be a universal set, An L-subset on X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X [10]. *All algebraic operations on L can be extended pointwise to the powerset L^X as follows: for all $x \in X$*

(1) $\mu \leq \rho$ *iff* $\mu(x) \leq \rho(x)$,
(2) $(\mu * \rho)(x) = \mu(x) * \rho(x)$,
(3) $(\alpha * \rho)(x) = \alpha * \rho(x)$,
(4) $(\mu \to \rho)(x) = \mu \to \rho(x)$,
(5) $(\alpha \to \rho)(x) = \alpha \to \rho(x)$,
(6) $\mu'(x) = \mu(x) \to 0$.

Definition 2.2. [2] An L-fuzzy quasi-equivalence relation on a universe set X is a map $R : X \times X \to L$ satisfies the following conditions

(1) reflexive if $R(x, x) = 1, \ \forall x \in X$,

(2) transitive if $\bigvee_{y\in X} R(x, y) * R(y, z) \leq R(x, z), \ \forall \ x, y, z \in X$.

An L-fuzzy quasi-equivalence relation is called an L-fuzzy equivalence relation on X if it satisfies

(3) symmetric if $R(x, y) = R(y, x), \ \forall \ x, y \in X$,

Sometimes we use the following condition instead of (2)

(4) $R(x, y) * R(y, z) \leq R(x, z), \ \forall \ x, y, z \in X$.

An L-fuzzy quasi-equivalence relation is called serial if

$$\bigvee_{y\in X} R(x, y) = 1, \ \forall \ x, y \in X.$$

The pair (X, R) is called an L- fuzzy quasi approximation (rep. approximation) space if R is an L- fuzzy quasi- equivalence (rep. equivalence) relation. Let (X, R_1) and (X, R_2) be two L-fuzzy approximation spaces. A mapping $f : X \to Y$ is called R-map if

$$R_1(x, y) \leq R_2(f(x), f(y)) \ \forall \ (x, y) \in X \times X.$$

Let R be an L-fuzzy quasi-equivalence relation on X. Define $R^{-1}(x, y) = R(y, x)$ for all $x \in X$. Then R^{-1} is an L-fuzzy quasi-equivalence relation on X.

Definition 2.3. [7] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(\lambda, \mu) = \bigwedge_{x\in X} (\lambda(x) \to \mu(x)), \ \forall \ \lambda, \mu \in L^X,$$

then S is an L-partial order on L^X. For $\lambda, \mu \in L^X, \ S(\lambda, \mu)$ can be interpreted as the degree to which λ is a subset of μ. It is called the subsethood degree or the fuzzy inclusion order.

Lemma 2.4. [7] *Let S be the fuzzy inclusion order, then $\forall \lambda, \mu, \rho, \nu \in L^X$ and $a \in L$ the following statements hold*

(1) $\mu \leq \rho \Leftrightarrow S(\mu, \rho) = 1$,
(2) $S(\mu, a \to \rho) = S(a * \mu, \rho) = a \to S(\mu, \rho)$,
(3) $\mu \leq \rho \Rightarrow S(\lambda, \mu) \leq S(\lambda, \rho)$ *and* $S(\mu, \lambda) \geq S(\rho, \lambda),\ \forall\ \lambda \in L^X$,
(4) $S(\lambda, \mu) * S(\rho, \nu) \leq S(\lambda * \rho, \mu * \nu)$ *and* $S(\lambda, \mu) \wedge S(\rho, \nu) \leq S(\lambda \wedge \rho, \mu \wedge \nu)$.

Definition 2.5. [9] Let X be the universe set, a family $\tau \subseteq L^X$ is called L-topology if it satisfies the following conditions

(1) $1_X, 1_\phi \in \tau$,
(2) $\lambda, \mu \in \tau$ implies $\lambda \wedge \mu \in \tau$,
(3) $\forall i\ \{\lambda_i : i \in I\} \subseteq \tau$ implies $\bigvee_{i \in I} \lambda_i \in \tau$.
An L-topology τ is called strongly if it further satisfies the following:
(4) $\overline{\alpha} * \lambda \in \tau$ for all $\lambda \in \tau,\ \alpha \in L$,
(5) $\overline{\alpha} \to \lambda \in \tau$ for all $\lambda \in \tau,\ \alpha \in L$.

A strongly L-topological space (X, τ) is called an Alexandrov L-topological space if τ also satisfies that $\{\lambda_i : \forall\ i\ \in I\} \subseteq \tau$ implies $\bigwedge_{i \in I} \lambda_i \in \tau$.

An L-fuzzy topology [9, 18, 20, 21] is given by a mapping $\mathcal{T} : L^X \to L$ satisfies the following conditions:

(O1) $\mathcal{T}(1_X) = \mathcal{T}(1_\phi) = 1$,
(O2) $\mathcal{T}(\lambda \wedge \mu) \geq \mathcal{T}(\lambda) \wedge \mathcal{T}(\mu)$,
(O3) $\mathcal{T}(\bigvee_i \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i)$.

If in addition, $\mathcal{T}$ satisfies

(H1) $\mathcal{T}(\overline{\alpha} * \lambda) \geq \mathcal{T}(\lambda)$,
(H2) $\mathcal{T}(\overline{\alpha} \to \lambda) \geq \mathcal{T}(\lambda)$.

Then $\mathcal{T}$ is a strong L-fuzzy topology on X, and the pair $(X, \mathcal{T})$ is a strong L-fuzzy topological space.

An L-fuzzy topological space $(X, \mathcal{T})$ is called Alexandrov L-fuzzy topological space if $\mathcal{T}$ also satisfies

(O4) $\mathcal{T}(\bigwedge_{i \in I} \lambda_i) \geq \bigwedge_{i \in I} \mathcal{T}(\lambda_i)$.

A mapping $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is LF-continuous if it satisfies that $\mathcal{T}_1(f^{\leftarrow}(\lambda)) \geq \mathcal{T}_2(\lambda)$ for all $\lambda \in L^Y$, where $f^{\leftarrow}(\lambda) = \lambda \circ f$.

Definition 2.6. [7, 11] A L-fuzzified set of all upper sets of an approximation space (X, R) is a map $\nabla(R) : L^X \to L$ defined by

$$\forall \lambda \in L^X,\ \nabla(R)(\lambda) = \bigwedge_{(x,y) \in (X,X)} R(x, y) \to (\lambda(x) \to \lambda(y)).$$

Dually, an L-fuzzified set of all lower sets of an approximation space (X, R) is a map $\Delta(R) : L^X \to L$ defined by

$$\forall \mu \in L^X,\ \Delta(R)(\mu) = \bigwedge_{(x,y) \in (X,X)} R(x, y) \to (\mu(y) \to \mu(x)).$$

A fuzzy subset λ is called an upper set (resp. a lower set) if

$$\nabla(R)(\lambda) = 1 \ (resp. \ \Delta(R)(\lambda) = 1).$$

3. L-fuzzy Rough (Closure, Interior) Approximation Operators

The notion of fuzzy rough sets based on residuated lattice was proposed by Radzikowska and Kerre in [17]. By taking complete residuated lattices instead of $[0, 1]$ as truth value structure, it differs from the concept of fuzzy rough sets in [6, 12, 13, 15].

Definition 3.1. [10, 23] Let (X, R) ba an L-fuzzy quasi-approximation space. The two maps $\mathcal{C}_R, \mathcal{I}_R : L^X \to L^X$ are called upper and lower L-fuzzy rough approximation operators on X defined respectively by

$$\mathcal{C}_R(\lambda)(x) = \bigvee_{y \in X} R(x, y) * \lambda(y),$$

$$\mathcal{I}_R(\lambda)(x) = \bigwedge_{y \in X} (R(x, y) \to \lambda(y)), \ \forall \ \lambda \in L^X, x \in X.$$

In the case when $L = \{0, 1\}$, λ and R can be reduced to crisp subsets of X and $X \times X$ respectively, and $\mathcal{C}_R(\lambda)$ and $\mathcal{I}_R(\lambda)$ are precisely the corresponding concepts in classical rough set theory. Nevertheless, $\mathcal{I}_R(\lambda) \leq \mathcal{C}_R(\lambda)$, which is true in the classical case, is not in the fuzzy setting.

The following theorem provides some basic properties of the lower and upper L-fuzzy rough approximation operators.

Theorem 3.2. [23] *Let $\mathcal{C}_R, \mathcal{I}_R : L^X \to L^X$ be the lower and upper L-fuzzy rough approximation operators on X. Then they satisfy the following properties*

(1) $S(\lambda, \mathcal{C}_R(\lambda)) \geq 1$ and $S(\mathcal{I}_R(\lambda), \lambda) \geq 1 \ \forall \ \lambda \in L^X$,
(2) $S(\lambda, \mu) \leq S(\mathcal{C}_R(\lambda), \mathcal{C}_R(\mu))$ and $S(\lambda, \mu) \leq S(\mathcal{I}_R(\lambda), \mathcal{I}_R(\mu)) \ \forall \ \lambda, \mu \in L^X$,
(3) $\mathcal{C}_R(\bigvee_{i \in I} \lambda_i) = \bigvee_{i \in I} \mathcal{C}_R(\lambda_i)$ and $\mathcal{I}_R(\bigwedge_{i \in I} \lambda_i) = \bigwedge_{i \in I} \mathcal{I}_R(\lambda_i) \ \forall \ \lambda_i \in L^X$,
(4) $\mathcal{C}_R(\overline{\alpha} * \lambda) = \overline{\alpha} * \mathcal{C}_R(\lambda)$ and $\mathcal{I}_R(\overline{\alpha} \to \lambda) = \overline{\alpha} \to \mathcal{I}_R(\lambda)$, where $\alpha \in L$,
(5) $\mathcal{C}_R(\mathcal{C}_R(\lambda)) \leq \mathcal{C}_R(\lambda)$ and $\mathcal{I}_R(\mathcal{I}_R(\lambda)) \geq \mathcal{I}_R(\lambda)$,
(6) $\mathcal{C}_R(\overline{\alpha}) \leq \overline{\alpha}$ and $\mathcal{I}_R(\overline{\alpha}) \geq \overline{\alpha}$,
(7) $S(\mathcal{C}_R(\lambda), \mu) = S(\lambda, \mathcal{I}_{R^{-1}}(\mu))$.

In the following definitions 3.3, 3.4, we define a strongly stratified upper(lower) L-fuzzy quasi-approximation space which induce L-fuzzy topological spaces in the next section.

Definition 3.3. A map $\mathcal{C} : L^X \to L^X$ is called an upper L- fuzzy quasi-approximation operator on X iff $\mathcal{C}$ satisfies the following conditions

(UA1) $S(\lambda, \mathcal{C}(\lambda)) = 1$ and $\mathcal{C}(\overline{\alpha}) = \overline{\alpha}$,
(UA2) If $S(\mu, \lambda) \leq S(\mathcal{C}(\mu), \mathcal{C}(\lambda))$,
(UA3) $\mathcal{C}(\bigvee_{i \in I} \lambda_i) = \bigvee_{i \in I} \mathcal{C}(\lambda_i)$,
(UA4) $\mathcal{C}(\mathcal{C}(\lambda)) \leq \mathcal{C}(\lambda)$.

The pair $(X, \mathcal{C})$ is called an L-upper quasi-approximation operator. An upper L-fuzzy quasi-approximation operator is called:

(1) strongly stratified iff it satisfies (LS) $\mathcal{C}(\overline{\alpha} * \lambda) = \overline{\alpha} * \mathcal{C}(\lambda)$,

(2) upper L-fuzzy approximation if it satisfies (LA) $\mathcal{C}(1_x)(y) = \mathcal{C}(1_y)(x)$.

Definition 3.4. A map $\mathcal{I} : L^X \to L^X$ is called a lower L- fuzzy quasi-approximation operator on X iff $\mathcal{I}$ satisfies the following conditions:

(LA1) $S(\mathcal{I}(\lambda), \lambda) = 1$ and $\mathcal{I}(\overline{\alpha}) = \overline{\alpha}$,
(LA2) If $S(\mu, \lambda) \leq S(\mathcal{I}(\mu), \mathcal{I}(\lambda))$,
(LA3) $\mathcal{I}(\bigwedge_{i\in I} \lambda_i) = \bigwedge_{i\in I} \mathcal{I}(\lambda_i)$,
(LA4) $\mathcal{I}(\mathcal{I}(\lambda)) \geq \mathcal{I}(\lambda)$.

The pair $(X, \mathcal{I})$ is called a lower L-fuzzy quasi-approximation operator. A lower L-fuzzy quasi-approximation operator is called:

(1) strongly stratified iff it satisfies (LS) $\mathcal{I}(\overline{\alpha} \to \lambda) = \overline{\alpha} \to \mathcal{I}(\lambda)$,
(2) lower L-fuzzy approximation operator if it satisfies (LA) $\mathcal{I}(1'_x)(y) = \mathcal{I}(1'_y)(x)$.

Theorem 3.5. *If the mapping $\mathcal{C} : L^X \to L^X$ satisfies the following conditions*

(1) $1_x \leq \mathcal{C}(1_x)$,
(2) $\bigvee_{z\in X}(\mathcal{C}(1_x)(z) * \mathcal{C}(1_z)(y)) \leq \mathcal{C}(1_x)(y)$,
(3) $\mathcal{C}(\bigvee_{i\in I} \lambda_i) = \bigvee_{i\in I} \mathcal{C}(\lambda_i)$,
(4) $\mathcal{C}(\overline{\alpha} * 1_x) = \overline{\alpha} * \mathcal{C}(1_x)$.

Then we have the following properties

(a) $\mathcal{C}(\lambda) = \bigvee_{y\in X} \mathcal{C}(1_y) * \lambda(y)$,

(b) $\mathcal{C}$ satisfies the above conditions iff $\mathcal{C}$ is a strongly stratified upper L-fuzzy quasi-approximation operator.

Proof. (a) Since $\lambda(x) = \bigvee_{y\in X}(1_y(x) * \lambda(y))$,

$$\mathcal{C}(\lambda)(x) = \mathcal{C}(\bigvee_{y\in X}(1_y * \lambda(y)))(x) = \bigvee_{y\in X} \mathcal{C}(1_y * \lambda(y))(x) = \bigvee_{y\in X} \mathcal{C}(1_y)(x) * \lambda(y) \ (by(4)).$$

(b) ($\Rightarrow$) $\mathcal{C}$ is strongly stratified upper L-fuzzy quasi-approximation operator from the following statements

(UA1) By (a) and condition (1), we have

$$\mathcal{C}(\lambda)(x) = \bigvee_{y\in X} \mathcal{C}(1_y)(x) * \lambda(y) \geq \bigvee_{y\in X} (1_y(x) * \lambda(y)) = \lambda(x).$$

Since $\mathcal{C}(\overline{1}) = \overline{1}$ and $\overline{\alpha} * \overline{1} = \overline{\alpha}$, $\mathcal{C}(\overline{\alpha} * \overline{1}) = \overline{\alpha} * \overline{1} = \overline{\alpha}$.

(UA2) By (a), it is easy.

(UA4)

$$\mathcal{C}(\lambda)(x) = \bigvee_{y\in X} \mathcal{C}(1_y)(x) * \lambda(y) \geq \bigvee_{y\in X} (\bigvee_{z\in X} \mathcal{C}(1_y)(z) * \mathcal{C}(1_z)(x) * \lambda(y)) \ (by(2))$$

$$= \bigvee_{z\in X} \mathcal{C}(1_z)(x) * (\bigvee_{y\in X} \mathcal{C}(1_y)(z) * \lambda(y)) = \bigvee_{z\in X} \mathcal{C}(1_z)(x) * \mathcal{C}(\lambda)(z) = \mathcal{C}(\mathcal{C}(\lambda))(x).$$

($\Leftarrow$) We only show condition (2). By (UA4) we have

$$\mathcal{C}(1_x)(y) \geq \mathcal{C}(\mathcal{C}(1_x))(y) = \bigvee_{z\in X} \mathcal{C}(1_z)(y) * \mathcal{C}(1_x)(z) \ (by(a)).$$

The following two Theorems in [10] provides some basic properties of the lower and upper L-fuzzy rough approximation operators. □

Theorem 3.6. *Let (X, R) ba an L-fuzzy quasi-approximation space. Then the upper L-fuzzy rough approximation operators on X has the following properties*

(1) $\mathcal{C}_R(\overline{\alpha}) = \overline{\alpha}$ if and only if $\mathcal{C}_R(\overline{1}) = \overline{1}$,
(2) $\mathcal{C}_R(1_y * \overline{\alpha})(x) = R(x, y) * \alpha$,
(3) $\mathcal{C}_R(1_y)(x) = R(x, y)$,
(4) R is serial iff one of the following properties holds
(i) $\mathcal{C}_R(\overline{\alpha}) = \overline{\alpha}$ (ii) $\mathcal{C}_R(\overline{1}) = \overline{1}$, $\forall\alpha \in L$.

Theorem 3.7. *Let (X, R) be an L-fuzzy quasi-approximation space. Then the lower L-fuzzy rough approximation operators on X has the following properties:*

(1) $\mathcal{I}_R(\overline{\alpha}) = \overline{\alpha}$ if and only if $\mathcal{I}_R(\overline{0}) = \overline{0}$,
(2) $\mathcal{I}_R(1_y \to \overline{\alpha})(x) = R(x, y) \to \alpha$,
(3) $\mathcal{I}_R(1'_y)(x) = R(x, y) \to 0$,
(4) R is serial iff $\mathcal{I}_R(\overline{\alpha}) = \overline{\alpha}\ \forall\alpha \in L$,
(5) $\mathcal{I}(\lambda) = \bigwedge_{y\in X} \lambda'(y) \to \mathcal{I}(1'_y)$,
(6) $\mathcal{I}_R(\lambda) = \mathcal{C}'_R(\lambda')$ and $\mathcal{C}_R(\lambda) = \mathcal{I}'_R(\lambda')$,
(7) If $\mathcal{C}_R(1_x) = \mathcal{I}'_R(1'_x)$, then $\mathcal{C}_R(\lambda') = \mathcal{I}'_R(\lambda)$.

Proposition 3.8. *Let (X, R) be an L-fuzzy approximation space. Then for all $\lambda, \mu \in L^X$, we have the following properties*

(1) $\mathcal{I}_R(\mathcal{C}_R(\lambda)) = \mathcal{C}_R(\lambda)$,
(2) $\mathcal{C}_R(\mathcal{I}_R(\lambda)) = \mathcal{I}_R(\lambda)$,
(3) $S(\mathcal{C}_R(\lambda), \rho) = S(\lambda, \mathcal{I}_R(\rho))$,
(4) $S(\mathcal{I}_R(\lambda), \mathcal{I}_R(\rho)) = S(\mathcal{I}_R(\lambda), \rho)$,
(5) $S(\mathcal{C}_R(\lambda), \mathcal{C}_R(\rho)) = S(\lambda, \mathcal{C}_R(\rho))$.

Proof. (1) By (UA1), we have

$$\begin{aligned}
\mathcal{I}_R(\mathcal{C}_R(\lambda))(x) &= \bigwedge_{y\in X} (R(x, y) \to \mathcal{C}_R(\lambda)(y)) = \bigwedge_{y\in X} (R(x, y) \to \bigvee_{z\in X} (R(y, z) * \lambda(z))) \\
&= \bigwedge_{y\in X} \bigvee_{z\in X} (R(x, y) \to R(y, z) * \lambda(z)) \ (by(I4)) \\
&\geq \bigwedge_{y\in X} \bigvee_{z\in X} (R(x, y) \to (R(y, x) * R(x, z) * \lambda(z)) \\
&\geq \bigvee_{z\in X} (R(x, z) * \lambda(z)) = \mathcal{C}_R(\lambda)(x) \ (by(I8)).
\end{aligned}$$

(2) By (LA1),we have

$$\begin{aligned}\mathcal{C}_R(\mathcal{I}_R(\lambda))(x) &= \bigvee_{y\in X}(R(x,y)*\mathcal{I}_R(\lambda)(y)) = \bigvee_{y\in X}(R(x,y)*\bigwedge_{z\in X}(R(y,z)\to\lambda(z)))\\ &= \bigvee_{y\in X}\bigwedge_{z\in X}(R(x,y)*(R(y,z)\to\lambda(z)))\ (by(I11))\\ &\le \bigvee_{y\in X}\bigwedge_{z\in X}(R(x,y)*(R(y,x)*R(x,z)\to\lambda(z)))\\ &\le \bigwedge_{z\in X}(R(x,z)*\lambda(z)) = \mathcal{I}_R(\lambda)(x)\ (by(I5)).\end{aligned}$$

(3)

$$\begin{aligned}S(\mathcal{C}_R(\lambda),\rho) &= \bigwedge_{x\in X}(\mathcal{C}_R(\lambda)(x)\to\rho(x)) = \bigwedge_{x\in X}((\bigvee_{y\in X}R(x,y)*\lambda(y))\to\rho(x))\\ &= \bigwedge_{x\in X}(\bigwedge_{y\in X}(R(x,y)*\lambda(y)\to\rho(x)))\ (by(I5))\\ &= \bigwedge_{x\in X}\bigwedge_{y\in X}(\lambda(y)\to(R(x,y)\to\rho(x)))\ (by(I6))\\ &= \bigwedge_{y\in X}(\lambda(y)\to\bigwedge_{x\in X}(R(y,x)\to\rho(x)))\\ &= \bigwedge_{y\in X}(\lambda(y)\to\mathcal{I}_R(\rho)(y)) = S(\lambda,\mathcal{I}_R(\rho)).\end{aligned}$$

(4) By using (1), (3), we have $S(\mathcal{I}_R(\lambda),\mathcal{I}_R(\rho)) = S(\mathcal{C}_R(\mathcal{I}_R(\lambda)),\rho) = S(\mathcal{I}_R(\lambda),\rho)$.

(5) By using (2), (3), we have $S(\mathcal{C}_R(\lambda),\mathcal{C}_R(\rho)) = S(\lambda,\mathcal{I}_R(\mathcal{C}_R(\rho))) = S(\lambda,\mathcal{C}_R(\rho))$. □

Definition 3.9. Let R be a quasi-equivalence relation on X. A fuzzy set $\lambda\in L^X$ is called compatible with R (or left-extensional with respect to R on X) if for any $x,y\in X$ it holds $\lambda(x)*R(x,y)\le\lambda(y)$.

A fuzzy set λ is called right-extensional with respect to R if $\lambda(y)*R(x,y)\le\lambda(x)$ for all $x,y\in X$. i.e, $R(x,y)\le\lambda(y)\to\lambda(x)$.

Lemma 3.10. *(1) $\lambda=\mathcal{I}_R(\lambda)$ iff λ is compatible with R,*

(2) $\mathcal{C}_R$ is compatible with R if R is an equivalence relation,

(3) $\mathcal{C}_{R^{-1}}$ is right-extensional with respect to R.

Proof. (1) Let $\lambda=\mathcal{I}_R(\lambda)$, then

$$\begin{aligned}\lambda(x)*R(x,y) = \mathcal{I}_R(\lambda)*R(x,y) &= (\bigwedge_{z\in X}R(x,z)\to\lambda(z))*R(x,y)\\ &\le R(x,y)*(R(x,y)\to\lambda(y))\le\lambda(y)\ (by(I7)).\end{aligned}$$

If λ is compatible with R, then $\lambda(x)*R(x,y)\le\lambda(y)\Leftrightarrow\lambda(x)\le R(x,y)\to\lambda(y)$. So, $\lambda(x)\le\bigwedge_{y\in X}R(x,y)\to\lambda(y)=\mathcal{I}_R(\lambda)(x)$. Thus, $\lambda=\mathcal{I}_R(\lambda)$.

(2)

$$\begin{aligned}\mathcal{C}_R(\mu)(x)*R(x,y) &= (\bigvee_{z\in X}R(x,z)*\mu(z))*R(x,y) = \bigvee_{z\in X}(R(x,z)*R(x,y))*\mu(z)\\ &= \bigvee_{z\in X}(R(y,x)*R(x,z))*\mu(z)\le\bigvee_{z\in X}R(y,z)*\mu(z)=\mathcal{C}_R(\mu)(y).\end{aligned}$$

(3) By similar way. □

4. L-fuzzy Topological Spaces

In this section we will show that an upper(lower) L- fuzzy quasi-approximation operator on a set X induces a new kind of strongly stratified L-fuzzy topology in natural way and hence induces a strong L-topology on a set X.

Let $(X,\mathcal{C}_R)$ be an upper L-fuzzy quasi-approximation space, there exists a method to induce a strongly stratified L-fuzzy topological space, In fact, define $\mathcal{T}_R : L^X \to L$ as follows: $\forall\ \lambda \in L^X$,

$$\begin{aligned}\mathcal{T}_R(\lambda) = S(\mathcal{C}_R(\lambda),\lambda) &= \bigwedge_{x\in X}(\mathcal{C}_R(\lambda)(x) \to \lambda(x))\\ &= \bigwedge_{y\in X}(\bigvee_{x\in X} R(x,y) * \lambda(y) \to \lambda(x)) = \bigwedge_{x,y\in X} R(x,y) \to (\lambda(y) \to \lambda(x)).\end{aligned}$$

Observe that for all $\alpha \in L, \lambda \in L^X$, we have

$$\begin{aligned}\mathcal{T}_{\mathcal{C}_R}(\alpha*\lambda) = S(\mathcal{C}_R(\alpha*\lambda),\alpha*\lambda) &= S(\alpha*\mathcal{C}_R(\lambda),\alpha*\lambda)\ (by\ Theorem\ 3.2(4))\\ &= S(\mathcal{C}_R(\lambda),\alpha \to (\alpha*\lambda))\ (by\ Lemma\ 5.2(2))\\ &\geq S(\mathcal{C}_R(\lambda),\lambda) = \mathcal{T}_{\mathcal{C}_R}(\lambda)\ (by\ (I10)).\end{aligned}$$

From this, we obtain the following Lemma.

Lemma 4.1. *If $(X,\mathcal{C}_R)$ is an upper L-fuzzy quasi-approximation space, then $\mathcal{T}_{\mathcal{C}_R}$ satisfies*

$$(H1)\ \mathcal{T}_R(\overline{\alpha}*\lambda) \geq \mathcal{T}_R(\lambda).$$

Theorem 4.2. *$\mathcal{T}_R$ is an strongly stratified Alexandrov L-fuzzy topology.*

Proof. (O1) $\mathcal{T}_R(1_X) = S(\mathcal{C}_R(1_X),1_X) = S(1_X,1_X) = 1$, $\mathcal{T}_R(1_\phi) = S(\mathcal{C}_R(1_\phi),1_\phi) = S(1_\phi,1_\phi) = 1$.

(O3) Let $\{\lambda_i : i \in I\}$ be any family of fuzzy subset in X, then

$$\begin{aligned}\mathcal{T}_R(\bigvee_i \lambda_i) = S(\mathcal{C}_R(\bigvee_i \lambda_i),\bigvee_i \lambda_i) &= S(\bigvee_i \mathcal{C}_R(\lambda_i),\bigvee_i \lambda_i)\ (by\ Theorem\ 3.2(3))\\ &\geq \bigwedge_i S(\mathcal{C}_R(\lambda_i,\bigvee_i \lambda_i)) \geq \bigwedge_i S(\mathcal{C}_R(\lambda_i),\lambda_i) = \bigwedge_i \mathcal{T}_R(\lambda_i).\end{aligned}$$

(O4)

$$\begin{aligned}\mathcal{T}_R(\bigwedge_i \lambda_i) &= \bigwedge_{x,y\in X} R(x,y) \to \big((\bigwedge_i \lambda_i)(y) \to (\bigwedge_i \lambda_i)(x)\big)\\ &= \bigwedge_{x,y\in X} R(x,y) \to \bigwedge_i \big((\bigwedge_i \lambda_i)(y) \to (\lambda_i(x))\big)\\ &\geq \bigwedge_i \bigwedge_{x,y\in X} R(x,y) \to (\lambda_i(y) \to \lambda_i(x)) = \bigwedge_i \mathcal{T}_R(\lambda_i).\end{aligned}$$

(H2)

$$\begin{aligned}\mathcal{T}_R(\overline{\alpha} \to \lambda) = S(\mathcal{C}_R(\overline{\alpha} \to \lambda),\overline{\alpha} \to \lambda) &= S(\overline{\alpha}*\mathcal{C}_R(\overline{\alpha} \to \lambda),\lambda)\ (by\ Lemma\ 5.2(2))\\ &= S(\mathcal{C}_R(\overline{\alpha}*(\overline{\alpha} \to \lambda)),\lambda)\ (by\ Theorem\ 3.2(4))\\ &\geq S(\mathcal{C}_R(\lambda),\lambda) = \mathcal{T}_R(\lambda)\ (by (I7)).\end{aligned}$$

□

Corollary 4.3. *$\tau_R = \{\lambda \in L^X \mid \tau_R(\lambda) = 1\} = \{\lambda \in L^X \mid \mathcal{C}_R(\lambda) = \lambda\}$ is a strongly stratified Alexandrov L-fuzzy topology.*

Theorem 4.4. *Let $(X, \mathcal{C}_R)$ be a strongly stratified upper L-fuzzy quasi-approximation space. Define an operator $R_{\mathcal{C}} : X \times X \to L$ as follows*

$$R_{\mathcal{C}}(x, y) = \mathcal{C}(1_x)(y).$$

Then,

(1) $(X, R_{\mathcal{C}})$ is an L-fuzzy quasi-approximation space,

(2) $R_{\mathcal{C}_R} = R$ and $\mathcal{C}_{R_{\mathcal{C}}} = \mathcal{C}$.

Proof. (1) Since $R_{\mathcal{C}}(x, y) = \mathcal{C}(1_x)(y) \geq 1_x(y)$, then $R_{\mathcal{C}}(x, x) = 1$. From Theorem 3.5, we have $\mathcal{C}(\lambda)(x) = \bigvee_{z \in X}(\lambda(z) * \mathcal{C}(1_z)(x))$.

$$\begin{aligned} R_{\mathcal{C}}(x, y) &= \mathcal{C}(1_x)(y) \geq \mathcal{C}(\mathcal{C}(1_x))(y) \\ &= \bigvee_{z \in X}(\mathcal{C}(1_x)(z) * \mathcal{C}(1_z)(y)) = \bigvee_{z \in X}(R_{\mathcal{C}}(x, z) * R_{\mathcal{C}}(z, y)). \end{aligned}$$

(2) $R_{\mathcal{C}_R}(x, y) = \mathcal{C}_R(1_x)(y) = \bigvee_{z \in X}(1_x(z) * R(z, y)) = R(x, y)$. □

Corollary 4.5. *Let $(X, \mathcal{C}_R)$ be a strongly stratified upper L-fuzzy quasi-approximation space. Define an operator $R^{-1}_{\mathcal{C}} : X \times X \to L$ as follows*

$$R^{-1}_{\mathcal{C}}(x, y) = \mathcal{C}(1_y)(x).$$

Then,

(1) $(X, R^{-1}_{\mathcal{C}})$ is an L-quasi-approximation space,

(2) $R^{-1}_{\mathcal{C}_{R^{-1}}} = R$ and $\mathcal{C}_{R^{-1}_{\mathcal{C}}} = \mathcal{C}$.

Similarly, Let $(X, \mathcal{I}_R)$ be a lower L-fuzzy quasi-approximation space, there exists a method to induce a strongly stratified L-fuzzy topological space, In fact, define $\mathcal{T} : L^X \to L$ as follows

$$\mathcal{T}_R(\lambda) = S(\lambda, \mathcal{I}_R(\lambda)) = \bigwedge_{x, y \in X} R(x, y) \to (\lambda(x) \to \lambda(y)).$$

Observe that for all $\alpha \in L, \lambda \in L^X$, we have

$$\begin{aligned} \mathcal{T}_R(\overline{\alpha} \to \lambda) &= S(\overline{\alpha} \to \lambda, \mathcal{I}_R(\overline{\alpha} \to \lambda)) = S(\overline{\alpha} \to \lambda, \overline{\alpha} \to \mathcal{I}_R(\lambda)) \ (by\ Theorem\ 3.2(4)) \\ &= S(\overline{\alpha} * (\overline{\alpha} \to \lambda), \mathcal{I}_R(\lambda)) \ (by\ Lemma\ 5.2(2)) \\ &\geq S(\lambda, \mathcal{I}_R(\lambda)) = \mathcal{T}_R(\lambda) \ (by (I7)). \end{aligned}$$

From this, we obtain the following Lemma.

Lemma 4.6. *If $(X, \mathcal{I}_R)$ is a lower L-fuzzy quasi-approximation space, then $\mathcal{T}_R$ satisfies*

$$\mathcal{T}_R(\overline{\alpha} \to \lambda) \geq \mathcal{T}_R(\lambda).$$

Theorem 4.7. *τ_R is a strongly stratified Alexandrov L-topology.*

Proof. By a dual sense of Theorem 4.2, it is easily proved. □

Corollary 4.8. *$\tau_R = \{\lambda \in L^X \mid \mathcal{I}_R(\lambda) = \lambda\}$ is a strongly stratified Alexandrov L-topology.*

Corollary 4.9. $\tau_R = \{\lambda \in L^X \mid \lambda(x) * R(x, y) \leq \lambda(y) \ \forall \ x, y \in X\}$.

Proof. By Corollary 4.8 and Lemma 3.10, it is easily proved. □

Corollary 4.10. *Let τ_R be the L-topology defined as Corollary 4.8, then for every $x, y \in X$, we have*

$$R(x, y) = \bigwedge_{\lambda \in \tau_R} \lambda(x) \to \lambda(y).$$

Theorem 4.11. *Let τ be an L-topology on a set X, define the relation R_τ on X as for all $x, y \in X$,*

$$R_\tau(x, y) = \bigwedge_{\lambda \in \tau_R} \lambda(x) \to \lambda(y).$$

Then R_τ is reflexive and transitive. Moreover, $\tau \subseteq \tau_{R_\tau}$.

Proof. For each $x \in X$, we have $R_\tau(x, x) = \bigwedge_{\lambda \in \tau}(\lambda(x) \to \lambda(x)) = 1$, hence R is reflexive.

For each $x, y, z \in X$, we have

$$\begin{aligned} R_\tau(x, y) * R_\tau(y, z) &= \big(\bigwedge_{\lambda \in \tau} (\lambda(x) \to \lambda(y))\big) * \big(\bigwedge_{\lambda \in \tau} (\lambda(y) \to \lambda(z))\big) \\ &\leq \big(\lambda(x) \to \lambda(y)\big) * (\lambda(y) \to \lambda(z)) \leq \lambda(x) \to \lambda(z) \ (by \ (I3)). \end{aligned}$$

Hence, $R_\tau(x, y) * R_\tau(y, z) \leq \bigvee_{y \in X} \bigwedge_{\lambda \in \tau}(\lambda(x) \to \lambda(z)) = R_\tau(x, z)$. Thus,

$R_\tau(x, y) * R_\tau(y, z) \leq R_\tau(x, z)$, i.e. R_τ is transitive. For each $\lambda \in L^X$, we have

$$\lambda(x) * R_\tau(x, y) \leq \lambda(y) \ \forall \ x, y \in X.$$

Hence, $\lambda(x) \leq R_\tau(x, y) \to \lambda(y)$ for each $y \in X$. So, $\lambda(x) \leq \mathcal{I}_R(\lambda)(x)$. Then we have $\mathcal{I}_R(\lambda) = \lambda$, because R_τ is reflexive, i.e. $\lambda \in \tau_R$. □

Theorem 4.12. *If τ is a strongly stratified Alexandrov L-topology on X and R_τ is a L-relation, then $\tau = \{\lambda \in L^X \mid \lambda(x) * R_\tau(x, y) \leq \lambda(y) \ \forall x, y \in X\}$. Moreover,*

$$\tau = \tau_{R_\tau}.$$

Proof. We show that for each $\lambda \in L^X$, with $\lambda(x) * R_\tau(x, y) \leq \lambda(y)$ implies $\lambda \in \tau$. For $x \in X$, define $\mu_x : X \to L$ as

$$\mu_x(z) = \lambda(x) * R_\tau(x, z).$$

Then, $\mu_x(x) = \lambda(x)$ and $\mu_x(z) \leq \lambda(z)$ for all $z \in X$ which implies

$$\lambda = \bigvee_{x \in X} \mu(x).$$

For each $\rho \in \tau$, define $f_\rho(z) = \rho(x) \to \rho(z)$. It follows that f_ρ, because τ is strongly stratified. Since τ is Alexandrov L-topology, then $\bigwedge_{\rho \in \tau}(f_\rho) \in \tau$.

Now, $\mu_x(z) = \lambda(x) * R_\tau(x, z) = \lambda(x) * (\bigwedge_{\rho \in \tau}(\rho(x) \to \rho(z))) = \lambda(x) * \bigwedge_{\rho \in \tau}(f_\rho)$.

So, $\mu_x \in \tau$, because τ is strongly stratified. Then $\lambda \in \tau$. On the other hand, since for each $\lambda \in L^X$ with $\mathcal{I}_R(\lambda) = \lambda$, we have

$$\lambda(x) * R_\tau(x,y) = \mathcal{I}_R(\lambda)(x) * R_\tau(x,y) = (\bigwedge_{y \in X} (R_\tau(x,y) \to \lambda(y)) * R_\tau(x,y)$$
$$\leq (R_\tau(x,y) \to \lambda(y))) * R_\tau(x,y) \leq \lambda(y) \ (by\ (I7)).$$

Therefore, $\lambda \in \tau$, i.e. $\tau_{R_\tau} \subseteq \tau$. By Theorem 4.11, we have $\tau = \tau_{R_\tau}$. □

Example 4.13. Let $(L = [0,1], *, \to, 1)$ be a complete residuated lattice with the law of double negation defined by

$$x * y = (x + y - 1) \vee 0,\ x \to y = (1 - x + y) \wedge 1,\ x' = 1 - x.$$

Let $X = \{a, b, c\}$ and $\lambda \in L^X$ as follows $\lambda(a) = 1,\ \lambda(b) = 0.1,\ \lambda(c) = 0.4.$ Define $R \in L^{X \times X}$ as follows

$$R = \begin{pmatrix} 1 & 0.2 & 0.9 \\ 0.8 & 1 & 0.7 \\ 0.6 & 0.5 & 1 \end{pmatrix}$$

So, we have

(1) By Definition 3.4, we could obtain an L-fuzzy strongly stratified interior space

$$\mathcal{I}(\lambda)(x) = \begin{cases} 0.5, & if\ x = a, \\ 0.1, & if\ x = b, \\ 0.4, & if\ x = c. \end{cases}$$

So, $\mathcal{I}(0.25 \to \lambda) = 0.25 \to \mathcal{I}(\lambda) = 0.85.$

(2) By Theorem 3.5, we could obtain a L-fuzzy strongly stratified closure space

$$\mathcal{C}(\lambda)(x) = \begin{cases} 0.3, & if\ x = a, \\ 0.8, & if\ x = b, \\ 0.6, & if\ x = c. \end{cases}$$

So, $\mathcal{C}(0.25 * \lambda) = 0.25 * \mathcal{C}(\lambda) = 0.25.$

3) By Theorem 4.2, we could obtain a strongly stratified Alexandrov L-fuzzy topology

$$\mathcal{T}_R(\mu) = \begin{cases} 1, & if\ \mu = \bar{0}\ or \bar{1}, \\ 0.3, & if\ \mu = \lambda \end{cases}$$

So, $\mathcal{T}_R(0.25 \to \lambda) = 1 \geq \mathcal{T}_R(\lambda).$

5. Continuous Mappings

Definition 5.1. Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two upper L-fuzzy quasi-approximation spaces, then a mapping $f : (X, \mathcal{C}_1) \to (Y, \mathcal{C}_2)$ is a $\mathcal{C}$-map if $f(\mathcal{C}_1(\lambda)) \leq \mathcal{C}_2(f(\lambda))$ for each $\lambda \in L^X$.

Definition 5.2. Let $(X, \mathcal{I}_1)$ and $(Y, \mathcal{I}_2)$ be two lower L-fuzzy quasi-approximation spaces, then a mapping $f : (X, \mathcal{I}_1) \to (Y, \mathcal{I}_2)$ is a $\mathcal{I}$-map if $f^{\leftarrow}(\mathcal{I}_2(\lambda)) \leq \mathcal{I}_1(f^{\leftarrow}(\lambda))$ for each $\lambda \in L^X$.

Theorem 5.3. *If* $f : (X, R_1) \to (Y, R_2)$ *is a* R*-map, then* $f : (X, \mathcal{C}_{R_1}) \to (Y, \mathcal{C}_{R_2})$ *is a* $\mathcal{C}$*-map.*

Proof.

$$f(\mathcal{C}_{R_1}(\mu))(y) = \bigvee_{x \in f^{\leftarrow}(\{y\})} \mathcal{C}_{R_1}(\mu)(x) = \bigvee_{x \in f^{\leftarrow}(\{y\})} \bigvee_{z \in X} (R_1(x, z) * \mu(z))$$
$$\leq \bigvee_{z \in X} R_2(f(x), f(z)) * f(\mu)(f(z)) \leq \mathcal{C}_{R_2}(f(\mu))(y).$$

□

Theorem 5.4. *If a mapping* $f : (X, \mathcal{C}_1) \to (Y, \mathcal{C}_2)$ *is a* $\mathcal{C}$*-map, then* $f : (X, R_{\mathcal{C}_1}) \to (Y, R_{\mathcal{C}_2})$ *is* R*-map.*

Proof. $R_{\mathcal{C}_1}(x, y) = \mathcal{C}_1(1_x)(y) \leq f(\mathcal{C}_1(1_x))(f(y)) \leq R_2(1_{f(x)})(f(y)) = R_{\mathcal{C}_2}(f(x), f(y)).$ □

Theorem 5.5. *If a mapping* $f : (X, \mathcal{C}_{R_1}) \to (Y, \mathcal{C}_{R_2})$ *is* $\mathcal{C}$*-map, then* $f : (X, \mathcal{T}_{\mathcal{C}_{R_1}}) \to (Y, \mathcal{T}_{\mathcal{C}_{R_2}})$ *is* LF*-continuous.*

Proof. The continuity of $f : (X, \mathcal{T}_{\mathcal{C}_{R_1}}) \to (Y, \mathcal{T}_{\mathcal{C}_{R_2}})$ can be shown as follows

$$\mathcal{T}_{\mathcal{C}_{R_1}}(f^{\leftarrow}(\lambda)) = S(\mathcal{C}_{R_1}(f^{\leftarrow}(\lambda)), f^{\leftarrow}(\lambda)) \geq S(f^{\leftarrow}(\mathcal{C}_{R_2}(\lambda)), f^{\leftarrow}(\lambda))$$
$$= \bigwedge_{x \in X} (\mathcal{C}_{R_2}(\lambda)(f(x)) \to \lambda(f(x))) \geq \bigwedge_{y \in X} (\mathcal{C}_{R_2}(\lambda)(y) \to \lambda(y))$$
$$= S(\mathcal{C}_{R_2}(\lambda), \lambda) = \mathcal{T}_{\mathcal{C}_{R_2}}(\lambda).$$

□

Theorem 5.6. *If* $f : (X, R_1) \to (Y, R_2)$ *is a* R*-map, then* $f : (X, \mathcal{I}_{R_1}) \to (Y, \mathcal{I}_{R_2})$ *is an* $\mathcal{I}$*-map.*

Proof.

$$f^{\leftarrow}(\mathcal{I}_{R_2}(\lambda))(x) = \mathcal{I}_{R_2}(\lambda)(f(x)) = \bigwedge_{z \in Y} (R_2(f(x), z) \to \lambda(z))$$
$$\leq \bigwedge_{f(y) = z \in Y} (R_2(f(x), f(y)) \to \lambda(f(y)))$$
$$\leq \bigwedge_{y \in X} (R_1(x, y) \to f^{\leftarrow}(\lambda)(y)) = \mathcal{I}_{R_1}(f^{\leftarrow}(\lambda))(x).$$

□

Theorem 5.7. *If a mapping* $f : (X, \mathcal{I}_{R_1}) \to (Y, \mathcal{I}_{R_2})$ *is continuous, then* $f : (X, \mathcal{T}_{\mathcal{I}_{R_1}}) \to (Y, \mathcal{T}_{\mathcal{I}_{R_2}})$ *is* LF*-continuous.*

Proof. The continuity of $f : (X, \mathcal{T}_{\mathcal{I}_{R_1}}) \to (Y, \mathcal{T}_{\mathcal{I}_{R_2}})$ can be shown as follows

$$\mathcal{T}_{\mathcal{I}_{R_1}}(f^{\leftarrow}(\lambda)) = S(f^{\leftarrow}(\lambda), \mathcal{I}_{R_1}(f^{\leftarrow}(\lambda))) \geq S(f^{\leftarrow}(\lambda), f^{\leftarrow}(\mathcal{I}_{R_2}(\lambda)))$$
$$= \bigwedge_{x \in X} (\lambda(f(x)) \to \mathcal{I}_{R_2}(\lambda)(f(x))) \geq \bigwedge_{y \in X} (\lambda(y) \to \mathcal{I}_{R_2}(\lambda)(y))$$
$$= S(\lambda, \mathcal{I}_{R_2}(\lambda)) = \mathcal{T}_{\mathcal{I}_{R_2}}(\lambda).$$

□

Corollary 5.8. *If a mapping* $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ *is continuous, then* $f : (X, R_{\mathcal{T}_2}) \to (Y, R_{\mathcal{T}_1})$ *is* R*-map.*

Proof. Since $\lambda \in \mathcal{T}_2$ implies $f^{\leftarrow}(\lambda) \in \mathcal{T}_1$, then by Theorem 4.11 we have

$$R_{\mathcal{T}_2}(f(x), f(y)) = \bigwedge_{\lambda \in \mathcal{T}_2} (\lambda(f(x)) \to \lambda(f(y))) = \bigwedge_{\lambda \in \mathcal{T}_2} (f^{\leftarrow}(\lambda)(x) \to f^{\leftarrow}(\lambda)(y))$$
$$= \bigwedge_{\mu \in \mathcal{T}_1} (\mu(x) \to \mu(y)) = R_{\mathcal{T}_1}(x, y).$$

□

Theorem 5.9. *Let $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ be two upper L-fuzzy quasi-approximation spaces. If $(X, \mathcal{C}_1)$ is strongly stratified, then $f : (X, R_1) \to (Y, R_2)$ is $\mathcal{C}$-map iff*

$$f(\mathcal{C}_1(1_x)) \leq \mathcal{C}_2(1_{f(x)}) \ \forall\ x\ \in X.$$

Proof. Since $\lambda = \bigvee_{z\in X} \lambda(z) * 1_z$, we have

$$\begin{aligned} f(\mathcal{C}_1(\lambda))(y) = & \bigvee_{x\in f^{\leftarrow}(\{y\})} \mathcal{C}_1(\bigvee_{z\in X} \lambda(z) * 1_z)(x) = \bigvee_{x\in f^{\leftarrow}(\{y\})} \bigvee_{z\in X} \lambda(z) * \mathcal{C}_1(1_z)(x) \\ & = \bigvee_{z\in X} \lambda(z) * (\bigvee_{x\in f^{\leftarrow}(\{y\})} \mathcal{C}_1(1_z)(x)) = \bigvee_{z\in X} \lambda(z) * \big(f(\mathcal{C}_1(1_z))(y)\big) \\ & \leq \bigvee_{z\in X} \lambda(z) * \big(\mathcal{C}_2(1_{f(z)})(y)\big) \leq \bigvee_{z\in X} f(\lambda)f(z) * \big(\mathcal{C}_2(1_{f(z)})(y)\big) \leq \mathcal{C}_2(f(\lambda))(y). \end{aligned}$$

□

6. Conclusion

In this paper, we have proved that a pair of L-fuzzy quasi-approximation operators can induce an L-fuzzy topological space (L-topological space) if and only if the fuzzy relation is reflexive and transitive. On the other hand, under certain conditions an L-fuzzy interior (closure) operator derived from an L-fuzzy topological space can be associated with a reflexive and transitive fuzzy relation such that the induced L-fuzzy lower (upper) approximation operator is the L-fuzzy interior (closure) operator.

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