

ON METRIC SPACES INDUCED BY FUZZY METRIC SPACES

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ABSTRACT. For a class of fuzzy metric spaces (in the sense of George and Veeramani) with an H-type t-norm, we present a method to construct a metric on a fuzzy metric space. The induced metric space shares many important properties with the given fuzzy metric space. Specifically, they generate the same topology, and have the same completeness. Our results can give the constructive proofs to some problems for fuzzy metric spaces in the literature, which are shown by examples in this paper.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [30] in 1965. After that, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications [7, 24, 29]. In the theory of fuzzy topological spaces, one of the main problems is to obtain an appropriate and consistent notion of a fuzzy metric space. This problem was investigated by many authors from different points of view [3, 5, 11, 16]. Kramosil and Michalek [18] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [20]. Later, George and Veeramani [3] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek. The fuzzy metric space in the sense of George and Veeramani has become an area of active research in the last years since it not only provides rich topological structures which can be obtained, in many cases, from classical theorems, but also shows new and intriguing phenomena [9, 12, 13]. From now on, when we talk about fuzzy metric spaces we refer to this type. It is worth mentioning that the prestigious Spanish scholar Gregori and his collaborators have made remarkable contribution to developing the theory of fuzzy metric space (see [6, 7, 8, 9, 10, 11, 12, 13, 14]).

The study of the relationship between fuzzy metric spaces and metric spaces constitutes a natural and interesting question in the theory of fuzzy metric space. In [3], George and Veeramani proved that every metric space can induced a standard fuzzy metric space, moreover, the topology generated by the metric d coincides with the topology generated by the induced fuzzy metric. Conversely, in [11] Gregori and Romaguera showed that every topological space generated by a fuzzy metric space is metrizable. To induce a metric by a fuzzy metric, in [9], Gregori, Morillas

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and Sapena studied the class of strong fuzzy metric spaces. This class includes the class of stationary fuzzy metrics, and in particular, when the fuzzy metric is principal, they obtained a family of metrics which are compatible with the topology generated by the fuzzy metric. In [19], Li introduced a formula to induce metrics from a fuzzy ultrametric, and consider consistency of the metrics.

In this paper, for a class of fuzzy metric spaces with an H-type t-norm, we present a method to construct a metric on a fuzzy metric space. The induced metric space shares many important properties with the given fuzzy metric space. Specifically, they generate the same topology, and have the same completeness. Our results can give the constructive proofs to some problems for fuzzy metric spaces, which are shown by some examples.

2. Preliminaries

For the sake of completeness, we briefly recall some notions from metric spaces and fuzzy metric spaces. Our basic reference on metric spaces is [27]. Throughout this paper the letters \mathbb{N} , \mathbb{R} , \mathbb{R}^+ and \mathbb{R}_∞^+ will denote the set of all positive integer numbers, of all real numbers, of all nonnegative real numbers and of all nonnegative extended real numbers, respectively.

Definition 2.1. [27] Let X be a nonempty set. A function $d : X \times X \rightarrow \mathbb{R}^+$ is said to be a metric on X if d satisfies the following axioms:

- (i) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, x) = 0$ for all $x \in X$;
- (ii) if $d(x, y) = 0$ then $x = y$;
- (iii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iv) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is then called a metric space. If we consider the definition of a metric space given above and relax the second requirement, we arrive at the concept of a pseudo-metric space [2]. We will also consider extended metrics. They satisfy the usual axioms for a metric, except that we allow $d(x, y) = +\infty$.

Definition 2.2. [17] A triangular norm (t-norm for short) is a binary operation on the unit interval $[0, 1]$, i.e., a function $* : [0, 1]^2 \rightarrow [0, 1]$, such that for all $a, b, c, d \in [0, 1]$ the following four axioms are satisfied:

- (T-1) $a * 1 = a$. (boundary condition)
- (T-2) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$. (monotonicity)
- (T-3) $a * b = b * a$. (commutativity)
- (T-4) $a * (b * c) = (a * b) * c$. (associativity)

The t-norm minimum will be denoted by \wedge and the usual product by \bullet . A t-norm $*$ is said to be continuous if it is a continuous function in $[0, 1]^2$; a t-norm $*$ is said to be strictly monotone if $a * b < a * d$ whenever $a > 0$ and $b < d$; a t-norm $*$ is called strict if it is continuous and strictly monotone; a t-norm $*$ is called Archimedean if for all $a, b \in (0, 1)$ there is an integer $n \in \mathbb{N}$ such that $a^{(n)} < b$ [17], where $a^{(n)}$ is defined by $a^{(1)} = a$, $a^{(n)} = a^{(n-1)} * a$, $n \geq 2$. The t-norm \bullet is a strict Archimedean t-norm.

Lemma 2.3. [17] Let $(*_k)_{k \in K}$ be a family of t -norms and let $((\alpha_k, \beta_k))_{k \in K}$ be a family of pairwise disjoint open subintervals of the unit interval $[0, 1]$ (i.e., K is an at most countable index set). Consider the linear transformations $\{\varphi_k : [\alpha_k, \beta_k] \rightarrow [0, 1]\}_{k \in K}$ given by

$$\varphi_k(u) = \frac{u - \alpha_k}{\beta_k - \alpha_k}.$$

Then the function $* : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$a * b = \begin{cases} \varphi_k^{-1}(\varphi_k(a) *_k \varphi_k(b)), & \text{if } a, b \in (\alpha_k, \beta_k), \\ a \wedge b, & \text{otherwise,} \end{cases}$$

is a t -norm.

Definition 2.4. [17] Let $(*_k)_{k \in K}$ be a family of t -norms and let $((\alpha_k, \beta_k))_{k \in K}$ be a family of pairwise disjoint open subintervals of the unit interval $[0, 1]$. The t -norm $*$ defined in Lemma 2.3 is called the ordinal sum of the summands $((\alpha_k, \beta_k), *_k)_{k \in K}$, and we shall write $* = ((\alpha_k, \beta_k), *_k)_{k \in K}$.

Lemma 2.5. [17] A function $* : [0, 1]^2 \rightarrow [0, 1]$ is a continuous t -norm if and only if $*$ is an ordinal sum of continuous Archimedean t -norms.

Definition 2.6. [22] A t -norm $*$ is of H-type if the function family $(x^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$, where $x^{(n)}$ is defined by $x^{(1)} = x$, $x^{(n)} = x^{(n-1)} * x$, $n \geq 2$, $x \in [0, 1]$.

Lemma 2.7. [15] Let $*$ be a continuous t -norm. Then $*$ is of H-type if and only if there exists a strictly increasing sequence $(a_n)_{n \in \mathbb{N}}$ from the interval $[0, 1]$ such that $\lim_{n \rightarrow \infty} a_n = 1$ and $a_n * a_n = a_n$.

It is obvious that \wedge is an H-type t -norm. In fact, there are innumerable H-type t -norms [22].

Example 2.8. Let $(a_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence from the interval $[0, 1]$ such that $\lim_{n \rightarrow \infty} a_n = 1$, and let $(\alpha_n, \beta_n) = (a_n, a_{n+1})$, for all $n \in \mathbb{N}$. Then by Lemma 2.7, the ordinal sum $* = ((\alpha_n, \beta_n), \bullet)_{n \in \mathbb{N}}$ is an H-type t -norm.

Definition 2.9. [3] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X, t, s > 0$:

- (FM-1) $M(x, y, t) > 0$,
- (FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
- (FM-3) $M(x, y, t) = M(y, x, t)$,
- (FM-4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
- (FM-5) $M(x, y, -) : (0, \infty) \rightarrow (0, 1]$ is continuous.

A simple but useful fact is that for every $x, y \in X$, the function $M(x, y, -) : (0, \infty) \rightarrow (0, 1]$ is nondecreasing [3]. In [3], George and Veeramani proved that every fuzzy metric M generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) : r \in (0, 1), x \in X\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$. Since the family of sets

$$\left\{ B_M\left(x, \frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

is a local base at x , the above topology is first countable. In general, it is easy to prove the following lemma.

Lemma 2.10. *Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two strictly decreasing sequences in $(0, 1]$ such that $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} b_n = 0$. Then the family of open sets of the form $\{B_M(x, a_n, b_n) : n \in \mathbb{N}, x \in X\}$, is a base for the topological space (X, τ_M) , where $B_M(x, a_n, b_n) = \{y \in X : M(x, y, b_n) > 1 - a_n\}$.*

Definition 2.11. [12] Let $(X, M, *)$ be a fuzzy metric space. Then

(a) a sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be Cauchy if for each $\varepsilon \in (0, 1)$ and $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

(b) the fuzzy metric space $(X, M, *)$ is called complete if every Cauchy sequence is convergent with respect to τ_M .

3. Main Results

In this section, we always suppose that the t-norm $*$ is a continuous H-type t-norm which is the ordinal sum of the summands $((\alpha_k, \beta_k), *_k)_{k \in K}$, where $*_k$ are continuous strict Archimedean t-norms.

Definition 3.1. [25] Let $(X, M, *)$ be a fuzzy metric space and $(a_n)_{n \in \mathbb{N}}$ an strictly increasing sequence in $[0, 1)$ with the property in Lemma 2.7. Then we define a function sequence $(\rho_n)_{n \in \mathbb{N}}$ from $X \times X$ to \mathbb{R}_∞^+ by

$$\rho_n(x, y) = \begin{cases} \inf\{t : t \in A_n(x, y)\} & \text{if } A_n(x, y) \neq \emptyset, \\ +\infty, & \text{if } A_n(x, y) = \emptyset, \end{cases}$$

where $A_n(x, y)$ is a subset of \mathbb{R}^+ defined by $A_n(x, y) = \{t : M(x, y, t) > a_n\}$.

Theorem 3.2. *For any $n \in \mathbb{N}$, the function ρ_n in Definition 3.1 is an extended pseudo-metric on X .*

Proof. It is obvious that the function ρ_n satisfies Axioms (i) and (iii) in Definition 2.1. To prove the triangle inequality, for any $x, y, z \in X$, we distinguish the following two cases.

Case 1: $A_n(x, y) = \emptyset$ or $A_n(y, z) = \emptyset$. In this case, we have $\rho_n(x, y) = \infty$ or $\rho_n(y, z) = \infty$, which implies the triangle inequality is true.

Case 2: $A_n(x, y) \neq \emptyset$ and $A_n(y, z) \neq \emptyset$. Suppose $t \in A_n(x, y)$ and $s \in A_n(y, z)$. Then we have $M(x, y, t) > a_n$ and $M(y, z, s) > a_n$. It follows from the monotonicity of $*$ and Lemma 2.7 that $M(x, y, t) * M(y, z, s) \geq a_n * a_n = a_n$. In fact, we can prove that the above inequality is a strict inequality. Without loss of generality, suppose $M(x, y, t) \leq M(y, z, s)$. We distinguish the following two cases.

Case i: there exists a $k_0 \in K$ such that $M(x, y, t) \in (\alpha_{k_0}, \beta_{k_0})$ and $\alpha_{k_0} \geq a_n$. Since $*_{k_0}$ is a strict Archimedean t-norm and φ_{k_0} is strictly monotone, by Lemma 2.3 and Lemma 2.5 we get that

$$\begin{aligned} M(x, y, t) * M(y, z, s) &\geq M(x, y, t) * M(x, y, t) \\ &= \varphi_{k_0}^{-1}(\varphi_{k_0}(M(x, y, t)) *_{k_0} \varphi_{k_0}(M(x, y, t))) \\ &> \varphi_{k_0}^{-1}(\varphi_{k_0}(\alpha_{k_0}) *_{k_0} \varphi_{k_0}(\alpha_{k_0})) = \alpha_{k_0} * \alpha_{k_0} \geq a_n. \end{aligned}$$

Case ii:

$$M(x, y, t) \in [0, 1] \setminus \bigcup_{k \in K} (\alpha_k, \beta_k),$$

where $[0, 1] \setminus \bigcup_{k \in K} (\alpha_k, \beta_k)$ is the complementary set of $\bigcup_{k \in K} (\alpha_k, \beta_k)$ in $[0, 1]$.

In this case, by Lemma 2.3 and Lemma 2.5 we have that

$$M(x, y, t) * M(y, z, s) \geq M(x, y, t) * M(x, y, t) = M(x, y, t) \wedge M(x, y, t) > a_n.$$

Consequently, from (FM-4) of Definition 2.9, it follows that $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s) > a_n$, which implies $t + s \in A_n(x, z)$. Thus we have $t + s \geq \inf\{v : v \in A_n(x, z)\} = \rho_n(x, z)$. By the arbitrariness of $t \in A_n(x, y)$ and $s \in A_n(y, z)$, we get that

$$\rho_n(x, y) + \rho_n(y, z) = \inf\{t : t \in A_n(x, y)\} + \inf\{s : s \in A_n(y, z)\} \geq \rho_n(x, z).$$

Consequently, ρ_n is an extended pseudo-metric on X . \square

Theorem 3.3. For any $n, m \in \mathbb{N}$ with $m < n$, $\rho_m(x, y) \leq \rho_n(x, y)$ for all $x, y \in X$.

Proof. Since $a_n > a_m$, we have

$$A_n(x, y) = \{t : M(x, y, t) > a_n\} \subseteq \{t : M(x, y, t) > a_m\} = A_m(x, y).$$

Thus we have

$$\rho_m(x, y) = \inf\{t : t \in A_m(x, y)\} \leq \inf\{t : t \in A_n(x, y)\} = \rho_n(x, y). \quad \square$$

Theorem 3.4. A sequence $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$ if and only if it is a Cauchy sequence in (X, ρ_n) for all $n \in \mathbb{N}$.

Proof. Suppose $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$. According to the hypothesis, for each $\varepsilon \in (0, 1)$ and $t > 0$, there is an $i_0 \in \mathbb{N}$ such that $M(x_i, x_j, t) > 1 - \varepsilon$ for all $i, j \geq i_0$. Thus for any fixed $n \in \mathbb{N}$ and any $\theta > 0$, let $\varepsilon \in (0, 1 - a_n)$ and $t \in (0, \theta)$. There is an $i_0 \in \mathbb{N}$ such that $M(x_i, x_j, t) > 1 - \varepsilon > a_n$ for all $i, j \geq i_0$, which implies $t \in A_n(x_i, x_j)$ for all $i, j \geq i_0$. Consequently, we have $\rho_n(x_i, x_j) = \inf\{s : s \in A_n(x_i, x_j)\} \leq t < \theta$, for all $i, j \geq i_0$. Thence $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in (X, ρ_n) .

Conversely, for any fixed $\varepsilon \in (0, 1)$ and $t > 0$, take $n \in \mathbb{N}$ and $\theta > 0$ such that $a_n > 1 - \varepsilon$ and $\theta < t$. Since $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in (X, ρ_n) , there exists an $i_0 \in \mathbb{N}$ such that $\rho_n(x_i, x_j) < \theta$ for all $i, j \geq i_0$. Consequently, we have that $M(x_i, x_j, t) \geq M(x_i, x_j, \theta) > a_n > 1 - \varepsilon$ for all $i, j \geq i_0$. Thus $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$. \square

Let $(x_i)_{i \in \mathbb{N}}$ be a convergent sequence in (X, ρ_n) . Since the limit points are not unique, we denoted the set of all the limit points in (X, ρ_n) by C_n .

Theorem 3.5. If the sequence $(x_j)_{j \in \mathbb{N}}$ is convergent in (X, ρ_n) for all $n \in \mathbb{N}$, then the family of sets of limit points $(C_n)_{n \in \mathbb{N}}$ is a family of nested closed sets, i.e., C_n is a closed set in (X, ρ_n) and

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots.$$

Proof. First, we show that C_n is a closed set in (X, ρ_n) for any $n \in \mathbb{N}$. Let $(y_i)_{i \in \mathbb{N}}$ be a sequence in C_n and let it converge to $y \in X$ with respect to ρ_n . Consequently, for each $\varepsilon > 0$, there is an $i_0 \in \mathbb{N}$ such that $\rho_n(y_i, y) < \varepsilon$ for all $i \geq i_0$. Thus for any fixed $i_1 \geq i_0$, since $(x_j)_{j \in \mathbb{N}}$ converges to y_{i_1} , there exists a $j_0 \in \mathbb{N}$ such that $\rho_n(x_j, y_{i_1}) < \varepsilon$, for all $j \geq j_0$. Consequently, we get that $\rho_n(x_j, y) \leq \rho_n(x_j, y_{i_1}) + \rho_n(y_{i_1}, y) \leq 2\varepsilon$, for all $j \geq j_0$. Thus, we get $(x_j)_{j \in \mathbb{N}}$ converges to y , i.e., $y \in C_n$.

Then for any $y \in C_n$ and any $x_i \in (x_j)_{j \in \mathbb{N}}$, by Theorem 3.3, we have $\rho_1(y, x_i) \leq \rho_2(y, x_i) \leq \dots \leq \rho_n(y, x_i)$, which implies that $\lim_{i \rightarrow \infty} \rho_1(y, x_i) \leq \lim_{i \rightarrow \infty} \rho_2(y, x_i) \leq \dots \leq \lim_{i \rightarrow \infty} \rho_n(y, x_i) = 0$. Thus the family of sets $(C_n)_{n \in \mathbb{N}}$ are nested closed sets, i.e.,

$$C_1 \supseteq C_2 \supseteq \dots \supseteq C_n \supseteq \dots.$$

□

Theorem 3.6. $(X, M, *)$ is a complete metric space if and only if every Cauchy sequence in $(X, M, *)$ is convergent in (X, ρ_n) for all $n \in \mathbb{N}$ and the intersection of the sets of limit points $(C_n)_{n \in \mathbb{N}}$ is a singleton set.

Proof. If $(X, M, *)$ is a complete metric space, then for any Cauchy sequence $(x_i)_{i \in \mathbb{N}}$ in $(X, M, *)$, suppose $x \in X$ is the unique limit point with respect to M .

Thus for each $\varepsilon \in (0, 1)$ and $t > 0$, there is an $i_0 \in \mathbb{N}$ such that $M(x_i, x, t) > 1 - \varepsilon$ for all $i \geq i_0$. Thus for any fixed $n \in \mathbb{N}$ and any $\theta > 0$, let $\varepsilon \in (0, 1 - a_n)$ and $t \in (0, \theta)$. There is an $i_0 \in \mathbb{N}$ such that $M(x_i, x, t) > 1 - \varepsilon > a_n$, for all $i \geq i_0$, which implies $\rho_n(x_i, x) \leq t < \theta$, for all $i \geq i_0$. Thence $(x_i)_{i \in \mathbb{N}}$ converges to x in (X, ρ_n) for all $n \in \mathbb{N}$, which implies $x \in \bigcap_{n \in \mathbb{N}} C_n$.

In addition, let $y \in \bigcap_{n \in \mathbb{N}} C_n$. For any fixed $\varepsilon \in (0, 1)$ and $t > 0$, take $n \in \mathbb{N}$ and $\theta > 0$ such that $a_n > 1 - \varepsilon$ and $\theta < t$. Since $(x_i)_{i \in \mathbb{N}}$ converges to y with respect to ρ_n , there exists an $i_0 \in \mathbb{N}$ such that $\rho_n(x_i, y) < \theta$ for all $i \geq i_0$. Consequently, we have that $M(x_i, y, t) \geq M(x_i, y, \theta) > a_n > 1 - \varepsilon$, for all $i \geq i_0$. Thus $\lim_{i \rightarrow \infty} M(x_i, y, t) = 1$ for any $t > 0$. By the uniqueness of limit point in $(X, M, *)$, we have $x = y$.

Conversely, for any Cauchy sequence $(x_i)_{i \in \mathbb{N}}$ in $(X, M, *)$, by hypothesis, $(x_i)_{i \in \mathbb{N}}$ is convergent in (X, ρ_n) for all $n \in \mathbb{N}$ and let $\{x\} = \bigcap_{n \in \mathbb{N}} C_n$. From the above discussion, we can get that $\lim_{i \rightarrow \infty} M(x_i, x, t) = 1$ for any $t > 0$. Thus $(X, M, *)$ is a complete metric space. □

Theorem 3.7. For any $n \in \mathbb{N}$, the extended pseudo-metric ρ_n as a function from $(X, M, *) \times (X, M, *)$ to \mathbb{R}_∞^+ is continuous.

Proof. Let $(x, y) \in X \times X$ and let $(x_i, y_i)_{i \in \mathbb{N}}$ be a sequence in $X \times X$ that converges to (x, y) with respect to the fuzzy metric M . Thus we have that

$$\lim_{i \rightarrow \infty} M(x_i, x, t) = \lim_{i \rightarrow \infty} M(y_i, y, t) = 1,$$

for any $t > 0$. Then for each $\varepsilon \in (0, 1)$ and $t > 0$, there is an $i_0 \in \mathbb{N}$ such that $M(x_i, x, t) > 1 - \varepsilon$ for all $i \geq i_0$. Thus for any fixed $n \in \mathbb{N}$ and any $\theta > 0$, let $\varepsilon \in (0, 1 - a_n)$ and $t \in (0, \theta)$. There is an $i_0 \in \mathbb{N}$ such that $M(x_i, x, t) > 1 - \varepsilon > a_n$ for all $i \geq i_0$, which implies $\rho_n(x_i, x) \leq t < \theta$ for all $i \geq i_0$. Consequently, we have $\lim_{i \rightarrow \infty} \rho_n(x_i, x) = 0$, and similarly $\lim_{i \rightarrow \infty} \rho_n(y_i, y) = 0$.

We now distinguish the following two cases.

Case 1: $\rho_n(x, y) < +\infty$. For any $i \in \mathbb{N}$, we have $\rho_n(x_i, y_i) - \rho_n(x, y) \leq \rho_n(x_i, x) + \rho_n(y, y_i)$, and $\rho_n(x, y) - \rho_n(x_i, y_i) \leq \rho_n(x_i, x) + \rho_n(y, y_i)$, which implies that $|\rho_n(x, y) - \rho_n(x_i, y_i)| \leq \rho_n(x_i, x) + \rho_n(y, y_i)$. Thus we get that

$$|\rho_n(x, y) - \lim_{i \rightarrow \infty} \rho_n(x_i, y_i)| \leq \lim_{i \rightarrow \infty} \rho_n(x_i, x) + \lim_{i \rightarrow \infty} \rho_n(y, y_i) = 0.$$

Case 2: $\rho_n(x, y) = +\infty$. On the one hand, we have $\rho_n(x, y) \leq \rho_n(x_i, x) + \rho_n(x_i, y_i) + \rho_n(y, y_i)$, for all $i \in \mathbb{N}$. On the other hand, since $\lim_{i \rightarrow \infty} \rho_n(x_i, x) = 0$ and $\lim_{i \rightarrow \infty} \rho_n(y, y_i) = 0$, there exists an $i_0 \in \mathbb{N}$ such that $\rho_n(x_i, x) + \rho_n(y, y_i) < +\infty$ whenever $i \geq i_0$. Thus we have $\rho_n(x_i, y_i) = +\infty$ for $i \geq i_0$. \square

From Theorem 3.2, it follows that the extended pseudo-metric ρ_n can generate a topology τ_{ρ_n} on X which has as a base the family of open balls of the form

$$\left\{ B_{\rho_n}\left(x, \frac{1}{i}\right) : i \in \mathbb{N}, x \in X \right\},$$

where

$$B_{\rho_n}\left(x, \frac{1}{i}\right) = \left\{ y \in X : \rho_n(x, y) < \frac{1}{i} \right\}.$$

Theorem 3.8. *The topology τ_M is finer than τ_{ρ_n} , i.e., $\tau_M \supseteq \tau_{\rho_n}$ for any $n \in \mathbb{N}$.*

Proof. In fact, we can prove that for all $i \in \mathbb{N}$ and $x \in X$,

$$B_{\rho_n}\left(x, \frac{1}{i}\right) = B_M\left(x, 1 - a_n, \frac{1}{i}\right).$$

If $y \in B_M(x, 1 - a_n, 1/i)$, then $M(x, y, 1/i) > 1 - (1 - a_n) = a_n$. By Remark 2.6 of [3], we can find a $t_0, 0 < t_0 < 1/i$ such that $M(x, y, t_0) > a_n$ which implies $t_0 \in A_n(x, y)$. Thus we have $\rho_n(x, y) \leq t_0 < 1/i$, i.e., $y \in B_{\rho_n}(x, 1/i)$.

Conversely, for any $y \in B_{\rho_n}(x, 1/i)$, we have $1/i > \rho_n(x, y)$, i.e.,

$$\frac{1}{i} > \inf \{t : t \in A_n(x, y)\}.$$

Thus we get $1/i \in A_n(x, y)$ which means $M(x, y, 1/i) > a_n = 1 - (1 - a_n)$. Consequently, $y \in B_M(x, 1 - a_n, 1/i)$. \square

In what follows, we will present a metric d on X which generates the same topology as the fuzzy metric does.

Definition 3.9. For any extended pseudo-metric ρ_n , define a function d_n from $X \times X$ to $[0, 1]$ by

$$d_n(x, y) = \begin{cases} \frac{\rho_n(x, y)}{1 + \rho_n(x, y)}, & \text{if } \rho_n(x, y) < +\infty, \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3.10. *For any $n \in \mathbb{N}$, the function d_n in Definition 3.9 is a pseudo-metric on X .*

Proof. It is obvious that the function d_n satisfies Axiom (i) and (iii) in Definition 2.1 because ρ_n satisfies these axioms. To prove the triangle inequality, for any $x, y, z \in X$, we distinguish the following two cases.

Case 1: $\rho_n(x, y) = +\infty$ or $\rho_n(y, z) = +\infty$. In this case, we have $d_n(x, y) = 1$ or $d_n(y, z) = 1$, which implies the triangle inequality is true.

Case 2: $\rho_n(x, y) \neq +\infty$ and $\rho_n(y, z) \neq +\infty$. Since $\rho_n(x, z) \leq \rho_n(x, y) + \rho_n(y, z)$, we have $\rho_n(x, z) < +\infty$ which implies $d_n(x, z) < 1$.

In this case, since the function $f(x) = x/(1+x)$, $x \in \mathbb{R}^+$, is monotonically increasing, we have that

$$d_n(x, z) = \frac{\rho_n(x, z)}{1 + \rho_n(x, z)} \leq \frac{\rho_n(x, y) + \rho_n(y, z)}{1 + (\rho_n(x, y) + \rho_n(y, z))}.$$

Furthermore, we get that

$$\begin{aligned} d_n(x, z) &\leq \frac{\rho_n(x, y)}{1 + (\rho_n(x, y) + \rho_n(y, z))} + \frac{\rho_n(y, z)}{1 + (\rho_n(x, y) + \rho_n(y, z))} \\ &\leq \frac{\rho_n(x, y)}{1 + \rho_n(x, y)} + \frac{\rho_n(y, z)}{1 + \rho_n(y, z)} = d_n(x, y) + d_n(y, z). \end{aligned}$$

Thus d_n is a pseudo-metric on X . \square

Theorem 3.11. *For any $n \in \mathbb{N}$, the pseudo-metric d_n as a function from $(X, M, *) \times (X, M, *)$ to $[0, 1]$ is continuous.*

Proof. Let $(x, y) \in X \times X$ and let $(x_i, y_i)_{i \in \mathbb{N}}$ be a sequence in $X \times X$ that converges to (x, y) with respect to the fuzzy metric M .

If $d_n(x, y) < 1$, then we have $\rho_n(x, y) < +\infty$. By Theorem 3.7, and the continuity of the function $f(x) = x/(1+x)$, we get that $\lim_{i \rightarrow \infty} d_n(x_i, y_i) = d_n(x, y)$.

If $d_n(x, y) = 1$, then we have $\rho_n(x, y) = +\infty$. From Case 2 in the proof of Theorem 3.7, it follows that there exists an $i_0 \in \mathbb{N}$ such that $\rho_n(x_i, y_i) = +\infty$ for $i \geq i_0$. Thus $d_n(x_i, y_i) = 1$ for $i \geq i_0$ which implies $\lim_{i \rightarrow \infty} d_n(x_i, y_i) = d_n(x, y)$. \square

Similarly, the pseudo-metric d_n can generate a topology τ_{d_n} on X which has as a base the family of open balls of the form

$$\left\{ B_{d_n}\left(x, \frac{1}{i}\right) : i \in \mathbb{N}, x \in X \right\},$$

where

$$B_{d_n}\left(x, \frac{1}{i}\right) = \left\{ y \in X : d_n(x, y) < \frac{1}{i} \right\}.$$

Theorem 3.12. *For any $n \in \mathbb{N}$, $\tau_{d_n} = \tau_{\rho_n}$.*

Proof. In fact, it is easy to check that for all $i \in \mathbb{N}$ and $x \in X$,

$$B_{\rho_n}\left(x, \frac{1}{i}\right) = B_{d_n}\left(x, \frac{1}{i+1}\right).$$

\square

Definition 3.13. In the fuzzy metric space $(X, M, *)$, let A be a subset of X . For $x \in X$ and $t > 0$, let

$$M(x, A, t) = \begin{cases} \sup\{M(x, y, t) : y \in A\}, & \text{if } A \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.14. *For every set A in the fuzzy metric space $(X, M, *)$, we have $\bar{A} = \{x : M(x, A, t) = 1 \text{ for all } t > 0\}$, where \bar{A} is the closure of A with respect to τ_M .*

Proof. If $A = \emptyset$, then for any $x \in X$, we have $M(x, \emptyset, t) = 0$ for all $t > 0$. Consequently, the set $\{x : M(x, A, t) = 1, \text{ for all } t > 0\} = \emptyset = \bar{A}$. Thus we can suppose that $A \neq \emptyset$.

If $x \in \bar{A}$, then there exists a sequence $(x_i)_{i \in \mathbb{N}}$ in A such that for any $t > 0$, $\lim_{i \rightarrow \infty} M(x, x_i, t) = 1$. Thus we have $M(x, A, t) = 1$ for all $t > 0$.

Conversely, if $M(x, A, t) = 1$ for all $t > 0$, then for every $i \in \mathbb{N}$, there exists an $x_i \in A$ such that $M(x, x_i, 1/i) > 1 - 1/i$. For any $\varepsilon \in (0, 1)$ and any $t > 0$, take an $i_0 \in \mathbb{N}$ such that $i_0 > \max\{1/\varepsilon, 1/t\}$. Then we have that $M(x, x_i, t) \geq M(x, x_i, \frac{1}{i}) > 1 - \frac{1}{i} > 1 - \varepsilon$, whenever $i \geq i_0$. Thus we have that the sequence $(x_i)_{i \in \mathbb{N}}$ converges to x with respect to M , i.e., $x \in \bar{A}$. \square

Definition 3.15. In the pseudo-metric space (X, d_n) , the distance $d_n(x, A)$ from a point x to a set A is defined by

$$d_n(x, A) = \begin{cases} \inf\{d_n(x, y) : y \in A\}, & \text{if } A \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 3.16. [2] *For a pair of points x, y and a set A in (X, d_n) , we have $|d_n(x, A) - d_n(y, A)| \leq d_n(x, y)$.*

Theorem 3.17. *For any $n \in \mathbb{N}$ and for every set A in the fuzzy metric space $(X, M, *)$ assigning to each point $x \in X$ the distance $d_n(x, A)$ defines a continuous function from $(X, M, *)$ to $[0, 1]$.*

Proof. Let $x \in X$ and let $(x_i)_{i \in \mathbb{N}}$ be a sequence in X such that $\lim M(x, x_i, t) = 1$ for all $t > 0$. By Theorem 3.6, we have $\lim_{i \rightarrow \infty} \rho_n(x, x_i) = 0$. Thus from Definition 3.9, we get $\lim_{i \rightarrow \infty} d_n(x, x_i) = 0$. Then by Lemma 3.16, we have that $|d_n(x, A) - d_n(x_i, A)| \leq d_n(x, x_i)$. Consequently, we get $\lim_{i \rightarrow \infty} d_n(x_i, A) = d_n(x, A)$. \square

Theorem 3.18. *For every set A in the fuzzy metric space $(X, M, *)$, $x \in \bar{A}$ if and only if $x \in \{y : d_n(y, A) = 0\}$ for all $n \in \mathbb{N}$.*

Proof. If $A = \emptyset$, then for any $x \in X$, we have $d_n(x, \emptyset) = 1$ for all $n \in \mathbb{N}$. Consequently, the set $\{x : d_n(x, A) = 0\} = \emptyset$ for all $n \in \mathbb{N}$.

Suppose that $A \neq \emptyset$. If $x \in \bar{A}$, then there exists a sequence $(x_i)_{i \in \mathbb{N}}$ in A such that for any $t > 0$, $\lim_{i \rightarrow \infty} M(x, x_i, t) = 1$. By Theorem 3.6, we have $\lim_{i \rightarrow \infty} \rho_n(x, x_i) = 0$ for all $n \in \mathbb{N}$, which implies that $x \in \{y : d_n(y, A) = 0\}$ for all $n \in \mathbb{N}$.

Conversely, if $x \in \{y : d_n(y, A) = 0\}$ for all $n \in \mathbb{N}$. Then for every $n \geq 2$, there exists an $x_n \in A$ such that $d_n(x, x_n) < 1/n$. From the proof of Theorem 3.12, it follows that $\rho_n(x, x_n) < 1/(n-1)$. Thus we have $M(x, x_n, 1/(n-1)) > a_n$.

For any $\varepsilon \in (0, 1)$ and any $t > 0$, take an $n_0 \in \mathbb{N}$ such that $n_0 > 1 + 1/t$ and $a_n > 1 - \varepsilon$. Then we have that $M(x, x_n, t) \geq M(x, x_n, 1/(n-1)) > a_n > 1 - \varepsilon$, whenever $n \geq n_0$. Thus we have that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to x with respect to M , i.e., $x \in \bar{A}$. \square

Definition 3.19. For any fuzzy metric space $(X, M, *)$, define a function d from $X \times X$ to $[0, 1]$ by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x, y).$$

Theorem 3.20. *The function d in Definition 3.19 is a metric on X .*

Proof. Since $(d_n)_{n \in \mathbb{N}}$ is a family of pseudo-metrics, we have that d satisfies Axiom (i), (iii) and (iv) in Definition 2.1. Since the fuzzy metric space $(X, M, *)$ is a Hausdorff space [3], then for every $x \in X$, $\{x\}$ is closed. Let $y \in X$ with $y \neq x$. Then $y \notin \overline{\{x\}} = \{x\}$. Thus by Theorem 3.18, there exists an $n_0 \in \mathbb{N}$ such that $d_{n_0}(x, y) > 0$. Consequently, $d(x, y) > 0$. Hence d is a metric on X . \square

Theorem 3.21. *For the fuzzy metric space $(X, M, *)$ and the induced metric space (X, d) , we have $\tau_M = \tau_d$.*

Proof. For every nonempty set A in $(X, M, *)$, if $x \in \bar{A}$, then by Theorem 3.18 we have $d_n(x, A) = 0$ for all $n \in \mathbb{N}$, which implies that

$$\begin{aligned} d(x, A) &= \sup \{d(x, y) : y \in A\} \\ &= \sup \left\{ \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x, y) : y \in A \right\} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x, A) = 0. \end{aligned}$$

Thus x is in the closure of A with respect to the metric d .

Conversely, suppose $d(x, A) = 0$. Since for any $n \in \mathbb{N}$, we have $d_n(x, y) \leq 2^n d(x, y)$, which implies that $d_n(x, A) \leq 2^n d(x, A)$. Thus we have $d_n(x, A) = 0$ for all $n \in \mathbb{N}$. By Theorem 3.18, we get $x \in \bar{A}$. Thus we get that the closure operator in $(X, M, *)$ is the same as the one in (X, d) , i.e., $\tau_M = \tau_d$. \square

Theorem 3.22. *$(X, M, *)$ is complete if and only if (X, d) is complete.*

Proof. Suppose $(X, M, *)$ is complete. Let $(x_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in (X, d) . For any fixed $n_0 \in \mathbb{N}$ and any $\varepsilon > 0$, there exists an $i_0 \in \mathbb{N}$ such that

$$\frac{\varepsilon}{2^{n_0}} > d(x_i, x_j) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_i, x_j) \geq \frac{1}{2^{n_0}} d_{n_0}(x_i, x_j).$$

for all $i, j \geq i_0$. Thus we have $d_{n_0}(x_i, x_j) < \varepsilon$ for all $i, j \geq i_0$. By the arbitrariness of n_0 , we get that $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in (X, d_n) for all $n \in \mathbb{N}$. From Theorem 3.4 and Theorem 3.12, it follows that $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $(X, M, *)$. Then there exists a limit point $x \in X$. By Theorem 3.6 and Theorem 3.12, we get $\lim_{i \rightarrow \infty} d_n(x_i, x) = 0$ for all $n \in \mathbb{N}$. For any $\varepsilon > 0$, since the series $\sum_{n=1}^{\infty} 1/2^n$ is convergent, there exists an $m \in \mathbb{N}$ such that

$$\sum_{n=m}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

Furthermore, for every $n = 1, 2, \dots, m-1$, there exists i_n , such that $d_n(x_i, x) < \varepsilon/2$ for $i, j \geq i_n$. Let $i' = \max\{i_1, i_2, \dots, i_{m-1}\}$. Now we get that

$$d(x_i, x) = \sum_{n=1}^{m-1} \frac{1}{2^n} d_n(x_i, x) + \sum_{n=m}^{\infty} \frac{1}{2^n} d_n(x_i, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $i \geq i'$. Thence $\lim_{i \rightarrow \infty} d(x_i, x) = 0$ which implies (X, d) is complete.

Conversely, suppose (X, d) is complete. Let $(x_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in $(X, M, *)$. From Theorem 3.4 and Theorem 3.12, it follows that $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in (X, d_n) for all $n \in \mathbb{N}$. By a similar discussion with the above, we can get that $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in (X, d) . Thus there exists a unique $x \in X$ such that $\lim_{i \rightarrow \infty} d(x_i, x) = 0$. Since for any $n \in \mathbb{N}$, we have $d_n(x_i, x) \leq 2^n d(x_i, x)$, which implies that $\lim_{i \rightarrow \infty} d_n(x_i, x) = 0$, for all $n \in \mathbb{N}$. By Theorem 3.6 and Theorem 3.12, we get that $\lim_{i \rightarrow \infty} M(x_i, x, t) = 1$ for all $t > 0$. Thence $(X, M, *)$ is complete. \square

Now we present some examples as applications of the obtained main results.

Example 3.23. Let (X, ρ) be a metric space. Define

$$M(x, y, t) = \frac{t}{t + \rho(x, y)},$$

for all $x, y \in X$ and $t > 0$. Then (X, M, \wedge) is a fuzzy metric space [28].

Since \wedge is an H-type t-norm, we can take a strictly increasing sequence $(a_n)_{n \in \mathbb{N}}$ in $[0, 1)$ by letting $a_n = 1 - 1/n$ for all $n \in \mathbb{N}$. From Definition 3.1, for every $n \in \mathbb{N}$ we get the extended pseudo-metric $\rho_n(x, y) = (n - 1)\rho(x, y)$ for all $x, y \in X$. Then from Definition 3.15, for every $n \in \mathbb{N}$, we have the pseudo-metric

$$d_n(x, y) = \frac{(n - 1)\rho(x, y)}{1 + (n - 1)\rho(x, y)},$$

for all $x, y \in X$. It follows from Definition 3.19 the induced metric d is defined by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{(n - 1)\rho(x, y)}{1 + (n - 1)\rho(x, y)}.$$

We have that the metric ρ and d are equivalent. In fact, for any $x \in X$ and any sequence $(x_i)_{i \in \mathbb{N}}$, suppose $\lim_{i \rightarrow \infty} \rho(x, x_i) = 0$. By a similar discussion with the one in the proof of Theorem 3.22, we can get $\lim_{i \rightarrow \infty} d(x_i, x) = 0$. Conversely, if $\lim_{i \rightarrow \infty} d(x_i, x) = 0$, then for any $\varepsilon > 0$ there exists an $i_0 \in \mathbb{N}$ such that

$$\frac{\rho(x, x_i)}{2^2(1 + \rho(x, x_i))} \leq d(x, x_i) \leq \frac{\varepsilon}{2^2(1 + \varepsilon)},$$

for all $i \geq i_0$ which implies that $\rho(x, x_i) \leq \varepsilon$ for all $i \geq i_0$. Thus $\lim_{i \rightarrow \infty} \rho(x, x_i) = 0$. Consequently, we have (X, d) is complete if and only if (X, ρ) is complete. By Theorem 3.22, we get $(X, M, *)$ is complete if and only if (X, ρ) is complete. Thus we have given a constructive proof to Result 2.9 in [4] and to Theorem 2 in [11] for the t-norms satisfying the assumptions at the beginning of this section.

Example 3.24. Let $X = [0, 1]$ and let $*$ be the H-type t-norm in Example 2.8 with $a_n = 1 - 1/2^n$ for all $n \in \mathbb{N}$. Define a fuzzy set M on $X^2 \times (0, \infty)$ by

$$M(x, y, t) = \begin{cases} x * y, & \text{if } x \neq y, \\ 1, & \text{if } x = y, \end{cases}$$

for all $x, y \in X, t > 0$. Then $(X, M, *)$ is a fuzzy metric space [10]. Furthermore, $*$ is the strongest t-norm which makes $(X, M, *)$ to be a fuzzy metric space [23].

From Definition 3.1, for every $n \in \mathbb{N}$ we get that the extended pseudo-metric $\rho_n(x, y)$ is defined by

$$\rho_n(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 0, & \text{if } x \neq y \text{ and } x * y > 1 - \frac{1}{2^n}, \\ \infty, & \text{if } x \neq y \text{ and } x * y \leq 1 - \frac{1}{2^n}, \end{cases}$$

for all $x, y \in X$.

Then from Definition 3.15, for every $n \in \mathbb{N}$, we have that the pseudo-metric d_n is defined by

$$d_n(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 0, & \text{if } x \neq y \text{ and } x * y > 1 - \frac{1}{2^n}, \\ 1, & \text{if } x \neq y \text{ and } x * y \leq 1 - \frac{1}{2^n}, \end{cases}$$

for all $x, y \in X$.

Thus the induced metric d is defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{1}{2^{n_0-1}}, & \text{if } x \neq y, \end{cases}$$

where n_0 is the smallest positive integer satisfying

$$\frac{1}{1 - x * y} \leq 2^{n_0}.$$

Now, we prove that $[0, 1)$ is the set of all isolated points in (X, d) . In fact, we can prove that

$$d(x, y) \geq \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2^n}, & \text{if } x \in (1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}], n \in \mathbb{N}, \end{cases}$$

for all $y \in X$.

If $x \in [0, 1/2]$, then for any $y \in X$, we have

$$x * y \leq x \wedge y \leq \frac{1}{2},$$

which implies

$$\frac{1}{1 - x * y} \leq 2.$$

Thus we have $d(x, y) = 1$ for all $y \in X$.

If $x \in (1 - 1/2^n, 1 - 1/2^{n+1}]$, then for any $y \in X$, we have

$$x * y \leq 1 - \frac{1}{2^{n+1}},$$

which implies

$$\frac{1}{1 - x * y} \leq 2^{n+1}.$$

Thus we have $d(x, y) \geq 1/2^n$ for all $y \in X$.

Next, we will prove that if $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence, then either there exists an $i_0 \in \mathbb{N}$ such that $x_i = x_{i_0}$ for all $i \geq i_0$, or the sequence $(x_i)_{i \in \mathbb{N}}$ converges to 1.

In fact, on the one hand, if there exists an $i' \in \mathbb{N}$ such that $\sup\{x_i : i \geq i'\} < 1$, then there exists a $k_0 \in \mathbb{N}$ such that $x_i \leq 1 - 1/2^{k_0+1}$ for all $i \geq i'$. Thus we have $d(x_i, x_j) \geq 1/2^{k_0}$ for all $i, j \geq i'$ with $x_i \neq x_j$. Since $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence, there must exist an $i_0 \in \mathbb{N}$ such that $x_i = x_{i_0}$ for all $i \geq i_0$.

On the other hand, if for all $i' \in \mathbb{N}$, $\sup\{x_i : i \geq i'\} = 1$, then without loss of generality, we suppose $(x_i)_{i \in \mathbb{N}}$ is strictly increasing and converges to 1 with respect to the Euclidean metric. For any $\varepsilon > 0$, there exists an $n' \in \mathbb{N}$ such that $1/2^{n'-1} < \varepsilon$. In addition, there exists an $i_0 \in \mathbb{N}$ such that $x_i \geq 1 - 1/2^{n'}$ for all $i \geq i_0$. We get that $x_i * 1 = x_i \geq 1 - 1/2^{n'}$ which implies $d(x_i, 1) \leq 1/2^{n'-1} \leq \varepsilon$ for all $i \geq i_0$. Thence the sequence $(x_i)_{i \in \mathbb{N}}$ converges to 1.

Thus we can get that (X, d) is complete. It follows from Theorem 3.22 that $(X, M, *)$ is complete.

Example 3.25. Let $X = \{1 - 1/i : i \in \mathbb{N}\}$. Define a fuzzy set M on $X^2 \times (0, \infty)$ by

$$M(1 - 1/i, 1 - 1/i, t) = 1 \text{ for all } i \in \mathbb{N}, t > 0;$$

$$M(1 - 1/2i, 1 - 1/2j, t) = (1 - 1/2i) \wedge (1 - 1/2j) \text{ for all } i, j \in \mathbb{N} \text{ with } i \neq j, t > 0;$$

$$M(1 - 1/(2i - 1), 1 - 1/(2j - 1), t) = (1 - 1/(2i - 1)) \wedge (1 - 1/(2j - 1)) \text{ for all } i, j \in \mathbb{N} \text{ with } i \neq j, t > 0;$$

$$M(1 - 1/2i, 1 - 1/(2j - 1), t) = M(1 - 1/(2j - 1), 1 - 1/2i, t) = (1 - 1/2i) \wedge (1 - 1/(2j - 1)) \text{ for all } i, j \in \mathbb{N}, t \geq 1;$$

$$M(1 - 1/2i, 1 - 1/(2j - 1), t) = M(1 - 1/(2j - 1), 1 - 1/2i, t) = (1 - 1/2i) \wedge (1 - 1/(2j - 1)) \wedge t \text{ for all } i, j \in \mathbb{N}, 0 < t < 1.$$

Then (X, M, \wedge) is a fuzzy metric space [28]. Take an strictly increasing sequence $(a_n)_{n \in \mathbb{N}}$ in $[0, 1)$ by letting $a_n = 1 - 1/n$ for all $n \in \mathbb{N}$. From Definition 3.1, for every $n \in \mathbb{N}$ we get the extended pseudo-metric $\rho_n(x, y)$ is defined by

$$\rho_n(1 - \frac{1}{i}, 1 - \frac{1}{i}) = 0$$

for all $i \in \mathbb{N}$;

$$\rho_n\left(1 - \frac{1}{2i}, 1 - \frac{1}{2j}\right) = \begin{cases} 0, & \text{if } 2i \wedge 2j > n, \\ \infty, & \text{if } 2i \wedge 2j \leq n, \end{cases}$$

for all $i, j \in \mathbb{N}$ with $i \neq j$;

$$\rho_n\left(1 - \frac{1}{2i - 1}, 1 - \frac{1}{2j - 1}\right) = \begin{cases} 0, & \text{if } (2i - 1) \wedge (2j - 1) > n, \\ \infty, & \text{if } (2i - 1) \wedge (2j - 1) \leq n, \end{cases}$$

for all $i, j \in \mathbb{N}$ with $i \neq j$;

$$\rho_n\left(\frac{2i - 1}{2i}, \frac{2j - 2}{2j - 1}\right) = \rho_n\left(\frac{2j - 2}{2j - 1}, \frac{2i - 1}{2i}\right) = \begin{cases} 1 - \frac{1}{n}, & \text{if } 2i \wedge (2j - 1) > n, \\ \infty, & \text{if } 2i \wedge (2j - 1) \leq n, \end{cases}$$

for all $i, j \in \mathbb{N}$.

Then from Definition 3.15, for every $n \in \mathbb{N}$, we have that the pseudo-metric d_n is defined by

$$d_n(1 - \frac{1}{i}, 1 - \frac{1}{i}) = 0$$

for all $i \in \mathbb{N}$;

$$d_n \left(1 - \frac{1}{2i}, 1 - \frac{1}{2j} \right) = \begin{cases} 0, & \text{if } 2i \wedge 2j > n, \\ 1, & \text{if } 2i \wedge 2j \leq n, \end{cases}$$

for all $i, j \in \mathbb{N}$ with $i \neq j$;

$$d_n \left(1 - \frac{1}{2i-1}, 1 - \frac{1}{2j-1} \right) = \begin{cases} 0, & \text{if } (2i-1) \wedge (2j-1) > n, \\ 1, & \text{if } (2i-1) \wedge (2j-1) \leq n, \end{cases}$$

for all $i, j \in \mathbb{N}$ with $i \neq j$;

$$d_n \left(\frac{2i-1}{2i}, \frac{2j-2}{2j-1} \right) = d_n \left(\frac{2j-2}{2j-1}, \frac{2i-1}{2i} \right) = \begin{cases} \frac{n-1}{2n-1}, & \text{if } 2i \wedge (2j-1) > n, \\ 1, & \text{if } 2i \wedge (2j-1) \leq n, \end{cases}$$

for all $i, j \in \mathbb{N}$.

Thus it follows from Definition 3.19 the induced metric d is defined by

$$d \left(1 - \frac{1}{i}, 1 - \frac{1}{i} \right) = 0$$

for all $i \in \mathbb{N}$;

$$d \left(1 - \frac{1}{2i}, 1 - \frac{1}{2j} \right) = \frac{1}{2^{(2i \wedge 2j) - 1}},$$

for all $i, j \in \mathbb{N}$ with $i \neq j$;

$$d \left(1 - \frac{1}{2i-1}, 1 - \frac{1}{2j-1} \right) = \frac{1}{2^{((2i-1) \wedge (2j-1)) - 1}},$$

for all $i, j \in \mathbb{N}$ with $i \neq j$;

$$d \left(\frac{2i-1}{2i}, \frac{2j-2}{2j-1} \right) = \frac{1}{2^{(2i \wedge (2j-1)) - 1}} + \sum_{n=1}^{(2i \wedge (2j-1)) - 1} \frac{n-1}{2^n(2n-1)},$$

for all $i, j \in \mathbb{N}$.

It is easy to see that τ_d is a discrete topology and $(1 - 1/2i)_{i \in \mathbb{N}}$ is a Cauchy sequence in (X, d) . Thus it is not convergent. Consequently, by Theorem 3.22, the fuzzy metric space $(X, M, *)$ is not complete.

4. Conclusions

In this paper, for a fuzzy metric space, we introduced a family of extended pseudo-metrics to equivalently characterize the analysis properties of the fuzzy metric by Theorem 3.4 and Theorem 3.6. And then we induced a metric by the given fuzzy metric. For these two metrics, by Theorem 3.21 and Theorem 3.22 we got that they shares many important properties. As applications of the main results, it is shown that our results can give the constructive proofs to some problems for fuzzy metric spaces in the literature.

Since the fuzzy fixed point theory is a very important issue in the theory of fuzzy metric spaces [1, 4, 5, 21, 26], it could be an interesting problem what are the relationships between the fixed point theory in fuzzy metric spaces and the ones in the induced metric spaces.

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