ALGEBRAIC PROPERTIES OF INTUITIONISTIC FUZZY RESIDUATED LATTICES

F. GHANAVIZI MAROOF AND E. ESLAMI

Abstract. In this paper, we investigate more relations between the symmetric residuated lattices L with their corresponding intuitionistic fuzzy residuated lattice $\tilde{L}$. It is shown that some algebraic structures of L such as Heyting algebra, Glivenko residuated lattice and strict residuated lattice are preserved for $\tilde{L}$. Examples are given for those structures that do not remain the same. Also some special subsets of $\tilde{L}$ such as regular elements $Rg(\tilde{L})$, dense elements $D(\tilde{L})$, infinitesimal elements $Inf(\tilde{L})$, boolean elements $B(\tilde{L})$ and $Rad_{BL}(\tilde{L})$ are characterized. The relations between these and corresponding sets in L will be investigated.

1. Introduction

H. Ono considered residuated lattices as an algebraic structure of substructural logics in [17]. P. Hájek introduced the notion of BL-algebra as a residuated lattice with two more conditions, namely divisibility and prelinearity to prove the completeness of Łukasiewicz logic as a many valued logic [13]. He showed that these algebras are the best algebraic counterparts of fuzzy logics generated by continuous t-norms [13]. K. T. Atanassov [1, 3] introduced the notion of an intuitionistic fuzzy set as a generalization of a fuzzy set. There are controversy ideas [8, 5] on the word intuitionistic fuzzy conflicting with the intuitionistic fuzzy logic introduced by G.Takeuti and S.Titani in [22]. Despite of our understanding the different opinions on the terminological aspects, we are not going to enter in these old discussions. But to make the distinction, we may choose the A-IFS to denote the concept of intuitionistic fuzzy set introduced by K. T. Atanassov. Many researchers have been working on the theory of this subject and developed it in interesting different branches [4, 14, 19]. K. T. Atanassov and S. Stoeva generalized the concept of A-IFS to intuitionstic L-fuzzy sets [2] where L is an appropriate lattice. A. Tepavcevic and T. Gerstenkorn give a new definition of lattice valued intuitionstic fuzzy sets [21]. Glad Deschrijver, et al. [7, 9] considered the intuitionistic fuzzy connectives and defined negator, t-norms, t-conorms and implicators on the lattice of membership values. Andreja Tepavcevic et al. considered the general form of lattice valued IFSs in [20]. E. Eslami introduced the notion of Intuitionistic Fuzzy Residuated Lattices (A-IFRL) built from symmetric residuated lattices. In fact he constructed lattice $\tilde{L}$ as a part of $L \times L$, where L is a symmetric residuated lattice [10, 11]. The

Received: November 2014; Revised: December 2015; Accepted: January 2016

Key words and phrases: Intuitionstic fuzzy residuated lattice, Heyting algebra, Relative Stone lattice, Glivenko residuated lattice, MV (MTL, SRL)-algebra.

residuated lattice $\tilde{L}$ is called the Intuitionistic fuzzy residuated lattice corresponding to L. We may also think of this as a two dimensional residuated lattice. In [12] we showed that if L is a BL-algebra as an algebraic counter part of fuzzy logic, it is not necessarily true that $\tilde{L}$ is a BL-algebra. This shows that algebraic semantics of fuzzy logic and A-IFL are essentially different, as expected. This motivates us to consider different cases of residuated lattices for L and examine whether they are preserved for $\tilde{L}$.
This paper is organized as follows: in the next section we give the preliminaries including the basic definitions and theorems that are needed in the other parts. In section 3, we prove some connections between the structure L and $\tilde{L}$ such as if L is a Heyting algebra (Glivenko residuated lattice, SRL-algebra) , then $\tilde{L}$ is a Heyting algebra (Glivenko residuated lattice, SRL-algebra), respectively. Also by giving some examples we will show that the structure $\tilde{L}$ is not an MV(MTL)-algebra or a relative Stone lattice even though we assume that L is an MV(MTL)-algebra or a relative Stone lattice. In section 4, we characterize some special subsets of $\tilde{L}$ such as $Rg(\tilde{L})$, $D(\tilde{L})$, $Rad_{BL}(\tilde{L})$, $B(\tilde{L})$ and $Inf(\tilde{L})$ and their connections.

2. **Preliminaries**

In this section we give some definitions and theorems that we need in the sequel.

2.1. **A-IFSs and RLs.** In this part we recall original definition of an intuitionistic fuzzy set given by K. Atanassov and the definition of a residuated lattice together with their basic properties:

Definition 2.1. [1] *Let $\boldsymbol{U}$ be the universe. By an Intuitionistic Fuzzy set (IFS) in $\boldsymbol{U}$ we mean a set of ordered triples $\boldsymbol{A} = \{(x, \mu_A(x), \nu_A(x)) | x \in \boldsymbol{U}\}$, where $\mu_A(x)$ is the membership degree of x to $\boldsymbol{A}$ and $\nu_A(x)$ is the non-membership degree of x to $\boldsymbol{A}$ such that $\mu : \boldsymbol{U} \longrightarrow [0, 1]$ and $\nu : \boldsymbol{U} \longrightarrow [0, 1]$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in \boldsymbol{U}$. The complement of an IFS $\boldsymbol{A}$ is defined by $\boldsymbol{A}^c = \{(x, \nu_A(x), \mu_A(x)) | x \in \boldsymbol{U}\}$.*

We recall from [9] that $\mathbf{L}^* = \{(x, y) \in [0, 1]^2 | 0 \leq x + y \leq 1\}$ is a complete lattice with the order defined by

$$(x_1, x_2) \preceq (y_1, y_2) \; if \; and \; only \; if \; x_1 \leq y_1 \; and \; y_2 \leq x_2.$$

The notion of a residuated lattice is defined as follows:

Definition 2.2. [13, 18] *A residuated lattice is an algebra $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:*

(i) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(ii) $(L, *, 1)$ is a commutative monoid, and
(iii) the operation $*$ and $\rightarrow$ form an adjoint pair,i.e.,

$$x * y \leq z \; if \; and \; only \; if \; x \leq y \rightarrow z$$

for all $x, y, z \in L$. We denote $x \rightarrow 0$ by $\neg x$ for all $x \in L$.

- *A residuated lattice L is called a **Heyting algebra** in which $x * y = x \wedge y$, for all $x, y \in L$.*

- *A residuated lattice L is called an* ***MTL-algebra*** *if it satisfies the following equation for all $x, y \in L$:*
  $(x \to y) \vee (y \to x) = 1$ *(Prelinearity).*
- *An MTL-algebra L is called a* ***BL-algebra*** *if it satisfies the following equation for all $x, y \in L$:*
  $x * (x \to y) = x \wedge y$ *(Divisibility).*
- *A residuated lattice L is called an* ***MV-algebra*** *if it satisfies the following equation for all $x, y \in L$:*
  $(x \to y) \to y = (y \to x) \to x$.

Based on the above Definition, the basic properties of residuated lattices are summarized in the following theorem:

Theorem 2.3. [18] *In any residuated lattice $L = (L, \wedge, \vee, *, \to, 0, 1)$ the following properties hold for all $x, y, z \in L$:*
(1) $x \leq y$ if and only if $x \to y = 1$,
(2) $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$.

We recall from [18] that an element $a \in L$ is called complemented if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exist it is called a complement of a. We will denote the set of all complemented elements in L by $B(L)$. For any symmetric residuated lattice L, we have

$$B(L) = \{a \in L |\ a \vee \neg a = 1\}. \quad (1)$$

Proposition 2.4. [18] *Let L be a residuated lattice. For $a \in L$ and $n \geq 1$, the following assertions are equivalent:*

(i) $a^n \in B(L)$,
(ii) $a \vee \neg(a)^n = 1$.

We know from [10] that if $L = (L, \wedge, \vee, *, \to, 0, 1)$ is a residuated lattice and $\neg : L \to L$ is defined by $\neg x = x \to 0$, then $\neg$ is a negator on L which is not necessarily involutive.

Definition 2.5. [6] *A residuated lattice is called an ivolutive residuated lattice if the negation $\neg$ is involutive, i.e., when the following equation holds:*

$$\neg\neg x = (x \to 0) \to 0 = x.$$

But in the case where $\neg$ is not involutive, if it is possible to add an involutive negation on the given residuated lattice, we will get a symmetric residuated lattice defined as follows:

Definition 2.6. [6] *A residuated lattice L is called a symmetric residuated lattice if it is equipped with a unary operation $\sim$ satisfying:*

$$\sim\sim x = x,$$
$$\sim (x \vee y) = \sim x \wedge \sim y,$$
$$\sim (x \wedge y) = \sim x \vee \sim y.$$

for all $x, y \in L$.

From the above definition, we can easily check that if $x \leq y$, then $\sim y \leq \sim x$. Also we have $\sim x \leq 1$, for all $x \in L$, so $\sim 1 \leq \sim\sim x = x$, for all $x \in L$. Hence $\sim 1 = 0$ and therefore $\sim 0 = \sim\sim 1 = 1$.

2.2. **A-IFRLs.** We recall the process of constructing an Intuitionistic Fuzzy Residuated Lattice (Now A-IFRL) given by E. Eslami in [10]. Let $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ be a symmetric residuated lattice. Define the algebra

$$\tilde{L} = (\tilde{L}, \bigwedge, \bigvee, T, I, \tilde{0}, \tilde{1}),$$

where:

(i) $\tilde{L} = \{(x, y) \in L^2 | x \leq \sim y\}$,
(ii) $(x, y) \bigwedge (u, v) = (x \wedge u, y \vee v)$,
(iii) $(x, y) \bigvee (u, v) = (x \vee u, y \wedge v)$,
(iv) $T((x, y), (u, v)) = (x * u, S(y, v))$, where $S(y, v) = \sim (\sim y * \sim v)$.
(v) $I((x, y), (u, v)) = ((x \rightarrow u) \wedge (\sim y \rightarrow \sim v), \sim (\sim y \rightarrow \sim v))$,
(vi) $\tilde{0} = (0, 1)$, $\tilde{1} = (1, 0)$.
It is proved that:

Theorem 2.7. [10] *Let $L, \tilde{L}, T$ and I be as above. Then $\tilde{L} = \big(\tilde{L}, \bigwedge, \bigvee, T, I, \tilde{0}, \tilde{1}\big)$ is a residuated lattice.*

Definition 2.8. *The algebra $\tilde{L}$ with the corresponding properties is called an A-Intuitionistic Fuzzy Residuated Lattice (A-IFRL) corresponding to the symmetric residuated lattice L.*

Example 2.9. [10] *Let $L_0 = \{0, a, b, 1\}$, where $0 < a < b < 1$. $L_0 = (L_0, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ is a symmetric residuated lattice with the operators defined in TABLE 1.*

x	$\sim x$
0	1
a	b
b	a
1	0

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

TABLE 1. Operators $\sim$, $*$ and $\rightarrow$

Now

$$\tilde{L_0} = \{(0, 0), (0, a), (0, b), (0, 1), (a, b), (a, a), (a, 0), (b, a), (b, 0), (1, 0)\},$$

with the order $\preceq$ shown in FIGURE 1 and corresponding T and I, is an A-Intuitionistic Fuzzy Residuated Lattice.

Note that the tables for operators T and I in this example are obtained by easy but long computations using (iv) and (v) respectively. To save time and space, we omit them.

3. Algebraic Connection Between L and $\tilde{L}$

Let us fix some notations and terminologies in this section for some conveniences. We assume that the lattice $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$ is a symmetric residuated complete lattice (Which makes $\tilde{L}$ a complete lattice, by Lemma 3.2 in [10]) and $(a, b)' = I\big((a, b), \tilde{0}\big)$, for all $(a, b) \in \tilde{L}$.

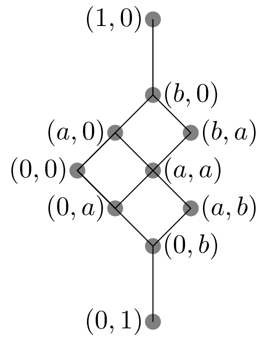

FIGURE 1. Lattice Structure of $\tilde{L_0}$

To have a common and conventional form for $\tilde{L}$, from now on, we use $\bigodot$ and $\longrightarrow$, instead of T and I in the Definition 2.8, respectively.

3.1. **Some Heritable Conditions.** In this subsection we prove some algebraic properties that $\tilde{L}$ inherits from L. In fact, we choose some algebras which are residuated lattices with additional properties.

Theorem 3.1. *Let L and $\tilde{L}$ be as in Definition 2.8. Then L is a Heyting algebra if and only if $\tilde{L}$ is a Heyting algebra.*

Proof. Suppose L is a Heyting algebra. We show that $(a,b)\bigodot(c,d) = (a,b)\bigwedge(c,d)$, for all $(a,b),(c,d) \in \tilde{L}$.

By Definition 2.8, $(a,b)\bigodot(c,d) = \big(a * c, S(b,d)\big)$ and from definition of S we have $(a,b)\bigodot(c,d) = \big(a*c, \sim(\sim b * \sim d)\big)$. By assumption and properties of $\sim$ we get:

$$(a,b)\bigodot(c,d) = \big(a \wedge c, \sim(\sim b \wedge \sim d)\big)$$

$$= \big(a \wedge c, \sim\sim b \vee \sim\sim d\big)$$

$$= \big(a \wedge c, b \vee d\big) = (a,b)\bigwedge(c,d).$$

Conversely, assume that $\tilde{L}$ is a Heyting algebra. Let $a, b \in L$ be arbitrary elements. Then $(a,\sim a),(b,\sim b) \in \tilde{L}$. Since $\tilde{L}$ is a Heyting algebra,

$$\big(a,\sim a\big)\bigwedge\big(b,\sim b\big) = \big(a,\sim a\big)\bigodot\big(b,\sim b\big).$$

Now by Definition 2.8 we have

$$\big(a\wedge b, \sim a \vee \sim b\big) = \big(a*b, \sim(\sim\sim a * \sim\sim b)\big).$$

By using the properties of $\sim$ we get

$$\big(a\wedge b, \sim(a\wedge b)\big) = \big(a*b, \sim(a*b)\big),$$

which implies that $a \wedge b = a * b$. □

Recall that from [15] a strict residuated lattice (SRL-algebra) is a residuated lattice L which satisfies the following equation:
$\neg(x * y) = \neg x \vee \neg y$, for all $x, y \in L$.

Theorem 3.2. *Let L and $\tilde{L}$ be as in Definition 2.8. Then L is a strict residuated lattice (SRL-algebra) if and only if $\tilde{L}$ is a strict residuated lattice.*

Proof. We show that

$$A \bigodot B \longrightarrow \big(0,1\big) = (A \longrightarrow \tilde{0}) \bigvee (B \longrightarrow \tilde{0})$$

for all $A, B \in \tilde{L}$ and $\tilde{0} = (0, 1)$. Here we employ brackets for some outer paranteses. Let $A = (a, b), B = (c, d) \in \tilde{L}$ be arbitrary elements.

We have

$$Q = A \bigodot B \longrightarrow \big(0,1\big) = \big(a,b\big) \bigodot \big(c,d\big) \longrightarrow \big(0,1\big)$$

$$= \big(a * c, \sim (\sim b * \sim d)\big) \longrightarrow \big(0,1\big)$$

$$= \Big(\neg \sim\sim \big(\sim b * \sim d\big), \sim \neg \sim\sim \big(\sim b * \sim d\big)\Big).$$

Using strict and symmetric properties of L, we conclude

$$Q = \Big(\neg\big(\sim b * \sim d\big), \sim \neg\big(\sim b * \sim d\big)\Big)$$

$$= \Big(\neg \sim b \vee \neg \sim d, \sim \big(\neg \sim b \vee \neg \sim d\big)\Big)$$

$$= \Big(\neg \sim b \vee \neg \sim d, \sim \neg \sim b \wedge \sim \neg \sim d\Big).$$

Thus by Definition 2.8,

$$Q = (\neg \sim b, \sim \neg \sim b) \bigvee (\neg \sim d, \sim \neg \sim d)$$

$$= (A \longrightarrow \tilde{0}) \bigvee (B \longrightarrow \tilde{0}).$$

Conversely, assume that $\tilde{L}$ is an SRL-algebra. Let $a, b \in L$ be arbitrary elements. Then $(a, \sim a), (b, \sim b) \in \tilde{L}$. Since $\tilde{L}$ is a SRL-algebra,

$$\big(a, \sim a\big) \bigodot \big(b, \sim b\big) \longrightarrow \big(0,1\big) = \Big(\big(a, \sim a\big) \longrightarrow \big(0,1\big)\Big) \bigvee \Big(\big(b, \sim b\big) \longrightarrow \big(0,1\big)\Big).$$

Now by Definition 2.8 we have

$$\Big(a * b, \sim \big(\sim\sim a * \sim\sim b\big)\Big) \longrightarrow \big(0,1\big)$$
$$= \Big(\neg \sim\sim a, \sim \neg \sim\sim a\Big) \bigvee \Big(\neg \sim\sim b, \sim \neg \sim\sim b\Big).$$

Using the properties of $\sim$ we get

$$\Big(\neg \sim\sim \big(a * b\big), \neg \sim\sim \big(a * b\big)\Big) = \Big(\neg a \vee \neg b, \sim \big(\neg a \vee \neg b\big)\Big),$$

which implies that $\neg(a * b) = \neg a \vee \neg b$. $\square$

Recall from [16] that a Glivenko residuated lattice is a residuated lattice L that fulfils $\neg\neg(\neg\neg x \to x) = 1$, for all $x \in L$.

Theorem 3.3. *If L is a Glivenko residuated lattice, then $\tilde{L}$ is a Glivenko residuated lattice.*

Proof. It is sufficient to prove that $\Big(\big((A')' \longrightarrow A\big)'\Big)' = \tilde{1}$, for all $A = (a, b) \in \tilde{L}$.

We have,

$$A' = (a, b) \longrightarrow \tilde{0} = (a, b) \longrightarrow (0, 1)$$

$$= \big((a \to 0) \wedge (\sim b \to \sim 1), \sim (\sim b \to \sim 1)\big)$$

$$= (\neg a \wedge \neg \sim b, \sim \neg \sim b),$$

Since $a \leq \sim b$, using Theorem 2.3 part (2), $\neg \sim b \leq \neg a$ and from this it follows that $\neg a \wedge \neg \sim b = \neg \sim b$. Therefore

$$A' = (\neg \sim b, \sim \neg \sim b).$$

Hence

$$\big(A'\big)' = A' \longrightarrow \tilde{0} = (\neg \sim b, \sim \neg \sim b) \longrightarrow (0, 1)$$

$$= \big((\neg \sim b \to 0) \wedge (\neg \sim b \to \sim 1), \sim (\neg \sim b \to \sim 1)\big)$$

$$= (\neg\neg \sim b, \sim \neg\neg \sim b).$$

Since $(a, b) \in \tilde{L}$, we have $a \leq \sim b$ and therefore from Theorem 2.3, part (2) , we have
$\neg\neg \sim b \to a \leq \neg\neg \sim b \to \sim b$.

Thus we get

$$\big(A'\big)' \longrightarrow A$$

$$= \big((\neg\neg \sim b \to a) \wedge (\neg\neg \sim b \to \sim b), \sim (\neg\neg \sim b \to \sim b)\big)$$

$$= \big(\neg\neg \sim b \to a, \sim (\neg\neg \sim b \to \sim b)\big).$$

Finally we have,

$$\Big(\big((A')' \longrightarrow A\big)'\Big)'$$
$$= \Big(\neg\neg \sim\sim \big(\neg\neg \sim b \to \sim b\big), \sim \neg\neg \sim\sim \big(\neg\neg \sim b \to \sim b\big)\Big)$$

$$= \Big(\neg\neg\big(\neg\neg \sim b \to \sim b\big), \sim \neg\neg\big(\neg\neg \sim b \to \sim b\big)\Big)$$

$$= (1, 0) = \tilde{1}.$$

□

3.2. **Some Counter Examples.** In the current section we show that the A-IFRL, $\tilde{L}$, does not inherit some algebraic properties from L.

Example 3.4. *Consider the given algebra L_0 in Example 2.9. We observe that L_0 is an MTL-algebra (even more it is a BL-algebra), but $\tilde{L_0}$ is not an MTL-algebra.*

In fact let take the elements $(0,0)$ and (a,b) of $\tilde{L_0}$, then $((0,0) \longrightarrow (a,b)) \bigvee ((a,b) \longrightarrow (0,0)) = (a,b) \bigvee (a,0) = (a,0) \neq (1,0)$, which shows that prelinearity condition does not hold in $\tilde{L_0}$.

Example 3.5. *Consider the symmetric MV-algebra $L_1 = (L_1, \wedge, \vee, \rightarrow, \sim, 0, 1)$ with the operators defined in TABLE 2, where $L_1 = \{0, a, b, c, d, 1\}$, $0 < a, b < c < 1$ and $0 < b < d < 1$.*

x	$\sim x$
0	1
a	d
b	c
c	b
d	a
1	0

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

TABLE 2. Operators $\sim$, $*$ and $\rightarrow$

Construct

$$\tilde{L}_1 = \{(0,0), (0,a), (0,b), (0,c), (0,d), (0,1), (a,0), (a,b), (a,d), (b,0), (b,a), (b,b), (b,c), (c,0), (c,b), (d,0), (d,a), (1,0)\}.$$

We see that $\tilde{L_1} = (\tilde{L_1}, \bigwedge, \bigvee, \bigodot, \longrightarrow, \tilde{0}, \tilde{1})$ with the order $\preceq$ and corresponding $\bigodot$ and $\longrightarrow$, is an A-IFRL which is not an MV-algebra.

The reasons are as follows:
We know that an MV-algebra L_1 is a residuated lattice which satisfies the equation

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

for all $x, y \in L_1$. So it is sufficient to show that it doesn't hold in $\tilde{L_1}$.

Put $A = (b,c), B = (b,a) \in \tilde{L_1}$. We have

$$\begin{aligned}
\big(A \longrightarrow B\big) \longrightarrow B &= \big((b,c) \longrightarrow (b,a)\big) \longrightarrow (b,a) \\
&= \big((b \rightarrow b) \wedge (\sim c \rightarrow \sim a), \sim (\sim c \rightarrow \sim a)\big) \longrightarrow (b,a) \\
&= \big(1 \wedge (b \rightarrow d), \sim (b \rightarrow d)\big) \longrightarrow (b,a) \\
&= (1,0) \longrightarrow (b,a) \\
&= (b,a).
\end{aligned}$$

Also we have,

$$\begin{aligned}
\big(B \longrightarrow A\big) \longrightarrow A &= \big((b,a) \longrightarrow (b,c)\big) \longrightarrow (b,c) \\
&= \big((b \rightarrow b) \wedge (\sim a \rightarrow \sim c), \sim (\sim a \rightarrow \sim c)\big) \longrightarrow (b,c) \\
&= \big(1 \wedge (d \rightarrow b), \sim (d \rightarrow b)\big) \longrightarrow (b,c) \\
&= (c,b) \longrightarrow (b,c) \\
&= (c \rightarrow b) \wedge (\sim b \rightarrow \sim c) \longrightarrow \sim (\sim b \rightarrow \sim c) \\
&= d \wedge d \longrightarrow \sim d = (d,a).
\end{aligned}$$

Hence,

$$\Big(A \longrightarrow B\Big) \longrightarrow B \neq \Big(B \longrightarrow A\Big) \longrightarrow A.$$

We recall from [18] that a Heyting algebra L is called a relative Stone lattice if it satisfies the prelinearity equation.

Example 3.6. *Consider the symmetric relative Stone lattice* $L_2 = (L_2, \wedge, \vee, \rightarrow, \sim, 0, 1)$ *with the operators defined in TABLE* 3*, where* $L_2 = \{0, a, b, c, d, 1\}$,$0 < a, b < c < 1$ *and* $0 < a < d < 1$*.*

x	$\sim x$
0	1
a	c
b	d
c	a
d	b
1	0

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	0	b
c	0	a	b	c	a	c
d	0	a	0	a	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	d	d	1	1	d	1
c	0	d	b	1	d	1
d	b	c	b	c	1	1
1	0	a	b	c	d	1

TABLE 3. Operators $\sim$, $*$ and $\rightarrow$

Construct

$$\tilde{L}_2 = \{(0,0),(0,a),(0,b),(0,c),(0,d),(0,1),(a,0),(b,0),(c,0),$$
$$(d,0),(a,a),(b,a),(c,a),(a,b),(d,b),(a,c),(b,d),(1,0)\}.$$

We see that $\tilde{L_2} = (\tilde{L_2}, \bigwedge, \bigvee, \bigodot, \longrightarrow, \tilde{0}, \tilde{1})$ *With the order* $\preceq$ *and corresponding* $\bigodot$ *and* $\longrightarrow$ *is an A-IFRL which is not a relative Stone lattice.*

The reasons are as follows:

We know that a relative Stone lattice L_2 is a Heyting algebra that fulfils the equation

$$(x \rightarrow y) \vee (y \rightarrow x) = 1,$$

for all $x, y \in L_2$. So it is sufficient to show that it doesn't hold in $\tilde{L_2}$. Put $A = (0, b), B = (a, a) \in \tilde{L_2}$. We have

$$\begin{aligned} A \longrightarrow B &= (0,b) \longrightarrow (a,a) \\ &= \big((0 \rightarrow a) \wedge (d \rightarrow c), \sim (d \rightarrow c)\big) \\ &= (1 \wedge c, a) = (c, a). \end{aligned}$$

Also we have,

$$\begin{aligned} B \longrightarrow A &= (a,a) \longrightarrow (0,b) \\ &= \big((a \rightarrow 0) \wedge (c \rightarrow d), \sim (c \rightarrow d)\big) \\ &= (b \wedge d, b) = (0, b). \end{aligned}$$

So we conclude that

$$\begin{aligned} (A \longrightarrow B) \bigvee (B \longrightarrow A) &= (c,a) \bigvee (0,b) \\ &= (c \vee 0, a \wedge b) = (c, 0) \neq (1, 0). \end{aligned}$$

4. **Fuzzy Part and Some Special Subsets**

Since A-IFSs are extensions of FSs, It is important to find the fuzzy part of an A-IFRL $\tilde{L}$ and distinguish the roles of it in identifying special subsets of $\tilde{L}$.

From now on our $\tilde{L}$ is a symmetric A-IFRL in which the added involutive negation is standard negation. $N(a, b) = (b, a)$ for all $(a, b) \in \tilde{L}$.

Definition 4.1. [11] *Let $F(\tilde{L}) = \{(a,b) \in \tilde{L} | b = \sim a\}$. We call $F(\tilde{L})$ the fuzzy part of $\tilde{L}$.*

For any symmetric residuated lattice L, we recall four subsets associated with L from [18]:

$$Rg(L) = \{a \in L | \neg\neg a = a\}, \tag{2}$$

$$D(L) = \{a \in L | \neg a = 0\}, \tag{3}$$

$$K(L) = \{a \in L | \neg a = \sim a\}, \tag{4}$$

$$Rad_{BL}(L) = \{a \in L | \neg(a^n) \leq a, \; for\; all\; n \in \mathbb{N}\}. \tag{5}$$

The elements of $Rg(L)$ are called regular and those of $D(L)$ dense elements of L.

Now we find $K(\tilde{L})$, $Rg(\tilde{L})$, $B(\tilde{L})$, $Rad(\tilde{L})$ and $D(\tilde{L})$. By $(1)-(5)$ we have

$$\begin{aligned} K(\tilde{L}) &= \{A \in \tilde{L} | \; A' = NA\} = \{(a,b) \in \tilde{L} | \; (a,b)' = N\big((a,b)\big)\} \\ &= \{(a,b) \in \tilde{L} | \; (\neg \sim b, \sim \neg \sim b) = (b,a)\} \\ &= \{(a,b) \in \tilde{L} | \; a = \sim b, \neg \sim b = b\}. \end{aligned}$$

So,

$$K(\tilde{L}) = \{(a,b) \in \tilde{L} | \; a = \sim b, \neg \sim b = b\}. \tag{6}$$

Regarding the above notions we state the following lemma from [11]:

Lemma 4.2. *Let L be a normal symmtric residuated lattice. Then $K(\tilde{L}) = F(\tilde{L}) \bigcap \big(K(L) \times K(L)\big)$. (By a normal symmtric residuated lattice, we mean a symmetric residuated lattice L in which $\neg x = \neg x \wedge \sim x$ [6].)*

Now we characterize the regular elements of $\tilde{L}$ as follow:

$$\begin{aligned} Rg(\tilde{L}) &= \{A \in \tilde{L} | \; (A')' = A\} = \{(a,b) \in \tilde{L} | \; \big((a,b)'\big)' = (a,b)\} \\ &= \{(a,b) \in \tilde{L} | \; (\neg\neg \sim b, \sim \neg\neg \sim b) = (a,b)\} \\ &= \{(a,b) \in \tilde{L} | \; a = \sim b, \neg\neg \sim b = a\}. \end{aligned}$$

Hence

$$Rg(\tilde{L}) = \{(a,b) \in \tilde{L} | \; a = \sim b, \neg\neg \sim b = a\}. \tag{7}$$

Proposition 4.3. *Let L be a symmetric residuated lattice. Then $Rg(\tilde{L}) = F(\tilde{L}) \bigcap \big(Rg(L) \times L\big)$.*

Proof. Let $(a,b) \in Rg(\tilde{L})$ be an arbitrary element. By (7), we have $a = \sim b$ and $\neg\neg \sim b = a$. From $a = \sim b$ we can conclude that $(a,b) \in F(\tilde{L})$, also by putting a in $\neg\neg \sim b = a$ instead of $\sim b$ we have $\neg\neg a = a$. So $a \in Rg(L)$ and therefore $(a,b) \in F(\tilde{L}) \bigcap \big(Rg(L) \times L\big)$.

It is easy to check the converse. □

Lemma 4.4. *Let L be a symmetric residuated lattice. Then $F(\tilde{L}) \bigcap \big(Rg(L) \times Rg(L)\big) \subsetneq Rg(\tilde{L})$.*

Proof. Considering Example 2.9, we have $Rg(L) = \{0, a, 1\}$. So

$$\begin{aligned} T &= Rg(L) \times Rg(L) \\ &= \{(0,0), (0,a), (0,1), (a,0), (a,a), (a,1), (1,0), (1,a), (1,1)\}, \end{aligned}$$

and $F(\tilde{L}) = \{(0,1), (a,b), (b,a), (1,0)\}$.

By (7), we have $Rg(\tilde{L}) = \{(0,1), (a,b), (1,0)\}$. Hence $T \bigcap F(\tilde{L}) = \{(0,1), (1,0)\} \subsetneq Rg(\tilde{L})$. The proof of the inclusion part is similar to the proof of the above lemma. □

Proposition 4.5. *Let L be a normal symmtric residuated lattice and $\tilde{L} = K(\tilde{L})$. Then $Rg(\tilde{L}) = \tilde{L}$. In other words, all the elements of $\tilde{L}$ are regular under these conditions.*

Proof. Let $A = (a, b)$ be an arbitrary element in $\tilde{L}$. We have

$$(A')' = (\neg\neg \sim b, \sim \neg\neg \sim b)$$

Since $(a, b) \in K(\tilde{L})$ by using Lemma 4.2, we can put *an* instead of $\sim b$ in the above equation. Hence $(A')' = (\neg\neg a, \sim \neg\neg a)$. Again by using Lemma 4.2 we conclude $(A')' = (\sim\sim a, \sim\sim\sim a)$, since $(a, b) \in \big(K(L) \times K(L)\big)$. Now by symmetry property, $(A')' = (a, \sim b)$ that is $(A')' = A$, since $(a, b) \in F(\tilde{L})$. □

Now we characterize $B(\tilde{L})$ and $D(\tilde{L})$. By (1) and (3) we have

$$\begin{aligned} B(\tilde{L}) &= \{A \in \tilde{L} |\ A \bigvee A' = \tilde{1}\} \\ &= \{(a,b) \in \tilde{L} |\ (a \vee \neg \sim b, b \wedge \sim \neg \sim b) = (1,0)\} \\ &= \{(a,b) \in \tilde{L} |\ a \vee \neg \sim b = 1, b \wedge \sim \neg \sim b = 0\} \\ &= \{(a,b) \in \tilde{L} |\ a \vee \neg \sim b = 1\}. \end{aligned}$$

Hence

$$B(\tilde{L}) = \{(a,b) \in \tilde{L} |\ a \vee \neg \sim b = 1\}. \tag{8}$$

Also we have

$$\begin{aligned} D(\tilde{L}) &= \{A \in \tilde{L} |\ A' = \tilde{0}\} = \{(a,b) \in \tilde{L} |\ (\neg \sim b, \sim \neg \sim b) = (0,1)\} \\ &= \{(a,b) \in \tilde{L} |\ \neg \sim b = 0\}. \end{aligned}$$

So

$$D(\tilde{L}) = \{(a,b) \in \tilde{L} | \neg \sim b = 0\}. \tag{9}$$

It is worth to express that $D(L)$ is a filter in a symmetric residuated lattice L, So $D(\tilde{L})$ is also a filter in $\tilde{L}$ and as in [11], $\tilde{D(L)} = \{(a,b) \in \tilde{L} | a \in D(L)\}$ is a filter in $\tilde{L}$, and we have:

Proposition 4.6. *Let L and $\tilde{L}$ be as in Definition 2.8. Then $\tilde{D(L)} \subsetneq D(\tilde{L})$.*

Proof. Let $A = (a, b) \in \tilde{L}$ be an arbitrary element in $\tilde{D(L)}$. We have $a \leq \sim b$ and $a \in D(L)$ (or equivalently, $\neg a = 0$). Therefore $0 \leq \neg \sim b \leq \neg a = 0$ and so $A = (a, b) \in D(\tilde{L})$. To show that equality does not hold, we compute $\tilde{D(L)}$ and $D(\tilde{L})$ in Example 2.9, that is, $\tilde{D(L)} = \{(b, a), (b, 0), (1, 0)\} \subsetneq D(\tilde{L}) = \{(0, 0), (0, a), (a, a), (a, 0), (b, a), (b, 0), (1, 0)\}$. □

Lemma 4.7. *Let L be a symmtric residuated lattice. Then* $F(\tilde{L}) \bigcap \big(D(L) \times D(L)\big) \subsetneq D(\tilde{L})$.

Proof. In fact it is easy to see that $F(\tilde{L}) \bigcap \big(D(L) \times D(L)\big) \subset D(\tilde{L})$. Also we have $(1, 0) \in D(\tilde{L})$ but $(1, 0) \notin F(\tilde{L}) \bigcap \big(D(L) \times D(L)\big)$. □

Now we characterize $Rad_{BL}(\tilde{L})$. By (5) we have

$$\begin{aligned} Rad_{BL}(\tilde{L}) &= \{A \in \tilde{L} |\ (A^n)' \preceq A,\ for\ all\ n \in \mathbb{N}\} \\ &= \{(a, b) \in \tilde{L} |\ ((a, b)^n)' \preceq (a, b),\ for\ all\ n \in \mathbb{N}\} \\ &= \{(a, b) \in \tilde{L} |\ (a^n, \sim (\sim b)^n)' \preceq (a, b),\ for\ all\ n \in \mathbb{N}\} \\ &= \{(a, b) \in \tilde{L} |\ (\neg(\sim b)^n, \sim \neg(\sim b)^n) \preceq (a, b),\ for\ all\ n \in \mathbb{N}\}. \end{aligned}$$

Hence after some computations we obtain:

$$Rad_{BL}(\tilde{L}) = \{(a, b) \in \tilde{L} |\ \neg(\sim b)^n \leq a,\ for\ all\ n \in \mathbb{N}\}. \tag{10}$$

Proposition 4.8. *Let L be a symmetric residuated lattice. Then* $F(\tilde{L}) \bigcap \big(Rad_{BL}(L) \times L\big) \subseteq Rad_{BL}(\tilde{L})$.

Proof. Let $(a, b) \in F(\tilde{L}) \bigcap \big(Rad_{BL}(L) \times L\big)$ be an arbitrary element. By (10), we have $a = \sim b$ and $\neg(a)^n \leq a$ for every $n \in \mathbb{N}$. By putting $\sim b$ in $\neg(a)^n \leq a$ instead of a we have $\neg(\sim b)^n \leq a$ for every $n \in \mathbb{N}$. So $(a, b) \in Rad_{BL}(\tilde{L})$. □

It is easy to see that the converse of Proposition 4.8 is not true. It is sufficient to choose $(0, 0) \in \tilde{L_2}$ in Example 3.6. So we have $(0, 0) \notin F(\tilde{L_2})$, but $(0, 0) \in Rad_{BL}(\tilde{L_2})$.
From [11] we know that $B(\tilde{L}) = (B(L) \times B(L)) \bigcap F(\tilde{L})$.

Theorem 4.9. *Let (a, b) be an arbitrary element in $\tilde{L}$. If* $(a, b) \in B(\tilde{L}) \bigcap Rad_{BL}(\tilde{L})$*, then $a^n \in B(L)$ for every $n \in \mathbb{N}$.*

Proof. By (8) and (10) we have $a \vee \neg \sim b = 1$ and $\neg(\sim b)^n \leq a$ for every $n \in \mathbb{N}$. On the other hand $(\sim b)^n \leq \sim b$ for every $n \in \mathbb{N}$, so by Theorem 2.3 part (2) we get $\neg \sim b \leq \neg(\sim b)^n$ for every $n \in \mathbb{N}$. By hypothesise $1 = a \vee \neg \sim b \leq a \vee \neg(\sim b)^n \leq 1$ and therefore $a \vee \neg(\sim b)^n = 1$ for every $n \in \mathbb{N}$. Since $B(\tilde{L}) \subseteq F(\tilde{L})$, we have $a = \sim b$. Hence $a \vee \neg(a)^n = 1$ for every $n \in \mathbb{N}$ and so by Proposition 2.4, we obtain $a^n \in B(L)$ for every $n \in \mathbb{N}$. □

Obiviously, the converse of Theorem 4.9 doesn't hold. Since by choosing $(0,0) \in \tilde{L_2}$ in Example 3.6, we have $0^n \in B(L_2)$ for every $n \in \mathbb{N}$, but $(0,0) \notin B(\tilde{L_2})$.
We recall from [18] that an element a of a residuated lattice L is called infinitesimal if $a \neq 1$ and $a^n \geq \neg a$ for any $n \in \mathbb{N}$. We denote by $Inf(L)$ the set of all infinitesimals of L. Now we characterize $Inf(\tilde{L})$. We have

$$\begin{aligned} Inf(\tilde{L}) &= \{A \in \tilde{L} |\ A \neq \tilde{1}, A' \preceq A^n,\ for\ all\ n \in \mathbb{N}\} \\ &= \{(a,b) \in \tilde{L} |\ (a,b) \neq (1,0), (a,b)' \preceq (a,b)^n,\ for\ all\ n \in \mathbb{N}\} \\ &= \{(a,b) \in \tilde{L} |\ (a,b) \neq (1,0), (\neg \sim b, \sim \neg \sim b) \preceq (a^n, \sim (\sim b)^n),\ for\ all\ n \in \mathbb{N}\}. \end{aligned}$$

So after some computations we get at the conclusion that:

$$Inf(\tilde{L}) = \{(a,b) \in \tilde{L} |\ (a,b) \neq (1,0), \neg \sim b \leq a^n,\ for\ all\ n \in \mathbb{N}\}. \tag{11}$$

Proposition 4.10. *Let L be a symmetric residuated lattice. Then* $F(\tilde{L}) \bigcap \big(Inf(L) \times L\big) \subsetneq Inf(\tilde{L})$.

Proof. Let $(a,b) \in F(\tilde{L}) \bigcap \big(Inf(L) \times L\big)$ be an arbitrary element. We have $a = \sim b$, $a \neq 1$ and $\neg a \leq a^n$ for every $n \in \mathbb{N}$. By putting $\sim b$ in $\neg a \leq a^n$ instead of a we have $\neg \sim b \leq a^n$ for every $n \in \mathbb{N}$. So $(a,b) \in Inf(\tilde{L})$. In Example 3.5, we have $F(\tilde{L}) = \{(0,1),(a,d),(d,a),(b,c),(c,b),(1,0)\}$ and $Inf(L) \times L = \{(1,0),(1,a),(1,b),(1,c),(1,d),(1,1)\}$. So $F(\tilde{L}) \cap \big(Inf(L) \times L\big) = \{(1,0)\}$, which is not in $Inf(\tilde{L})$. □

Based on the above we may summarize basic results in the following corollary:

Corollary 4.11. *Let L be a symmtric residuated lattice. Then we have:*

(i) $D(\tilde{L}) \bigcap B(\tilde{L}) = \{(1,0)\}$,
(ii) $D(\tilde{L}) \bigcap Rg(\tilde{L}) = \{(1,0)\}$,
(iii) $D(\tilde{L}) \bigcap K(\tilde{L}) = \{(1,0)\}$,
(iv) $Rg(\tilde{L}) \bigcap B(\tilde{L}) = B(\tilde{L})$,
(v) $Inf(\tilde{L}) \bigcap D(\tilde{L}) = D(\tilde{L}) \setminus \{(1,0)\}$,
(vi) $Inf(\tilde{L}) \bigcap B(\tilde{L}) = \emptyset$.

Proof. We just prove (v) and (vi), the others are similar.
(v): It is sufficient to show that $D(\tilde{L}) \setminus \{(1,0)\} \subseteq Inf(\tilde{L}) \bigcap D(\tilde{L})$. Let (a,b) be an arbitrary element in $D(\tilde{L}) \setminus \{(1,0)\}$. Then by (11) and (9), we have $(a,b) \neq (1,0)$, $\neg \sim b = 0$, so we obtain $\neg \sim b = 0 \leq a^n$ for every $n \in \mathbb{N}$. Hence $(a,b) \in Inf(\tilde{L}) \bigcap D(\tilde{L})$.
(vi): Let $(a,b) \in Inf(\tilde{L}) \bigcap B(\tilde{L})$. Then by (11) and (8), we have $a \vee \neg \sim b = 1$, $(a,b) \neq (1,0)$ and $\neg \sim b \leq a^n$ for every $n \in \mathbb{N}$. So $1 = a \vee \neg \sim b \leq a \vee a^n = a \leq 1$, hence $a = 1$ and therefore $(a,b) = (1,0)$. That is a contradiction. □

5. Conclusion

Since the A-IFRL $\tilde{L} = (\tilde{L}, \bigwedge, \bigvee, \bigodot, \longrightarrow, \tilde{0}, \tilde{1})$ is obtained from a residuated lattice $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$, their relation is very important. Some algebraic structures are preserved. For instance if L is a Heyting algebra, Glivenko residuated lattice and Strict residuated lattice so is $\tilde{L}$. One of the most noticed point is that even if L is a BL-algebra as the algebraic counterpart of fuzzy logic, $\tilde{L}$ is not a Bl-algebra. This shows that Intuitionistic fuzzy logic is a proper extension of fuzzy logic as expected. In this regards we defined the fuzzy part of $\tilde{L}$ and investigated some relations among the fuzzy part and other special sets of $\tilde{L}$. We showed by counter examples that some algebraic structures are not preserved. For instance MTL-algebra, MV-algebra and relative Stone lattices. There are probably more algebraic structures in either case. Depending on the needs and applications, one can obtain different results. Most of the applications fall into lattice theory and algebraic structure theory along with algebraic logic as our examples and counter examples show. There are probably more algebraic properties that are heritable from L to $\tilde{L}$. Since the negation induced by $\longrightarrow$ in $\tilde{L}$ is never involutive, we added standard negation to $\tilde{L}$ to get a symmetric A-IFRL. It might be possible to choose another involutive negations rather than standard one. Then there are still many open questions on choosing $*, \sim$ in L and N in $\tilde{L}$. So we can obtain different algebras and as a result different logics are discovered.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87–96.

[2] K. T. Atanassov and S. Stoeva, *Intuitionistic L-fuzzy sets*, in:R.Trapple (ed.), Elsevier Science Publishers B.V., North Holland, 1984.

[3] K. T. Atanassov and G. Gargov, *Elements of intuitionistic fuzzy logic. part I*, Fuzzy Sets and Systems, **145** (1998), 267–277.

[4] P. Burillo and H. Bustince, *Intuitionistic fuzzy relations. effects of Atanassov's operators on the properties of Intuitionistic Fuzzy relations*, Mathware and Soft Computing, **2** (1995), 117–148.

[5] G. Cattaneo and D. Ciucci, *Basic intuitionistic principles in fuzzy set theories and its extensions (A terminological debate on Atanassov IFS)*, Fuzzy Sets and Systems, **24** (2006), 3198–3219.

[6] R. Cignoli and F. Esteva, *Commutative integral bounded residuated lattices with an added involution*, Annals of Pure and Applied Logic, **171** (2009), 150–170.

[7] C. Cornelis and G. Deschrijver and E. E. Kerre, *Classification on intuitionistic fuzzy implicators: an algebraic approach*, In Proceedings of the FT & T' 02, Durham, North Carolina, 105–108.

[8] D. Dubois and S. Gottwald and P. Hájek and J. Kacprzyk and H. Prade, *Terminological difficulties in fuzzy set theory- The case of "Intuitionistic Fuzzy Sets"*, Fuzzy Sets and Systems, **156** (2005), 485–491.

[9] G. Deschrijver and C. Cornelis and E. E. Kerre, *Intuitionistic fuzzy connectives revisited*, In proceedings of IPMU'02, 2002.

[10] E. Eslami, *An algebraic structure for Intuitionistic Fuzzy Logic*, Iranian Journal of Fuzzy Systems, **9(6)** (2012), 31–41.

[11] E. Eslami and W. Peng-Yung, *More on intutionistic fuzzy residuated lattices*, Journal of Multiple-Valued Logic and Soft Computing, **20(3)** (2013), 335–352.

[12] E. Eslami and F. Ghanavizi Maroof, *A Proposed axiomatic system for atanassov intuitionistic fuzzy logic (A-IFL)*, Notes on Intuitionistic Fuzzy Sets, **19(3)** (2013), 1–14.
[13] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic, Kluwer Academic Publishers, Drdrecht, 1998.
[14] Y. Hong and X. Ruiping and F. Xianwen, *Characterizing ordered semigroups by means of Intuitionistic Fuzzy Bi- ideals*, Mathware and Soft Computing, **14** (2007), 57–66.
[15] M. Kondo, *Note on strict residuated lattices with an involutive negation*, AAA80 Workshop on General Algebra& Workshop on Non- classical algebraic Structures, Bedlewo, Poland, 1-6 june, 2010.
[16] C. Muresan, *Dense elements and classes of a residuated lattices*, Bull. Math. Soc. Sci. Math. Roumanie Tome, **53(1)** (2010), 11–24.
[17] H. Ono, *Substructural logics and residuated lattices - an introduction*, Trends in Logic, (2003), 177–212.
[18] D. Piciu, *Algebras of fuzzy logic*, Craiova: Ed universtaria, 2007.
[19] E. Szmidt and K. Marta, *Atanassov's intuitionistic fuzzy sets in classification of imbalanced and overlapping classes. intelligent techniques and tools for novel system architectures*, Studies in Computational Intelligence (SCI), **109** (2008), 455–471.
[20] A. Tepavcevic and M. G. Ranitovic, *General form of lattice valued intuitionistic fuzzy sets*, Computational Intelligence, Theory and Applications, Springer Berlin Heidelberg, Germany, (2006), 375–381.
[21] A. Tepavcevic and T. Gerstenkorn, *Lattice valued intuitionistic fuzzy sets*, Central European Journal of Mathematics, **2(3)** (2004), 388–398.
[22] G. Takeuti and S. Titani, *Intuitionistic fuzzy logic and intuitionistic fuzzy set theory*, Journal of Symbolic Logic, **49(3)** (1984), 851–866.

Farnaz Ghanavizi Maroof, Department of Mathematics, Faculty of Mathematics and Compute, Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran
E-mail address: `farnaz.ghanavizi@yahoo.com`

Esfandiar Eslami*, Department of Mathematics, Faculty of Mathematics and Compute, Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran
E-mail address: `esfandiar.eslami@uk.ac.ir`

*Corresponding author