UNIFORM CONNECTEDNESS AND UNIFORM LOCAL CONNECTEDNESS FOR LATTICE-VALUED UNIFORM CONVERGENCE SPACES

G. JÄGER

ABSTRACT. We apply Preuß' concept of $\mathbb{E}$-connectedness to the categories of lattice-valued uniform convergence spaces and of lattice-valued uniform spaces. A space is uniformly $\mathbb{E}$-connected if the only uniformly continuous mappings from the space to a space in the class $\mathbb{E}$ are the constant mappings. We develop the basic theory for $\mathbb{E}$-connected sets, including the product theorem. Furthermore, we define and study uniform local $\mathbb{E}$-connectedness, generalizing a classical definition from the theory of uniform convergence spaces to the lattice-valued case. In particular it is shown that if the underlying lattice is completely distributive, the quotient space of a uniformly locally $\mathbb{E}$-connected space and products of locally uniformly $\mathbb{E}$-connected spaces are locally uniformly $\mathbb{E}$-connected.

1. Introduction

Connectedness was first defined by G. Cantor in [2]. In the more modern setting of metric spaces, it can be expressed as follows. A metric space (X, d) is connected if for all $\epsilon > 0$ and all $x, y \in X$ there are finitely many points $x = t_1, t_2, ..., t_n = y$ such that $d(t_k, t_{k+1}) \leq \epsilon$ for all $k = 1, 2, ..., n-1$. This notion bears nowadays the name *well-chainedness* or *chain-connectedness*. It was shown later, that for bounded, closed subsets, this definition is equivalent to the requirement that the space cannot be separated into two non-empty, disjoint closed subsets. The latter characterization does not need a metric and was subsequently considered as the "proper" definition of connectedness in topology, see e.g. [8]. Cantor's concept reappeared after the introduction of uniform spaces. A uniform space $(X, \mathcal{U})$ is *well-chained* if for all $x, y \in X$ and all $U \in \mathcal{U}$, there is a natural number n such that $(x, y) \in U^n$, see e.g. [22]. It was shown in [19] that a uniform space is well-chained if and only if each uniformly continuous mapping from $(X, \mathcal{U})$ into the discrete two-point uniform space is constant. (The latter is called *uniform connectedness* in [19].) It is well-known that, similarly, a topological space is connected if each continuous mapping into the discrete two-point topological space is constant. These characterizations were subsequently generalized by Preuß [20, 21] and the concept of $\mathbb{E}$-connectedness. A (uniform, resp. topological) space X is $\mathbb{E}$-connected if, for

Received: July 2015; Revised: January 2016; Accepted: February 2016
Key words and phrases: L-topology, L-uniform convergence space, Uniform connectedness, Local connectedness.

each (uniform, resp. topological) space E in $\mathbb{E}$, the only (continuous resp. uniformly continuous) mappings from X to E are the constant ones.

In the realm of (uniform) convergence spaces, Vainio [23, 24, 25] developed the theory of connectedness along Preuß' lines. He also introduced a notion of local connectedness [24]. Also Gähler [5] contributed to the theory. For uniform convergence spaces, Kneis [18] generalized Cantor's connectedness in order to prove a fixed point theorem, generalizing a similar result by Taylor [22] from uniform spaces to uniform convergence spaces.

In this paper, we use Preuß' concept of $\mathbb{E}$-connectedness and apply it to lattice-valued uniform convergence spaces. We develop the basic theory for uniformly $\mathbb{E}$-connected sets. Further, we define a suitable notion of uniform local $\mathbb{E}$-connectedness, generalizing Vainio's approach [24] to the lattice-valued case.

The paper is organised as follows. In the second section, we provide the necessary notation, definitions and results on lattices, lattice-valued sets and lattice-valued filters needed later on. Section 3 collects the definitions and results regarding lattice-valued uniform convergence spaces and lattice-valued limit spaces. Section 4 discusses the concepts of uniform $\mathbb{E}$-connectedness and Section 5 then collects the results about uniformly $\mathbb{E}$-connected sets. Section 6 is devoted to uniform local $\mathbb{E}$-connectedness and in the last section, we finally draw some conclusions.

2. **Preliminaries**

We consider in this paper *frames*, i.e. complete lattices L (with bottom element $\bot$ and top element $\top$) for which the infinite distributive law $\bigvee_{j\in J}(\alpha\wedge\beta_j) = \alpha\wedge\bigvee_{j\in J}\beta_j$ holds for all $\alpha, \beta_j \in L$ ($j \in J$). In a frame L, we can define an implication operator by $\alpha \to \beta = \bigvee\{\gamma \in L \ : \ \alpha \wedge \gamma \le \beta\}$. This implication is then right-adjoint to the meet operation, i.e. we have $\delta \le \alpha \to \beta$ iff $\alpha \wedge \delta \le \beta$. A complete lattice L is *completely distributive* if the following distributive laws are true.

$$(CD1) \quad \bigvee_{j\in J}\left(\bigwedge_{i\in I_j}\alpha_{ji}\right) = \bigwedge_{f\in\prod_{j\in J} I_j}\left(\bigvee_{j\in J}\alpha_{jf(j)}\right),$$

$$(CD2) \quad \bigwedge_{j\in J}\left(\bigvee_{i\in I_j}\alpha_{ji}\right) = \bigvee_{f\in\prod_{j\in J} I_j}\left(\bigwedge_{j\in J}\alpha_{jf(j)}\right).$$

It is well known that, in a complete lattice, (CD1) and (CD2) are equivalent. In any complete lattice we can define the *wedge-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \le \bigvee D$ there is $\delta \in D$ such that $\alpha \le \delta$. Then $\alpha \le \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j\in J}\beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. In a completely distributive lattice we have $\alpha = \bigvee\{\beta \ : \ \beta \lhd \alpha\}$ for any $\alpha \in L$. An element $\alpha \in L$ in a lattice is called *prime* if $\beta \wedge \gamma \le \alpha$ implies $\beta \le \alpha$ or $\gamma \le \alpha$.

For notions from category theory, we refer to the textbook [1].

For a frame L and a set X, we denote the set of all L-sets $a, b, c, ... : X \longrightarrow L$ by L^X. We define, for $\alpha \in L$ and $A \subseteq X$, the L-set α_A by $\alpha_A(x) = \alpha$ if $x \in A$ and $\alpha_A(x) = \bot$ else. In particular, we denote the constant L-set with value $\alpha \in L$ by

α_X and $\top_A$ is the characteristic function of $A \subseteq X$. The operations and the order are extended pointwisely from L to L^X. For $a \in L^X$ we define $[a > \bot] = \{x \in X : a(x) > \bot\}$.

For $a, b \in L^{X\times X}$ we define $a^{-1} \in L^{X\times X}$ by $a^{-1}(x,y) = a(y,x)$ and $a \circ b \in L^{X\times X}$ by $a \circ b(x,y) = \bigvee_{z\in X}(a(x,z) \wedge b(z,y))$, for all $(x,y) \in X \times X$, see [12]. Then, for $A, B \subseteq X \times X$, $(\top_A)^{-1} = \top_{A^{-1}}$ with $A^{-1} = \{(x,y) : (y,x) \in A\}$ and $\top_A \circ \top_B = \top_{A\circ B}$, where $A \circ B = \{(x,y) :$ there is $z \in X$ s.t. $(x,z) \in A, (z,y) \in B\}$. Further, we denote $\Delta_X = \{(x,x) : x \in X\}$.

A mapping $\mathcal{F} : L^X \longrightarrow L$ is called a *stratified L-filter on* X [9] if (LF1) $\mathcal{F}(\top_X) = \top$ and $\mathcal{F}(\bot_X) = \bot$, (LF2) $\mathcal{F}(a) \leq \mathcal{F}(b)$ whenever $a \leq b$, (LF3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$ and (LFs) $\mathcal{F}(\alpha_X) \geq \alpha$ for all $a, b \in L^X$ and all $\alpha \in L$. A typical example is, for $x \in X$, the *point L-filter* $[x]$ defined by $[x](a) = a(x)$ for all $a \in L^X$. We denote the set of all stratified L-filters on X by $\mathcal{F}^s_L(X)$ and order it by $\mathcal{F} \leq \mathcal{G}$ if for all $a \in L^X$ we have $\mathcal{F}(a) \leq \mathcal{G}(a)$. For a family of stratified L-filters $\mathcal{F}_i$ $(i \in J)$, the infimum in the order is given by $(\bigwedge_{i\in J} \mathcal{F}_i)(a) = \bigwedge_{i\in J} \mathcal{F}_i(a)$ for all $a \in L^X$. The supremum, however, only exists if $\mathcal{F}_{i_1}(a_1) \wedge \mathcal{F}_{i_2}(a_2) \wedge ... \wedge \mathcal{F}_{i_n}(a_n) = \bot$ whenever $a_1 \wedge a_2 \wedge ... \wedge a_n = \bot_X$. In this case the supremum is given by $(\bigvee_{i\in J} \mathcal{F}_i)(a) = \bigvee\{\mathcal{F}_{i_1}(a_1) \wedge \mathcal{F}_{i_2}(a_2) \wedge ... \wedge \mathcal{F}_{i_n}(a_n) : a_1 \wedge a_2 \wedge ... \wedge a_n \leq a\}$, see [9]. Consider now a mapping $f : X \longrightarrow Y$. For $\mathcal{F} \in \mathcal{F}^s_L(X)$ then $f(\mathcal{F}) \in \mathcal{F}^s_L(Y)$ is defined by $f(\mathcal{F})(b) = \mathcal{F}(f^{\leftarrow}(b))$ with $f^{\leftarrow}(b) = b \circ f$ for $b \in L^X$, [9]. For $\mathcal{G} \in \mathcal{F}^s_L(Y)$ we define $f^{\leftarrow}(\mathcal{G})(a) = \bigvee\{\mathcal{G}(b) : f^{\leftarrow}(b) \leq a\}$. If $\mathcal{G}(b) = \bot$ whenever $f^{\leftarrow}(b) = \bot_X$, then $f^{\leftarrow}(\mathcal{G}) \in \mathcal{F}^s_L(X)$, see [10]. We will need the following two examples later. Firstly, if $M \subseteq X$ we define $i_M : M \longrightarrow X$, $i_M(x) = x$. In case of existence, we denote, for $\mathcal{F} \in \mathcal{F}^s_L(X)$, $\mathcal{F}_M = i_M^{\leftarrow}(\mathcal{F})$. Secondly, for sets X_i $(i \in J)$, we denote the projections $p_j : \prod_{i\in J} X_i \longrightarrow X_j$ and define the *stratified L-product filter* $\prod_{i\in J} \mathcal{F}_i = \bigvee_{i\in J} p_i^{\leftarrow}(\mathcal{F}_i)$, see [3, 10]. The following result follows directly from the definition.

Lemma 2.1. *Let $\mathcal{F}_i \in \mathcal{F}^s_L(X_i)$ for $i \in J$. Then, for $U \subseteq \prod_{i\in J} X_i$,*
$$\prod_{i\in J} \mathcal{F}_i(\top_U) = \bigvee\{\bigwedge_{i\in J} \mathcal{F}_i(\top_{U_i}) : \prod_{i\in J} U_i \subseteq U \text{ and only finitely many } U_i \neq X_i\}.$$

We denote stratified L-filters on $X \times X$ by $\Phi, \Psi, ...$. In [12] we defined the following constructions. For $\Phi, \Psi \in \mathcal{F}^s_L(X \times X)$ we define $\Phi^{-1} \in \mathcal{F}^s_L(X \times X)$ by $\Phi^{-1}(a) = \Phi(a^{-1})$ for all $a \in L^{X\times X}$. We further define $\Phi \circ \Psi : L^{X\times X} \longrightarrow L$ by $\Phi \circ \Psi(a) = \bigvee\{\Phi(b) \wedge \Psi(c) : b \circ c \leq a\}$. Then $\Phi \circ \Psi \in \mathcal{F}^s_L(X \times X)$ if and only if $b \circ c = \bot_{X\times X}$ implies $\Phi(b) \wedge \Psi(c) = \bot$. In this case we also say that $\Phi \circ \Psi$ *exists.* Lastly, we denote $[\Delta_X] = \bigwedge_{x\in X}[(x,x)]$.

Lemma 2.2. *Let $\bot \in L$ be prime and let $a, b \in L^X$ and $B \subseteq X$. If $a \circ b \leq \top_B$ then $\top_{[a>\bot]} \circ \top_{[b>\bot]} \leq \top_B$.*

Proof. The proof is easy and left for the reader. □

Corollary 2.3. *Let $\bot \in L$ be prime, let $\Phi, \Psi \in \mathcal{F}^s_L(X \times X)$ and let $B \subseteq X \times X$. Then $\Phi \circ \Psi(\top_B) = \bigvee\{\Phi(\top_C) \wedge \Psi(\top_D) : C \circ D \subseteq B\}$.*

Lemma 2.4. *Let $\Psi \in \mathcal{F}^s_L(X \times X)$ and let $x \in X$. We define $\Psi(x) : L^X \longrightarrow L$ by $\Psi(x)(a) = \bigvee\{\Psi(\psi) \ : \ \psi(\cdot, x) \leq a\}$. Then $\Psi(x) \in \mathcal{F}^s_L(X)$ if and only if $\Psi(\psi) = \perp$ whenever $\psi(\cdot, x) = \perp_X$.*

Proof. We omit the straightforward proof and only mention that the condition is used to ensure $\Psi(x)(\perp_X) = \perp$. □

We note that if $\Psi \leq [\Delta_X]$, then $\psi(\cdot, x) = \perp_X$ implies $\Psi(\psi) \leq \bigwedge_{y \in X} \psi(y, y) \leq \psi(x, x) = \perp$. Hence, in this case, $\Psi(x) \in \mathcal{F}^s_L(X)$.

Lemma 2.5. *Let $\Phi, \Psi \in \mathcal{F}^s_L(X \times X)$, $\mathcal{F} \in \mathcal{F}^s_L(X)$ and let $x \in X$ and $\Phi(x), \Psi(x) \in \mathcal{F}^s_L(X)$. The following hold.*
(1) If $\Phi \leq \Psi$, then $\Phi(x) \leq \Psi(x)$.
(2) $(\Phi \wedge \Psi)(x) \leq \Phi(x) \wedge \Psi(x)$.
(3) $[\Delta_X](x) = [x]$.
(4) $\Psi = \Psi(x) \times [x]$.
(5) $(\mathcal{F} \times [x])(x) \leq \mathcal{F}$.

Proof. (1) and (2) are easy and left for the reader.

(3) We have $[\Delta_X](x)(a) = \bigvee\{\bigwedge_{y \in X} \phi(y, y) \ : \ \phi(\cdot, x) \leq a\} \leq \bigvee\{\phi(x, x) \ : \ \phi(\cdot, x) \leq a\} \leq a(x) = [x](a)$. On the other hand, for $a \in L^X$, we define $\phi_a(u, v) = \top$ if $v \neq x$ and $\phi_a(u, v) = a(u)$ if $v = x$. Then $\phi_a(\cdot, x) = a$ and hence $[\Delta](x)(a) \geq \bigwedge_{y \in X} \phi_a(y, y) = \phi_a(x, x) = a(x) = [x](a)$.

(4) For $\phi \in L^{X \times X}$ we have $\phi(\cdot, x) \times \top_{\{x\}} \leq \phi$ and hence $\Psi(x) \times [x](\psi) = \bigvee\{\Psi(x)(c) \wedge [x](d) \ : \ c \times d \leq \psi\} \geq \bigvee\{\Psi(\phi) \wedge d(x) \ : \ \phi(\cdot, x) \times d \leq \psi\} \geq \Psi(\psi) \wedge \top_{\{x\}}(x) = \Psi(\psi)$. For the converse inequality, we note that $c \times d \leq \psi$ and $\phi(\cdot, x) \leq c$ implies $\phi(\cdot, x) \times d \leq \psi$. Hence it follows with (LFs) that if $c \times d \leq \psi$, then $\Psi(x)(c) \wedge d(x) \leq \bigvee\{\Psi(\phi \wedge (d(x))_X) \ : \ \phi(\cdot, x) \leq c\} \leq \bigvee\{\Psi(\phi \wedge (d(x))_X) \ : \ \phi \wedge (d(x))_X \leq \psi\} \leq \Psi(\psi)$. Hence $(\Psi(x) \times [x])(\psi) = \bigvee\{\Psi(x)(c) \wedge [x](d) \ : \ c \times d \leq \psi\} \leq \Psi(\psi)$.

(5) If $\phi(\cdot, x) \leq a$ then if $c \times d \leq \phi$ we have, for all $y \in X$, that $c(y) \wedge d(x) \leq \phi(y, x) \leq a(y)$. Hence it follows $(\mathcal{F} \times [x])(\phi) \leq \{\mathcal{F}(c \wedge (d(x))_X) \ : \ c \wedge (d(x))_X \leq a\} \leq \mathcal{F}(a)$ and therefore $(\mathcal{F} \times [x])(x)(a) = \bigvee\{(\mathcal{F} \times [x])(\phi) \ : \ \phi(\cdot, x) \leq a\} \leq \mathcal{F}(a)$. □

We will later need a further construction. We describe the situation. Let X_i be sets ($i \in J$). We denote the projections $\pi_j : \prod_{i \in J}(X_i \times X_i) \longrightarrow X_j \times X_j$, $((x_i, y_i)) \longmapsto (x_j, y_j)$, the mapping $\nu : \prod_{i \in J}(X_i \times X_i) \longrightarrow \prod_{i \in J} X_i \times \prod_{i \in J} X_i$ defined by $\nu((x_i, y_i)) = ((x_i), (y_i))$ and the product of the projections $p_j : \prod_{i \in J} X_i \longrightarrow X_j$, $p_j \times p_j : \prod_{i \in J} X_i \times \prod_{i \in J} X_i \longrightarrow X_j \times X_j$. Then $(p_j \times p_j) \circ \nu = \pi_j$ for all $j \in J$. For $\Psi_i \in \mathcal{F}^s_L(X_i \times X_i)$, ($i \in J$) we define

$$\bigotimes_{i \in J} \Psi_i = \nu(\prod_{i \in J} \Psi_i) \in \mathcal{F}^s_L(\prod_{i \in J} X_i \times \prod_{i \in J} X_i).$$

Following Gähler [5], we call $\bigotimes_{i \in J} \Psi_i$ the *stratified relation product L-filter of the Ψ_i ($i \in J$).*

Proposition 2.6. *Let* $\Psi_i \in \mathcal{F}_L^s(X_i \times X_i)$ *for* $i \in J$ *and* $X = \prod_{i\in J} X_i$. *Let* $\Phi \in \mathcal{F}_L^s(X \times X)$. *Then*
(1) $(p_j \times p_j)(\bigotimes_{i\in J} \Psi_i) \geq \Psi_j$;
(2) $\bigotimes_{i\in J}((p_i \times p_i)(\Phi)) \leq \Phi$;
(3) $\bigotimes_{i\in J}[\Delta_{X_i}] \leq [\Delta_{\prod_{i\in J} X_i}]$.

Proof. (1) We use $(p_j \times p_j)\circ\nu = \pi_j$. Then $(p_j \times p_j)(\bigotimes_{i\in J} \Psi_i) = \pi_j(\prod_{i\in J} \Psi_i) \geq \Psi_j$.

(2) It is not difficult to show that for $a \in L^{X\times X}$ and $a_1 \in L^{X_{j_1}\times X_{j_1}}$,...,$a_n \in L^{X_{j_n}\times X_{j_n}}$ we have $(p_{j_1}\times p_{j_1})^{\leftarrow}(a_1) \wedge ... \wedge (p_{j_n}\times p_{j_n})^{\leftarrow}(a_n) \leq a$ whenever $\pi_{j_1}^{\leftarrow}(a_1) \wedge ... \wedge \pi_{j_n}^{\leftarrow}(a_n) \leq \nu^{\leftarrow}(a)$. Hence $\nu(\prod_{i\in J}(p_i \times p_i)(\Phi))(a) = \bigvee\{\Phi((p_{j_1}\times p_{j_1})^{\leftarrow}(a_1) \wedge ... \wedge (p_{j_n}\times p_{j_n})^{\leftarrow}(a_n)) \;:\; \pi_{j_1}^{\leftarrow}(a_1) \wedge ... \wedge \pi_{j_n}^{\leftarrow}(a_n) \leq \nu^{\leftarrow}(a)\} \leq \Phi$.

(3) For $a \in L^{X\times X}$ and $a_1 \in L^{X_{j_1}\times X_{j_1}}$,...,$a_n \in L^{X_{j_n}\times X_{j_n}}$, if $\pi_{j_1}^{\leftarrow}(a_1) \wedge ... \wedge \pi_{j_n}^{\leftarrow}(a_n)((x_i, x_i)) = a_1(x_{j_1}, x_{j_1}) \wedge ... \wedge a_n(x_{j_n}, x_{j_n}) \leq \nu^{\leftarrow}(a)((x_i, x_i)) = a((x_i), (x_i))$, then $\bigwedge_{x_{j_1}\in X_{j_1}} a_1(x_{j_1}, x_{j_1}) \wedge ... \wedge \bigwedge_{x_{j_n}\in X_{j_n}} a_n(x_{j_n}, x_{j_n}) \leq \bigwedge_{(x_i)\in X} a((x_i), (x_i))$. Hence, $\bigotimes_{i\in J}[\Delta_{X_i}](a) = \bigvee\{[\Delta_{X_{j_1}}](a_1) \wedge ... \wedge [\Delta_{X_{j_n}}](a_n) \;:\; \pi_{j_1}^{\leftarrow}(a_1) \wedge ... \wedge \pi_{j_n}^{\leftarrow}(a_n) \leq \nu^{\leftarrow}(a)\} \leq \bigwedge_{(x_i)\in X} a((x_i), (x_i)) = [\Delta_X](a)$. □

3. Lattice-valued Uniform Convergence Spaces and Lattice-valued Limit Spaces

Let $X \neq \emptyset$. A mapping $\Lambda : \mathcal{F}_L^s(X \times X) \longrightarrow L$ is called a *stratified L-uniform convergence structure* and the pair (X, Λ) a *stratified L-uniform convergence space* [3, 12] if for all $x \in X$ and all $\Phi, \Psi \in \mathcal{F}_L^s(X \times X)$,

(UC1) $\Lambda([(x, x)]) = \top \quad \forall x \in X$;
(UC2) $\Phi \leq \Psi \implies \Lambda(\Phi) \leq \Lambda(\Psi)$;
(UC3) $\Lambda(\Phi) \leq \Lambda(\Phi^{-1})$;
(UC4) $\Lambda(\Phi) \wedge \Lambda(\Psi) \leq \Lambda(\Phi \wedge \Psi)$;
(UC5) $\Lambda(\Phi) \wedge \Lambda(\Psi) \leq \Lambda(\Phi \circ \Psi)$ whenever $\Phi \circ \Psi$ exists.

A mapping $f : (X, \Lambda) \longrightarrow (X', \Lambda')$, where $(X, \Lambda), (X', \Lambda')$ are stratified L-uniform convergence spaces, is called *uniformly continuous* iff $\Lambda(\Phi) \leq \Lambda'((f \times f)(\Phi))$ for all $\Phi \in \mathcal{F}_L^s(X \times X)$. The category *SL-UCS* has as objects the stratified L-uniform convergence spaces and as morphisms the uniformly continuous mappings. Then *SL-UCS* is a well-fibred topological construct and has natural function spaces, i.e. *SL-UCS* is Cartesian closed [12]. In particular, constant mappings are uniformly continuous. We describe the initial constructions. Let $(f_i : X \longrightarrow (X_i, \Lambda_i))_{i\in I}$ be a source. Define for $\Phi \in \mathcal{F}_L^s(X \times X)$ the *initial stratified L-uniform convergence structure on* X by $\Lambda(\Phi) = \bigwedge_{i\in I} \Lambda_i((f_i \times f_i)(\Phi))$. In particular, we can define subspaces and product spaces.

- *Subspace:* Let $(X, \Lambda) \in |SL\text{-}UCS|$ and let $T \subseteq X$ and $i_T : T \longrightarrow X$ be the embedding mapping defined by $i_T(x) = x$ for $x \in T$. Then the *subspace* $(T, \Lambda|_T)$ is defined by $\Lambda|_T(\Phi) = \Lambda((i_T \times i_T)(\Phi))$ for $\Phi \in \mathcal{F}_L^s(T \times T)$.
- *Product space:* Let $(X_i, \Lambda_i) \in |SL\text{-}UCS|$ for all $i \in J$ and let $X = \prod_{i\in J} X_i$ be the Cartesian product and consider the projections $p_j : X \longrightarrow X_j$. Then

the *product space* $(X, \pi\text{-}\Lambda)$ is defined by $\pi\text{-}\Lambda(\Phi) = \bigwedge_{i\in J} \Lambda_i((p_i \times p_i)(\Phi))$ for all $\Phi \in \mathcal{F}_L^s(X \times X)$.

Subspaces and product spaces are well behaved. Let $T_i \subseteq X_i$ and $(X_i, \Lambda_i) \in |SL\text{-}UCS|$ for all $i \in J$. We denote $X = \prod_{i\in J} X_i$ and $T = \prod_{i\in J} T_i$ and the projections $p_j : X \longrightarrow X_j$ and $q_j : T \longrightarrow T_j$ and the embeddings $i_T : T \longrightarrow X$ and $i_{T_j} : T_j \longrightarrow X_j$. Then we have $(p_j \times p_j) \circ (i_T \times i_T) = (i_{T_j} \times i_{T_j}) \circ (q_j \times q_j)$. It follows that if we denote the product structure on X w.r.t. the projections p_j by $\pi\text{-}\Lambda_i$ and the product structure on T w.r.t. the projections q_j and the spaces $(T_i, \Lambda|_{T_i})$ by $\pi\text{-}(\Lambda|_{T_i})$, then we have $\pi\text{-}(\Lambda|_{T_i}) = (\pi\text{-}\Lambda_i)|_T$. Moreover, we have the following result.

Lemma 3.1. *Let $(X_i, \Lambda_i) \in |SL\text{-}UCS|$ for all $i \in J$ and let $(z_i) \in \prod_{i\in J} X_i$ be fixed. Define the slice $\widetilde{X}_j = \{(x_i) \in \prod_{i\in J} X_i \ : \ x_i = z_i \forall i \neq j\} = \prod_{i\in J} T_i$ with $T_i = \{z_i\}$ if $i \neq j$ and $T_j = X_j$. Then $(\widetilde{X}_j, \pi\text{-}\Lambda|_{\widetilde{X}_j})$ is isomorphic to (X_j, Λ_j).*

Proof. We use the notations from above and define $h : \widetilde{X}_j \longrightarrow X_j$ by $h((x_i)) = x_j$. Then $h = p_j \circ i_{\widetilde{X}_j}$ is uniformly continuous. Clearly h is a bijection and its inverse is defined by $h^{-1}(x_j) = (x_i)$ with $x_i = z_i$ for $i \neq j$. Then $q_i \circ h^{-1}(x_j) = z_i$ for $i \neq j$, i.e. $q_i \circ h^{-1}$ is a constant mapping for $i \neq j$. For $i = j$, we have $q_j \circ h^{-1}(x_j) = x_j$, i.e. it is the identity mapping. Hence all compositions $q_i \circ h^{-1}$ are uniformly continuous and therefore also h^{-1} is uniformly continuous. □

In $SL\text{-}UCS$, also final structures exist. They are, however, complicated and we will use only quotient spaces later. Let $(X, \Lambda) \in |SL\text{-}UCS|$ and let $f : X \longrightarrow X'$ be a surjective mapping. We define the following stratified L-uniform convergence structure Λ_f on X'. Let $\Phi' \in \mathcal{F}_L^s(X' \times X')$. Then

$$\Lambda_f(\Phi') = \bigvee\{\bigwedge_{k=1}^{m} \Lambda(\Phi_{k1}) \wedge ... \wedge \Lambda(\Phi_{kn_k}) \ : \ \bigwedge_{k=1}^{m} (f\times f)(\Phi_{k1}) \circ \cdots \circ (f\times f)(\Phi_{kn_k}) \leq \Phi'\}.$$

Lemma 3.2. *Let $(X, \Lambda) \in |SL\text{-}UCS|$ and let $f : X \longrightarrow X'$ be a surjective mapping. Then $(X', \Lambda_f) \in |SL\text{-}UCS|$ and for a further mapping $g : (X', \Lambda_f) \longrightarrow (Y, \Lambda_Y)$ we have that g is uniformly continuous if and only if $g \circ f$ is uniformly continuous.*

Proof. We first show, that $(X', \Lambda_f) \in |SL\text{-}UCS|$. The axioms (UC1) and (UC2) are easy. (UC3) follows from $((f \times f)(\Phi))^{-1} = (f \times f)(\Phi^{-1})$ and (UC3) for (X, Λ). (UC4) is again clear by construction and (UC5) follows as $\Theta \leq \Phi$ and $\Upsilon \leq \Psi$ implies $\Theta \circ \Upsilon \leq \Phi \circ \Psi$. It is furthermore clear that $f : (X, \Lambda) \longrightarrow (X', \Lambda_f)$ is uniformly continuous. Let now $g : (X', \Lambda_f) \longrightarrow (Y, \Lambda_Y)$ be a mapping such that $g \circ f$ is uniformly continuous. Then, for $\Phi' \in \mathcal{F}_L^s(X' \times X')$ we have

$$\begin{aligned}
\Lambda_f(\Phi') &= \bigvee\{\bigwedge_{k=1}^{m} \Lambda(\Phi_{k1}) \wedge ... \wedge \Lambda(\Phi_{kn_k}) \ : \\
&\qquad \bigwedge_{k=1}^{m} (f \times f)(\Phi_{k1}) \circ \cdots \circ (f \times f)(\Phi_{kn_k}) \leq \Phi'\} \\
&\leq \bigvee\{\bigwedge_{k=1}^{m} \Lambda_Y((g \times g)((f \times f)(\Phi_{k1}))) \wedge ... \wedge \Lambda_Y((g \times g)((f \times f)(\Phi_{kn_k}))) \ : \\
&\qquad \bigwedge_{k=1}^{m} (f \times f)(\Phi_{k1}) \circ \cdots \circ (f \times f)(\Phi_{kn_k}) \leq \Phi'\}.
\end{aligned}$$

With $\Psi_{kl} = (f\times f)(\Phi_{kl})$ then

$$\begin{aligned}\Lambda_f(\Phi') &\le \bigvee\{\bigwedge_{k=1}^{m}\Lambda_Y((g\times g)(\Psi_{k1}))\wedge ...\wedge\Lambda_Y((g\times g)(\Psi_{kn_k})) \ : \\ &\qquad \bigwedge_{k=1}^{m}\Psi_{k1}\circ\cdots\circ\Psi_{kn_k}\le\Phi'\} \\ &\le \bigvee\{\bigwedge_{k=1}^{m}\Lambda_Y((g\times g)(\Psi_{k1}))\wedge ...\wedge\Lambda_Y((g\times g)(\Psi_{kn_k})) \ : \\ &\qquad \bigwedge_{k=1}^{m}(g\times g)(\Psi_{k1})\circ\cdots\circ(g\times g)(\Psi_{kn_k})\le(g\times g)(\Phi')\} \\ &\le \Lambda_Y((g\times g)(\Phi').\end{aligned}$$

Therefore g is uniformly continuous. □

Hence, Λ_f is the final structure and (X',Λ_f) is the *quotient space* for the sink $f:(X,\Lambda)\longrightarrow X'$.

For $(X,\Lambda)\in|SL\text{-}UCS|$ we define the *stratified L-entourage filter* by $\mathcal{N}_\Lambda(a)=\bigwedge_{\Phi\in\mathcal{F}_L^s(X\times X)}(\Lambda(\Phi)\to\Phi(a))$, see [12]. We further define, for $\alpha\in L$, the *stratified α-level L-entourage filter* by $\mathcal{N}_\alpha(a)=\bigwedge_{\Lambda(\Phi)\ge\alpha}\Phi$, see [14].

Lemma 3.3. [12] *A mapping $f:(X,\Lambda)\longrightarrow(X',\Lambda')$ satisfies $\mathcal{N}_{\Lambda'}\le(f\times f)(\mathcal{N}_\Lambda)$ whenever it is uniformly continuous.*

In [12] we defined the *discrete stratified L-uniform convergence structure* on X, Λ_δ, by $\Lambda_\delta(\Phi)=\top$ if $\Phi\ge\bigwedge_{x\in A}[(x,x)]$ for some finite set $A\subseteq X$ and $\Lambda_\delta(\Phi)=\bot$ else. It is not difficult to see that in case that X is a finite set, then $\Lambda_\delta(\Phi)=\top$ if $\Phi\ge[\Delta_X]$ and $\Lambda_\delta(\Phi)=\bot$ else.

We further consider the following stratified L-uniform convergence structure, which we shall call the *strong discrete stratified L-uniform convergence structure*

$$\Lambda_\delta^s(\Phi)=\bigwedge_{a\in L^{X\times X}}([\Delta_X](a)\to\Phi(a)).$$

Whenever $X=\{0,1\}$, then we denote $[\Delta]=[\Delta_{\{0,1\}}]$ for simplicity.

A pair $(X,\mathcal{U})$ of a non-void set X and a stratified L-filter $\mathcal{U}\in\mathcal{F}_L^s(X\times X)$ is called a *stratified L-uniform space* [6, 7] if $\mathcal{U}$ satisfies the following axioms (LU1) $\mathcal{U}\le[\Delta_X]$, (LU2) $\mathcal{U}\le\mathcal{U}^{-1}$ and (LU3) $\mathcal{U}\le\mathcal{U}\circ\mathcal{U}$. A mapping $f:(X,\mathcal{U})\longrightarrow(X',\mathcal{U}')$ is called *uniformly continuous* if $\mathcal{U}'\le(f\times f)(\mathcal{U})$. The category $SL\text{-}UNIF$ has as objects the stratified L-uniform spaces and as morphisms the uniformly continuous mappings. This category can be embedded into $SL\text{-}UCS$ by defining, for $(X,\mathcal{U})\in|SL\text{-}UNIF|$, the stratified L-uniform convergence structure $\Lambda_\mathcal{U}$ by $\Lambda_\mathcal{U}(\Phi)=\bigwedge_{a\in L^{X\times X}}(\mathcal{U}(a)\to\Phi(a))$. Then a mapping $f:(X,\mathcal{U})\longrightarrow(X',\mathcal{U}')$ is uniformly continuous if and only if $f:(X,\Lambda_\mathcal{U})\longrightarrow(X',\Lambda_{\mathcal{U}'})$ is uniformly continuous. $SL\text{-}UNIF$ is then isomorphic to a reflective subcategory of $SL\text{-}UCS$, see [3]. We define $\mathcal{U}_\alpha=\bigwedge_{\Lambda_\mathcal{U}(\Phi)\ge\alpha}\Phi$. Then $\Lambda_\mathcal{U}(\mathcal{U}_\alpha)\ge\alpha$, cf. [14].

A pair $(X,\lim)$ of a non-void set X and a mapping $\lim:\mathcal{F}_L^s(X)\longrightarrow L^X$ is called a *stratified L-limit space*, if the axioms (LC1) $\lim[x](x)=\top$; (LC2) $\lim\mathcal{F}\le\lim\mathcal{G}$

whenever $\mathcal{F} \leq \mathcal{G}$ and (LC3) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X) \ : \ \lim \mathcal{F} \wedge \lim \mathcal{G} \leq \lim \mathcal{F} \wedge \mathcal{G}$ are satisfied, [10]. A mapping $f : X \longrightarrow X'$ between the stratified L-limit spaces $(X, \lim), (X', \lim')$ is called *continuous* if and only if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have $\lim \mathcal{F}(x) \leq \lim' f(\mathcal{F})(f(x))$. The category of all stratified L-limit spaces with the continuous mappings as morphisms is denoted by $SL\text{-}LIM$. The category $SL\text{-}LIM$ is topological and Cartesian closed [11].

In [13] we defined the following two *separation axioms* in $SL\text{-}LIM$. We call $(X, \lim) \in |SL\text{-}LIM|$ a *T1-space* if for all $x, y \in X$, $x = y$ whenever $\lim[y](x) = \top$ and we call $(X, \lim)$ a *T2-space* if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$, $x = y$ whenever $\lim \mathcal{F}(x) = \lim \mathcal{F}(y) = \top$.

Let $(X, \Lambda) \in |SL\text{-}UCS|$. Then $(X, \lim(\Lambda)) \in |SL\text{-}LIM|$, where the limit map $\lim(\Lambda) : \mathcal{F}_L^s(X) \longrightarrow L^X$ is defined by $\lim(\Lambda)\mathcal{F}(x) = \Lambda(\mathcal{F} \times [x])$, see [12]. Furthermore, if $f : (X, \Lambda) \longrightarrow (X', \Lambda')$ is uniformly continuous then $f : (X, \lim(\Lambda)) \longrightarrow (X', \lim(\Lambda'))$ is continuous. Hence we can define a functor $H : SL\text{-}UCS \longrightarrow SL\text{-}LIM$. This functor preserves initial constructions.

Lemma 3.4. [12] *Let $(f_i : X \longrightarrow (X_i, \Lambda_i))_{i \in I}$ be a source in $SL\text{-}UCS$ and let Λ be the initial $SL\text{-}UCS$ structure on X. Then $\lim(\Lambda)$ is the initial $SL\text{-}LIM$ structure with respect to the source $(f_i : X \longrightarrow (X_i, \lim(\Lambda_i)))_{i \in I}$.*

In particular, for subspaces $(A, \Lambda|_A)$ of (X, Λ) we have $\lim(\Lambda|_A) = \lim(\Lambda)|_A$ and for product spaces $(\prod_{i \in J} X_i, \pi - \Lambda)$ we have $\lim(\pi - \Lambda) = \pi - \lim(\Lambda_i)$.

For a stratified L-uniform space $(X, \mathcal{U})$ and $x \in X$ we define the *stratified L-neighbourhood filter of x*, $\mathcal{N}_\mathcal{U}^x \in \mathcal{F}_L^s(X)$, by $\mathcal{N}_\mathcal{U}^x = \mathcal{U}(x)$ [6, 7] and with this the limit map $\lim(\mathcal{U})\mathcal{F}(x) = \bigwedge_{a \in L^X} (\mathcal{N}_\mathcal{U}^x(a) \to \mathcal{F}(a))$. Then $(X, \lim_\mathcal{U}) \in |SL\text{-}LIM|$ and, moreover, $\lim(\mathcal{U}) = \lim(\Lambda_\mathcal{U})$, see [3, 12].

We further call $(X, \Lambda) \in |SL\text{-}UCS|$ a T1-space (resp. a T2-space) if $(X, \lim(\Lambda))$ is a T1-space (resp. is a T2-space). It was shown in [16] that if L is a complete Boolean algebra, then (X, Λ) is a T2-space if and only if it is a T1-space.

In [17] we defined, for $(X, \lim) \in |SL\text{-}LIM|$, the *$\top$-closure of* $A \subseteq X$, $\overline{A}^{\lim} = \overline{A}$, by $x \in \overline{A}$ if there is $\mathcal{F} \in \mathcal{F}_L^s(X)$ such that $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_A) = \top$. In [15] a subset $A \subseteq X$ is called *$\top$-closed* if for $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\lim \mathcal{F}(x) = \top$ and $\mathcal{F}(\top_A) = \top$ implies $x \in A$. It is then not difficult to show that A is $\top$-closed if and only if $\overline{A} \subseteq A$. It was shown in [15] that in a T2-space, one-point sets $\{x\}$ are $\top$-closed. Hence, for a complete Boolean algebra L, in T1-spaces (X, Λ), the one-point sets are $\top$-closed.

Proposition 3.5. [17] *Let $(X, \lim^X), (Y, \lim^Y) \in |SL\text{-}LIM|$ and let $A \subseteq M \subseteq X$, $B \subseteq Y$ and let $f : X \longrightarrow Y$ be continuous.*
(1) $\overline{A}^M = \overline{A} \cap M$, where $\overline{A}^M$ is the $\top$-closure of A in the subspace $(M, \lim|_M)$.
(2) If $\lim \leq \lim'$, then $\overline{A}^{\lim'} \subseteq \overline{A}^{\lim}$.
(3) If B is $\top$-closed, then $f^{\leftarrow}(B)$ is $\top$-closed.

Proposition 3.6. [17] *Let $(X_i, \lim_i) \in |SL\text{-}LIM|$ for all $i \in j$ and let $(x_i) \in \prod_{i \in J} X_i$ be fixed. Define*

$$A = A((x_i)) = \{(y_i) \in \prod_{i \in J} X_i \ : \ x_j \neq y_j \text{ for at most finitely many } j \in J\}.$$

Then $\overline{A}^{\pi-\lim} = \prod_{i\in J} X_i$.

Let $\mathbb{E}$ be a class of stratified L-limit spaces. A space $(X, \lim) \in |SL\text{-}LIM|$ is called $\mathbb{E}$-*connected* [17] if, for any $(E, \lim_E) \in \mathbb{E}$, a continuous mapping $f : X \longrightarrow E$ is constant. A subset $A \subseteq X$ is called $\mathbb{E}$-connected if the subspace $(A, \lim|_A)$ is $\mathbb{E}$-connected.

Proposition 3.7. [17] *Let* $(X, \lim), (X', \lim'), (X_i, \lim_i) \in |SL\text{-}LIM|$, *(*$i \in J$*). Then*
(1) If $\mathbb{E}$ *is a class of T2-spaces and* $A \subseteq X$ *is* $\mathbb{E}$*-connected, then so is* $\overline{A}$*;*
(2) If $A, A_i \subseteq X$ *(*$i \in J$*) are* $\mathbb{E}$*-connected and* $A \cap A_i \neq \emptyset$ *for all* $i \in J$*, then* $A \cup \bigcup_{i\in J} A_i$ *is* $\mathbb{E}$*-connected.*
(3) If $\mathbb{E}$ *is a class of T2-spaces and all* $A_i \subseteq X_i$ *are* $\mathbb{E}$*-connected, then so is* $\prod_{i\in J} A_i$ *(as a subset of the product space).*
(4) If $A \subseteq X$ *is* $\mathbb{E}$*-connected and* $f : X \longrightarrow X'$ *is uniformly continuous, then* $f(A)$ *is* $\mathbb{E}$*-connected.*

For $\mathcal{F} \in \mathcal{F}_L^s(X)$, a set $\mathbb{B}$ of subsets of X is called a δ-*base of* $\mathcal{F}$ [17] if for $\mathcal{F}(\top_U) \geq \delta$ there is $B \in \mathbb{B}$, $B \subseteq U$ such that $\mathcal{F}(\top_B) \geq \delta$. A space $(X, \lim) \in |SL\text{-}LIM|$ is called *locally* $\mathbb{E}$-*connected* [17] if for all $\alpha \in L$, if $\lim \mathcal{F}(x) \geq \alpha$, there is $\mathcal{G} \leq \mathcal{F} \wedge [x]$ with $\lim \mathcal{G}(x) \geq \alpha$ and with a δ-base of $\mathbb{E}$-connected sets, whenever $\bot < \delta \leq \alpha$.

4. Uniform $\mathbb{E}$-connectedness

Let $\mathbb{E}$ be a class of stratified L-uniform convergence spaces (E, Λ_E) which contains a space with at least two points.

Definition 4.1. A space $(X, \Lambda) \in |SL\text{-}UCS|$ is called *uniformly* $\mathbb{E}$-*connected* if, for any $(E, \Lambda_E) \in \mathbb{E}$, every uniformly continuous mapping $f : (X, \Lambda) \longrightarrow (E, \Lambda_E)$ is constant.

In particular, we call (X, Λ) *uniformly connected* if it is uniformly $\mathbb{E}$-connected for $\mathbb{E} = \{(\{0, 1\}, \Lambda_\delta)\}$ and *strongly uniformly connected* if it is uniformly $\mathbb{E}$-connected for $\mathbb{E} = \{(\{0, 1\}, \Lambda_\delta^s)\}$.

Clearly, a strongly uniformly connected space (X, Λ) is uniformly connected. The converse is not true in general, as the following example shows.

Example 4.2. Let $L = \{\bot, \alpha, \top\}$ with $\bot < \alpha < \top$. We show that $(\{0, 1\}, \Lambda_\delta^s)$ is uniformly connected. There are two non-constant mappings $f : \{0, 1\} \longrightarrow \{0, 1\}$, namely $f = id_{\{0,1\}}$ and $f = 1 - id_{\{0,1\}}$. We will show that both are not uniformly continuous as mappings $f : (\{0, 1\}), \Lambda_\delta^s) \longrightarrow (\{0, 1\}, \Lambda_\delta)$. For $f = id_{\{0,1\}}$, consider the stratified L-filter

$$\mathcal{F}^*(a) = \begin{cases} \top & \text{if } a = \top_{\{0,1\}} \\ \alpha & \text{if } a(0) = \top, a(1) \neq \top \\ \alpha & \text{if } a(0) = \alpha \\ \bot & \text{if } a(0) = \bot \end{cases},$$

see [11]. It was shown in [4] that $\Lambda_\delta^s(\mathcal{F}^* \times \mathcal{F}^*) \geq \bigwedge_{a\in L^{\{0,1\}}}([(0, 0)](a) \rightarrow (\mathcal{F}^* \times \mathcal{F}^*)(a)) \geq \alpha$. However, $\Lambda_\delta(\mathcal{F}^* \times \mathcal{F}^*) = \bot$, because $\mathcal{F}^* \times \mathcal{F}^* \not\geq [\Delta] = [(0, 0)] \wedge [(1, 1)]$.

This can be seen using $a(x,y) = \begin{cases} \top & \text{if } x = y \\ \alpha & \text{if } x \neq y \end{cases}$. Then $[(0,0)] \wedge [(1,1)](a) = \top$ but $(\mathcal{F}^* \times \mathcal{F}^*)(a) \leq \alpha$, see [4]. Hence $f = id_{\{0,1\}}$ is not uniformly continuous.

For $f = 1 - id_{\{0,1\}}$ we define, for $a \in L^{\{0,1\}}$, $a^* = f^{\leftarrow}(a)$ and with this $\mathcal{F}_* \in \mathcal{F}_L^s(\{0,1\})$ by $\mathcal{F}_*(a) = \mathcal{F}^*(a^*)$. Then $\Lambda_\delta^s(\mathcal{F}_* \times \mathcal{F}_*) \geq \alpha$ but $\Lambda_\delta((f \times f)(\mathcal{F}_* \times \mathcal{F}_*)) = \Lambda_\delta(\mathcal{F}^* \times \mathcal{F}^*) = \bot$. Hence $f = 1 - id_{\{0,1\}}$ is not uniformly continuous too and the only continuous mappings are the constant ones. Therefore $(\{0,1\}, \Lambda_\delta^s)$ is uniformly connected. As clearly the identity mapping $f = id_{\{0,1\}} : (\{0,1\}, \Lambda_\delta^s) \longrightarrow (\{0,1\}, \Lambda_\delta^s)$ is uniformly continuous, $(\{0,1\}, \Lambda_\delta^s)$ is not strongly uniformly connected.

For a class of stratified L-uniform convergence spaces, $\mathbb{E}$, we denote $L(\mathbb{E}) = \{(E, \lim(\Lambda_E)) \,:\, (E, \Lambda_E) \in \mathbb{E}\}$.

Lemma 4.3. *Let $(X,\Lambda) \in |SL\text{-}UCS|$. If $(X, \lim(\Lambda))$ is $L(\mathbb{E})$-connected, then (X,Λ) is uniformly $\mathbb{E}$-connected.*

Lemma 4.4. *Let $\mathbb{E}$ be a class of stratified L-uniform convergence spaces which contains a space $(E, \lim_E)$ with $|E| \geq 2$. If (X,Λ) is uniformly $\mathbb{E}$-connected, then it is uniformly connected.*

Proof. Let $f : (X,\Lambda) \longrightarrow (\{0,1\}, \Lambda_\delta)$ be uniformly continuous and let $(E, \Lambda_E) \in \mathbb{E}$ with $x, y \in E$, $x \neq y$. We define $h : \{0,1\} \longrightarrow E$ by $h(0) = x$ and $h(1) = y$. We show that h is uniformly continuous. Let $\Lambda_\delta(\Phi) = \top$. Then $\Phi \geq [\Delta]$ and hence $(h \times h)(\Phi) \geq (h \times h)[\Delta]$. For $a \in L^{E \times E}$ we then have $(h \times h)([\Delta])(a) = [\Delta]((h \times h)^{\leftarrow}(a)) = (h \times h)^{\leftarrow}(a)(0,0) \wedge (h \times h)^{\leftarrow}(a)(1,1) = a(h(0),h(0)) \wedge a(h(1),h(1)) = a(x,x) \wedge a(y,y) = [(x,x)](a) \wedge [(y,y)](a)$. Hence $(h \times h)(\Phi) \geq [(x,x)] \wedge [(y,y)]$ and we conclude $\Lambda_E((h \times h)(\Phi)) \geq \Lambda_E([(x,x)]) \wedge \Lambda_E([(y,y)]) = \top$. Consequently h is uniformly continuous and therefore $h \circ f$ is also uniformly continuous and hence constant. As h is not constant, then f must be so. □

Uniform $\mathbb{E}$-connectedness often also entails strong uniform connectedness. However, we need a stronger assumption on the class $\mathbb{E}$.

Lemma 4.5. *Let $\mathbb{E}$ be a class of stratified L-uniform convergence spaces which contains a space $(E, \lim_E)$ with $|E| \geq 2$ and $\Lambda_E \leq \Lambda_{\delta,E}^s$. If (X,Λ) is uniformly $\mathbb{E}$-connected, then it is strongly uniformly connected.*

Proof. Let $f : (X,\Lambda) \longrightarrow (\{0,1\}, \Lambda_\delta^s)$ be uniformly continuous and let $(E, \Lambda_E) \in \mathbb{E}$ with $x, y \in E$, $x \neq y$. Again we define $h : \{0,1\} \longrightarrow E$ by $h(0) = x$ and $h(1) = y$. We show that h is $(\Lambda_\delta^s, \Lambda_E)$-uniformly continuous. Then $\Lambda_E((h \times h)(\Phi)) \geq \lambda_{\delta,E}^s((h \times h)(\Phi)) = \bigwedge_{a \in L^{E \times E}} ([\Delta_E](a) \rightarrow (h \times h)(\Phi)(a))$. For $a \in L^{E \times E}$ we have $[\Delta_E](a) \leq [(x,x)] \wedge [(y,y)](a) = a(x,x) \wedge a(y,y) = (h \times h)^{\leftarrow}(a)(0,0) \wedge (h \times h)^{\leftarrow}(a)(1,1) = [(0,0)] \wedge [(1,1)]((h \times h)^{\leftarrow}(a)) = [\Delta]((h \times h)^{\leftarrow}(a))$. Hence $\bigwedge_{a \in L^{E \times E}} ([\Delta_E](a) \rightarrow (h \times h)(\Phi)(a)) \geq \bigwedge_{a \in L^{E \times E}} ([\Delta]((h \times h)^{\leftarrow}(a) \rightarrow \Phi((h \times h)^{\leftarrow}(a))) \geq \bigwedge_{b \in L^{\{0,1\} \times \{0,1\}}} ([\Delta](b) \rightarrow \Phi(b)) = \Lambda_\delta^s(\Phi)$. Hence, together with h, also $h \circ f$ is uniformly continuous and therefore constant. As h is not constant, then f must be so. □

Strong uniform connectedness can be characterized by a "chaining condition".

Theorem 4.6. *A space* $(X,\Lambda) \in |SL\text{-}UCS|$ *is strongly uniformly connected if and only if for all* $x, y \in X$ *and all* $N \subseteq X \times X$ *with* $\mathcal{N}_\Lambda(\top_N) = \top$ *there is a natural number* n *such that* $(x, y) \in N^n$.

Proof. Let first (X,Λ) be strongly uniformly connected and assume that there is $(p,q) \in X \times X$ and $N \subseteq X \times X$ with $\mathcal{N}_\Lambda(\top_N) = \top$ but $(p,q) \notin N^n$ for all natural numbers n. We define $A = \{x \in X \ : \ (p,x) \in N^n$ for some natural number $n\}$ and $B = X \setminus A$. As $\top = \mathcal{N}_\Lambda(\top_N) \leq [(p,p)](\top_N)$ we see that $(p,p) \in N$ and hence A is non-empty. Clearly $q \notin A$, i.e. B is non-empty. We define the mapping $f : X \longrightarrow \{0,1\}$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. For $(x,y) \in N$ then, if $x \in A$ also $y \in A$ and if $x \in B$ then also $y \in B$. Hence $N \subseteq (A \times A) \cup (B \times B)$ and, because $\top = \mathcal{N}_\Lambda(\top_N) \leq \mathcal{N}_\Lambda(\top_{(A\times A)\cup(B\times B)})$, we conclude $\Lambda(\Phi) \leq \Phi(\top_N) \leq \Phi(\top_{(A\times A)\cup(B\times B)})$ for all $\Phi \in \mathcal{F}^s_L(X \times X)$. Furthermore, for $a \in L^{\{0,1\}\times\{0,1\}}$,

$$(f \times f)^{\leftarrow}(a) \wedge \top_{(A\times A)\cup(B\times B)}(x,y) = \begin{cases} a(0,0) & \text{if } (x,y) \in A \times A \\ a(1,1) & \text{if } (x,y) \in B \times B \\ \bot & \text{else} \end{cases} .$$

Hence $(f \times f)^{\leftarrow}(a) \wedge \top_{(A\times A)\cup(B\times B)} \geq [\Delta](a) \wedge \top_N$ and therefore, by stratification, $(f \times f)(\Phi)(a) \geq [\Delta](a) \wedge \Phi(\top_N) \geq [\Delta](a) \wedge \Lambda(\Phi)$. As $a \in L^{\{0,1\}\times\{0,1\}}$ was arbitrary, we conclude $\Lambda(\Phi) \leq \bigwedge_{a \in L^{\{0,1\}\times\{0,1\}}}([\Delta](a) \to (f \times f)(\Phi)(a)) = \Lambda^s_\delta((f \times f)(\Phi))$. Hence, f is uniformly continuous and not constant, a contradiction.

Let now $x \neq y$ and let $f : (X,\Lambda) \longrightarrow (\{0,1\}, \Lambda^s_\delta)$ be uniformly continuous. Then $[\Delta] = \mathcal{N}_{\Lambda^s_\delta} \leq (f \times f)(\mathcal{N}_\Lambda)$. Therefore, $\top = [\Delta](\top_\Delta) \leq \mathcal{N}_\Lambda(\top_{(f\times f)^{\leftarrow}(\Delta)})$ and there is a natural number, n, such that $(x,y) \in ((f \times f)^{\leftarrow}(\Delta))^n$, i.e. there are $x = x_0, x_1, ..., x_n = y$ such that $(x_k, x_{k+1}) \in (f \times f)^{\leftarrow}(\Delta)$ for $k = 0,1,2,...,n-1$. This means that $(f(x_k), f(x_{k+1})) \in \Delta$, i.e. $f(x_k) = f(x_{k+1})$ for $k = 0,1,2,...,n-1$. Hence $f(x) = f(y)$ and f is constant. □

For a class $\mathbb{E}$ of stratified L-uniform spaces, we call $(X,\mathcal{U}) \in |SL\text{-}UNIF|$ *uniformly* $\mathbb{E}$*-connected* if, for any $(E,\mathcal{U}_E) \in \mathbb{E}$, a uniformly continuous mapping $f : (X,\mathcal{U}) \longrightarrow (E,\mathcal{U}_E)$ is constant. If we denote $\Lambda(\mathbb{E}) = \{(E, \Lambda_{\mathcal{U}_E})) \ : \ (E,\mathcal{U}_E) \in \mathbb{E}\}$, then a stratified L-uniform space $(X,\mathcal{U})$ is uniformly $\mathbb{E}$-connected if and only if $(X, \Lambda_\mathcal{U}))$ is uniformly $\Lambda(\mathbb{E})$-connected. For $\mathbb{E} = \{(\{0,1\}, [\Delta])\}$, we call a uniformly $\mathbb{E}$-connected stratified L-uniform space *uniformly connected*. Hence $(X,\mathcal{U}) \in |SL\text{-}UNIF|$ is uniformly connected if and only if $(X, \Lambda_\mathcal{U})$ is strongly uniformly connected. We obtain as a direct consequence of Theorem 4.6 the following characterization.

Theorem 4.7. *A space* $(X,\mathcal{U}) \in |SL\text{-}UNIF|$ *is uniformly connected if and only if for all* $x, y \in X$ *and all* $N \subseteq X \times X$ *with* $\mathcal{U}(\top_N) = \top$ *there is a natural number* n *such that* $(x,y) \in N^n$.

For $L = \{0,1\}$, a uniform space that satisfies the condition of the above theorem is called *well-chained* [22].

5. Properties of Uniformly $\mathbb{E}$-connected Subsets

In the sequel, let $\mathbb{E}$ be a class of stratified L-uniform convergence spaces which contains a space (E, Λ^E) with at least two points. We call $A \subseteq X$, where $(X,\Lambda) \in$

$|SL\text{-}UCS|$, *uniformly* $\mathbb{E}$*-connected (in* (X,Λ)*)* if the subspace $(A,\Lambda|_A)$ is uniformly $\mathbb{E}$-connected. Uniform $\mathbb{E}$-connectedness of $A \subseteq X$ then becomes an *absolute property*, i.e. for $A \subseteq B \subseteq X$ we have that A is uniformly $\mathbb{E}$-connected in $(B,\Lambda|_B)$ iff A is uniformly $\mathbb{E}$-connected in (X,Λ).

Lemma 5.1. *Let* $(X,\Lambda^X),(Y,\Lambda^Y) \in |SL\text{-}UCS|$ *and let* $f : (X,\Lambda^X) \longrightarrow (Y,\Lambda^Y)$ *be uniformly continuous. If* $A \subseteq X$ *is uniformly* $\mathbb{E}$*-connected, then* $B = f(A)$ *is uniformly* $\mathbb{E}$*-connected.*

Proof. For $\Phi \in \mathcal{F}^s_L(A \times A)$ we have $\Lambda^X|_A(\Phi) = \Lambda^X((i_A \times i_A)(\Phi)) \leq \Lambda^Y((f\times f)\circ(i_A \times i_A)(\Phi))$. As $(f\times f)\circ(i_A\times i_A) = (i_B \times i_B)\circ(f\times f)$ we obtain $(f\times f)\circ(i_A\times i_A)(\Phi) = (i_B\times i_B)\circ(f\times f)(\Phi)$, and therefore $\Lambda^X|_A(\Phi) \leq \Lambda^Y|_B((f\times f)(\Phi))$. Hence, we may assume $A = X$, $B = Y = f(X)$ and $f : X \longrightarrow Y$ surjective. Let now $(E,\Lambda^E) \in \mathbb{E}$ and $h : (Y,\Lambda^Y) \longrightarrow (E,\Lambda^E)$ be uniformly continuous. Then $h \circ f : (X,\Lambda^X) \longrightarrow (E,\Lambda^E)$ is uniformly continuous and hence constant. As f is surjective, then also h must be constant. □

Lemma 5.2. *Let* $\mathbb{E}$ *be a class of T2-spaces,* $(X,\Lambda) \in |SL\text{-}UCS|$ *and let* $A \subseteq X$ *be uniformly* $\mathbb{E}$*-connected. Then also* $\overline{A} = \overline{A}^{\lim(\Lambda)}$ *is uniformly* $\mathbb{E}$*-connected.*

Proof. Let $(E,\Lambda^E) \in \mathbb{E}$ and $f : (\overline{A},\Lambda|_{\overline{A}}) \longrightarrow (E,\Lambda^E)$ be uniformly continuous. Then also $f|_A : (A,\Lambda|_A) \longrightarrow (E,\Lambda^E)$ is uniformly continuous and hence constant, i.e. $f|_A(A) = f(A) = \{e\}$ with some $e \in E$. As $(E,\lim(\Lambda^E))$ is a T2-space, $\{e\}$ is $\top$-closed and hence $M = f^{\leftarrow}(\{e\})$ is $\top$-closed in $(\overline{A},\lim(\Lambda)|_{\overline{A}}) = (\overline{A},\lim(\Lambda|_{\overline{A}}))$. We note that $A \subseteq M \subseteq \overline{A}$. Hence $\overline{A} = \overline{M\cap A} \subseteq \overline{M}\cap\overline{A} = \overline{M}^{\lim(\Lambda)|_{\overline{A}}} \subseteq M$, i.e. $M = \overline{A}$. Therefore $f(\overline{A}) = f(M) = \{e\}$ and f is constant. □

Lemma 5.3. *Let* $(X,\Lambda) \in |SL\text{-}UCS|$ *and let* $A_i, A \subseteq X$ *be uniformly* $\mathbb{E}$*-connected* $(i \in I)$ *with* $A \cap A_i \neq \emptyset$ *for all* $i \in I$*. Then* $A \cup \bigcup_{i\in I} A_i$ *is uniformly* $\mathbb{E}$*-connected.*

Proof. Let $(E,\lim^E) \in \mathbb{E}$ and let $f : A \cup \bigcup_{i\in I} A_i \longrightarrow E$ be uniformly continuous. Then all restrictions $f|_A : A \longrightarrow E$ and $f|_{A_i} : A_i \longrightarrow E$ are uniformly continuous and hence constant. As $A \cap A_i \neq \emptyset$ for all $i \in I$, all function values must be the same. □

Lemma 5.3 allows the definition of maximal uniformly $\mathbb{E}$-connected subsets of X.

Definition 5.4. Let $(X,\Lambda) \in |SL\text{-}UCS|$ and $C \subseteq X$ be uniformly $\mathbb{E}$-connected. C is called a *uniform* $\mathbb{E}$*-component* of X if $C = B$ whenever $C \subseteq B \subseteq X$ and B is uniformly $\mathbb{E}$-connected.

It follows immediately from Lemma 5.3 that the uniform $\mathbb{E}$-components form a partition of X.

Lemma 5.5. *Let* $\mathbb{E}$ *be a class of T2-spaces and let* $(X,\Lambda) \in |SL\text{-}UCS|$*. If* C *is a uniform* $\mathbb{E}$*-component of* X*, then* C *is* $\top$*-closed.*

Proof. With C also $\overline{C}$ is uniformly $\mathbb{E}$-connected. $C \subseteq \overline{C}$ and the maximality of C implies $\overline{C} = C$ and hence C is $\top$-closed. □

We finally state the important product theorem.

Theorem 5.6. *Let $\mathbb{E}$ be a class of T2-spaces and let $(X_i, \Lambda_i)_{i\in J}$ be a family in $|SL\text{-}UCS|$. Then the product space $(\prod_{i\in J} X_i, \pi\text{-}\Lambda)$ is uniformly $\mathbb{E}$-connected if and only if all (X_i, Λ_i) are uniformly $\mathbb{E}$-connected.*

Proof. Using Lemma 3.1, Lemma 5.2 and Proposition 3.7, the proof of Theorem 5.8 in [17] can be copied word-by-word. □

6. Uniform Local $\mathbb{E}$-connectedness

In the sequel, let $\mathbb{E}$ be a class of stratified L-limit spaces. For $\delta \in L$, a set of subsets $\mathbb{B} \subseteq P(X \times X)$ is called a *δ-base* of $\Phi \in \mathcal{F}_L^s(X \times X)$ if for all $U \subseteq X \times X$ with $\Phi(\top_U) \geq \delta$ there is $B \in \mathbb{B}$ such that $B \subseteq U$ and $\Phi(\top_B) \geq \delta$. For a subset $B \subseteq X \times X$ and $x \in X$ we denote $B(x) = \{y \in X \ : \ (y, x) \in B\}$. It is not difficult to see that then $\top_B(\cdot, x) = \top_{B(x)}$.

Definition 6.1. We call $(X, \Lambda) \in |SL\text{-}UCS|$ *uniformly locally $\mathbb{E}$-connected* if for all $\alpha \in L$, for all $\Phi \in \mathcal{F}_L^s(X\times X)$ with $\Lambda(\Phi) \geq \alpha$ there is $\Psi \in \mathcal{F}_L^s(X\times X)$, $\Psi \leq \Phi \wedge [\Delta]$, $\Lambda(\Psi) \geq \alpha$ with a δ-base $\mathbb{B}$ such that for all $x \in X$ the sets $B(x)$ with $B \in \mathbb{B}$ are $\mathbb{E}$-connected (in $(X, \lim(\Lambda))$), whenever $\bot < \delta \leq \alpha$.

For $L = \{0, 1\}$ this definition is slightly stronger than the definition of uniform local connectedness in Vanio [24]. In [24] it is only demanded that $\Psi \leq \Phi$. Our stronger requirement $\Psi \leq \Phi \wedge [\Delta]$ comes in handy lateron.

A stratified L-uniform space $(X,\mathcal{U})$ is called *uniformly locally $\mathbb{E}$-connected* if $(X, \Lambda_{\mathcal{U}})$ is uniformly locally $\mathbb{E}$-connected.

Proposition 6.2. *Let $(X,\mathcal{U}) \in |SL\text{-}UNIF|$. Then $(X,\mathcal{U})$ is uniformly locally $\mathbb{E}$-connected if and only if for all $\alpha \in L$, $\mathcal{U}_\alpha$ has a δ-base $\mathbb{B}$ such that the sets $B(x)$ with $B \in \mathbb{B}$ are $\mathbb{E}$-connected for all $x \in X$, whenever $\bot < \delta \leq \alpha$.*

Proof. Let first $(X,\mathcal{U})$ be uniformly locally $\mathbb{E}$-connected. Then $\Lambda_{\mathcal{U}}(\mathcal{U}_\alpha) \geq \alpha$. Hence there is $\Psi \leq \mathcal{U}_\alpha \wedge [\Delta] \leq \mathcal{U}_\alpha$ with $\Lambda_{\mathcal{U}}(\Psi) \geq \alpha$ and a δ-base $\mathbb{B}$ such that the sets $B(x)$ with $B \in \mathbb{B}$ are $\mathbb{E}$-connected for all $x \in X$ whenever $\bot < \delta \leq \alpha$. From $\Lambda(\Psi) \geq \alpha$ we conclude that $\Psi \geq \mathcal{U}_\alpha$ and hence $\Psi = \mathcal{U}_\alpha$ has a δ-base as desired whenever $\bot < \delta \leq \alpha$.

For the converse, let $\Lambda_{\mathcal{U}}(\Phi) \geq \alpha$. Then $\Phi \geq \mathcal{U}_\alpha$ and as always $\mathcal{U}_\alpha \leq [\Delta]$, we have $\mathcal{U}_\alpha \leq \Phi \wedge [\Delta]$. As $\Lambda_{\mathcal{U}}(\mathcal{U}_\alpha) \geq \alpha$ the claim follows if we choose $\Psi = \mathcal{U}_\alpha$. □

Proposition 6.3. *If $(X, \Lambda) \in |SL\text{-}UCS|$ is uniformly locally $\mathbb{E}$-connected, then $(X, \lim(\Lambda))$ is locally $\mathbb{E}$-connected.*

Proof. Let $\alpha \in L$, $\mathcal{F} \in \mathcal{F}_L^s(X)$ and let $x \in X$ such that $\lim(\Lambda)\mathcal{F}(x) \geq \alpha$. Then $\Lambda(\mathcal{F} \times [x]) \geq \alpha$. Hence there is $\Psi \in \mathcal{F}_L^s(X \times X)$ such that $\Psi \leq (\mathcal{F} \times [x]) \wedge [\Delta]$, $\Lambda(\Psi) \geq \alpha$ and, if $\bot < \delta \leq \alpha$, Ψ has a δ-base $\mathbb{B}$ with $B(x)$ $\mathbb{E}$-connected for all $x \in X$ and all $B \in \mathbb{B}$. Then $\Psi(x) \in \mathcal{F}_L^s(X)$. From Lemma 2.5 we conclude that $\Psi(x) \leq \mathcal{F} \wedge [x]$. We show that $\Psi(x)$ has a δ-base of $\mathbb{E}$-connected sets. If $U \subseteq X$ such that $\Psi(x)(\top_U) \geq \delta$, then $\Psi(T_{U\times\{x\}}) = (\Psi(x) \times [x])(\top_U \times \top_{\{x\}}) \geq \Psi(x)(\top_U) \wedge [x](\top_{\{x\}}) \geq \delta$. Hence there is $B \in \mathbb{B}$, $B \subseteq U \times \{x\}$ such that $\Psi(\top_B) \geq \delta$.

Clearly $B(x) \subseteq U$ and $\Psi(x)(\top_{B(x)}) \geq \Psi(\top_B) \geq \delta$ because $\top_B(\cdot, x) = \top_{B(x)}$. Therefore $\mathbb{B}(x) = \{B(x) \,:\, B \in \mathbb{B}\}$ is the required δ-base for $\Psi(x)$. □

Proposition 6.4. *Let $(X, \Lambda), (X', \Lambda') \in |SL\text{-}UCS|$ and let $f : (X, \Lambda) \longrightarrow (X', \Lambda')$ be a uniform isomorphism (i.e. f is bijective and both f and f^{-1} are uniformly continuous). If (X, Λ) is uniformly locally $\mathbb{E}$-connected, then so is (X', Λ').*

Proof. Let $\alpha \in L$ and $\Phi' \in \mathcal{F}_L^s(X' \times X')$ and $\Lambda'(\Phi') \geq \alpha$. Then, by uniform continuity of f^{-1}, $\Lambda((f^{-1} \times f^{-1})(\Phi')) \geq \alpha$. Hence there is $\Psi \leq (f^{-1} \times f^{-1})(\Phi') \wedge [\Delta_X]$ with $\Lambda(\Psi) \geq \alpha$ which has, for $\perp < \delta \leq \alpha$, a δ-base $\mathbb{B}$ such that for all $x \in X$ and all $B \in \mathbb{B}$, $B(x)$ is $\mathbb{E}$-connected. By uniform continuity of f, then $\Lambda'((f \times f)(\Psi)) \geq \alpha$ and $(f \times f)(\Psi) \leq (f \times f)((f^{-1} \times f^{-1})(\Phi)) \wedge [(f \times f)(\Delta_X)] = \Phi \wedge [\Delta_{X'}]$. We show that $(f \times f)(\Psi)$ has a δ-base $\mathbb{B}'$ with $B'(x')$ $\mathbb{E}$-connected for all $x' \in X'$ and all $B' \in \mathbb{B}'$. Let $(f \times f)(\Psi)(\top_U) \geq \delta$. Then $\Psi(\top_{(f^{-1} \times f^{-1})(U)}) \geq \delta$ and hence there is $B \subseteq (f^{-1} \times f^{-1})(U)$ with $\Psi(\top_B) \geq \delta$, $B(x)$ $\mathbb{E}$-connected for all $x \in X$. It follows that $B' = (f \times f)(B) \subseteq U$ and $(f \times f)(\Psi)(\top_{(f \times f)(B)}) \geq \Psi(\top_B) \geq \delta$. For $x' \in X'$ we have that $(f \times f)(B)(x') = f(B(f^{-1}(x')))$ is $\mathbb{E}$-connected, as f is continuous as a mapping from $(X, \lim(\Lambda))$ to $(X', \lim(\Lambda'))$ and $B(f^{-1}(x'))$ is $\mathbb{E}$-connected. □

We now look at the behaviour of uniform local $\mathbb{E}$-connectedness with respect to quotient spaces and product spaces. First we need two lemmas.

Lemma 6.5. *Let $(X, \lim) \in |SL\text{-}LIM|$ and let $A, B \subseteq X \times X$ with $\Delta_X \subseteq A$. If $B(x)$ and $A(z)$ are $\mathbb{E}$-connected for all $z \in X$, then $(A \circ B)(x)$ is $\mathbb{E}$-connected.*

Proof. This proof goes back to Vainio [24]. It is not difficult to show that $(A \circ B)(x) = \bigcup_{z \in B(x)} A(z)$. As $\Delta_X \subseteq A$, we moreover conlcude $B(x) \subseteq (A \circ B)(x)$ and hence $(A \circ B)(x) = \bigcup_{z \in B(x)} (A(z) \cup B(x))$. Again, as $\Delta_X \subseteq A$, we conclude that $A(z) \cap B(x) \neq \emptyset$ and hence $A(z) \cup B(x)$ is $\mathbb{E}$-connected for all $z \in B(x)$. Consequently also $(A \circ B)(x) = \bigcup_{z \in B(x)} A(z)$ is $\mathbb{E}$-connected. □

Lemma 6.6. *Let $B \subseteq X \times X$, $x \in X$ and let $f : X \longrightarrow Y$ be a mapping. Then $(f \times f)(B)(f(x)) = \bigcup_{z : f(z) = f(x)} f(B(z))$. Moreover, if $\Delta_X \subseteq B$, then $f(x) \in f(B(z))$ whenever $f(z) = f(x)$.*

Proof. Let first $y \in f(B(z))$ and $f(z) = f(x)$. Then there is $b \in X$ such that $(b, z) \in B$ and $f(b) = y$. Hence $(y, f(x)) = (f(b), f(z)) \in (f \times f)(B)$, i.e. $y \in (f \times f)(B)(f(x))$. Conversely, let $y \in (f \times f)(B)(f(x))$. Then $(y, f(x)) \in (f \times f)(B)$. Hence there is $(a, b) \in B$ such that $f(a) = y$ and $f(b) = f(x)$. We conclude $a \in B(b)$ and, consequently, $y = f(a) \in f(B(b))$. From $f(b) = f(x)$ we conclude $y \in \bigcup_{z : f(z) = f(x)} f(B(z))$. □

Theorem 6.7. *Let the lattice L be completely distributive and let $\perp \in L$ be prime. Let $(X, \Lambda) \in |SL\text{-}UCS|$ be uniformly locally $\mathbb{E}$-connected and let $f : X \longrightarrow X'$ be surjective. Then the quotient space (X', Λ_f) is uniformly locally $\mathbb{E}$-connected.*

Proof. Let $\alpha \in L$ and let $\Lambda_f(\Phi') \geq \alpha$. Let $\beta \lhd \alpha$. Then there are $\Phi_{k1}^\beta, ..., \Phi_{kn_k}^\beta$ $(k = 1, 2, ..., m)$ with $\bigwedge_{k=1}^m (f \times f)(\Phi_{k1}^\beta) \circ \cdots \circ (f \times f)(\Phi_{kn_k}^\beta) \leq \Phi'$ such that $\bigwedge_{k=1}^m \Lambda(\Phi_{k1}^\beta) \wedge$

... $\wedge \Lambda(\Phi^\beta_{kn_k}) \geq \beta$. For each Φ^β_{kl} there is $\Psi^\beta_{kl} \leq \Phi^\beta_{kl} \wedge [\Delta_X]$ such that $\Lambda(\Psi^\beta_{kl}) \geq \beta$ and which has, for $\perp < \delta \leq \beta$, a δ-base $\mathbb{B}_{kl}$ such that $B(x)$ is $\mathbb{E}$-connected for each $x \in X$ and each $B \in \mathbb{B}_{kl}$. In particular, $(f \times f)(\Psi^\beta_{kl}) \leq (f \times f)([\Delta_X]) = [\Delta_{X'}]$, as f is surjective. We define $\Psi^\beta = \bigwedge_{k=1}^m (f \times f)(\Psi^\beta_{k1}) \circ \cdots \circ (f \times f)(\Psi^\beta_{kn_k})$. Then $\Psi^\beta \leq \Phi \wedge [\Delta_{X'}]$ and $\Lambda_f(\Psi^\beta) \geq \beta$, as f is uniformly continuous.

We show that Ψ^β also has, for $\perp < \delta \leq \alpha$, a δ-base $\mathbb{B}^\beta$ with $B(x')$ $\mathbb{E}$-connected for all $x' \in X'$ and all $B \in \mathbb{B}^\beta$. Let $\Psi(\top_B) \geq \delta$. Then $(f \times f)(\Psi^\beta_{kl})(\top_B) = \Psi^\beta_{kl}(\top_{(f\times f)^\leftarrow(B)}) \geq \delta$ for all $k = 1, ..., m$ and $l = 1, ..., n_k$. Hence there are sets $C^\beta_{kl} \subseteq (f \times f)^\leftarrow(B)$ with $\Psi^\beta_{kl}(\top_{C_{kl}}) \geq \delta$. From $[\Delta_X] \geq \Psi^\beta_{kl}$ we conclude that $\Delta_X \subseteq C^\beta_{kl}$ and, by the surjectivity of f, then $\Delta_{X'} \subseteq (f \times f)(C^\beta_{kl}) \subseteq B$. Hence $\delta \leq (f \times f)(\Psi^\beta_{k1}) \circ \cdots \circ (f \times f)(\Psi^\beta_{kn_k})(\top_{(f\times f)(C_{k1})} \circ \cdots \circ \top_{(f\times f)(C_{kn_k})}) = (f \times f)(\Psi^\beta_{k1}) \circ \cdots \circ (f \times f)(\Psi^\beta_{kn_k})(\top_{(f\times f)(C_{k1}) \circ \cdots \circ (f\times f)(C_{kn_k})})$. By Lemma 6.5 and Lemma 6.6, the sets $((f \times f)(C_{k1}) \circ \cdots \circ (f \times f)(C_{kn_k}))(x')$ are $\mathbb{E}$-connected for all $x' \in X'$ and, as all these sets contain $\Delta_{X'}$ as a subset, so are $D^\beta(x') = (\bigcup_{k=1}^m (f \times f)(C_{k1}) \circ \cdots \circ (f \times f)(C_{kn_k})))(x')$ and $\Psi^\beta(\top_{D^\beta}) \geq \delta$.

We define now $\Psi = \bigvee_{\beta \lhd \alpha} \Psi^\beta$. This stratified L-filter exists and is $\leq \Phi \wedge [\Delta_{X'}]$. Moreover, $\Lambda_f(\Psi) \geq \Lambda_f(\Psi^\beta) \geq \beta$ for all $\beta \lhd \alpha$, and hence $\Lambda_f(\Psi) \geq \alpha$. We show that for $\perp < \delta \leq \alpha$, Ψ has a δ-base $\mathbb{B}$ with $B(x')$ $\mathbb{E}$-connected for all $x' \in X'$ and all $B \in \mathbb{B}$. Let $\Psi(\top_B) \geq \delta \rhd \eta$. Then there are $\beta^\eta_1, ..., \beta^\eta_n \lhd \alpha$ and $B^\eta_1, ..., B^\eta_n \subseteq X' \times X'$ such that $B^\eta_1 \cap ... \cap B^\eta_n \subseteq B$ and $\Psi^{\beta^\eta_1}(\top_{B^\eta_1}) \wedge ... \wedge \Psi^{\beta^\eta_n}(\top_{B^\eta_n}) \geq \eta$. We have seen above that each $\Psi^{\beta^\eta_l}$ has a suitable η-base and hence there are $C^\eta_1 \subseteq B^\eta_1, ..., C^\eta_n \subseteq B^\eta_n$ such that $\Psi^{\beta^\eta_1}(\top_{C^\eta_1}) \geq \eta, ..., \Psi^{\beta^\eta_n}(\top_{C^\eta_n}) \geq \eta$ and $C^\eta_1(x'), ..., C^\eta_n(x')$ are $\mathbb{E}$-connected for all $x' \in X'$. Again, $\Delta_{X'} \subseteq C^\eta_1, ..., C^\eta_n$. We define $C_1 = \bigcup_{\eta \lhd \delta} C^\eta_1, ..., C_n = \bigcup_{\eta \lhd \delta} C^\eta_n$. Then, for $l = 1, ..., n$ we have $\Psi^{\beta^\eta_l}(\top_{C_l}) \geq \eta$ for all $\eta \lhd \delta$, i.e. $\Psi^{\beta^\eta_l}(\top_{C_l}) \geq \delta$ and $C_l(x')$ is $\mathbb{E}$-connected for all $x' \in X'$. The set $C = C_1 \cup ... \cup C_n \subseteq B$ satisfies that $C(x')$ is $\mathbb{E}$-connected for all $x' \in X'$ and $\Psi(\top_C) \geq \Psi^{\beta^\eta_1}(\top_{C_1}) \wedge ... \wedge \Psi^{\beta^\eta_n}(\top_{C_n}) \geq \delta$. Hence Ψ has a δ-base as desired and (X', Λ_f) is uniformly locally $\mathbb{E}$-connected. □

Theorem 6.8. *Let the lattice L be completely distributive and let $\mathbb{E}$ be a class of T2-spaces. Let $(X_i, \Lambda_i) \in |SL\text{-}UCS|$ for all $i \in J$. If all (X_i, Λ_i) are uniformly locally $\mathbb{E}$-connected and all but finitely many $(X_i, \lim(\Lambda_i))$ are $\mathbb{E}$-connected, then the product space $(\prod_{i\in J} X_i, \pi - \Lambda)$ is uniformly locally $\mathbb{E}$-connected.*

Proof. We denote $X = \prod_{i\in J} X_i$. Let $\alpha \in L$ and let $\Phi \in \mathcal{F}^s_L(X \times X)$ such that $\pi - \Lambda(\Phi) \geq \alpha$. Then, for all $i \in J$, $\Lambda_i((p_i \times p_i)(\Phi)) \geq \alpha$ and hence, for each $i \in J$, there is $\Psi_i \in \mathcal{F}^s_L(X_i)$ with $\Psi_i \leq (p_i \times p_i)(\Phi) \wedge [\Delta_{X_i}]$ and $\Lambda_i(\Psi_i) \geq \alpha$ which has, for $\perp < \delta \leq \alpha$, a δ-base $\mathbb{B}_i$ such that $B_i(x_i)$ is $\mathbb{E}$-connected for each $B_i \in \mathbb{B}_i$ and each $x_i \in X_i$. We define $\Psi = \bigotimes_{i\in J} \Psi_i \in \mathcal{F}^s_L(X \times X)$. Then $\pi - \Lambda(\Psi) = \bigwedge_{i\in J} \Lambda_i((p_i \times p_i)(\bigotimes_{i\in J} \Psi_i)) \geq \bigwedge_{i\in J} \Lambda_i(\Psi_i) \geq \alpha$ and $\Psi \leq \bigotimes_{i\in J}((p_i \times p_i)(\Phi)) \leq \Phi$ and $\Psi \leq \bigotimes_{i\in J}[\Delta_{X_i}] \leq [\Delta_X]$, i.e. $\Psi \leq \Phi \wedge [\Delta_X]$. We show that, for $\perp < \delta \leq \alpha$, Ψ has a δ-base $\mathbb{B}$ with $B((x_i))$ $\mathbb{E}$-connected for all $B \in \mathbb{B}$ and all $(x_i) \in X$. Let $\Psi(\top_B) \geq \delta$ and let $\eta \lhd \delta$. We may assume $\eta > \perp$. Then $\prod_{i\in J} \Psi_i(\top_{\nu^\leftarrow(B)}) \rhd \eta$ and by Lemma 2.1 there are $U^\eta_i \subseteq X_i \times X_i$, $U^\eta_i \neq X_i \times X_i$ for only finitely many $i \in J$ with

$\prod_{i\in J} U_i^\eta \subseteq \nu^{\leftarrow}(B)$ and $\bigwedge_{i\in J} \Psi_i(\top_{U_i^\eta}) \geq \eta$. Hence, for all $i \in J$, $\Psi_i(\top_{U_i^\eta}) \geq \eta$ and there are sets $B_i^\eta \subseteq U_i^\eta$ such that $B_i^\eta(x_i)$ is $\mathbb{E}$-connected for all $x_i \in X_i$. We may assume that for all but finitely many $i \in J$, $B_i^\eta = X_i \times X_i$. Moreover we have $\Delta_{X_i} \subseteq B_i^\eta$ for all $i \in J$. It is not difficult to show that $\prod_{i\in J} B_i^\eta(x_i) = \nu(\prod_{i\in J} B_i^\eta)((x_i))$ and, as $\mathbb{E}$ consists of T2-spaces, these sets are $\mathbb{E}$-connected. Moreover, we have $\nu(\prod_{i\in J} B_i^\eta) \subseteq \nu(\prod_{i\in J} U_i^\eta) \subseteq \nu(\nu^{\leftarrow}(B)) \subseteq B$ and we have $\bigotimes_{i\in J} \Psi_i(\nu(\top_{\prod_{i\in J} B_i^\eta})) \geq \prod_{i\in J} \Psi_i(\top_{\prod_{i\in J} B_i^\eta}) \geq \bigwedge_{i\in J} \Psi_i(\top_{B_i^\eta}) \geq \eta$. From $\Delta_{X_i} \subseteq B_i^\eta$ we conclude that $\Delta_X \subseteq \nu(\prod_{i\in J} B_i^\gamma)$. Hence, if we define $B = \bigcup_{\eta\lhd\delta} \nu(\prod_{i\in J} B_i^\eta)$, then $B((x_i)) = \bigcup_{\eta\lhd\delta} \nu(\prod_{i\in J} B_i^\eta)((x_i))$ is $\mathbb{E}$-connected. As $\Psi(\top_B) \geq \eta$ for all $\eta \lhd \delta$, we obtain $\Psi(\top_B) \geq \delta$ and the proof is complete. □

7. Conclusions

We extended in this paper Preuß' $\mathbb{E}$-connectedness to stratified L-uniform convergence spaces and studied a suitable definition of uniform local $\mathbb{E}$-connectedness for such spaces, generalizing a definition and results from Vainio [24]. The preservation of local $\mathbb{E}$-connectedness under products (even for $L = \{0,1\}$) has not been shown before.

In the theory of classical uniform convergence spaces there is a further connectedness notion that plays a role in fixed point theorems, see Kneis [18]. Generalizing a definition from [18] we call a stratified L-uniform convergence space *well-chained* if for all $x, y \in X$ there is $\Phi_{xy} \in \mathcal{F}_L^s(X \times X)$ such that for $N \subseteq X \times X$, there is a natural number n with $(x,y) \in N^n$ whenever $\Lambda(\Phi_{xy}) \leq \Phi_{xy}(\top_N)$. For $L = \{0,1\}$ this definition coincides with the definition given by Kneis [18]. In $SL\text{-}UNIF$, then $(X,\mathcal{U})$ is well-chained if and only if it is strongly uniformly connected. In general, we only have that a well-chained space $(X,\Lambda) \in |SL\text{-}UCS|$ is strongly uniformly connected. This can be seen with Theorem 4.6. It would be interesting to know if the class WC of well-chained uniform convergence spaces coincides with the class $UC\mathbb{E}$ of uniformly $\mathbb{E}$-connected spaces for a suitable class $\mathbb{E}$. The following result sheds some light into this question. We call a space (X,Λ) *totally unchained* if the only well-chained sets $A \subseteq X$ (i.e. well-chained subspaces $(A, \Lambda|_A)$) are one-point sets. For instance, the space $(\{0,1\}, \Lambda_\delta^s)$ is totally unchained.

Lemma 7.1. *We have $WC \subseteq UC\mathbb{E}$ if and only if all spaces in $\mathbb{E}$ are totally unchained.*

Proof. Let $WC \subseteq UC\mathbb{E}$ and let $(E,\Lambda_E) \in \mathbb{E}$ and $A \subseteq E$ be well-chained. Then the inclusion mapping $i_A : A \longrightarrow E$ is uniformly continuous and hence constant, i.e. A is a one-point set. Conversely, let (X,Λ) be well-chained and let $f : (X,\Lambda) \longrightarrow (E,\Lambda_E)$ be uniformly continuous. It is not difficult to see that then $f(X) \subseteq E$ is well-chained too and hence, by assumption, $f(X) = \{a\}$, i.e. f is constant. □

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Gunther Jäger, School of Mechanical Engineering, University of Applied Sciences Stralsund, 18435 Stralsund, Germany
E-mail address: gunther.jaeger@fh-stralsund.de