

ASSESSING PROCESS PERFORMANCE WITH INCAPABILITY INDEX BASED ON FUZZY CRITICAL VALUE

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ABSTRACT. Process capability indices are considered as an important concept in statistical quality control. They have been widely used in the manufacturing industry to provide numerical measures on process performance. Moreover, some incapability indices have been introduced to account the process performance. In this paper, we focus on the one proposed by Chen [5]. In today's modern world, accurate and flexible information is needed. So, we apply fuzzy logic to measure the process incapability. Buckley's approach is used to fuzzify this index and to make a decision on process incapability, we utilize fuzzy critical value. Numerical examples are presented to demonstrate the performance and effectiveness of the proposed index.

1. Introduction

To assess manufacturing process capability, it have been proposed more indices. Those indices are effective tools for both process capability analysis and quality assurance. Numerous process capability indices (PCIs) have been proposed such as C_p , C_{pk} , C_{pm} and C_{pmk} . Vannman [33] presented a superstructure $C_p(u, v)$ containing these four basic indices, where $u, v \geq 0$. But when the tolerance interval is asymmetric, i.e., the target value is not the midpoint of the tolerance interval, these indices may fail to account the capability of the process. For these cases, some indices have been introduced such as C_{pm}^* . Furthermore, an incapability index, C_{pp} , was proposed by considering a simple transformation of the index C_{pm}^* . For asymmetric tolerances, the index C_{pp}'' was introduced that we will mention it in the next section.

In the literature, numerous papers have been published for statistical inference about process capability/incapability indices since 1980s. Especially there are many authors have studied process capability indices together with fuzzy set theory, such as Yongting [34], Lee, Wei, and Chang [24], Lee [25], Sadeghpour-Gildeh [31], Sadeghpour-Gildeh and Angoshtari [32], Hsu and Shu [8], Parchami, Mashinchi, Yavari and Maleki [26], Parchami, Mashinchi and Maleki [27], Parchami and Mashinchi [28, 29], Ramezani, Parchami and Mashinchi [30], Kahraman and Kaya [10, 11], Kaya [13], Kaya and Baracli [14], Kaya and Kahraman [15, 16, 17, 18, 19, 20, 21, 22, 23]. But only few authors investigated the process incapability

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index. So, this paper is aimed to estimate C_{pp}'' and fuzzify it by applying Buckley's approach and make a decision by using fuzzy critical value.

The structure of the rest of this paper is as follows. In the following section, some preliminary of capability indices and in the subsequent section, estimation of the index C_{pp}'' are presented. In section 4, we obtain cumulative distribution function of this index. Section 5 discusses fuzzy logic and reviews some basic definitions. We fuzzify this index in Section 6 and then, fuzzy decision-making procedure for detecting whether the process satisfies the current requirement is presented in section 7. Finally, two numerical examples are presented in section 8 to demonstrate the applicability of the proposed index and section 9 presents conclusions.

2. Preliminary

The two most commonly used process capability indices are C_p and C_{pk} [9, 12]. These two indices are defined as the following

$$C_p = \frac{USL - LSL}{6\sigma}, \quad (1)$$

$$C_{pk} = \min \left\{ \frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma} \right\}, \quad (2)$$

where the process follows normal distribution and μ is the process mean and σ is the process standard deviation, USL is the upper specification limit, LSL is the lower specification limit. The index C_p measures the process variation relative to the production tolerance, which reflects only the process potential. The index C_{pk} measures process performance based on the process loss. In fact, the design of C_{pk} is independent of the target value T , which can fail to account for process targeting. For this reason, Chan et al. [4] proposed the index C_{pm} that is based on Taguchi loss function [7]

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \quad (3)$$

where $d = (USL - LSL)/2$ and T is the target value which usually set to the midpoint of the specification limits ($T = M = (USL + LSL)/2$), i.e., the tolerance interval is symmetric. For the processes which the target value is not the midpoint of the specification limits, that is, the tolerance interval is asymmetric, C_{pm} index has some shortcoming.

Also, the index C_{pmk} was proposed which is a combination of two indices C_{pk} and C_{pm} as the following

$$C_{pmk} = \min \left\{ \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}} \right\}. \quad (4)$$

Vannman [33] introduced the superstructure $C_p(u, v)$ as

$$C_p(u, v) = \frac{d - u |\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \quad (5)$$

It is obvious that $C_p(0, 0) = C_p$, $C_p(1, 0) = C_{pk}$, $C_p(0, 1) = C_{pm}$ and $C_p(1, 1) = C_{pmk}$.

There have been some generalization of C_{pm} to deal with the processes with asymmetric tolerances. The first generalization shifts one of the two specification limits, so that the new tolerance interval is symmetric as $(T - d^*, T + d^*)$, where $d^* = \min\{D_l, D_u\}$, $D_l = T - LSL$ and $D_u = USL - T$ [4, 12]. Hence, the index C_{pm}^* is defined as:

$$C_{pm}^* = \frac{\min\{D_l, D_u\}}{3\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d^*}{3\sqrt{\sigma^2 + (\mu - T)^2}}. \quad (6)$$

Greenwich and Jahr-Schaffrath [6] considered a simple transformation of the C_{pm}^* index as follows:

$$C_{pp} = \left(\frac{\mu - T}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2, \quad (7)$$

where $D = d^*/3$. They defined the inaccuracy index as $C_{ia} = (\mu - T)^2/D^2$ and imprecision index as $C_{ip} = \sigma^2/D^2$. Thus, $C_{pp} = C_{ia} + C_{ip}$. They called C_{pp} , ‘‘incapability index’’. A small value of C_{pp} indicates that a process is more capable of meeting the required specifications than a process with a larger value of C_{pp} .

For asymmetric tolerances, Chen [5] considered a simple transformation of the index C_{pp} called C_{pp}'' which was defined as:

$$C_{pp}'' = \left(\frac{A}{D}\right)^2 + \left(\frac{\sigma}{D}\right)^2, \quad (8)$$

where $A = \max\{(\mu - T)d/D_u, (T - \mu)d/D_l\}$. By setting $C_{ia}'' = (A/D)^2$, we have $C_{pp}'' = C_{ia}'' + C_{ip}$. For symmetric tolerances, $A = |\mu - T|$. So, C_{ia}'' reduces to C_{ia} and then, $C_{pp}'' = C_{pp}$.

3. Estimation of the Index C_{pp}''

Let X_1, X_2, \dots, X_n be a random sample of size n from a normal distribution with mean μ and variance σ^2 measuring the characteristic under investigation. To estimate the index C_{pp}'' , it must be estimated A^2 and σ^2 and is denoted as:

$$\hat{C}_{pp}'' = \frac{\hat{A}^2}{D^2} + \frac{\hat{\sigma}^2}{D^2} = \frac{\hat{A}^2}{D^2} + \frac{S_n^2}{D^2}, \quad (9)$$

where $\hat{A} = \max\{d(\bar{X} - T)/D_u, d(T - \bar{X})/D_l\}$, the mean μ is estimated by the sample mean $\bar{X} = \sum_{i=1}^n X_i/n$ and the variance σ^2 is estimated by the sample variance $S_n^2 = \sum_{i=1}^n (X_i - \bar{X})^2/n$. The statistics \bar{X} and S_n^2 are the maximum likelihood estimators (MLEs) of μ and σ^2 . So, \hat{A} is MLE of A and then, \hat{C}_{pp}'' is MLE of C_{pp}'' .

4. Cumulative Distribution Function (CDF) of \hat{C}_{pp}''

Equation (9) can be expressed as

$$\hat{C}_{pp}'' = \frac{\sigma^2}{nD^2} \left(\frac{Yd^2}{D_u^2} + K \right), \quad (10)$$

where $K = nS_n^2/\sigma^2$, $Z = \sqrt{n}(\bar{X} - T)/\sigma$ and $Y = Z^2$. Therefore, K is distributed as chi-square with $n - 1$ degree of freedom and Z is distributed normally by non-central parameter $\delta = \sqrt{n}(\mu - T)/\sigma$, then Y is distributed as non-central chi-square. Hence for $\bar{X} > T$, it can be denoted as following

$$F_{\hat{C}_{pp}''}(x) = \int_0^{\frac{\sqrt{n}D_u D \sqrt{x}}{d\sigma}} G\left(\frac{nD^2}{\sigma^2}x - \frac{t^2 d^2}{D_u^2}\right) [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})] dt, \quad (11)$$

where $G(\cdot)$ is the CDF of the χ^2 distribution with $n - 1$ degree of freedom and $\phi(\cdot)$ is the PDF (probability density function) of the standard normal distribution. For the recommended sample sizes of $n \geq 20$ by Boyles [1], the PDF and CDF of \hat{C}_{pp}'' are approximations if we replace ξ by $\hat{\xi} = (\bar{X} - T)/S_n$.

Proof.

$$\begin{aligned} F_{\hat{C}_{pp}''}(x) &= P(\hat{C}_{pp}'' \leq x) \\ &= \int_0^\infty P(\hat{C}_{pp}'' \leq x | Y = y) f_Y(y) dy \\ &= \int_0^\infty P\left(\frac{yd^2}{D_u^2} + K < \frac{nD^2}{\sigma^2}x\right) f_Y(y) dy \\ &= \int_0^{\frac{nD_u^2 D^2}{d^2 \sigma^2}x} F_K\left(\frac{nD^2}{\sigma^2}x - \frac{yd^2}{D_u^2}\right) f_Y(y) dy \\ &= \int_0^{\frac{\sqrt{n}D_u D \sqrt{x}}{d\sigma}} G\left(\frac{nD^2}{\sigma^2}x - \frac{t^2 d^2}{D_u^2}\right) [\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})] dt \end{aligned}$$

□

5. Fuzzy Set Theory and Basic Definitions

Crisp means dichotomous, that is, yes-or-no type rather than more-or-less type. In traditional dual logic, for instance, a statement can be true or false-and nothing in between. Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known and that there are no doubts about their values or their occurrence [36]. But, in the real world there are many situations that we cannot cluster the parameters exactly.

Fuzzy sets were introduced by Zadeh [35] to manipulate data and information possessing nonstatistical uncertainties. Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. Here, we present some basic definitions.

Definition 5.1. Suppose X is a nonempty set. A fuzzy set \tilde{A} in X is characterized by its membership function $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ and $\mu_{\tilde{A}}(x)$ is interpreted as membership grade of element x in fuzzy set A for each $x \in X$. Also this function can be denoted by \tilde{A} , that is, $\tilde{A} : X \rightarrow [0, 1]$. In this paper, we use the second notation.

Definition 5.2. Let \tilde{A} be a fuzzy subset of X ; the support of \tilde{A} , denoted by $supp(\tilde{A})$, is the crisp subset of X whose elements all have nonzero membership grades in \tilde{A} as the following

$$supp(\tilde{A}) = \{x \in X \mid \tilde{A}(x) > 0\}.$$

Definition 5.3. A fuzzy set \tilde{A} is called

- (a) Normal, if there exist an $x \in X$ such that $\tilde{A}(x) = 1$.
- (b) Convex, if $\tilde{A}(\lambda x + (1 - \lambda)y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$, $\forall x, y \in R$, $\forall \lambda \in [0, 1]$.

Definition 5.4. An α -level set of a fuzzy set \tilde{A} of X is a non-fuzzy set denoted by \tilde{A}_α as follows

$$\tilde{A}_\alpha = [l_{\tilde{A}}(\alpha), \psi_{\tilde{A}}(\alpha)] = \{x \in X \mid \tilde{A}(x) \geq \alpha\}.$$

Definition 5.5. A fuzzy number is a normal and convex fuzzy set of R which its membership function is piecewise continuous.

Definition 5.6. A fuzzy number \tilde{A} is an $L-R$ fuzzy number if it can be expressed as

$$\tilde{A}(x) = \begin{cases} L(\frac{\alpha-x}{\alpha}) & x \leq \alpha, \\ R(\frac{x-\alpha}{\beta}) & x \geq \alpha. \end{cases}$$

where L and R are strictly decreasing functions which are defined on $[0, 1]$ and satisfy the following conditions

- (1) $L(0) = R(0) = 1$.
- (2) $L(1) = R(1) = 0$.
- (3) $0 < L(x) < 1, 0 < R(x) < 1, x \neq \{0, 1\}$.

Buckley [2, 3] proposed a new method in fuzzy statistics to estimate the parameters. The fuzzy situation of the estimators of the parameters is obtained by a set of confidence intervals of the parameters, one on top of the other, and so, the triangular shaped fuzzy numbers of the estimators are formed.

6. Fuzzifying the Index \hat{C}_{pp}''

In order to estimate process incapability index in real world applications, it is needed to estimate parameters based on the information from the sample data. In this section, we apply Buckley's approach to find the fuzzy estimators of process mean and variance and then, process incapability index.

6.1. Membership Function of Fuzzy Sample Mean and Variance. By the notion of Buckley's approach, we consider the confidence interval of μ as β -cut intervals of fuzzy \bar{X} and the confidence interval of σ^2 as β -cut intervals of fuzzy S_n^2 (for $\beta \in (0, 1]$). Therefore, we have

$$\begin{aligned} \tilde{\bar{X}}(\beta) &= [l_{\tilde{\bar{X}}}(\beta), \psi_{\tilde{\bar{X}}}(\beta)] \\ &= [\bar{X} - t_{1-\frac{\beta}{2}, n-1} \frac{S_n}{\sqrt{n-1}}, \bar{X} + t_{1-\frac{\beta}{2}, n-1} \frac{S_n}{\sqrt{n-1}}], \end{aligned} \quad (12)$$

and

$$\widetilde{S}_n^2(\beta) = [l_{\widetilde{S}_n^2}(\beta), \psi_{\widetilde{S}_n^2}(\beta)] = \left[\frac{nS_n^2}{\chi_{1-\frac{\beta}{2}, n-1}^2}, \frac{nS_n^2}{\chi_{\frac{\beta}{2}, n-1}^2} \right], \quad (13)$$

where $t_{1-\frac{\beta}{2}, n-1}$ and $\chi_{1-\frac{\beta}{2}, n-1}^2$ are, respectively, the points on the left side of the t-student density and chi-square density with $n - 1$ degrees of freedom where the probability of exceeding it is $(1 - \frac{\beta}{2})$.

Now place these β -cut intervals, one on top up of the other, to produce triangular shaped fuzzy numbers \widetilde{X} and \widetilde{S}_n^2 , respectively as shown in Figure 1.

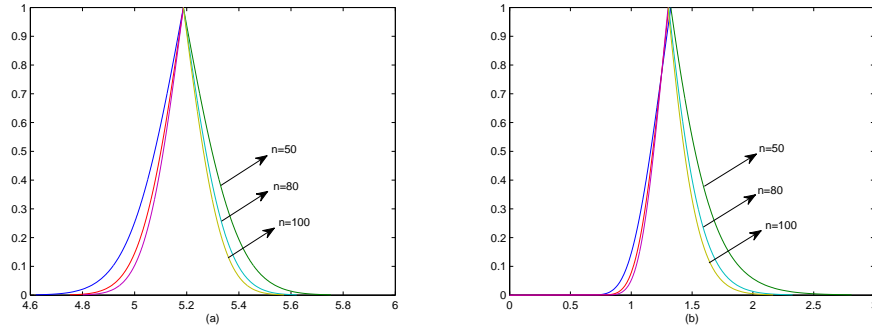


FIGURE 1. The Membership Function of (a) Fuzzy Sample Mean, (b) Fuzzy Sample Variance, at $\bar{x} = 5.187$, $s_n^2 = 1.281$ and for Various Values of Sample Size

6.2. Membership Function of Fuzzy Incapability Index Estimator. Based on equation (8) and the method proposed by Hsu and Shu [8], by choosing $g \in \widetilde{X}(\beta)$ and $h \in \widetilde{S}_n^2(\beta)$ we obtain

$$\widehat{C}_{pp}''(g, h) = \left(\frac{A(g)}{D}\right)^2 + \left(\frac{h}{D}\right)^2, \quad (14)$$

where $A(g) = \max\{(g-T)d/D_u, (T-g)d/D_l\}$. Then, β -cut intervals of estimator \widehat{C}_{pp}'' are as follows

$$\widetilde{C}_{pp}''(\beta) = [l_{\widetilde{C}_{pp}''}(\beta), \psi_{\widetilde{C}_{pp}''}(\beta)]. \quad (15)$$

Hence, we find the lower and upper limits of β -cut intervals for two following cases as the following

Case 1: For $\bar{X} \leq T$, we get

$$l_{\widetilde{C}_{pp}''}(\beta) = \frac{[(T - (\bar{X} + t_{1-\frac{\beta}{2}, n-1} \frac{S_n}{\sqrt{n-1}}))d]^2}{D_l^2 D^2} + \frac{nS_n^2}{D^2 \chi_{1-\frac{\beta}{2}, n-1}^2}, \quad (16)$$

$$\psi_{\widetilde{C}_{pp}''}(\beta) = \frac{[(T - (\bar{X} - t_{1-\frac{\beta}{2}, n-1} \frac{S_n}{\sqrt{n-1}}))d]^2}{D_l^2 D^2} + \frac{nS_n^2}{D^2 \chi_{\frac{\beta}{2}, n-1}^2}. \quad (17)$$

Case 2: For $\bar{X} > T$, we get

$$l_{\widetilde{C}_{pp}''}(\beta) = \frac{[(\bar{X} - t_{1-\frac{\beta}{2}, n-1} \frac{S_n}{\sqrt{n-1}} - T)d]^2}{D_u^2 D^2} + \frac{nS_n^2}{D^2 \chi_{1-\frac{\beta}{2}, n-1}^2}, \quad (18)$$

$$\psi_{\widetilde{C}_{pp}''}(\beta) = \frac{[(\bar{X} + t_{1-\frac{\beta}{2}, n-1} \frac{S_n}{\sqrt{n-1}} - T)d]^2}{D_u^2 D^2} + \frac{nS_n^2}{D^2 \chi_{\frac{\beta}{2}, n-1}^2}. \quad (19)$$

Now we set these β -cut intervals, one on the top up the other, to make triangular shaped fuzzy number \widetilde{C}_{pp}'' as shown in Figure 2.

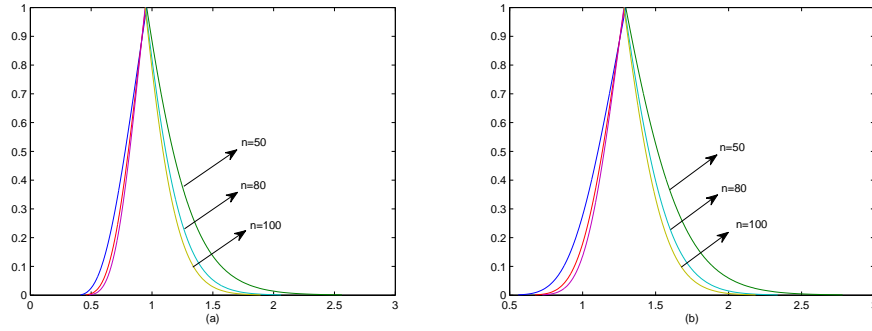


FIGURE 2. The Membership Function of \widetilde{C}_{pp}'' at (a) $\bar{x} = 3.510$, (b) $\bar{x} = 5.187, s_n^2 = 1.281, LSL = 0, T = 4, USL = 10$ and for Various Values of Sample Size

7. Fuzzy Decision-making

To determine whether a process meets the present capability requirement and runs under the desired quality condition, we can consider the following statistical hypothesis testing, $H_0 : C_{pp}'' \geq c$ versus $H_1 : C_{pp}'' < c$. Process fails to meet the capability requirement if $C_{pp}'' \geq c$, and meets the capability requirement if $C_{pp}'' < c$.

To investigate process capability and make a decision on manufacturing process capability, it can be done by critical value that we study this method for fuzzy situation.

The critical value c_0 can be determined by the following with α -risk $\alpha(c_0) = \alpha$ (the fortune of incorrectly judging an incapable process as capable). When $\bar{X} > T$, the critical value c_0 can be obtained by solving the following equation

$$\begin{aligned}
P(\hat{C}_{pp}'' \leq c_0 | C_{pp}'' = c) &= \int_0^{\frac{D_u \sqrt{n(\frac{\hat{\xi}^2 d^2}{D_u^2} + 1)} c_0}{d\sqrt{c}}} G\left(\frac{n(\frac{\hat{\xi}^2 d^2}{D_u^2} + 1)c_0}{c} - \frac{t^2 d^2}{D_u^2}\right) [\phi(t + \hat{\xi}\sqrt{n}) + \phi(t - \hat{\xi}\sqrt{n})] dt \\
&= \alpha.
\end{aligned} \tag{20}$$

To test process incapability with critical value for assessing process performance, it must be done by five steps as follows:

- **Step 1:** Set the value of c and the α -risk.
- **Step 2:** Place $\widetilde{C}_{pp}''(\beta) = [l_{\widetilde{C}_{pp}''}(\beta), \psi_{\widetilde{C}_{pp}''}(\beta)]$, one on the top up the other to construct the triangular shaped fuzzy number \widetilde{C}_{pp}'' and produce its the membership function.
- **Step 3:** Find the critical value c_0 , given c , α -risk and n from equation (20).
- **Step 4:** Determine β , a certain degree of vagueness on sample data.
- **Step 5:** Conclude as follows:
 - a. If $l_{\widetilde{C}_{pp}''}(\beta) > c_0$, then the process is incapable.
 - b. If $\psi_{\widetilde{C}_{pp}''}(\beta) < c_0$, then the process is capable.
 - c. If $l_{\widetilde{C}_{pp}''}(\beta) \leq c_0 \leq \psi_{\widetilde{C}_{pp}''}(\beta)$, then don't decide and further study is needed.

8. Numerical Examples

In this section, two examples are presented to demonstrate the performance of the proposed index. The first example is related to a norm given by a researcher, and the seconde one is a real-word case in a piston manufacturer that locates on Konya's industrial area, Turkey.

Example 8.1. Based on the norm given in the paper by Chen [5], we generate 50 data for one process that the lower and the upper specification limits are set to $LSL = 0$ and $USL = 10$, and the target value is $T = 7.5$. For this process, if $C_{pp}'' \geq 1.7$, then the process is called "incapable". The mean and standard deviation of the sample generated are as $\bar{x} = 8.25$ and $s_n = 1.25$, respectively. Then, β -cut intervals of fuzzy mean and fuzzy variance are as follows

$$\begin{aligned}
\widetilde{X}(\beta) &= [8.25 - 0.1786t_{1-\frac{\beta}{2},49}, 8.25 + 0.1786t_{1-\frac{\beta}{2},49}], \\
\widetilde{S}_n^2(\beta) &= \left[\frac{78.1250}{\chi_{1-\frac{\beta}{2},49}^2}, \frac{78.1250}{\chi_{\frac{\beta}{2},49}^2} \right].
\end{aligned}$$

Figure 3 shows the membership functions of the fuzzy numbers \widetilde{X} and \widetilde{S}_n^2 .

From two above equations, we calculate β -cuts of fuzzy estimator of C_{pp}'' as follows

$$l_{\widetilde{C}_{pp}''}(\beta) = 5.7605(0.75 - 0.1786t_{1-\frac{\beta}{2},49})^2 + \frac{112.5090}{\chi_{1-\frac{\beta}{2},49}^2},$$

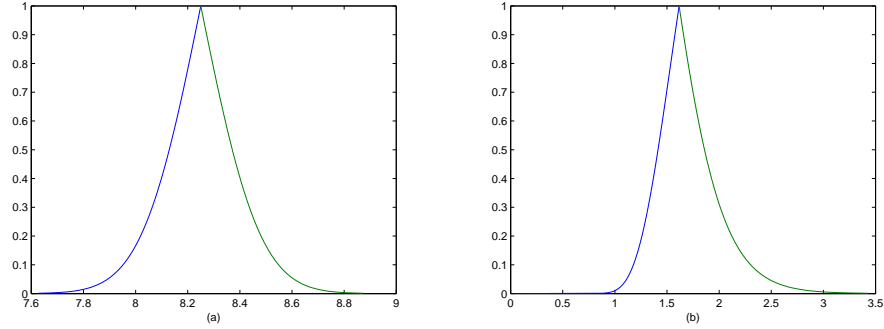


FIGURE 3. The Membership Function of (a) Fuzzy Mean, (b) Fuzzy Variance

$$\psi_{\widetilde{C}_{pp}''}(\beta) = 5.7605(0.75 + 0.1786t_{1-\frac{\beta}{2},49})^2 + \frac{112.5090}{\chi_{\frac{\beta}{2},49}^2}.$$

The membership function of the fuzzy number \widetilde{C}_{pp}'' is shown in Fig. 4.

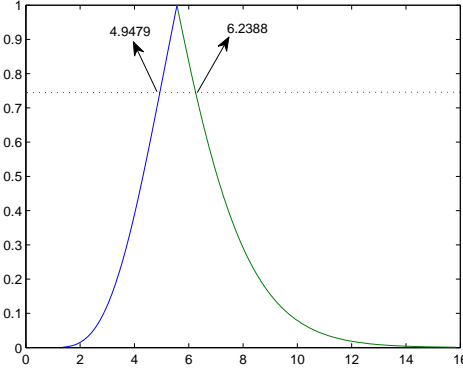


FIGURE 4. The Membership Function of Fuzzy Incapability Index

Based on equation (11), we obtain the cumulative distribution function of \widetilde{C}_{pp}'' as

$$F_{\widetilde{C}_{pp}''}(x) = \int_0^{2.3569\sqrt{y}} \int_0^{22.2204y-4t^2} \frac{1}{(23!)2^{24}\sqrt{2\pi}} x^{23} e^{-\frac{x}{2}} \times [e^{-\frac{(t+4.2426)^2}{2}} + e^{-\frac{(t-4.2426)^2}{2}}] dx dt.$$

As it was mentioned, the process is supposed to be incapable if $C_{pp}'' \geq 1.7$. So, the hypothesis are as $H_0 : C_{pp}'' \geq 1.7$ and $H_1 : C_{pp}'' < 1.7$. Then, to make a

decision, we set $\alpha = 0.05$ and $c = 1.7$ and obtain critical value by solving the following equation

$$\begin{aligned} P(\hat{C}_{pp}'' \leq c_0 | C_{pp}'' = c) &= \int_0^{\frac{5.5227\sqrt{c_0}}{\sqrt{c}}} \int_0^{\frac{122c_0}{c} - 4t^2} \left[\frac{1}{(23!)2^{24}\sqrt{2\pi}} x^{23} e^{-\frac{x}{2}} \right. \\ &\quad \left. \times \left[e^{-\frac{(t+4.2426)^2}{2}} + e^{-\frac{(t-4.2426)^2}{2}} \right] \right] dx dt \\ &= \alpha. \end{aligned}$$

By using Simpson quadrature integration technique, we get $c_0 = 3.11065$, which is the critical value. Based on the knowledge of the production engineers, a certain degree of imprecision on sample data is set to $\beta = 0.75$. Hence, the 0.75-level set of the fuzzy incapability index \widetilde{C}_{pp}'' is

$$[l_{\widetilde{C}_{pp}''}(0.75) \quad \psi_{\widetilde{C}_{pp}''}(0.75)] = [4.9479 \quad 6.2388],$$

as determined in Fig.4. Since $l_{\widetilde{C}_{pp}''}(0.75) > c_0$, we conclude that the process is “incapable”.

Furthermore, we can find the effect of β in our decision, as shown in Table 1. For $\beta \geq 0.15$, we conclude that the process is “incapable”, and for $\beta < 0.15$ we don't decide and further study is needed, i.e., we should take another sample and do the decision making procedure.

Now, suppose we want to do classical hypothesis testing on C_{pp}'' . The null hypothesis $H_0 : C_{pp}'' \geq 1.7$ will be rejected at significant level α if the estimated value of the index C_{pp}'' based on the random sample is less than the critical value c_0 , i.e., test function is as the following

$$\delta(X) = \begin{cases} 1 & \hat{C}_{pp}'' < c_0, \\ 0 & \text{O.W.} \end{cases}$$

Based on the equation (9), C_{pp}'' is obtained as 5.49. As we obtained in the above part, the critical value is $c_0 = 3.11065$, so, $C_{pp}'' > c_0$. Then, we can not reject the null hypothesis at level 0.05. Hence, the process is considered as “incapable”.

The above discussion shows that in the classical scheme, the process is incapable while in the fuzzy scheme, the process is incapable with the membership grade 0.15 or more, and for membership grade less than 0.15, we can not have a decision and more study is needed. Therefore, the new approach is sensitive to imprecision degree and has more information than the classical one.

Example 8.2. Here, we apply our proposed index on the data set which is used by Kaya and Kahraman [23]. The data are for a piston manufacturer. Its main products are piston, liner, and piston ring. In application, a Volvo Marin motor's piston for engine is selected. The measurable characteristics of the piston are such as piston diameter, compression height, bowl depth, total length, combustion chamber

β	$l_{\widehat{C}_{pp}''}$	$\psi_{\widehat{C}_{pp}''}$	β	$l_{\widehat{C}_{pp}''}$	$\psi_{\widehat{C}_{pp}''}$	β	$l_{\widehat{C}_{pp}''}$	$\psi_{\widehat{C}_{pp}''}$
0.95	5.4420	5.6959	0.60	4.5732	6.7004	0.25	3.5515	8.2544
0.90	5.3175	5.8263	0.55	4.4444	6.8706	0.20	3.3592	8.6109
0.85	5.1940	5.9598	0.50	4.3122	7.0519	0.15	3.1375	9.0557
0.80	5.0710	6.0970	0.45	4.1758	7.2468	0.10	2.8654	9.6600
0.75	4.9479	6.2388	0.40	4.0336	7.4586	0.05	2.4833	10.6491
0.70	4.8243	6.3859	0.35	3.8840	7.6920			
0.65	4.6996	6.5393	0.30	3.7245	7.9538			

 TABLE 1. β -cut Bounds of Fuzzy Estimator of \widehat{C}_{pp}''

diameter, pin diameter and pin length. In this application, the characteristic piston diameter is investigated by using fuzzy incapability index.

The specification limits are as $LSL = 130.150$ and $USL = 130.208$, and the target value is $T = 130.180$. A sample of 200 data is gathered and the sample mean and variance of piston diameters are obtained as 130.180 and 0.00009, respectively. The β -cut sets of fuzzy mean and fuzzy variance are as the following

$$\widetilde{X}(\beta) = [130.180 - 0.0006725 t_{1-\frac{\beta}{2}, 199}, 130.180 + 0.0006725 t_{1-\frac{\beta}{2}, 199}],$$

$$\widetilde{S}_n^2(\beta) = \left[\frac{0.018}{\chi_{1-\frac{\beta}{2}, 199}^2}, \frac{0.018}{\chi_{\frac{\beta}{2}, 199}^2} \right].$$

Fig. 5 shows the membership functions of these two fuzzy numbers.

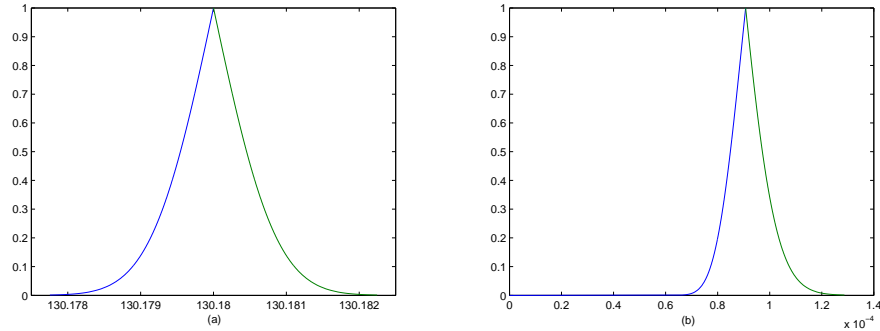


FIGURE 5. The Membership Function of (a) Fuzzy Mean, (b) Fuzzy Variance of Piston Diameter

Therefore, β -cut sets of the fuzzy estimator of C_{pp}'' are as what follows

$$l_{\widehat{C}_{pp}''}(\beta) = 0.00485 t_{1-\frac{\beta}{2}, 199}^2 + \frac{208.116}{\chi_{1-\frac{\beta}{2}, 199}^2},$$

$$\psi_{\widetilde{C}_{pp}''}(\beta) = 0.00485 t_{1-\frac{\beta}{2}, 199}^2 + \frac{208.116}{\chi_{\frac{\beta}{2}, 199}^2}.$$

The membership function of the fuzzy number \widetilde{C}_{pp}'' is shown in Fig. 6.

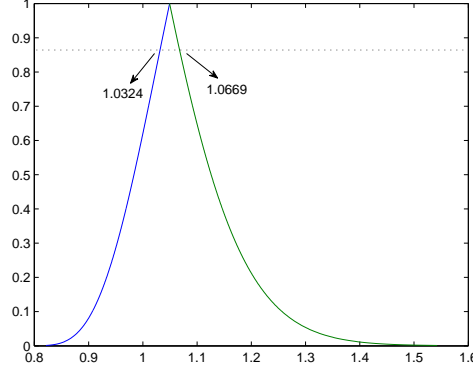


FIGURE 6. The Membership Function of Fuzzy Incapability Index of Piston Diameter

Now, let the process is supposed to be capable if $C_{pp}'' \geq 1.33$. Set $\alpha = 0.05$ and $c = 1.33$. Then, the critical value is obtained by the following equation

$$\begin{aligned} P(\widehat{C}_{pp}'' \leq c_0 | C_{pp}'' = c) \\ = \int_0^{\frac{13.6545\sqrt{c_0}}{\sqrt{c}}} \int_0^{\frac{200c_0}{c} - 1.073t^2} \frac{1}{(98!)2^{98}\sqrt{2\pi}} x^{98} e^{-\frac{x}{2} - \frac{t^2}{2}} dx dt = \alpha. \end{aligned}$$

By using Simpson quadrature integration technique, the critical value is obtained as $c_0 = 1.8466$. If we use 0.87 as the degree of impression on sample data, the 0.87-cut set of the fuzzy incapability index \widetilde{C}_{pp}'' is obtained as follows

$$[l_{\widetilde{C}_{pp}''}(0.87), \psi_{\widetilde{C}_{pp}''}(0.87)] = [1.0324, 1.0669],$$

as is determined in Fig.6. Since $\psi_{\widetilde{C}_{pp}''}(0.87) < c_0$, we conclude that the process is “capable”.

9. Conclusion

Process capability indices (PCIs) have been successfully developed and used by companies to dominate and compete in the high-profit markets for improving the quality and the productivity. Among the proposed PCIs, the process incapability index C_{pp}'' is one of the effective tools for the assessment of process capability. In this paper, a method for incorporating the fuzzy inference with a process incapability index is presented to evaluate production processes. A set of confidence interval of sample mean and variance is used to produce triangular shaped fuzzy number for the estimation of the process incapability index C_{pp}'' . Using Buckley’s approach

with some modification, the fuzzy hypothesis testing for process performance using the fuzzy estimator of C''_{pp} is proposed. Two examples are given to demonstrate that the method presented is effective and feasible.

As a whole, the new approach has some advantages. It is more sensitive than the crisp (traditional) method and also, can be reduced to crisp case in which $\beta = 1$. Furthermore, by using fuzzy set, It is much more simple to find a confidence set for C''_{pp} , but based on traditional methods it is difficult.

It is noted that there are some limitations about the new proposed index. Sometimes, one cannot make a decision about the proposed hypotheses based on the sample gathered, and further study is needed. Also, using Buckley's approach for some distributions to construct α -cut set and the membership function is not easy.

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