ON IMPULSIVE FUZZY FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we prove the existence and uniqueness of solution to the impulsive fuzzy functional differential equations under generalized Hukuhara differentiability via the principle of contraction mappings. Some examples are provided to illustrate the result.

1. Introduction

The theory of impulsive functional differential equations is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of many real processes and phenomena studied in optimal control, electronics, economics and so on. To further study on impulsive functional differential equations, we refer to monographs and paper (see [5, 17, 23, 25]). On the other hand, fuzzy differential equations (FDEs) forms a suitable setting for the mathematical modeling of real world problems in which uncertainty or vagueness pervades. Most practical problems can be modeled as FDEs ([4],[7],[15],[13],[9],[30],[29],[26]). We refer to the monographs [21] and references therein. Therefore, the construction of theories which combine suitably the theories of impulsive (functional) differential equations and fuzzy (functional) differential equations are essential. According to the best knowledge of us, there are not too many papers about the theory impulsive fuzzy (functional) differential equations, and impulsive fuzzy (functional) differential equations. Some basic results can be found in [6, 8, 10, 19, 27, 28] and [11]-[15].

Lakshmikantham et. al. [19] have initiated the study of fuzzy impulsive differential equations, base on combining suitably the theories of impulsive differential equations and fuzzy differential equations. M. Guo et. al. [10] studied some existence results for the impulsive functional differential inclusion and the fuzzy impulsive functional differential equation with some conditions, and studied the properties of the solution set and the attainable set. In [23], R. Rodríguez-López considered a boundary value problem associated with an impulsive fuzzy differential equation and approximate the extremal solutions in a fuzzy functional interval using the monotone method. Fuzzy comparison results of the solution for some impulsive periodic linear differential problems are also proved. In [27], author studied the existence of fuzzy solutions for impulsive differential equations. Lupulescu [20]

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proved the local existence and uniqueness results for fuzzy functional differential equations using the method of successive approximations and the global existence and uniqueness via the contraction principle.

In this paper, inspired and motivated by authors in papers , [1]-[3], [10, 19, 20, 23, 27]. We consider the impulsive fuzzy functional differential equations. The paper will be organized as follows. As preliminaries, we recall some basic results about fuzzy set space, fuzzy differentiation and integration. In section 3, we will prove the existence and uniqueness of solution to the impulsive fuzzy functional differential equations under generalized Hukuhara differentiability via the principle of contraction mappings. In Section 4, we provide some examples to illustrate the results.

2. **Preliminaries**

Let $\mathcal{K}_c(\mathbb{R}^d)$ denote the collection of all nonempty compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $\mathcal{K}_c(\mathbb{R}^d)$, we define as usual, i.e. $A, B \in \mathcal{K}_c(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, then we have $A + B = \{a + b \,|\, a \in A, b \in B\}$, $\lambda A = \{\lambda a \,|\, a \in A\}$. The Hausdorff-Pompeiu metric d_H in $\mathcal{K}_c(\mathbb{R}^d)$ is defined as follows

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|_{\mathbb{R}^d}, \sup_{b\in B} \inf_{a\in A} \|a - b\|_{\mathbb{R}^d}\},$$

where $A, B \in \mathcal{K}_c(\mathbb{R}^d)$. It is known that $(\mathcal{K}_c(\mathbb{R}^d), d_H)$ is a complete metric space.

Denote $E^d = \{u : \mathbb{R}^d \to [0, 1]$ such that $u(z)$ satisfies (i)-(iv) stated below$\}$

(i) u is normal, i.e, there exists $z_0 \in \mathbb{R}^d$ such that $u(z_0) = 1$;

(ii) u is fuzzy convex, that is, for $0 \le \lambda \le 1$,

$$u(\lambda z_1 + (1 - \lambda)z_2) \ge \min\{u(z_1), u(z_2)\},$$

for any $z_1, z_2 \in \mathbb{R}^d$;

(iii) u is upper semicontinuous;

(iv) $[u]^0 = cl\{z \in \mathbb{R}^d : u(z) > 0\}$ is compact.

Then E^d is called the space of fuzzy sets. For $\alpha \in (0, 1]$, denote $[u]^\alpha = \{z \in \mathbb{R}^d \,|\, u(z) \ge \alpha\}$. We will call this set an $\alpha-$cut ($\alpha-$ level set) of the fuzzy set u. For $u \in E^d$ one has that $[u]^\alpha \in \mathcal{K}_c(\mathbb{R}^d)$ for every $\alpha \in [0, 1]$. For two fuzzy sets $u_1, u_2 \in E^d$, we denote $u_1 \le u_2$ if and only if $[u_1]^\alpha \subseteq [u_2]^\alpha$. If $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function then, according to Zadeh's extension principle, one can extend g to $E^d \times E^d \to E^d$ by the formula $g(u_1, u_2)(z) = \sup_{z=g(z_1,z_2)} \min\{u_1(z_1), u_2(z_2)\}$. It is well known that if g is continuous then $[g(u_1, u_2)]^\alpha = g([u_1]^\alpha, [u_2]^\alpha)$ for all $u_1, u_2 \in E^d, \alpha \in [0, 1]$. Especially, for addition and scalar multiplication in fuzzy set space E^d, we have $[u_1 + u_2]^\alpha = [u_1]^\alpha + [u_2]^\alpha, [\lambda u_1]^\alpha = \lambda[u_1]^\alpha$.

Let us denote

$$D[u_1, u_2] = \sup\{d_H([u_1]^\alpha, [u_2]^\alpha) : 0 \le \alpha \le 1\}$$

the distance between u_1 and u_2 in E^d, where $d_H([u_1]^\alpha, [u_2]^\alpha)$ is the Hausdorff-Pompeiu distance between two sets $[u_1]^\alpha, [u_2]^\alpha$ of $\mathcal{K}_c(\mathbb{R}^d)$.

Definition 2.1. (see [16]) Let $f : [a,b] \to E^d$ be a measurable and integrably bounded. The integral of f over $[a,b]$, denote by $\int_a^b f(t)dt$, is defined levelwise by the expression

$$\left[\int_a^b f(t)dt\right]^\alpha := \int_a^b [f(t)]^\alpha dt$$
$$= \left\{\int_a^b \tilde{f}(t)dt \middle| \tilde{f} : [a,b] \to \mathbb{R}^d \text{ is a measurable selection for } [f(\cdot)]^\alpha\right\},$$

for every $\alpha \in [0,1]$.

Definition 2.2. [22] Let $u : [a,b] \to E^d$, $t \in [a,b]$. We say that u is differentiable at t, if there exists $\mathcal{D}_H^g u(t) \in E^d$ such that

(i) for all $h > 0$ sufficiently small, there exist $u(t+h) \ominus u(t)$, $u(t) \ominus u(t-h)$ and the limits

$$\lim_{h\to 0^+} \frac{u(t+h) \ominus u(t)}{h} = \lim_{h\to 0^+} \frac{u(t) \ominus u(t-h)}{h} = \mathcal{D}_H^g u(t)$$

or

(ii) for all $h > 0$ sufficiently small, there exist $u(t) \ominus u(t+h)$, $u(t-h) \ominus u(t)$ and the limits

$$\lim_{h\to 0^+} \frac{u(t) \ominus u(t+h)}{-h} = \lim_{h\to 0^+} \frac{u(t-h) \ominus u(t)}{-h} = \mathcal{D}_H^g u(t).$$

Theorem 2.3. [22] Let $u : [a,b] \to E^1$ and put $[u(t)]^\alpha = [u_\alpha^l(t), u_\alpha^r(t)]$ for each $\alpha \in [0,1]$.

(i) If u is (i)-differentiable then u_α^l and u_α^r are differentiable functions and

$$[\mathcal{D}_H^g u(t)]^\alpha = [(u_\alpha^l(t))', (u_\alpha^r(t))']. \tag{1}$$

(ii) If u is (ii)-differentiable then u_α^l and u_α^r are differentiable functions and

$$[\mathcal{D}_H^g u(t)]^\alpha = [(u_\alpha^r(t))', (u_\alpha^l(t))']. \tag{2}$$

For a positive number $\sigma > 0$, let $C_\sigma : C([-\sigma,0], E^d)$ denote the space of continuous mappings from $[-\sigma,0]$ to E^d. Define a metric D_σ in C_σ by

$$D_\sigma[u,v] = \sup_{t\in[-\sigma,0]} D[u(t), v(t)].$$

Let $u \in C([-\sigma,T], E^d)$. Then, for each $[-\sigma,T]$ we denote by u_t the element of C_σ defined by $u_t(s) = u(t+s)$, for $s \in [-\sigma,0]$.

3. Main Results

In this section, we will consider the impulsive fuzzy functional differential equations as follows:

$$\begin{cases} \mathcal{D}_H^g u(t) = f(t,u_t), & t \in [0,T],\ t \neq t_k, \\ u(t_k^+) = u(t_k) + I_k(t_k, u(t_k)), & k = 1,2,\dots m, \\ u(t) = \varphi(t), & t \in [-\sigma,0], \end{cases} \tag{3}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_k < t_m < t_{m+1} = T$, a mapping $f : [0,T] \times C_\sigma \to E^d$ is continuous on $(t_{k-1}, t_k]$ and for each $u \in C_\sigma$, $I_k : [0,T] \times E^d \to E^d$ are continuous, for $k = 1, 2, \ldots m$.

In this paper, we consider only (i)-differentiable type and (ii)-differentiable type solutions, i.e., such that there are no switching points on $[-\sigma, T]$.

Lemma 3.1. Assume that the function f is continuous. A fuzzy mapping $u : [-\sigma, T] \to E^d$ is a solution to the problem (3) on $[-\sigma, T]$ if and only if x is a continuous fuzzy mapping and it satisfies to one of the following fuzzy integral equations:

(**L1**) If u is (i)-differentiable, then

$$u(t) = \begin{cases} \varphi(t), & \text{for} \quad t \in [-\sigma, 0], \\ \varphi(0) + \int_0^t f(s, u_s)ds, & \text{for} \quad t \in [0, t_1], \\ \quad\vdots & \quad\vdots \\ \varphi(0) + \sum\limits_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(s, u_s)ds & \\ \qquad + \int_{t_k}^{t} f(s, u_s)ds + \sum\limits_{i=1}^{k} I_i(t_i, u(t_i)), & \text{for} \quad t \in (t_k, t_{k+1}]. \end{cases} \tag{4}$$

(**L2**) If u is (ii)-differentiable, then

$$u(t) = \begin{cases} \varphi(t), & \text{for} \quad t \in [-\sigma, 0], \\ \varphi(0) \ominus (-1) \int_0^t f(s, u_s)ds, & \text{for} \quad t \in [0, t_1], \\ \quad\vdots & \quad\vdots \\ \varphi(0) \ominus +(-1) \Bigg[\sum\limits_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(s, u_s)ds & \\ \qquad + \int_{t_k}^{t} f(s, u_s)ds \Bigg] + \sum\limits_{i=1}^{k} I_i(t_i, u(t_i)), & \text{for} \quad t \in (t_k, t_{k+1}]. \end{cases} \tag{5}$$

(In (5), we assume that the Hukuhara differences exist.)

Proof. In this proof, we only prove the case (ii)-differentiable of solution. The case (ii)-differentiable of solution can be proved similar. To prove the lemma, we divide it into two steps.

Step 1: If $x : [-\sigma, T]$ satisfies the problem (3), then it will be expressed as (5). Indeed, if $t \in [0, t_1]$ we have

$$\mathcal{D}_H^g u(t) = f(t, u_t).$$

By Lemma 3.2 in [15] we have

$$u(t) = \varphi(0) \ominus (-1) \int_0^t f(s, u_s)ds.$$

Then

$$u(t_1) = \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s)ds.$$

If $t \in (t_1, t_2]$ and Lemma 3.2 in [15] implies that

$$\begin{aligned} u(t) &= u(t_1^+) \ominus (-1) \int_{t_1}^{t} f(s, u_s) ds \\ &= [u(t_1) + I_1(t_1, u(t_1))] \ominus (-1) \int_{t_1}^{t} f(s, u_s) ds \\ &= I_1(t_1, u(t_1)) + \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t} f(s, u_s) ds. \end{aligned}$$

By mathematical induction, if $t \in (t_k, t_{k_1}]$, $k = 1, 2, \ldots, m$ and using Lemma 3.2 in [15] again, we have

$$\begin{aligned} u(t) &= u(t_k^+) \ominus (-1) \int_{t_k}^{t} f(s, u_s) ds \\ &= u(t_k) + I_k(t_k, u(t_k)) + \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t} f(s, u_s) ds \\ &\qquad \ominus (-1) \ldots \ominus (-1) \int_{t_{k-1}}^{t_k} f(s, u_s) ds \ominus (-1) \int_{t_k}^{t} f(s, u_s) ds \\ &= \sum_{i=1}^{k} I_i(t_i, u(t_i)) + \varphi(0) \ominus (-1) \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(s, u_s) ds \ominus (-1) \int_{t_k}^{t} f(s, u_s) ds. \end{aligned}$$

Step 2: If x satisfies (5), then it is equivalent to the problem (3). Indeed, if $t \in [0, t_1]$ it is easy to see that $u(0) = \varphi(0)$ and the Hukuhara difference $\varphi(0) \ominus (-1) \int_0^t f(s, u_s) ds$ exists. By Lemma 3.2 in [15] we have

$$\mathcal{D}_H^g u(t) = f(t, u_t), \text{ for } t \in [0, t_1].$$

If $t \in (t_1, t_2]$ and $t - h \in (t_1, t_2]$ with $h > 0$ enough small, we have

$$\begin{aligned} &u(t-h) \ominus u(t) \\ &\quad = \Big(I_1(t_1, u(t_1)) + \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t-h} f(s, u_s) ds \Big) \\ &\quad \ominus \Big(I_1(t_1, u(t_1)) + \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t} f(s, u_s) ds \Big) \\ &\quad = (-1) \int_{t_1}^{t} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t-h} f(s, u_s) ds \\ &\quad = (-1) \int_{t-h}^{t} f(s, u_s) ds \end{aligned} \tag{6}$$

and for $t + h \in (t_1, t_2)$,

$$\begin{aligned} &u(t) \ominus u(t+h) \\ &\quad = \Big(I_1(u(t_1)) + \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t} f(s, u_s) ds \Big) \\ &\quad \ominus \Big(I_1(u(t_1)) + \varphi(0) \ominus (-1) \int_0^{t_1} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t+h} f(s, u_s) ds \Big) \\ &\quad = (-1) \int_{t_1}^{t} f(s, u_s) ds \ominus (-1) \int_{t_1}^{t+h} f(s, u_s) ds \\ &\quad = (-1) \int_{t}^{t+h} f(s, u_s) ds. \end{aligned} \tag{7}$$

Multiplying both sides of the equations (6) and (7) by $\dfrac{1}{-h}$ and passing to the limit with $h \to 0^+$, we obtain

$$\lim_{h \to 0^+} \frac{u(t) \ominus u(t+h)}{-h} = f(t, u_t)$$

and

$$\lim_{h\to 0^+} \frac{u(t-h)\ominus u(t)}{-h} = f(t,u_t).$$

This allows us to claim that $u(t)$ is (ii)-differentiable on $(t_1,t_2]$ and consequently

$$\mathcal{D}_H^g u(t) = f(t,u_t), \text{ for each } t\in (t_1,t_2].$$

By mathematical induction, if $t\in (t_k,t_{k_1}]$, $k=1,2,\ldots,m$, we get

$$\mathcal{D}_H^g u(t) = f(t,u_t), \text{ for each } \ t\in (t_k,t_{k+1}].$$

Also, we can easily show that

$$u(t_k^+) = u(t_k) + I_k(t_k,u(t_k)), \quad k=1,2,\ldots,m.$$

The proof is complete. □

In the sequel, we will prove the existence and uniqueness of solution to the problem (3) via the principle of contraction mappings. Since the way of the proof is similar for two cases, we only consider case (ii)-differentiable of solution on $[0,T]$. For a given constant $a>0$, we consider the set E_a of all functions $u\in C([-\sigma,0],E^d)$ such that $u(t)=\varphi(t)$ on $[-\sigma,0]$ and $\sup_{t\geq 0} D[u(t),\hat{0}]\exp(-at)<\infty$. On E_a we can define the following metric

$$D_a[u,v] := \sup_{t\in[-\sigma,T]} D[u(t),v(t)]\exp(-at).$$

It is easy to see that (E_a,D_a) is a complete metric space (see detail in Lupulescu [20]).

Theorem 3.2. Suppose that the mapping $f:[0,T]\times C_\sigma\to E^d$ is jointly continuous and $I_k:[0,T]\times E^d\to E^d$ are continuous for $k=1,2,\ldots m$ and satisfy the following assumptions

(f1) there exists $L>0$ such that

$$D[f(t,\Psi),f(t,\Phi)] \leq LD_\sigma[\Psi,\Phi],$$

for all $\Psi,\Phi\in C_\sigma$ and for all $t\in[0,T]$;

(f2) there exist $M>0$ and $a>b\geq 0$ such that

$$D[F(t,\hat{0}),\hat{0}] \leq M\exp(bt)$$

for all $t\geq 0$;

(f3) there exists $L_k>0$ such that

$$D[I_k(t_k,\Psi),I_k(t_k,\Phi)] \leq L_k D[\Psi,\Phi],$$

for all $t_k\in[0,T]$, for all $\Psi,\Phi\in C_\sigma$.

Then the problem (3) has a unique solution for each case on $[0,T]$, provided that

$$L \leq \frac{a(2-k^2(k+1)\hat{L})}{2(1-\exp(-aT))}$$

and the Hukuhara differences in (5) exist for the case (ii)-differentiable of solution on $[0,T]$, where $\hat{L}=\max\{L_1,L_2,\ldots,L_k\}$ for $k=1,2,\ldots,m$ and a is a suitable positive constant.

Proof. We will use the principle of contraction mappings to prove the problem (3) has a unique solution on $[0,T]$. The proof will be given in two steps.

Step 1: If $f:[0,T]\times C_\sigma \to E^d$ and $I_k:[0,T]\times E^d \to E^d$ satisfy assumptions (f1)-(f3), then $\mathbb{P}(E_a)\subset E_a$, where a mapping $\mathbb{P}: C([-\sigma,T],E^d)\to C([-\sigma,T],E^d)$ given by

$$(\mathbb{P}u)(t)=\begin{cases}\varphi(t), & \text{for } t\in[-\sigma,0],\\ \varphi(0)\ominus(-1)\int_0^t f(s,u_s)ds, & \text{for } t\in[0,t_1],\\ \vdots & \vdots\\ \varphi(0)\ominus(-1)\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} f(s,u_s)ds & \\ \qquad\ominus(-1)\int_{t_k}^{t} f(s,u_s)ds+\sum_{i=1}^{k} I_i(t_i,u(t_i)), & \text{for } t\in(t_k,t_{k+1}].\end{cases}$$

Let $u\in E_a$, i.e, there exists $\rho>0$ such that $D[u(t),\hat{0}]\le \rho\exp(at)$ for any $t\in[-\sigma,T]$. It follows that $\sup_{\theta\in[-\sigma,0]} D[u(t+\theta)]\le \rho\exp(at)$ for any $t\in[0,T]$. For $t\in[0,t_1]$ we have

$$\begin{aligned}D[(\mathbb{P}u)(t),\hat{0}] &\le D[\varphi(0),\hat{0}]+L\int_0^t D_\sigma[u_s,\hat{0}]ds+\frac{M}{b}\exp(bt)\\ &\le D[\varphi(0),\hat{0}]+L\int_0^t \sup_{\theta\in[-\sigma,0]}[u(s+\theta),\hat{0}]ds+\frac{M}{b}\exp(bt)\\ &\le D[\varphi(0),\hat{0}]+L\int_0^t \rho\exp(as)ds+\frac{M}{b}\exp(bt)\\ &= D[\varphi(0),\hat{0}]+\frac{\rho L}{a}(\exp(at)-1)+\frac{M}{b}\exp(bt)\\ &\le D[\varphi(0),\hat{0}]+\frac{\rho L}{a}\exp(at)+\frac{M}{b}\exp(bt).\end{aligned}$$

Similarly, for $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$ we have

$$\begin{aligned}D[(\mathbb{P}u)(t),\hat{0}] &\le D[\varphi(0),\hat{0}]+D\left[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} f(s,u_s)ds,\hat{0}\right]\\ &\quad +D\left[\int_{t_k}^{t} f(s,u_s)ds,\hat{0}\right]+D\left[\sum_{i=1}^{k} I_i(t_i,u(t_i)),\hat{0}\right]\\ &\le D[\varphi(0),\hat{0}]+L\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}[D_\sigma[u_s,\hat{0}]ds+M\exp(bs)]\\ &\quad +L\int_{t_k}^{t}[D_\sigma[u_s,\hat{0}]ds+M\exp(bs)]ds+\sum_{i=1}^{k} L_i D[u,\hat{0}]\\ &= D[\varphi(0),\hat{0}]+L\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\sup_{\theta\in[-\sigma,0]}[u(s+\theta),\hat{0}]ds\\ &\quad +L\int_{t_k}^{t}\sup_{\theta\in[-\sigma,0]}[u(s+\theta),\hat{0}]ds+\frac{M}{b}(\exp(bt)-1)+k\rho\hat{L}\\ &\le D[\varphi(0),\hat{0}]+\frac{\rho L}{a}(\exp(at)-1)+\frac{M}{b}(\exp(bt)-1)+k\rho\hat{L}\end{aligned}$$

$$\leq D[\varphi(0),\hat{0}]+\frac{\rho L}{a}\exp(at)+\frac{M}{b}\exp(bt)+k\rho\hat{L}.$$

Thus, for $t\in[0,t_1]$ we have

$$\begin{aligned}\sup_{t\in[0,t_1]} & D[(\mathbb{P}u)(t),\hat{0}]\exp(-at)\\ &\leq \sup_{t\in[0,t_1]}\left(D[\varphi(0),\hat{0}]+\frac{\rho L}{a}\exp(at)+\frac{M}{b}\exp(bt)\right)\exp(-at)\\ &\leq D[\varphi(0),\hat{0}]+\frac{1}{b}(\rho L+M)<\infty,\end{aligned}$$

and for $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$

$$\begin{aligned}\sup_{t\in(t_k,t_{k+1}]} & D[(\mathbb{P}u)(t),\hat{0}]\exp(-at)\\ &\leq \sup_{t\in[0,t_1]}\big(D[\varphi(0),\hat{0}]+\frac{\rho L}{a}\exp(at)+\frac{M}{b}\exp(bt)+k\rho\hat{L}\big)\exp(-at)\\ &\leq D[\varphi(0),\hat{0}]+\frac{1}{b}(\rho L+M)+k\rho\hat{L}\exp(-aT)<\infty.\end{aligned}$$

If we take $K=\sup_{\theta\in[-\sigma,0]}D[\varphi(\theta),\hat{0}]$, then

$$\sup_{t\in[-\sigma,t_1]}D[(\mathbb{P}u)(t),\hat{0}]\exp(-at)\leq K+\frac{1}{b}(\rho L+M)<\infty$$

and

$$\sup_{t\in(t_k,t_{t_{k+1}}]}D[(\mathbb{P}u)(t),\hat{0}]\exp(-at)\leq K+\frac{1}{b}(\rho L+M)+k\rho\hat{L}\exp(-aT)<\infty.$$

Besides, for $t\in[-\sigma,0]$ we get

$$D[(\mathbb{P}u)(t),\hat{0}](-at)=D[\varphi(0),\hat{0}]\exp(-at)\leq K\exp(-at)\leq K\exp(a\sigma),$$

so that

$$\sup_{t\in[-\sigma,t_1]}D[(\mathbb{P}u)(t),\hat{0}]\exp(-at)\leq K\exp(a\sigma)+\frac{1}{b}(\rho L+M)<\infty$$

and

$$\sup_{t\in(t_k,t_{k+1}]}D[(\mathbb{P}u)(t),\hat{0}]\exp(-at)\leq K\exp(a\sigma)+\frac{1}{b}(\rho L+M)+k\rho\hat{L}\exp(-aT)<\infty$$

for $k=1,2,\dots,m$. Therefore, $\mathbb{P}u\in E_a$.

Step 2: If $f:[0,T]\times C_\sigma\to E^d$ and $I_k:[0,T]\times E^d\to E^d$ satisfy assumptions (f1)-f(3) and $L<a$ then $\mathbb{P}$ is a contraction on E_a. Indeed, let $u,v\in E_a$, for $t\in[0,t_1]$ we have

$$D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)]\leq L\int_0^t\sup_{r\in[s-\sigma,s]}D[u(r),v(r)]ds.$$

Furthermore, we have $D[u(t),v(t)]\leq D_a[u,v]\exp(at)$ for all $t\in[-\sigma,T]$. It follows that $\sup_{r\in[s-\sigma,s]}D[u(r),v(r)]\leq D_a[u,v]\exp(at)$ for all $t\in[0,T]$. Hence, for $t\in[0,t_1]$

we have

$$D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)] \le L\int_0^t D_a[u,v]\exp(as)ds = \frac{L}{a}D_a[u,v](\exp(at)-1)$$

and so

$$\begin{aligned} D_a[\mathbb{P}u,\mathbb{P}v] &= \sup_{t\in[-\sigma,T]} D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)]\exp(-at) = \sup_{t\in[0,T]} D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)]\exp(-at) \\ &\le \sup_{t\in[0,T]} \frac{L}{a}D_a[u,v](1-\exp(-at)) = \frac{L}{a}(1-\exp(-aT))D_a[u,v]. \end{aligned} \tag{8}$$

Similarly, for $t\in(t_k,t_{k+1}]$, $k=1,2,\dots,m$ we have

$$\begin{aligned} &D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)] \\ &\le L\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} D_\sigma[u_s,v_s]ds + L\int_{t_k}^{t} D_\sigma[u_s,v_s]ds + \sum_{i=1}^{k} L_i D[u(t_i),v(t_i)] \\ &= L\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \sup_{\theta\in[-\sigma,0]}[u(s+\theta),v(s+\theta)]ds \\ &+L\int_{t_k}^{t} \sup_{\theta\in[-\sigma,0]}[u(s+\theta),v(s+\theta)]ds + \sum_{i=1}^{k} L_i D[u(t_i),v(t_i)] \\ &\le L\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} D_a[u,v]\exp(as)ds + L\int_{t_k}^{t} D_a[u,v]\exp(as)ds \\ &+ \sum_{i=1}^{k} L_i D_a[u,v]\exp(at_i) \\ &\le \frac{L}{a}D_a[u,v](1-\exp(-at)) + D_a[u,v]\sum_{i=1}^{k} L_i\exp(at_i) \\ &\le \left(\frac{L}{a}(1-\exp(-at)) + k\hat{L}\sum_{i=1}^{k}\exp(at_i)\right)D_a[u,v] \end{aligned}$$

and so

$$\begin{aligned} D_a[\mathbb{P}u,\mathbb{P}v] &:= \sup_{t\in[-\sigma,T]} D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)]\exp(-at) \\ &= \sup_{t\in[0,T]} D[(\mathbb{P}u)(t),(\mathbb{P}v)(t)]\exp(-at) \\ &\le \sup_{t\in[0,T]}\left(\frac{L}{a}D_a[u,v](1-\exp(-at)) + k\hat{L}D_a[u,v]\sum_{i=1}^{k}\exp(at_i)\right)\exp(-at) \\ &\le \left(\frac{L}{a}(1-\exp(-aT)) + k\hat{L}\sum_{i=1}^{k}\exp(-a(T-t_i))\right)D_a[u,v]. \end{aligned} \tag{9}$$

From (8) and (9) we obtain

$$D_a[\mathbb{P}u,\mathbb{P}v] \le \frac{L}{a}(1-\exp(-aT))D_a[u,v], \quad \text{for } t\in[0,t_1]$$

and

$$D_a[\mathbb{P}u,\mathbb{P}v] \le \left(\frac{L}{a}(1-\exp(-aT)) + \frac{1}{2}k^2(k+1)\hat{L}\right)D_a[u,v] \quad \text{for } t\in(t_k,t_{k+1}].$$

Since for $k=1,2,\dots,m$,

$$L \le \frac{a(2-k^2(k+1)\hat{L})}{2(1-\exp\{-aT\})},$$

we infer that $\mathbb{P}$ is a contraction on E_a.

Applying Banach fixed point theorem provides the existence of a unique fixed point for $\mathbb{P}$ and the unique fixed of $\mathbb{P}$ is in the space E_a, that is a unique solution for the problem (3) in case (ii)-differentiable and similar to the case (i)- differentiable. □

Theorem 3.3. Assume that $f : [0,T] \times C_\sigma \to E^d$ and $I_k : [0,T] \times E^d \to E^d$, $k = 1, 2, \dots m$ satisfy the assumptions of Theorem 3.2. If $u(t)$ and $v(t)$ are any two solutions of the problem (3) on $[-\sigma, T]$ with $u(t) = \varphi(t)$ and $v(t) = \phi(t)$ on $[-\sigma, 0]$, then we have

$$D[u(t,\varphi), v(t,\phi)] \le D_\sigma[\varphi,\phi] \prod_{t_i \in [0,t]} (1 + L_i) \exp(Lt),$$

for $i = 1, 2, \dots k$.

Proof. Since $u(t)$ and $v(t)$ are any two solutions of the problem (3). For $t \in [0, t_1]$ we have

$$D[u(t,\varphi), v(t,\varphi)] \le D_\sigma[\varphi,\phi] + L \int_0^t \sup_{r \in [-\sigma,s]} D[u(r,\varphi), v(r,\phi)]ds.$$

For $t \in [t_k, t_{k+1})$, $k = 1, 2, \dots m$, we obtain

$$\begin{aligned}
D[u(t,\varphi), v(t,\phi)] &\le D[\varphi(0), \phi(0)] + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} D[f(s,u_s), f(s,v_s)]ds \\
&\quad + \int_{t_k}^{t} D[f(s,u_s), f(s,v_s)]ds + \sum_{i=1}^{k} D[I_i(t_i, u(t_i)), I_i(t_i, v(t_i))] \\
&\le D_\sigma[\varphi,\phi] + L \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \sup_{\theta \in [-\sigma,0]} D_\sigma[u(\theta+s), v(\theta+s)]ds \\
&\quad + L \int_{t_k}^{t} \sup_{\theta \in [-\sigma,0]} D_\sigma[u(\theta+s), v(\theta+s)]ds + \sum_{i=1}^{k} L_i D[u(t_i), v(t_i)] \\
&= D_\sigma[\varphi,\phi] + L \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \sup_{r \in [-\sigma,s]} D_\sigma[u(r), v(r)]ds \\
&\quad + L \int_{t_k}^{t} \sup_{r \in [-\sigma,s]} D_\sigma[u(r), v(r)]ds + \sum_{i=1}^{k} L_i D[u(t_i), v(t_i)].
\end{aligned}$$

If we put $w(s) = \sup_{r \in [-\sigma,s]} D_\sigma[u(r,\varphi), v(r,\phi)]$, $s \in [0, t_i]$, then we have

$$\begin{aligned}
D[u(t,\varphi), v(t,\varphi)] &\le D_\sigma[\varphi,\phi] + L \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} w(s)ds + L \int_{t_k}^{t} w(s)ds + \sum_{i=1}^{k} L_i D[u(t_i), v(t_i)] \\
&\le D_\sigma[\varphi,\phi] + L \int_0^t w(s)ds + \sum_{i=1}^{k} L_i w(t_i).
\end{aligned}$$

Applying Gronwall-Bellman inequality (see [25]-Lemma 1) we get

$$w(t) \le D_\sigma[\varphi,\phi] \prod_{t_i \in [0,t]} \exp(Lt), \quad i = 1, 2, \dots k.$$

Consequently, we have

$$D[u(t,\varphi), v(t,\phi)] \le D_\sigma[\varphi,\phi] \prod_{t_i \in [0,t]} (1 + L_i) \exp(Lt), \quad i = 1, 2, \dots k.$$

The proof is complete. □

4. Examples

Firstly, we consider the impulsive fuzzy functional differential equations as follows.

Example 4.1. Let us consider a class of impulsive fuzzy functional differential equations with distributed delay. For $l \in \mathbb{N}$ and times $0 < \sigma_1 < \ldots < \sigma_l < \sigma$. we consider the following type of impulsive fuzzy differential equations with delay:

$$\begin{cases} \mathcal{D}_H^g u(t) = \int_{-\sigma}^0 g_0(s, u(t+s))ds + \sum_{i=1}^{l} g_i(s, u(t-\sigma_i)) & \text{for } t \neq t_k, \\ u(t_k^+) = \dfrac{u(t_k)}{3 + u(t_k)}, & t = t_k, \\ u(t) = \varphi(t), & \text{for } t \in [-\sigma, 0], \end{cases} \tag{10}$$

where C_σ is the space $C([-\sigma, 0], E^1)$, $u : [-\sigma, T] \to E^1$, and $g_i : [0, T] \times C_\sigma \to E^1$, $i = \overline{1, l}$ are fuzzy-valued mappings.
Assume that $g_i : [0, T] \to E^1$, $i = \overline{1, l}$ satisfy the following hypotheses:

(g1) There exists $L_i > 0$ such that $D[g_i(t, u), g_i(t, v)] \leq L_i D[u, v]$ for all $u, v \in E^1$ and for all $t \in [0, T]$;
(g2) $g_i : [0, T] \times C_\sigma \to E^1$ are jointly continuous;
(g3) there exist $M_i > 0$ and $b_i > 0$ such that $D[g_i(t, \hat{0}), \hat{0}] \leq M_i e^{b_i} t$ for all $t \in [0, T]$;
(g4) there exists $L_k > 0$ such that $D[I_k(t_k, u(t_k)), I_k(t_k, v(t_k))] \leq L_k D[u, v]$ for all $u, v \in E^1$ and for all $t \in [0, T]$.

Then the fuzzy function $f : [0, T] \times C_\sigma \to E^1$ given by

$$f(t, \varphi) = \int_{-\sigma}^0 g_0(\tau, \varphi(\tau))d\tau + \sum_{i=1}^{l} g_i(\tau, \varphi(-\sigma_i)),$$

for $t \in [0, T]$, $t \neq t_k$.

Assume that the functions $g_i : [0, T] \times C_\sigma \to E^1$, $i = \overline{1, l}$ satisfy the assumptions (g1)-(g4). Then the problem (10) has a unique solution on $[0, T]$. Indeed, for every $u, v \in E^1$, for every $t \in [0, T]$, $t \neq t_k$ and let $L_i > 0$ be the Lipschitz constants of g_i we have

$$D[f(t, \varphi), f(t, \phi))] \leq \left(\sigma \sup_{t \in [-\sigma, 0]} + \sum_{i=1}^{l} L_i \right) D_\sigma[\varphi, \phi]$$

and so the assumption (g1) is satisfied.

The function $f(t, \varphi)$ satisfies the assumption (g3) since

$$D[f(t, \varphi), \hat{0}] \leq M_0 \sigma + \sum_{i=1}^{l} M_i$$

for all $t \in [0, T]$, $t \neq t_k$.

Moreover, we get that

$$\begin{aligned} D[I_k(t_k, u(t_k)), I_k(t_k, v(t_k))] &= D\left[\frac{u(t_k)}{3 + u(t_k)}, \frac{v(t_k)}{3 + v(t_k)} \right] \\ &= \sup_{\alpha \in [0,1]} \max \left\{ \left| \frac{u_\alpha^l(t_k)}{3 + u_\alpha^l(t_k)} - \frac{v_\alpha^l(t_k)}{3 + v_\alpha^l(t_k)} \right|, \left| \frac{u_\alpha^r(t_k)}{3 + u_\alpha^r(t_k)} - \frac{v_\alpha^r(t_k)}{3 + v_\alpha^r(t_k)} \right| \right\} \\ &\leq \frac{1}{3} \sup_{\alpha \in [0,1]} \max\{ |u_\alpha^l(t_k) - v_\alpha^l(t_k)|, |u_\alpha^r(t_k) - v_\alpha^r(t_k)| \} \\ &= \frac{1}{3} D[u(t_k), v(t_k)], \end{aligned}$$

which satisfies the assumption (g4) with $L_k = \dfrac{1}{3}$.

Therefore, the problem (10) has a unique solution on $[0, T]$.

In the next part, we solve the following impulsive fuzzy functional differential equations:

$$\begin{cases} \mathcal{D}_H^g u(t) = f(t, u(t-\sigma)), & t \in [0,T],\ t \neq t_k, \\ u(t_k^+) = u(t_k) + I_k(t_k, u(t_k)), & k = 1,2,\dots m, \\ u(t) = \varphi(t), & t \in [-\sigma, 0], \end{cases} \tag{11}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_k < t_m < t_{m+1} = T$ and a mapping $f : [0,T] \times C_\sigma \to E^1$ is continuous in $(t_{k-1}, t_k]$ and for each $u \in C_\sigma$, $I_k : [0,T] \times E^1 \to E^1$ are continuous, for $k = 1, 2, \ldots m$.

If the problem (11) satisfies Theorem 3.2, then problem (11) has a unique solution on $[0,T]$. Moreover, the solution of the problem (11) is dependent on independent of the choice of the derivative in Definition 2.2. The results in Theorem 2.3 are us an usefusl procedure to solve the problem (11).

For $t \in [0,T]$

$$[u(t)]^\alpha = [u_\alpha^l(t), u_\alpha^r(t)],$$

for $t \in [-\sigma, 0]$,

$$[u(t)]^\alpha = [\varphi_\alpha^l(t), \varphi_\alpha^r(t)],$$

for $k = 1, 2, \ldots m$

$$[I_k(t_k, u(t_k))]^\alpha = [I_{\alpha,k}^l(t_k, u_\alpha^l(t_k), u_\alpha^r(t_k)), I_{\alpha,k}^r(t_k, u_\alpha^r(t_k), u_\alpha^r(t_k))],$$

$$[u(t_k^+)]^\alpha = [(u_\alpha^l(t_k)), u_\alpha^r(t_k)],$$

and for $t \in [0,T]$

$$[f(t, u(t-\sigma))]^\alpha = [f_\alpha^l(t, u_\alpha^l(t-\sigma), u_\alpha^r(t-\sigma)), f_\alpha^r(t, u_\alpha^l(t-\sigma), u_\alpha^r(t-\sigma))].$$

Case 1: If u is (i)-differentiable, then from Theorem 2.3, we have

$$[\mathcal{D}_H^g u(t)]^\alpha = [(u_\alpha^l(t))', (u_\alpha^r(t))'], \quad \text{for } t \in [0,T].$$

Now, we proceed as follows:

(**b1**) Solve the impulsive functional differential system

$$\begin{cases} (u_\alpha^l(t))' = f_\alpha^l(t, u_\alpha^l(t-\sigma), u_\alpha^r(t-\sigma)), & t \in [0,T],\ t \neq t_k, \\ (u_\alpha^r(t))' = f_\alpha^r(t, u_\alpha^l(t-\sigma), u_\alpha^r(t-\sigma)), & t \in [0,T],\ t \neq t_k, \\ (u_\alpha^l(t_k^+)) = u_\alpha^l(t_k) + I_{\alpha,k}^l(t_k, u_\alpha^l(t_k), u_\alpha^r(t_k)), & k = 1,2,\dots m, \\ (u_\alpha^r(t_k^+)) = u_\alpha^r(t_k) + I_{\alpha,k}^r(t_k, u_\alpha^l(t_k), u_\alpha^r(t_k)), & k = 1,2,\dots m, \\ u_\alpha^l(t) = \varphi_\alpha^l(t), & t \in [-\sigma, 0], \\ u_\alpha^r(t) = \varphi_\alpha^r(t), & t \in [-\sigma, 0], \end{cases}$$

to find $u_\alpha^l(t)$ and $u_\alpha^r(t)$.

(**b2**) Ensure that $[u_\alpha^l(t), u_\alpha^r(t)]$ and $[(u_\alpha^l(t))', (u_\alpha^r(t))']$ are valid level sets.

(**b3**) Using the Stacking Theorem, pile up the levels $[u_\alpha^l(t), u_\alpha^r(t)]$ to a fuzzy solution $u(t)$.

Case 2: If u is (ii)-differentiable, then from Theorem 2.3, we have

$$[\mathcal{D}_H^g u(t)]^\alpha = [(u_\alpha^r(t))', (u_\alpha^l(t))'], \quad \text{for } t \in [0,T], \quad \alpha \in [0,1].$$

Similarly as Case 1, we also proceed as follows:

(c1) Solve the impulsive functional differential system

$$\begin{cases} (u_\alpha^l(t))' = f_\alpha^r(t, u_\alpha^l(t-\sigma), u_\alpha^r(t-\sigma)), & t \in [0,T],\ t \neq t_k, \\ (u_\alpha^r(t))' = f_\alpha^l(t, u_\alpha^l(t-\sigma), u_\alpha^r(t-\sigma)), & t \in [0,T],\ t \neq t_k, \\ (u_\alpha^l(t_k^+)) = u_\alpha^l(t_k) + I_{\alpha,k}^l(t_k, u_\alpha^l(t_k), u_\alpha^r(t_k)), & k = 1,2,\dots m, \\ (u_\alpha^r(t_k^+)) = u_\alpha^r(t_k) + I_{\alpha,k}^r(t_k, u_\alpha^l(t_k), u_\alpha^r(t_k)), & k = 1,2,\dots m, \\ u_\alpha^l(t) = \varphi_\alpha^l(t), & t \in [-\sigma, 0], \\ u_\alpha^r(t) = \varphi_\alpha^r(t), & t \in [-\sigma, 0], \end{cases}$$

to find $u_\alpha^l(t)$ and $u_\alpha^r(t)$.

(c2) Ensure that $[u_\alpha^l(t), u_\alpha^r(t)]$ and $[(u_\alpha^r(t))', (u_\alpha^l(t))']$ are valid level sets.

(c3) Using the Stacking Theorem, pile up the levels $[u_\alpha^l(t), u_\alpha^r(t)]$ to a fuzzy solution $u(t)$.

Example 4.2. Let us consider the following impulsive fuzzy functional differential equation:

$$\begin{cases} \mathcal{D}_H^g u(t) = ru(t-1), & t \in [0,4],\ t \neq t_k, \\ u(t_k^+) = 2u(t_k^-), & t_k = 2k,\ k = 1,2, \\ u(t) = (-1,0,1) \in E^1, & t \in [-1,0], \end{cases} \tag{12}$$

where r is a positive constant.

Case 1: If u is (i)-differentiable, then from Theorem 2.3, we have

$$[\mathcal{D}_H^g u(t)]^\alpha = [(u_\alpha^l(t))', (u_\alpha^r(t))'], \quad \text{for } t \in [0,4], \quad \alpha \in [0,1].$$

Now, we present the procedure for solving the impulsive fuzzy functional differential equations.

The problem (12) is translated into the following system of impulsive functional differential equations:

$$\begin{cases} (u_\alpha^l(t))' = ru_\alpha^l(t-1), & t \in [0,4],\ t \neq t_k, \\ (u_\alpha^r(t))' = ru_\alpha^r(t-1), & t \in [0,4],\ t \neq t_k, \\ u_\alpha^r(t_k^+) = 2u_\alpha^r(t_k^-), & t_k = 2k,\ k = 1,2, \\ u_\alpha^r(t_k^+) = 2u_\alpha^r(t_k^-), & t_k = 2k,\ k = 1,2, \\ u_\alpha^l(t) = \alpha - 1, & t \in [-1,0], \\ u_\alpha^r(t) = 1 - \alpha, & t \in [-1,0]. \end{cases} \tag{13}$$

Using the method of steps, we can obtain a solution of the system (13) on $[-1,4]$ and it reads

$$u_\alpha^l(t) = \begin{cases} \alpha - 1, & t \in [-1,0], \\ (\alpha-1)(1+rt), & t \in [0,1), \\ (\alpha-1)\big[(1+r) + r(t-1) + \frac{1}{2}r^2(t-1)^2\big], & t \in [1,2), \\ (\alpha-1)\big[1 + 2r + \dfrac{1}{2}r^2 + r(1+r)(t-2) \\ \qquad + \frac{1}{2}r^2(t-2)^2 + \frac{1}{6}r^3(t-2)^3\big], & t \in [2,3), \\ (\alpha-1)\big[\big(1 + 3r + r^2 + \frac{1}{6}r^3\big) + r(1 + 2r + \frac{1}{2}r^2)(t-3) \\ \qquad + \frac{1}{2}r^2(1+r)(t-3)^2 + \frac{1}{6}r^3(t-3)^3 + \frac{1}{18}r^4(t-3)^4\big], & t \in [3,4] \end{cases}$$

and

$$u^r_\alpha(t) = \begin{cases} 1-\alpha, & t\in[-1,0],\\ (1-\alpha)(1+rt), & t\in[0,1),\\ (1-\alpha)\big[(1+r)+r(t-1)+\frac{1}{2}r^2(t-1)^2\big], & t\in[1,2),\\ (1-\alpha)\big[1+2r+\dfrac{1}{2}r^2+r(1+r)(t-2) & \\ \qquad +\frac{1}{2}r^2(t-2)^2+\frac{1}{6}r^3(t-2)^3\big], & t\in[2,3),\\ (1-\alpha)\big[\big(1+3r+r^2+\frac{1}{6}r^3\big)+r(1+2r+\frac{1}{2}r^2)(t-3) & \\ \qquad +\frac{1}{2}r^2(1+r)(t-3)^2+\frac{1}{6}r^3(t-3)^3+\frac{1}{18}r^4(t-3)^4\big], & t\in[3,4], \end{cases}$$

for each $\alpha \in [0,1]$.

It is easy to check that the inequalities $u^l_\alpha(t) \le u^r_\alpha(t)$ and $(u^l_\alpha(t))' \le (u^r_\alpha(t))'$ are true for every $t \in [-1,4]$ and every $\alpha \in [0,1]$. Hence, by Stacking Theorem, we infer that the fuzzy solution $u(t)$ of the problem (12) has level sets

$$[u(t)]^\alpha = [u^l_\alpha(t), u^r_\alpha(t)],$$

for all $t \in [-1,4]$.

Case 2: If u is (ii)-differentiable, then from Theorem 2.3, we have

$$[\mathcal{D}^g_H u(t)]^\alpha = [(u^r_\alpha(t))', (u^l_\alpha(t))'], \quad \text{for } t\in[0,4], \quad \alpha\in[0,1].$$

Similarly as Case 1, the problem (12) is also translated into the following system of impulsive functional differential equations:

$$\begin{cases} (u^l_\alpha(t))' = ru^r_\alpha(t-1), & t\in[0,4],\ t\neq t_k,\\ (u^r_\alpha(t))' = ru^l_\alpha(t-1), & t\in[0,4],\ t\neq t_k,\\ u^r_\alpha(t_k^+) = 2u^r_\alpha(t_k^-), & t_k=2k,\ k=1,2,\\ u^r_\alpha(t_k^+) = 2u^r_\alpha(t_k^-), & t_k=2k,\ k=1,2,\\ u^l_\alpha(t) = \alpha-1, & t\in[-1,0],\\ u^r_\alpha(t) = 1-\alpha, & t\in[-1,0]. \end{cases} \tag{14}$$

By the method of steps, we also obtain a solution of the system (14) on $[-1,4]$ as follows

$$u^l_\alpha(t) = \begin{cases} \alpha-1, & t\in[-1,0],\\ (\alpha-1)(1-rt), & t\in[0,1),\\ (\alpha-1)\big[(1-r)-r(t-1)+\frac{1}{2}r^2(t-1)^2\big], & t\in[1,2),\\ (\alpha-1)\big[1-2r+\dfrac{1}{2}r^2+r(1+r)(t-2) & \\ \qquad +\frac{1}{2}r^2(t-2)^2-\frac{1}{6}r^3(t-2)^3\big], & t\in[2,3),\\ (\alpha-1)\big[\big(1-3r+r^2-\frac{1}{6}r^3\big)-r(1-2r+\frac{1}{2}r^2)(t-3) & \\ \qquad -\frac{1}{2}r^2(1-r)(t-3)^2-\frac{1}{6}r^3(t-3)^3+\frac{1}{18}r^4(t-3)^4\big], & t\in[3,4] \end{cases}$$

and

$$u^r_\alpha(t) = \begin{cases} 1-\alpha, & t\in[-1,0],\\ (1-\alpha)(1-rt), & t\in[0,1),\\ (1-\alpha)\big[(1-r)-r(t-1)+\frac{1}{2}r^2(t-1)^2\big], & t\in[1,2),\\ (1-\alpha)\big[1-2r+\dfrac{1}{2}r^2+r(1+r)(t-2) & \\ \qquad +\frac{1}{2}r^2(t-2)^2-\frac{1}{6}r^3(t-2)^3\big], & t\in[2,3),\\ (1-\alpha)\big[\big(1-3r+r^2-\frac{1}{6}r^3\big)-r(1-2r+\frac{1}{2}r^2)(t-3) & \\ \qquad -\frac{1}{2}r^2(1-r)(t-3)^2-\frac{1}{6}r^3(t-3)^3+\frac{1}{18}r^4(t-3)^4\big], & t\in[3,4], \end{cases}$$

for each $\alpha \in [0,1]$.

It is easy to check that the inequalities $u^l_\alpha(t) \leq u^r_\alpha(t)$ and $(u^r_\alpha(t))' \leq (u^l_\alpha(t))'$ are true for every $t \in [-1,4]$ and every $\alpha \in [0,1]$. Hence, by Stacking Theorem, we infer that the fuzzy solution $u(t)$ of the problem (12) has level sets

$$[u(t)]^\alpha = [u^l_\alpha(t), u^r_\alpha(t)],$$

for all $t \in [-1,4]$.

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