WIDTH INVARIANT APPROXIMATION OF FUZZY NUMBERS

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Abstract. In this paper, we consider the width invariant trapezoidal and triangular approximations of fuzzy numbers. The presented methods avoid the effortful computation of Karush-Kuhn-Tucker Theorem. Some properties of the new approximation methods are presented and the applicability of the methods is illustrated by examples. In addition, we show that the proposed approximations of fuzzy numbers preserve the expected value too.

1. Introduction

The set of fuzzy numbers, introduced by Dubois and Prade in [17], is an especial subclass of fuzzy sets on the real line and it plays a significant role in many important theoretical and practical considerations [16]. In processing of uncertain information presented by fuzzy numbers, the complicated membership functions have many drawbacks. Also handling complicated membership functions has some difficulties in interpretation of the results. These are the main reasons why we need a suitable approximation method for fuzzy numbers. In the last few years many papers have investigated the approximation of fuzzy numbers [1, 4, 11, 28, 30]. Chanas [14] and Grzegorzewski [20] have proposed the interval approximation of fuzzy numbers independently in 2001 and 2002, respectively. Abbasbandy and Asady have studied the trapezoidal approximation in [3]. Yeh [26] has proposed new algorithms for computing trapezoidal and triangular approximations. A triangular approximation of fuzzy numbers with respect to a weighted L_2-type metric is proposed by Zeng and Li in [31]. Unfortunately, this method does not always provide a triangular fuzzy number [7, 28]. Yeh in [28] has proposed a method for weighted approximations with trapezoidal or triangular fuzzy numbers and later he generalized these results by considering general weighted L_2-type metrics for weighted semi-trapezoidal approximations [29]. Grzegorzewski and Mrówka [21, 22] have proposed the expected interval invariant trapezoidal approximation. Then Ban [6] and Yeh [27] independently improved their results. Abbasbandy and Amirfakhrian have investigated the approximation problem of a generalized fuzzy number in [2]. Brandas has studied trapezoidal approximation preserving the core, the ambiguity and the value in [12]. Coroianu et al. have proposed a nearest approximation method by piecewise linear 1-knot fuzzy numbers [15]. Recently, Ban and Coroianu have presented the existence and uniqueness conditions to study of fuzzy number

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approximations under a general condition [11].
In this paper, we present new methods for trapezoidal and triangular approximations preserving the width. Similar to some recent papers [8, 10, 27], we avoid the effortful computation associated with the Karush-Kuhn-Tucker Theorem. Furthermore, we show that the proposed approximation preserves the expected value too.
The paper is organized as follows. In Section 2, we recall some basic notation and definitions. The width invariant trapezoidal approximation is presented in Section 3. The nearest symmetric trapezoidal approximation is studied in Section 4. In Section 5, we consider width invariant approximation by triangular fuzzy numbers. Some properties of the presented methods are given in Section 6. Finally, the conclusion is depicted in Section 7.

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper, see for example [16, 17].

Definition 2.1. Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0,1]$. Thus $u(x)$ is interpreted as the degree of membership of an element x in the fuzzy set u for each $x \in X$.

Let us denote by $F(\mathbb{R})$ the class of fuzzy subsets of the real axis (i.e., $u : \mathbb{R} \to [0,1]$) satisfying the following properties:
(i) u is normal, i.e., there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(ts+(1-t)r) \geq \min\{u(s), u(r)\},\ \forall t \in [0,1],\ s, r \in \mathbb{R}$),
(iii) u is upper semicontinuous on $\mathbb{R}$,
(iv) $cl\{s \in \mathbb{R} | u(s) > 0\}$ is compact where cl denotes the closure of a subset.

Then $F(\mathbb{R})$ is called the space of fuzzy numbers. Obviously $\mathbb{R} \subset F(\mathbb{R})$. For $0 < \alpha \leq 1$, we denote $u_\alpha = \{s \in \mathbb{R} | u(s) \geq \alpha\}$ and $u_0 = cl\{s \in \mathbb{R} | u(s) > 0\}$. From the conditions (i)-(iv), it follows that for any $u \in F(\mathbb{R})$, the $\alpha-$level set, u_α, is a nonempty compact interval, for all $0 \leq \alpha \leq 1$. The notation $u_\alpha = [u_L(\alpha), u_U(\alpha)]$ denotes $\alpha-$level set of u where

$$u_L(\alpha) = \inf\{x \in \mathbb{R} | u(x) \geq \alpha\}, \quad u_U(\alpha) = \sup\{x \in \mathbb{R} | u(x) \geq \alpha\}, \quad \forall \alpha \in (0,1].$$

For $u, v \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$, the sum $u+v$ and the scalar product λu are defined by

$$(u+v)_\alpha = u_\alpha + v_\alpha,\ (\lambda u)_\alpha = \lambda u_\alpha, \quad \forall \alpha \in [0,1],$$

where $u_\alpha + v_\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and λu_α means the usual product between an scalar and a subset of $\mathbb{R}$.
The expected interval EI and expected value EV of a fuzzy number A are defined as ([18, 23])

$$EI(A) = \left[\int_0^1 A_L(\alpha) d\alpha, \int_0^1 A_U(\alpha) d\alpha\right], \quad EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha) + A_U(\alpha) d\alpha\right).$$

The width of a fuzzy number A is defined as [14]

$$W(A)=\int_{-\infty}^{+\infty}A(x)dx=\int_0^1 A_U(\alpha)d\alpha-\int_0^1 A_L(\alpha)d\alpha. \tag{1}$$

Chanas [14] proved that for each fuzzy number A, we have

$$W(A)=W(EI(A)). \tag{2}$$

A metric structure on the set of fuzzy numbers is defined by [19]

$$d^2(A,B)=\int_0^1 (A_L(\alpha)-B_L(\alpha))^2 d\alpha+\int_0^1 (A_U(\alpha)-B_U(\alpha))^2 d\alpha. \tag{3}$$

A trapezoidal fuzzy number T, $T_\alpha=[T_L(\alpha),T_U(\alpha)]$, $\alpha\in[0,1]$, is given by

$$T_L(\alpha)=t_1+(t_2-t_1)\alpha, \quad \text{and} \quad T_U(\alpha)=t_4-(t_4-t_3)\alpha,$$

where $t_1,t_2,t_3,t_4\in\mathbb{R}$, $t_1\le t_2\le t_3\le t_4$. We denote a trapezoidal fuzzy number by $T=(t_1,t_2,t_3,t_4)$. The set of all trapezoidal fuzzy numbers is denoted by $F^T(\mathbb{R})$. For trapezoidal fuzzy number $T=(t_1,t_2,t_3,t_4)$, if we have $t_2=t_3$, then we call T a triangular fuzzy number. Also when $t_2-t_1=t_4-t_3$, then the fuzzy number T is called a symmetric trapezoidal fuzzy number. We denote by $F^S(\mathbb{R})$, the set of all symmetric trapezoidal fuzzy numbers.

Bodjanova in [13] has introduced a useful representation for trapezoidal fuzzy numbers by

$$T=[l,u,x,y], \quad l,u,x,y\in\mathbb{R},$$

where $x,y\ge 0, 2(u-l)\ge x+y$ and

$$T_L(\alpha)=l+x(\alpha-\frac{1}{2}), \quad T_U(\alpha)=u-y(\alpha-\frac{1}{2}), \quad \forall\alpha\in[0,1].$$

Then we can get easily

$$l=\frac{t_1+t_2}{2}, \quad u=\frac{t_3+t_4}{2}, \quad x=t_2-t_1, \quad y=t_4-t_3, \tag{4}$$

and $T\in F^S(\mathbb{R})$ if and only if $x=y$. Also, by direct calculation, we get

$$EV(A)=\frac{u+l}{2}, \tag{5}$$

$$W(A)=u-l. \tag{6}$$

By this notation, the distance between $T,T'\in F^T(\mathbb{R})$, $T=[l,u,x,y]$ and $T'=[l',u',x',y']$ becomes [26]

$$d^2(T,T')=(l-l')^2+(u-u')^2+\frac{1}{12}(x-x')^2+\frac{1}{12}(y-y')^2. \tag{7}$$

In [13], an important kind of fuzzy number is introduced as follows. Let $a,b,c,d\in\mathbb{R}$ and $a\le b\le c\le d$. We denote fuzzy number A by $A=(a,b,c,d)_{r,s}$ such that $r,s>0$ and

$$A_\alpha=[a+(b-a)\alpha^{1/r},d-(d-c)\alpha^{1/s}], \quad \alpha\in[0,1]. \tag{8}$$

When $r = s = 1$, then $A \in F^T(\mathbb{R})$. The set of all such fuzzy numbers is denoted by $F^{r,s}(\mathbb{R})$.
In [21], using Karush-Kuhn-Tucker Theorem, the authors have proposed a trapezoidal approximation preserving the expected interval, which is called later by Yeh [27], the extended trapezoidal approximation. Unfortunately, the extended trapezoidal approximation $T_e(A)$ of a fuzzy number A may not be a fuzzy number generally, see [5, 25].
We can consider extended trapezoidal fuzzy number $A = [A_L(\alpha), A_U(\alpha)]$ where $A_L(\alpha)$ and $A_U(\alpha)$ are polynomials defined on $[0, 1]$ with degree 0 or 1, without any other condition.

Definition 2.2. [27] An extended trapezoidal fuzzy number $A = [A_L(\alpha), A_U(\alpha)]$ is an order pair of polynomial functions of degree less than or equal to 1, i.e. $degA_L(\alpha) \leq 1$ and $degA_U(\alpha) \leq 1$.

It is obvious that in general, an extended trapezoidal fuzzy number may not be a fuzzy number. For instance, it is easy to check that the extended trapezoidal fuzzy number $A = [3\alpha + 1, 3 - \alpha]$ is not a fuzzy number.
Let $A = [A_L(\alpha), A_U(\alpha)]$ be an extended trapezoidal fuzzy number. Then analogous to (4), we can write A in the presentation $A = [l_e, u_e, x_e, y_e]$ where

$$l_e = \frac{A_L(0) + A_L(1)}{2}, \;\; u_e = \frac{A_U(1) + A_U(0)}{2}, \;\; x_e = A_L(1) - A_L(0), \;\; y_e = A_U(0) - A_U(1).$$

The set of all extended trapezoidal fuzzy numbers is denoted by $F_e^T(\mathbb{R})$. We may define a metric in $F_e^T(\mathbb{R})$ by (3) or (7) [27]. Additionally, the expected value and width of an extended trapezoidal fuzzy number can be defined by (5) and (6), respectively.
The extended trapezoidal approximation $T_e(A) = [l_e, u_e, x_e, y_e]$ of a given fuzzy number A is the extended trapezoidal fuzzy number which minimizes the distance $d(A, X)$, where $X \in F_e^T(\mathbb{R})$ [27]. In [26], the author proved that the extended trapezoidal approximation $T_e(A) = [l_e, u_e, x_e, y_e]$ of a fuzzy number A can be computed as

$$l_e = \int_0^1 A_L(\alpha)d\alpha, \qquad u_e = \int_0^1 A_U(\alpha)d\alpha, \tag{9}$$

$$x_e = 12\int_0^1 (\alpha - \frac{1}{2})A_L(\alpha)d\alpha, \qquad y_e = -12\int_0^1 (\alpha - \frac{1}{2})A_U(\alpha)d\alpha.$$

The real numbers x_e and y_e are non-negative and $u_e \geq l_e$. In [25], Yeh has proved two important properties related to the distance of the extended trapezoidal approximation operator, as follows.

Proposition 2.3. *Let $A, B \in F(\mathbb{R})$ and $T_e(A), T_e(B)$ be the extended trapezoidal approximations of A and B, respectively. Then we have*

$$d(T_e(A), T_e(B)) \leq d(A, B).$$

Proposition 2.4. *Let $A \in F(\mathbb{R})$. Then for any $B \in F^T(\mathbb{R})$, we have*

$$d^2(A, B) = d^2(A, T_e(A)) + d^2(T_e(A), B).$$

Remark 2.5. Let A and B be arbitrary fuzzy numbers and $T_e(A) = [l_e, u_e, x_e, y_e]$, $T_e(B) = [l'_e, u'_e, x'_e, y'_e]$ be their corresponding extended trapezoidal approximations. Then by Proposition 2.3 and (7), we have

$$(l_e - l'_e)^2 + (u_e - u'_e)^2 \leq d^2(A, B),$$
$$(x_e - x'_e)^2 + (y_e - y'_e)^2 \leq 12\ d^2(A, B).$$

Proposition 2.6. *Let $A \in F(\mathbb{R})$ and $T_e(A) = [l_e, u_e, x_e, y_e]$ be the extended trapezoidal approximation of A, then*

$$W(A) = W(T_e(A)).$$

Proof. By direct calculation, we obtain

$$W(T_e(A)) = u_e - l_e = \int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha = W(A).$$

□

3. **Width Invariant Trapezoidal Approximation**

In this section, we present a new method to find trapezoidal approximation of a given fuzzy number which preserve the width of fuzzy number. For any fuzzy number A, we compute the nearest trapezoidal fuzzy number $T(A)$ such that $W(A) = W(T(A))$. Let us denote by $T_e(A)$, the extended trapezoidal approximation of A. Then according to Propositions 2.4 and 2.6, finding the nearest width invariant trapezoidal approximation of A is equivalent to finding $T(A) \in F^T(\mathbb{R})$ where

$$W(T(A)) = W(T_e(A)),$$
$$d(T(A), T_e(A)) \leq d(B, T_e(A)), \quad \forall\ B \in F^T(\mathbb{R}),$$

such that $W(B) = W(T_e(A))$. Therefore, $T(A) = [l_T, u_T, x_T, y_T]$ if and only if $(l_T, u_T, x_T, y_T) \in \mathbb{R}^4$ satisfies

$$\min\left((l - l_e)^2 + (u - u_e)^2 + \frac{1}{12}(x - x_e)^2 + \frac{1}{12}(y - y_e)^2\right), \tag{10}$$

such that

$$x, y \geq 0, 2u - 2l \geq x + y, \tag{11}$$

$$u - l = u_e - l_e. \tag{12}$$

Then we have the optimization problem

$$\min\left(2(u - u_e)^2 + \frac{1}{12}(x - x_e)^2 + \frac{1}{12}(y - y_e)^2\right),$$

such that

$$x, y \geq 0, 2u - 2l \geq x + y,$$

where l_e, u_e, x_e, y_e are given by (9). It is easy to see that (10)-(12) is equivalent to problem

$$\min((x - x_e)^2 + (y - y_e)^2), \tag{13}$$

$$x, y \geq 0, 2u - 2l \geq x + y.$$

In addition

$$l = l_e, \quad u = u_e. \tag{14}$$

We define

$$M = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0, \quad x + y \leq 2u_e - 2l_e\},$$

and denote by $P_M(Z)$, the orthogonal projection of $Z \in \mathbb{R}^2$ on non-empty set M, with respect to d_E, the Euclidean metric on $\mathbb{R}^2$. Let us consider the inner product $<.,.>$ on $\mathbb{R}^2$ defined by

$$<(a_1, b_1), (a_2, b_2)> = a_1 a_2 + b_1 b_2.$$

If $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$ then

$$\begin{aligned} D^2((a_1, b_1), (a_2, b_2)) &= <(a_1 - a_2, b_1 - b_2), (a_1 - a_2, b_1 - b_2)> \\ &= 2(a_1 - a_2)^2 + (b_1 - b_2)^2, \end{aligned}$$

is the induced distance on $\mathbb{R}^2$ by the above inner product. We have the following theorem.

Theorem 3.1. *Let A be a given fuzzy number. Then there exists a unique width invariant trapezoidal approximation of A. Namely, there exists a unique trapezoidal fuzzy number $T(A)$ with $W(T(A)) = W(A)$ and satisfying*

$$d(T(A), A) \leq d(B, A), \quad \forall B \in F^T(\mathbb{R}),$$

where $W(B) = W(A)$.

Proof. To prove the statement of the theorem, we have to prove that (13)-(14) has a unique solution. First, according to (9), the set M is nonempty, since

$$2u_e - 2l_e = 2\int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha \geq 0.$$

On the other hand, M is a convex and closed subset of $\mathbb{R}^2$. Therefore for any $(x, y) \in \mathbb{R}^2$, there exists a unique $P_M(x, y)$ where [24]

$$D((x, y), P_M(x, y)) = \inf_{C \in M} D((x, y), C).$$

Then $(x_T, y_T) = P_M(x_e, y_e)$ is the unique solution of problem (13)-(14). We can also compute (l_T, u_T) accordingly to (14) uniquely by

$$l_T = l_e, \quad u_T = u_e. \tag{15}$$

□

Because $x_e, y_e \geq 0$, corresponding to (1)-(4) in Figure 1, the following cases are possible to find (x_T, y_T).

(1) If $(x_e, y_e) \in M$, i.e., $x_e + y_e \leq 2(u_e, l_e)$, then $P_M(x_e, y_e) = (x_e, y_e)$ and we have

$$x_T = x_e, \quad y_T = y_e.$$

If $(x_e, y_e) \notin M$, then we have 3 other possible cases.

(2) If $2l_e - 2u_e + x_e - y_e > 0$, then

$$x_T = 2u_e - 2l_e, \quad y_T = 0.$$

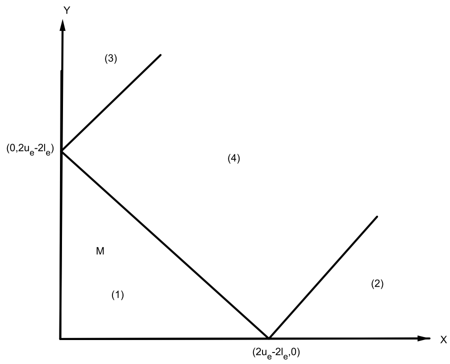

FIGURE 1. Corresponding Cases to Find x_e and y_e in T_A

(3) If $2u_e - 2l_e + x_e - y_e < 0$, then

$$x_T = 0, \quad y_T = 2u_e - 2l_e.$$

(4) If

$$\begin{aligned} x_e + y_e &> 2(u_e, l_e), \\ 2l_e - 2u_e + x_e - y_e &\leq 0, \\ 2u_e - 2l_e + x_e - y_e &\geq 0, \end{aligned}$$

then we have

$$x_T = u_e - l_e + \frac{1}{2}x_e - \frac{1}{2}y_e, \quad y_T = u_e - l_e - \frac{1}{2}x_e + \frac{1}{2}y_e.$$

Example 3.2. Let $A_\alpha = [2\alpha - 2, 1 - \sqrt{\alpha}]$. After elementary calculus from (9), we obtain

$$x_e = 2, \ y_e = \frac{4}{5}, \ l_e = -1, \ u_e = \frac{1}{3}.$$

Then we have $M = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, \frac{8}{3} \geq x + y\}$, where it is satisfied in case (4). Therefore we obtain

$$x_T = \frac{29}{15}, \ y_T = \frac{11}{15}, \ l_T = -1, \ u_T = \frac{1}{3},$$

$$W(A) = W(T_e(A)) = W(T(A)) = \frac{4}{3}.$$

By (4), we immediately obtain $t_1 = l - \frac{x}{2}, \ t_2 = l + \frac{x}{2}, \ t_3 = u - \frac{y}{2}, \ t_4 = u + \frac{y}{2}$. Then using (9) and (15), we easily get the following corollary.

Corollary 3.3. *Let $A \in F(\mathbb{R})$, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$, and $T(A) = (t_1, t_2, t_3, t_4)$ be the width invariant trapezoidal approximation of A.*
(1) If

$$\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha - \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \leq 0,$$

then

$$t_1 = \int_0^1 (4-6\alpha)A_L(\alpha)d\alpha, \quad t_2 = \int_0^1 (6\alpha-2)A_L(\alpha)d\alpha,$$

$$t_3 = \int_0^1 (6\alpha-2)A_U(\alpha)d\alpha, \quad t_4 = \int_0^1 (4-6\alpha)A_U(\alpha)d\alpha.$$

(2) If

$$\int_0^1 (3\alpha-1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-2)A_U(\alpha)d\alpha > 0,$$

then we obtain

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \quad t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha)d\alpha.$$

(3) If

$$\int_0^1 (3\alpha-2)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)d\alpha < 0,$$

then

$$t_1 = t_2 = t_3 = \int_0^1 A_L(\alpha)d\alpha, \quad t_4 = 2\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha.$$

(4) If

$$\int_0^1 (1-3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)d\alpha < 0,$$

$$\int_0^1 (3\alpha-1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-2)A_U(\alpha)d\alpha \leq 0,$$

$$\int_0^1 (3\alpha-2)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)d\alpha \geq 0,$$

then

$$t_1 = \int_0^1 (3-3\alpha)A_L(\alpha)d\alpha + \int_0^1 (1-3\alpha)A_U(\alpha)d\alpha,$$

$$t_2 = t_3 = \int_0^1 (3\alpha-1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)d\alpha,$$

$$t_4 = \int_0^1 (1-3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3-3\alpha)A_U(\alpha)d\alpha.$$

Example 3.4. Let $A = (1,2,3,4)_{2,2}$. Then using case (1) in Corollary 3.3, we deduce $T(A) = (\frac{19}{15}, \frac{31}{15}, \frac{44}{15}, \frac{56}{15})$. Also using case (4), the width invariant trapezoidal approximation of $B = (1,200,201,220)_{2,2}$ is $T(B) = (\frac{365}{6}, \frac{413}{2}, \frac{413}{2}, \frac{1249}{6})$.

4. **Width Invariant Symmetric Trapezoidal Approximation**

In this section, we propose a method to find the width invariant symmetric trapezoidal approximation based on the presented method in previous section. The fuzzy number $S_A = [l_S, u_S, x_S, y_S]$, is the width invariant symmetric trapezoidal approximation of A if and only if $(x_S, y_S) \in \mathbb{R}^2$ satisfies the minimization problem

$$\min(x - x_e)^2 + (y - y_e)^2, \tag{16}$$

$$x, y \geq 0, \quad 2u_e - 2l_e \geq x + y, \quad x = y,$$

and

$$u_S = u_e, \quad l_S = l_e. \tag{17}$$

The problem (16) is equivalent to the problem

$$\min(2x^2 + x_e^2 + y_e^2 - 2x(x_e + y_e)),$$

$$x \geq 0, \quad u_e - l_e \geq x.$$

Now, since $x_e, y_e \geq 0$ and $x = \dfrac{x_e + y_e}{2}$ is the minimum point of the function $h(x) = 2x^2 + x_e^2 + y_e^2 - 2x(x_e + y_e)$, then we have the following possible cases.

(1) If $x_e + y_e \leq 2(u_e - l_e)$ then

$$x_S = y_S = \frac{x_e + y_e}{2}, \quad u_S = u_e, \quad l_S = l_e.$$

(2) If $x_e + y_e > 2(u_e - l_e)$ then

$$x_S = y_S = u_e - l_e, \quad u_S = u_e, \quad l_S = l_e.$$

By (9) we get the following result to find the width invariant symmetric trapezoidal approximation $S_A(S_1, S_2, S_3, S_4)$ of a given fuzzy number A.

Theorem 4.1. *(1) If*

$$\int_0^1 (1 - 3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \geq 0,$$

then we obtain

$$S_1 = \int_0^1 (-3\alpha + \frac{5}{2})A_L(\alpha)d\alpha + \int_0^1 (3\alpha - \frac{3}{2})A_U(\alpha)d\alpha,$$

$$S_2 = \int_0^1 (3\alpha - \frac{1}{2})A_L(\alpha)d\alpha + \int_0^1 (-3\alpha + \frac{3}{2})A_U(\alpha)d\alpha,$$

$$S_3 = \int_0^1 (-3\alpha + \frac{3}{2})A_L(\alpha)d\alpha + \int_0^1 (3\alpha - \frac{1}{2})A_U(\alpha)d\alpha,$$

$$S_4 = \int_0^1 (3\alpha - \frac{3}{2})A_L(\alpha)d\alpha + \int_0^1 (-3\alpha + \frac{5}{2})A_U(\alpha)d\alpha.$$

(2) If

$$\int_0^1 (1 - 3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha < 0,$$

then we get

$$S_1 = \frac{3}{2}\int_0^1 A_L(\alpha)d\alpha - \frac{1}{2}\int_0^1 A_U(\alpha)d\alpha,$$
$$S_2 = S_3 = \frac{1}{2}\int_0^1 A_L(\alpha)d\alpha + \frac{1}{2}\int_0^1 A_U(\alpha)d\alpha,$$
$$S_4 = -\frac{1}{2}\int_0^1 A_L(\alpha)d\alpha + \frac{3}{2}\int_0^1 A_U(\alpha)d\alpha.$$

Example 4.2. Let $A = (1,2,3,4)_{2,2}$. Then using case (1) in Theorem 4.1, we obtain $S_A = (\frac{19}{15}, \frac{31}{15}, \frac{44}{15}, \frac{56}{15})$. The fuzzy number $B = (1,2,4,35)_{2,2}$ satisfies case (2) of Theorem 4.1, then we deduce $S_B = (-\frac{14}{3}, 8, 8, \frac{62}{3})$.

5. Width Invariant Triangular Approximation

In this section, we propose a method to compute the width invariant triangular approximation for a given fuzzy number. By (13)-(14), $t_A = [l_t, u_t, x_t, y_t]$ is the width invariant triangular approximation of A if and only if

$$u_t = u_e, \quad l_t = l_e, \tag{18}$$

and (x_t, y_t) satisfies

$$\min((x - x_e)^2 + (y - y_e)^2), \tag{19}$$

$$x, y \geq 0, x + y = 2u_e - 2l_e. \tag{20}$$

Let us define $N = \{(x,y) \in \mathbb{R}^2 | x, y \geq 0, x + y = 2u_e - 2l_e\}$. Then, similar to the demonstration of Theorem 3.1, we have the following theorem.

Theorem 5.1. *Let A be a given fuzzy number. Then there exists a unique width invariant triangular approximation of A. Namely, there exists a unique triangular fuzzy number t_A with $W(t_A) = W(A)$ and satisfying*

$$d(t_A, A) \leq d(B, A), \quad \forall B \in F^T(\mathbb{R}),$$

where $W(B) = W(A)$.

For a given fuzzy number A, Theorem 5.1 suggests the following method to compute the width invariant triangular approximation $t_A = [l_t, u_t, x_t, y_t]$ (Figure 2.)

(1) If $2l_e - 2u_e + x_e + y_e \leq 0$ and $6u_e - 6l_e - x_e - y_e \geq 0$, then we have

$$x_t = u_e - l_e + \frac{1}{2}x_e - \frac{1}{2}y_e, \quad y_t = u_e - l_e - \frac{1}{2}x_e + \frac{1}{2}y_e, \quad u_t = u_e, \quad l_t = l_e.$$

(2) If $2l_e - 2u_e + x_e - y_e > 0$, then

$$x_t = 2u_e - 2l_e, \quad y_t = 0, \quad u_t = u_e, \quad l_t = l_e.$$

(3) If $2u_e - 2l_e + x_e - y_e < 0$, then we get

$$x_t = 0, \quad y_t = 2u_e - 2l_e, \quad u_t = u_e, \quad l_t = l_e.$$

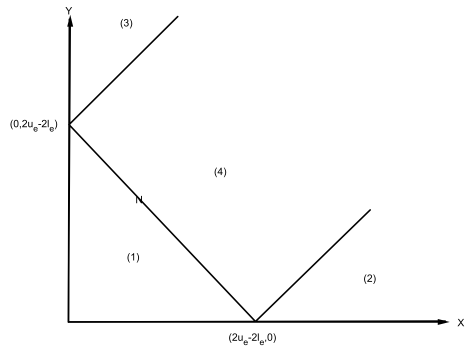

FIGURE 2. Corresponding Cases to Find x_e and y_e in t_A

In the following, using (9), we present a computation method to obtain the width invariant triangular approximation of A, in the representation $t_A = (t_1, t_2, t_3)$.

Theorem 5.2. *(1) If*

$$\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 2)A_U(\alpha)d\alpha \leq 0,$$

and

$$\int_0^1 (3\alpha - 2)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha \geq 0,$$

then we have

$$t_1 = \int_0^1 (3 - 3\alpha)A_L(\alpha)d\alpha + \int_0^1 (1 - 3\alpha)A_U(\alpha)d\alpha,$$
$$t_2 = \int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha,$$
$$t_3 = \int_0^1 (1 - 3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3 - 3\alpha)A_U(\alpha)d\alpha.$$

(2) If

$$\int_0^1 (3\alpha - 1)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 2)A_U(\alpha)d\alpha > 0,$$

then we get

$$t_1 = 2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha, \quad t_2 = t_3 = \int_0^1 A_U(\alpha)d\alpha.$$

(3) If

$$\int_0^1 (3\alpha - 2)A_L(\alpha)d\alpha + \int_0^1 (3\alpha - 1)A_U(\alpha)d\alpha < 0,$$

then we have

$$t_1 = t_2 = \int_0^1 A_L(\alpha)d\alpha, \quad t_3 = -\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha.$$

Example 5.3. Let $A = (1, 2, 3, 4)_{2,2}$ be a fuzzy number. Then using case (1) in Theorem 5.2, we get

$$t_{(1,2,3,4)_{2,2}} = (\frac{5}{6}, \frac{5}{2}, \frac{25}{6}).$$

For a given fuzzy number A, to obtain the width invariant symmetric triangular approximation $s_A = [l_s, u_s, x_s, y_s]$, using (16) and (17), we first calculate $(x_s, y_s) \in \mathbb{R}^2$ by solving minimization problem

$$\min((x - x_e)^2 + (y - y_e)^2), \tag{21}$$

$$x, y \geq 0, \tag{22}$$

$$x + y = 2u_e - 2l_e, \tag{23}$$

$$x = y, \tag{24}$$

and then we compute x_s, y_s by

$$u_s = u_e, \quad l_s = l_e.$$

According to (23) and (24), we have

$$x = y = u_e - l_e.$$

Then the minimization problem (21) has a unique solution.
Now, by (9), we get the following result to obtain the width invariant symmetric triangular approximation of the fuzzy number A, in the representation $s_A = (s_1, s_2, s_3)$.

Theorem 5.4. *Let A be a fuzzy number with α-level sets $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$. Then $s_A = (s_1, s_2, s_3)$ is given by*

$$s_1 = \frac{3}{2}\int_0^1 A_L(\alpha)d\alpha - \frac{1}{2}\int_0^1 A_U(\alpha)d\alpha,$$

$$s_2 = \frac{1}{2}\int_0^1 A_L(\alpha)d\alpha + \frac{1}{2}\int_0^1 A_U(\alpha)d\alpha,$$

$$s_3 = -\frac{1}{2}\int_0^1 A_L(\alpha)d\alpha + \frac{3}{2}\int_0^1 A_U(\alpha)d\alpha.$$

Example 5.5. It is easy to obtain $s_{(1,2,3,4)_{2,2}} = (\frac{5}{6}, \frac{5}{2}, \frac{25}{6})$ and $s_{(1,2,4,35)_{2,2}} = (-\frac{14}{3}, 8, \frac{62}{3})$.

6. **Properties**

Some important properties of fuzzy numbers approximation such as translation invariance, scale invariance and additivity have been studied in [6, 8, 28, 29]. Ban et al. have proved that the ambiguity invariant approximation operators possess the translation invariance's and scale invariance's properties and these operators generally do not have additivity's property [10]. The presented approaches in this section to study scale invariance, translation invariance, additivity and Lipschitz continuity are identical with those in [10]. We show that the new approximation operators possess the similar properties to the presented operators in [10] and they have smaller Lipschitz constants. Since we have

$$W(A+z) = W(A),\ \forall A \in F(\mathbb{R}),\quad z \in \mathbb{R},$$

and

$$d(A+z, B+z) = d(A,B), \forall A, B \in F(\mathbb{R}),\quad z \in \mathbb{R},$$

then by Theorem 1 in [9], we get the translation invariance of the presented operators in Corollary 3.3, Theorems 4.1, 5.2 and 5.4. On the other hand, since we have

$$W(\lambda.A) = |\lambda| W(A), \forall A \in F(\mathbb{R}),\ \lambda \in \mathbb{R},$$

and

$$d(\lambda.A, \lambda.B) = |\lambda| d(A,B), \forall A, B \in F(\mathbb{R}),\ \lambda \in \mathbb{R},$$

then by Theorem 4 in [9], we deduce the scale invariance property of the presented operators in Corollary 3.3, Theorems 4.1, 5.2 and 5.4.
It is immediate that the presented operators in Theorem 5.4 is additive,

$$s_A + s_B = s_{A+B},\quad \forall A, B \in F(\mathbb{R}),$$

while the presented operators in Corollary 3.3 and Theorems 4.1 and 5.2 are not.

Example 6.1. Let $A_\alpha = [\sqrt{\alpha}, 1]$ be a fuzzy number. Then using case (2) in Corollary 3.3, we have $T_A = (\frac{1}{3}, 1, 1, 1)$. Also for fuzzy number $B = (0,0,0,1)$, by case (1), we get $T_B = (0,0,0,1)$. Therefore

$$T_A + T_B = (\frac{1}{3}, 1, 1, 2).$$

On the other hand, $(A+B)_\alpha = [\sqrt{\alpha}, 2-\alpha]$ and we can apply case (4) in Corollary 3.3 to obtain

$$T_{A+B} = (\frac{3}{10}, \frac{31}{30}, \frac{31}{30}, \frac{59}{30}).$$

Therefore $T_A + T_B \neq T_{A+B}$. It is easy to get $t_A = (\frac{1}{3}, 1, 1, 1)$, $\quad t_B = (0,0,1)$ and $t_{A+B} = (\frac{3}{10}, \frac{31}{30}, \frac{31}{30}, \frac{59}{30})$. Therefore we have

$$t_A + t_B \neq t_{A+B}.$$

Example 6.2. Let $A_\alpha = [\sqrt{\alpha}, 1]$ and $B_\alpha = [0, 2-\sqrt{\alpha}]$ be two fuzzy numbers. Then applying cases (2) and (1) in Theorem 4.1, we get respectively

$$S_A = (\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{7}{6}) \text{ and } S_B = (-\frac{1}{5}, \frac{1}{5}, \frac{17}{15}, \frac{23}{15}).$$

On the other hand, by case (1) in Theorem 4.1 for $A+B$, we have

$$S_{A+B} = (\frac{4}{15}, \frac{16}{15}, \frac{29}{15}, \frac{41}{15}).$$

Therefore $S_A + S_B \neq S_{A+B}$.

The continuity is an essential property in the approximation of fuzzy numbers. In applications, sometimes it is indicated as robustness. Therefore, in the following, we investigate the continuity of the presented approximation operators in Sections 3-5. We prove the presented operators satisfy a stronger condition, i.e., they are Lipschitz.

Theorem 6.3. (1) *Let T be the width invariant trapezoidal approximation operator presented in Corollary* 3.3. *Then we have*

$$d(T_A, T_B) \leq \sqrt{\frac{8}{3} + \frac{2}{3}\sqrt{6}}\ d(A, B), \quad \forall A, B \in F(\mathbb{R}).$$

(2) *For the width invariant symmetric trapezoidal approximation operator presented in Theorem* 4.1, *we have*

$$d(S_A, S_B) \leq \sqrt{\frac{10}{3} + \frac{4}{3}\sqrt{3}}\ d(A, B), \quad \forall A, B \in F(\mathbb{R}).$$

(3) *The width invariant triangular approximation operator presented in Theorem* 5.2 *satisfies the inequality*

$$d(t_A, t_B) \leq \sqrt{\frac{8}{3} + \frac{2}{3}\sqrt{6}}\ d(A, B), \quad \forall A, B \in F(\mathbb{R}).$$

(4) *Let s be the width invariant symmetric triangular approximation operator presented in Theorem* 5.4. *Then*

$$d(s_A, s_B) \leq \sqrt{\frac{7}{3}}\ d(A, B), \quad \forall A, B \in F(\mathbb{R}).$$

Proof. (1) Let $A = [A_L(\alpha), A_U(\alpha)]$ and $B = [B_L(\alpha), B_U(\alpha)],\ \alpha \in [0, 1]$ be two fuzzy numbers where $T_e(A) = [l_e, u_e, x_e, y_e]$ and $T_e(B) = [l'_e, u'_e, x'_e, y'_e]$ are their extended trapezoidal approximations, respectively. Let us denote by $T_A = [l_T, u_T, x_T]$ and $T_B = [l'_T, u'_T, x'_T, y'_T]$, the width invariant trapezoidal approximations of A and B, respectively. Then by (7) we have

$$d^2(T_A, T_B) = (l_T - l'_T)^2 + (u_T - u'_T)^2 + \frac{1}{12}(x_T - x'_T)^2 + \frac{1}{12}(y_T - y'_T)^2. \tag{25}$$

Also using (15) and Remark 2.5 we get

$$\begin{aligned}(l_T - l'_T)^2 + (u_T - u'_T)^2 &= (l_e - l'_e)^2 + (u_e - u'_e)^2 \\ &\leq d^2(A, B).\end{aligned}$$

Substituting in (25), we obtain

$$d^2(T_A, T_B) \leq d^2(A, B) + \frac{1}{12}[(x_T - x'_T)^2 + (y_T - y'_T)^2],$$

or

$$d^2(T_A, T_B) \leq d^2(A, B) + \frac{1}{12}\ d_E^2(A_0, B_0), \tag{26}$$

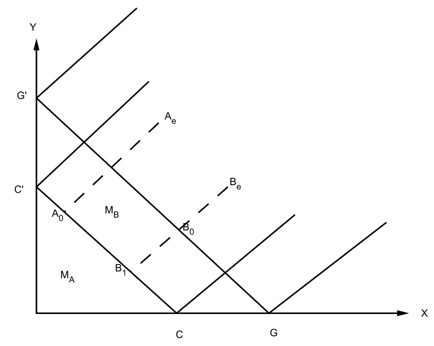

FIGURE 3. One Case in the Evaluating of the Lipschitz Constant of T_A

where $A_0 = (x_T, y_T)$, $B_0 = (x'_T, y'_T)$ and d_E denotes the Euclidean metric on $\mathbb{R}^2$. Without loss of generality, we assume $u'_e - l'_e \geq u_e - l_e$. Now, let us consider the sets

$$M_A = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0, 2u_e - 2l_e \geq x + y\},$$
$$M_B = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0, 2u'_e - 2l'_e \geq x + y\},$$

and the points

$$C(2u_e - 2l_e, 0), \quad C'(0, 2u_e - 2l_e), \quad G(2u'_e - 2l'_e, 0), \quad G'(0, 2u'_e - 2l'_e), \tag{27}$$

which define the convex closed sets M_A and M_B in $\mathbb{R}^2$ (Fig. 3). According to Theorem 3.1, we get $A_0 = P_{M_A}(A_e)$ and $B_0 = P_{M_B}(B_e)$, where $A_e = (x_e, y_e)$ and $B_e(x'_e, y'_e)$. We denote the projection of B_0 on M_A by B_1, which is the unique element in M_A that minimizes $d_E(B_0, Q)$, where $Q \in M_A$. Therefore B_1 is the projection of B_e on M_A, that is $B_1 \in M_A$ and $\min_{R \in M_A} \; d_E(B_e, R) = d_E(B_e, B_1)$. Also it is easy to check that

$$d_E(B_1, B_0) \leq d_E(C, G) = d_E(C', G').$$

We have

$$\begin{aligned} d_E^2(C, G) &= [2(u'_e - u_e) - 2(l'_e - l_e)]^2 \\ &= \left(2\int_0^1 (B_U(\alpha) - A_U(\alpha))d\alpha - 2\int_0^1 (B_L(\alpha) - A_L(\alpha))d\alpha\right)^2 \\ &\leq 8\left(\int_0^1 (B_U(\alpha) - A_U(\alpha))^2 d\alpha + \int_0^1 (B_L(\alpha) - A_L(\alpha))^2 d\alpha\right) \\ &\leq 8 \; d^2(A, B), \end{aligned}$$

and therefore we get

$$d_E^2(B_1, B_0) \leq 8 \; d^2(A, B).$$

Because M_A is a convex closed subset of $\mathbb{R}^2$, we get

$$d_E(P_{M_A}(A_e), P_{M_A}(B_e)) \leq d_E(A_e, B_e),$$

that is,

$$d_E(A_0, B_1) \leq d_E(A_e, B_e).$$

Since by Remark 2.5, we get $d_E(A_e, B_e) \leq 2\sqrt{3}\ d(A, B)$, it follows that

$$\begin{aligned} d_E(A_0, B_0) &\leq d_E(A_0, B_1) + d_E(B_1, B_0) \leq d_E(A_e, B_e) + \sqrt{8}\ d(A, B) \\ &\leq 2\sqrt{3}d(A, B) + 2\sqrt{2}\ d(A, B) \leq 2(\sqrt{2} + \sqrt{3})\ d(A, B). \end{aligned}$$

Substituting in (26), we obtain

$$d^2(T_A, T_B) \leq d^2(A, B) + \frac{4(5 + 2\sqrt{6})}{12}\ d^2(A, B).$$

Therefore we get

$$d(T_A, T_B) \leq \sqrt{\frac{8}{3} + \frac{2}{3}\sqrt{6}}\ d(A, B).$$

(2) Let $A, B \in F(\mathbb{R})$ be two fuzzy numbers where

$$S_A = [l_S, u_S, x_S, y_S], \qquad S_B = [l'_S, u'_S, x'_S, y'_S],$$

are their width invariant symmetric trapezoidal approximations, respectively. Let us denote by $T_e(A) = [l_e, u_e, x_e, y_e]$ and $T_e(B) = [L'_e, u'_e, x'_e, y'_e]$, the extended trapezoidal approximations of A and B. Also suppose

$$A_e = (x_e, y_e), \quad B_e = (x'_e, y'_e), \quad A_0 = (x_S, y_S), \quad B_0 = (x'_S, y'_S).$$

Similar to the proof of case (1), we obtain

$$d^2(S_A, S_B) \leq d^2(A, B) + \frac{d_E^2(A_0, B_0)}{12}. \tag{28}$$

Let us assume that $u'_e - l'_e \geq u_e - l_e$. We consider the points given in (27) and the convex closed sets

$$\begin{aligned} R_A &= \{(x, y) \in \mathbb{R}^2 | x \geq 0, x = y, 2u_e - 2l_e \geq x + y\}, \\ R_B &= \{(x, y) \in \mathbb{R}^2 | x \geq 0, x = y, 2u'_e - 2l'_e \geq x + y\}. \end{aligned}$$

We see that R_A and R_B represent the medians of the triangulares OCC' and OGG', respectively (Figure 3). By Theorem 3.1, we get $A_0 = P_{R_A}(A_e)$ and $B_0 = P_{R_B}(B_e)$. Similar to the proof of case (1), let us consider $B_1 = P_{R_A}(B_0)$. Therefore B_1 is the projection of B_e on R_A and we get

$$d_E(B_1, B_0) \leq \sqrt{2}\ d_E(C, G) = \sqrt{2}\ d_E(C', G').$$

Since R_A is a convex and closed set, then

$$d_E(A_0, B_1) \leq d_E(A_e, B_e).$$

On the other hand, since $d_E(C, G) \leq \sqrt{8}\ d(A, B)$, we have

$$\begin{aligned} d_E(A_0, B_0) &\leq d_E(A_0, B_1) + d_E(B_1, B_0) \leq d_E(A_e, B_e) + 4\ d(A, B) \\ &\leq 2\sqrt{3}d(A, B) + 4\ d(A, B) \leq 2(\sqrt{3} + 2)\ d(A, B). \end{aligned}$$

Substituting in (28), we obtain

$$d(S_A, S_B) \leq \sqrt{\frac{10}{3} + \frac{4}{3}\sqrt{3}}\ d(A, B).$$

(3) Similar, to the proof of (1), it is easy to obtain

$$d^2(t_A, t_B) \leq d^2(A, B) + \frac{d_E^2(A_0, B_0)}{12}. \tag{29}$$

Without loss of generality, we assume $u_e - l_e \leq u'_e - l'_e$. Consider the convex closed sets

$$N_A = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x + y = 2u_e - 2l_e\},$$
$$N_B = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x + y = 2u'_e - 2l'_e\},$$

and the points given in (27) (Figure 4). Here, N_A and N_B are the closed segments $[CC']$ and $[GG']$, respectively. According to Theorem 3.1, we get $A_0 = P_{N_A}(A_e)$ and $B_0 = P_{N_B}(B_e)$. We denote the projection of B_0 on N_A by B_1, i.e., $B_1 = P_{N_A}(B_0)$. Therefore B_1 is the projection of B_e on N_A, that is $B_1 \in N_A$ and $\min_{R \in N_A} d_E(B_e, R) = d_E(B_e, B_1)$. Also it is easy to get $d_E(B_1, B_0) \leq d_E(C, G) = d_E(C', G')$. So, the rest of the proof is similar to the proof of case (1). So we obtain

$$d(t_A, t_B) \leq \sqrt{\frac{8}{3} + \frac{2}{3}\sqrt{6}}\; d(A, B).$$

(4) Let $A, B \in F(\mathbb{R})$ and $s_A = [l_s, u_s, x_s, y_s], s_B = [l'_s, u'_s, x'_s, y'_s]$, be their width invariant triangular approximations, respectively. Let us denote by $T_e(A) = [l_e, u_e, x_e, y_e]$ and $T_e(B) = [l'_e, u'_e, x'_e, y'_e]$, the extended trapezoidal approximations of A and B. We consider the points $A_e(x_e, y_e),\ B_e(x'_e, y'_e),\ A_0(x_s, y_s),\ B_0(x'_s, y'_s)$. Similar to the proof of case (1), we obtain

$$d^2(s_A, s_B) \leq d^2(A, B) + \frac{d_E^2(A_0, B_0)}{12}. \tag{30}$$

Because $x_s = y_s$ and $x'_s = y'_s$, we get

$$d_E^2(A_0, B_0) = 2(x_s - x'_s)^2 = 2[(u_e - l_e) - (u'_e - l'_e)]^2 = 2\ d_E^2(C, G),$$

where $d_E^2(C, G) \leq 8d^2(A, B)$. Then we have

$$d_E^2(A_0, B_0) \leq 16\ d^2(A, B). \tag{31}$$

Therefore by (30) and (31) we get $d(s_A, s_B) \leq \sqrt{\frac{7}{3}}\, d(A, B)$ and the proof is complete. □

Remark 6.4. It is worth mentioning that the width invariant trapezoidal approximations of fuzzy numbers, preserves the expected value too. In the following, we consider the different cases in Corollary 3.3 to examine this property.

Case 1:

$$\begin{aligned} EV(T(A)) &= \frac{1}{4}\left[2\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha\right] \\ &= \frac{1}{2}\left[\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha\right] = EV(A). \end{aligned}$$

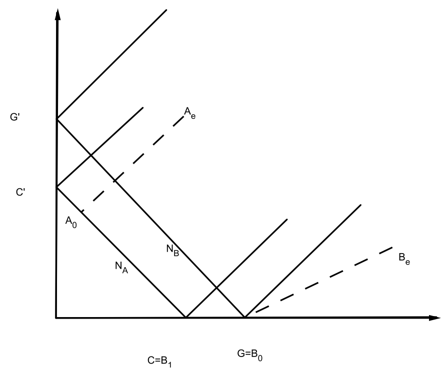

FIGURE 4. One Case in the Evaluating of the Lipschitz Constant of t_A

Case 2:

$$\begin{aligned} EV(T(A)) &= \frac{1}{4}\left[2\int_0^1 A_L(\alpha)d\alpha - \int_0^1 A_U(\alpha)d\alpha + 3\int_0^1 A_U(\alpha)d\alpha\right] \\ &= \frac{1}{4}\left[2\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha\right] = EV(A). \end{aligned}$$

Case 3:

$$\begin{aligned} EV(T(A)) &= \frac{1}{4}\left[\int_0^1 A_U(\alpha)d\alpha - \int_0^1 A_L(\alpha)d\alpha + 3\int_0^1 A_L(\alpha)d\alpha\right] \\ &= \frac{1}{4}\left[2\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha\right] = EV(A). \end{aligned}$$

Case 4:

$$\begin{aligned} EV(T(A)) &= \frac{1}{4}\left[\int_0^1 (3-3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3\alpha-1)A_U(\alpha)d\alpha + \int_0^1 (6\alpha-2)A_L(\alpha)d\alpha\right] \\ &+ \frac{1}{4}\left[\int_0^1 (6\alpha-2)A_U(\alpha)d\alpha + \int_0^1 (1-3\alpha)A_L(\alpha)d\alpha + \int_0^1 (3-3\alpha)A_U(\alpha)d\alpha\right] \\ &= \frac{1}{4}\left[2\int_0^1 A_L(\alpha)d\alpha + 2\int_0^1 A_U(\alpha)d\alpha\right] = EV(A). \end{aligned}$$

7. Conclusion

In this manuscript, we have investigated the width invariant approximations by (symmetric) trapezoidal and triangular fuzzy numbers. This method avoids the difficulties of Karush-Kuhn-Tucker Theorem in calculations. We have showed that this approximation satisfies Lipschitz condition and also it preserves the property of expected value.

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