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MINIMAL AND STATEWISE MINIMAL INTUITIONISTIC GENERAL L-FUZZY AUTOMATA

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This paper is dedicated to Professor L. A. Zadeh on the occasion of his 95th birthday and the 50th year of the birth of fuzzy logic

Abstract. In this note, by considering the notions of the intuitionistic general L-fuzzy automaton and (α, β)-language, we show that for any (α, β)-language $\mathcal{L}$, there exists a minimal intuitionistic general L-fuzzy automaton recognizing $\mathcal{L}$. We prove that the minimal intuitionistic general L-fuzzy automaton is isomorphic with threshold (α, β) to any (α, β)-reduced max-min intuitionistic general L-fuzzy automaton. Also, we show that for any strong deterministic max-min intuitionistic general L-fuzzy automaton there exists a statewise (α, β)-minimal intuitionistic general L-fuzzy automaton. In particular, a connection between the minimal and statewise (α, β)-minimal intuitionistic general L-fuzzy automaton is presented. Also, for a given intuitionistic general L-fuzzy automaton, we present two algorithms, which determines states of the minimal intuitionistic general L-fuzzy automaton and the statewise (α, β)-minimal intuitionistic general L-fuzzy automaton. Finally, by giving some examples, we comparison minimal intuitionistic general L-fuzzy automaton and statewise (α, β)-minimal intuitionistic general L-fuzzy automaton.

1. **Introduction**

Fuzzy automaton was introduced by Wee [24] in 1967 and Santos [19] in 1968. Fuzzy finite automata have many applications in different branches of science, for example in learning system, pattern recognition, neural networks and database theory [6, 8, 11, 13, 14, 17, 25]. The intuitionistic fuzzy sets introduced by Atanassov [1] have been found to be highly useful to deal with vagueness. Atanassov by adding non-membership value, which may express more accurate and flexible information as compared with fuzzy sets. Using the notion of intuitionistic fuzzy sets, Jun [9] introduced the notion of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines. Based on the papers [9, 10], Zhang and Li [27] discussed intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata. Atanassov and Stoeva generalized the concept of IFS to Intuitionistic L-fuzzy sets [2] where L is an appropriate lattice. Thus on the basis of lattice-valued intuitionistic fuzzy sets, Yang et al. [26] introduced the concept of lattice-valued intuitionistic fuzzy finite state machines. In 2004, Doostfatemeh and Kremer [5] extended the notion of fuzzy automaton and gave the notion of general fuzzy automaton. Their

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key motivation of introducing the notion general fuzzy automaton was the insufficiency of the current literature to handle the applications which rely on fuzzy automaton as a modeling tool, assigning membership values to active states of a fuzzy automaton, resolve the multi -membership. Another important insufficiency of the current literature is the lack of methodologies which enable us to define and analyze the continuous operation of fuzzy automaton. In 2015, Shamsizadeh and Zahedi gave the notion of max-min intuitionistic general fuzzy automaton [21, 20]. The present authors [22] gave the notion of max-min intuitionistic general L-fuzzy automaton (IGLFA) and (α, β)-language for an IGLFA.

Note that the state minimization is a fundamental problem in automaton theory. So, it is important to study in the intuitionistic fuzzy automaton. There are many papers on the minimization problem of fuzzy finite automaton. For example, minimization of the mealy type of fuzzy finite automaton is discussed in [4], minimization of fuzzy finite automaton with crisp final states without outputs is studied in [3], minimization of deterministic finite automaton with fuzzy (final) states in [12] for more information see [16, 15, 18, 23].

In this paper first, we prove that for any (α, β)-language, there exists a minimal intuitionistic general L-fuzzy automaton. Also, we give an algorithm which computes the states of the minimal automaton with time complexity $O(|Q|^{|Q|+3}|X|^{|Q|+1} |Z|)$, where X is the set of alphabet, Q is the set of states and Z is the set of outputs. We show that the minimal IGLFA is isomorphic with threshold (α, β) to any (α, β)-reduced, (α, β)-complete, (α, β)-accessible, deterministic max-min IGLFA. Also, we prove that the minimal IGLFA is an (α, β)-reduced.

In section 4, we present an algorithm to generate the statewise minimal of the max-min IGLFA with time complexity $O(|X||Q|^4)$. Furthermore, we show that for any strong deterministic max-min IGLFA there exists a statewise (α, β)-minimal IGLFA. Moreover, if $\tilde{F}^*$ is an (α, β)-complete, (α, β)-accessible, deterministic max-min IGLFA and it is recognizing (α, β)-language $\mathcal{L}$, then the minimal $\tilde{F}^*_{\mathcal{L}}$ is homomorphism with threshold (α, β) to statewise (α, β)-minimal $\tilde{F}^*_m$, where $\tilde{F}^*_m$ is statewise (α, β)-equivalent to $\tilde{F}^*$.

2. **Preliminaries**

We give some definitions which are necessary for the other parts.

Definition 2.1. [1] Let A in E is given. An intuitionistic fuzzy set (IFS) A^+ on E is an object of the following form

$$A^+ = \{\langle x, \mu_A(x), \nu_A(x)\rangle \big| x \in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the values of membership and non-membership of element x in E to the set A, respectively. For every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

In what follows, we denote the set A^+ by A.

Let $L = (L, \leq_L, 0, 1)$ be a bounded (complete) lattice. An L-fuzzy set A on E is function $A : E \to L$ [7].

Definition 2.2. [2] Let X be a nonempty set and L be a bounded lattice with an involutive order reversing unary operation $N : L \to L$. An intuitionistic L-fuzzy set (ILFS) is an object of the form $A = \{(x, \mu(x), \nu(x)) | x \in E\}$, where μ and ν are functions $\mu : E \to L, \nu : E \to L$ in which for every $x \in X$, $\mu(x) \leq N(\nu(x))$.

In the rest of paper, it is assumed that $L = (L, \leq_L, T, S, 0, 1)$ is a bounded lattice, where endowed with an Lt-norm T, an Lt-conorm S, the least element 0 and the greatest element 1, also with an involutive order reversing unary operation $N : L \to L$.

Notation 1. Let $A, B \in L$. We define $A <_L B$ if and only if $A \leq_L B$ and $A \neq B$. Also, suppose that $A \geq_L B$ if and only if $B \leq_L A$.

Definition 2.3. [22] An intuitionistic general L-fuzzy automaton (IGLFA) $\tilde{F}$ is a ten-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$, where

(1) Q is a set of states,
(2) X is a finite set of input symbols, $X = \{a_1, a_2, ..., a_m\}$,
(3) $\tilde{R}$ is the ILFS of start states, $\tilde{R} = \{(q, \mu^{t_0}(q), \nu^{t_0}(q)) \big| q \in R\}$, where R is a finite subset of Q,
(4) Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_l\}$,
(5) $\tilde{\delta} : (Q \times L \times L) \times X \times Q \to L \times L$ is the augmented transition function,
(6) $\tilde{\omega} : (Q \times L \times L) \times Z \to L \times L$ is the output function,
(7) $F_1 = (F_1^T, F_1^S)$, where $F_1^T : L \times L \to L$ is an Lt-norm which is called the membership assignment function. Furthermore, $F_1^S : L \times L \to L$ is an Lt-conorm, where is the dual of F_1^T respect to involutive negation and it is called non-membership assignment function.
(8) $F_2 = (F_2^T, F_2^S)$, where $F_2^T : L \times L \to L$ is an Lt-norm and which is called the membership assignment output function. Moreover, $F_2^S : L \times L \to L$ is an Lt-conorm which is called non-membership assignment output function, where it is the dual of F_2^T respect to the involutive negation.
(9) $F_3 = (F_3^{TS}, F_3^{ST})$, where $F_3^{ST} : L^* \to L$ is an Lt-norm and is called the multi-non-membership function. Also, $F_3^{TS} : L^* \to L$ is an Lt-conorm, where it is the dual of F_3^{ST} respect to the involutive negation and it is called the multi-membership function.
(10) $F_4 = (F_4^{TS}, F_4^{ST})$, where $F_4^{ST} : L^* \to L$ is an Lt-norm and is called the multi-non-membership output function. Moreover, $F_4^{TS} : L^* \to L$ is an Lt-conorm, where it is the dual of F_4^{ST} respect to the involutive negation and is multi-membership output function.

Let $Q_{act}(t_i)$ be the set of all active states at time t_i for every $i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q), \nu^{t_i}(q)) \big| \exists (q', \mu^{t_{i-1}}(q'), \nu^{t_{i-1}}(q')) \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta, \mu^{t_i}(q) >_L 0\}$, for every positive integer i.

Since $Q_{act}(t_i)$ is an ILFS, to show that a state q belongs to $Q_{act}(t_i)$, we write $q \in Domain(Q_{act}(t_i))$ and for simplicity of notation we denote it by $q \in Q_{act}(t_i)$.

We assume that $\mu^{t_0}(q) = 0$ and $\nu^{t_0}(q) = 1$, for every $q \in Q$ such that $q \notin \tilde{R}$ and $\mu^{t_0}(q) >_L 0$, for every $q \in \tilde{R}$.

In the sequel, we assume that the max-min IGLFA has a finite number of states.

Definition 2.4. [22] Let $\tilde{F}$ be an IGLFA. We define the max-min intuitionistic general L-fuzzy automaton $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$, where $\tilde{\delta}^* : Q_{act} \times X^* \times Q \to L \times L$, $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), ...\}$ and for every $i \geq 0$,

$$\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), \Lambda, p) = \begin{cases} 1 & \text{if p=q} \\ 0 & \text{otherwise} \end{cases}, \tag{1}$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), \Lambda, p) = \begin{cases} 0 & \text{if p=q} \\ 1 & \text{otherwise} \end{cases}. \tag{2}$$

Also, for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_1((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and $\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_2((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and recursively,

$$\begin{aligned} &\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1u_2...u_n, p) = \\ &\quad \vee \{\tilde{\delta}_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}_1((p_1, \mu^{t_1}(p_1), \nu^{t_1}(p_1)), u_2, p_2) \wedge ... \\ &\quad \wedge \tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{3}$$

$$\begin{aligned} &\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1u_2...u_n, p) = \\ &\quad \wedge \{\tilde{\delta}_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_2((p_1, \mu^{t_1}(p_1), \nu^{t_1}(p_1)), u_2, p_2) \vee ... \\ &\quad \vee \tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) \big| p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{4}$$

in which $u_i \in X$ for every $1 \leq i \leq n$ and u_{i+1} is the entered input at time t_i for every $0 \leq i \leq n-1$.

Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then the cardinality of $\tilde{F}^*$ is defined by $|\tilde{F}^*| = |Q|$. In the rest of this paper, we denote every max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ with $\tilde{F}^*$. Also, for every bounded lattice L, we suppose that $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.

Definition 2.5. [22] Let $\tilde{F}^*$ be a max-min IGLFA. Then

(1) $\tilde{F}^*$ is (α, β)-complete, if for each $q \in Q, a \in X$ there exists $p \in Q$ such that $\delta_1(q, a, p) >_L \alpha$ and $\delta_2(q, a, p) <_L \beta$,

(2) $q \in Q$ is (α, β)-accessible if there exist $p \in \tilde{R}, x \in X^*$ such that $\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) >_L \alpha$ and $\tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) <_L \beta$,

(3) $\tilde{F}^*$ is (α, β)-accessible if for every $q \in Q$, q is an (α, β)-accessible.

Definition 2.6. [22] Let $\tilde{F}^*$ be a max-min IGLFA. Then we say that $\tilde{F}^*$ is deterministic if there exists a unique $p_0 \in \tilde{R}$ such that $\mu^{t_0}(p_0) >_L 0$ and for every $q \in Q, a \in X$ there exists at most one $p \in Q$ such that $\delta_2(q, a, p) <_L 1$.

Definition 2.7. [22] Let $\tilde{F}^*$ be a max-min IGLFA. Then the (α, β)-language recognized by $\tilde{F}^*$ is a subset of X^* defined by:

$$\begin{aligned} \mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \{x \in X^* \big| &\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\omega}_1((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha, \\ &\tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \vee \tilde{\omega}_2((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b') <_L \beta, \\ &\text{for some } p \in \tilde{R}, q \in Q, b, b' \in Z\}. \end{aligned}$$

Definition 2.8. [22] Let X be a nonempty finite set. Then subset $\mathcal{L}$ of X^* is called recognizable (α,β)-language if there exists a max-min IGLFA $\tilde{F}^*$ such that $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$.

Definition 2.9. [22] Let $\tilde{F}^*$ be an (α,β)-accessible, (α,β)-complete, deterministic max-min IGLFA. We define a relation on Q by $q_1\rho^{\alpha,\beta}q_2$ if and only if

$$\begin{aligned}
&\{w \in X^* \big| \tilde{\delta}_1^*((q_1,\mu^{t_i}(q_1)),\nu^{t_i}(q_1)),w,q) \wedge \tilde{\omega}_1((q,\mu^{t_i+|w|}(q),\nu^{t_i+|w|}(q)),b) >_L \alpha,\\
&\qquad \tilde{\delta}_2^*((q_1,\mu^{t_i}(q_1)),\nu^{t_i}(q_1)),w,q) \vee \tilde{\omega}_2((q,\mu^{t_i+|w|}(q),\nu^{t_i+|w|}(q)),b') <_L \beta,\\
&\qquad \text{for some } b,b' \in Z, q \in Q\} =\\
&\{w \in X^* \big| \tilde{\delta}_1^*((q_2,\mu^{t_j}(q_2)),\nu^{t_j}(q_2)),w,q) \wedge \tilde{\omega}_1((q,\mu^{t_j+|w|}(q),\nu^{t_j+|w|}(q)),b) >_L \alpha,\\
&\qquad \tilde{\delta}_2^*((q_2,\mu^{t_j}(q_2)),\nu^{t_j}(q_2)),w,q) \vee \tilde{\omega}_2((q,\mu^{t_j+|w|}(q),\nu^{t_j+|w|}(q)),b') <_L \beta,\\
&\qquad \text{for some } b,b' \in Z, q \in Q\},
\end{aligned}$$

where $q_1 \in Q_{act}(t_i)$ and $q_2 \in Q_{act}(t_j)$.

Definition 2.10. [22] We say that the (α,β)-accessible, (α,β)-complete, deterministic max-min IGLFA $\tilde{F}^*$ is (α,β)-reduced if $q_1\rho^{\alpha,\beta}q_2$ implies that $q_1 = q_2$, for every $q_1,q_2 \in Q$.

Definition 2.11. [22] Let $\tilde{F}_1^* = (Q_1,X,\tilde{R}_1,Z_1,\tilde{\delta}^*,\tilde{\omega},\ F_1,F_2,F_3,F_4)$ and $\tilde{F}_2^* = (Q_2,X,\tilde{R}_2,Z_2,\tilde{\delta}'^*,\tilde{\omega}',F_1,F_2,F_3,F_4)$ be two max-min IGLFA. A homomorphism from $\tilde{F}_1^*$ onto $\tilde{F}_2^*$ with threshold (α,β), is a function ξ from Q_1 onto Q_2 such that for every $q',q'' \in Q_1$, $u \in X$ and $b_1,b_2 \in Z$ the following conditions hold:

(1) $(\mu_{Q_1}^{t_0}(q') >_L \alpha \ \&\ \nu_{Q_1}^{t_0}(q') <_L \beta \Longleftrightarrow \mu_{Q_2}^{t_0}(\xi(q')) >_L \alpha \ \&\ \nu_{Q_2}^{t_0}(\xi(q')) <_L \beta)$,

(2)
$$\begin{aligned}
&(\delta_1(q',u,q'') >_L \alpha \ \&\ \delta_2(q',u,q'') <_L \beta \Longleftrightarrow\\
&\delta_1'(\xi(q'),u,\xi(q'')) >_L \alpha \ \&\ \delta_2'(\xi(q'),u,\xi(q'')) <_L \beta),
\end{aligned}$$

(3) $(\omega_1(q',b_1) >_L \alpha \ \&\omega_2(q',b_2)<_L\beta \Longrightarrow \omega_1'(\xi(q'),b)>_L \alpha\&\ \omega_2(\xi(q'),b_2) <_L\beta)$, for some $b,b' \in Z'$.

We say that ξ is an isomorphism with threshold (α,β) if and only if ξ is a homomorphism with threshold (α,β) that is one-one and $(\omega_1(q',b_1) >_L \alpha \ \&\ \omega_2(q',b_2) <_L \beta)$ if and only if $(\omega_1'(\xi(q'),b) >_L \alpha \ \&\ \omega_2(\xi(q'),b_2) <_L \beta)$, for some $b,b' \in Z'$.

Definition 2.12. [22] Let $\mathcal{L}$ be an (α,β)-language. A relation $R_{\mathcal{L}}$ on X^* is defined as follow:
For any two strings x and y in X^*, $xR_{\mathcal{L}}y$ if for every $z \in X^*$ either $xz,yz \in \mathcal{L}$ or $xz,yz \notin \mathcal{L}$.

3. Minimal Intuitionistic General L-Fuzzy Automata with One Initial State

Definition 3.1. For any (α,β)-language $\mathcal{L} \subseteq X^*$ and $u \in X^*$, a subset $\mathcal{L}_u$ of X^* defined by $\mathcal{L}_u = \{v \in X^* \big| uv \in \mathcal{L}\}$ is called a right quotient of $\mathcal{L}$ with respect to u.

Let $Q_{\mathcal{L}} = \{\mathcal{L}_u \big| u \in X^*\}$ be the set of all right quotients of $\mathcal{L}$ and define $\tilde{R} = \{\mathcal{L}_\Lambda\}, \mu^{t_0}(\mathcal{L}_\Lambda) = 1, \nu^{t_0}(\mathcal{L}_\Lambda) = 0, Z = \{b\}$, a mapping $\delta_{\mathcal{L}} : Q_{\mathcal{L}} \times X \times Q_{\mathcal{L}} \to L \times L$ by:

$$\delta_{\mathcal{L}1}(\mathcal{L}_u, a, \mathcal{L}_v) = \begin{cases} \gamma_1 & if\ \mathcal{L}_{ua} = \mathcal{L}_v \\ 0 & otherwise \end{cases}, \tag{5}$$

$$\delta_{\mathcal{L}2}(\mathcal{L}_u, a, \mathcal{L}_v) = \begin{cases} \eta_1 & if\ \mathcal{L}_{ua} = \mathcal{L}_v \\ 1 & otherwise \end{cases}, \tag{6}$$

and $\omega_{\mathcal{L}} : Q_{\mathcal{L}} \times Z \to L \times L$ by:

$$\omega_{\mathcal{L}1}(\mathcal{L}_u, b) = \begin{cases} \gamma_2 & if\ u \in \mathcal{L} \\ 0 & otherwise \end{cases}, \tag{7}$$

$$\omega_{\mathcal{L}2}(\mathcal{L}_u, b) = \begin{cases} \eta_2 & if\ u \in \mathcal{L} \\ 1 & otherwise \end{cases}, \tag{8}$$

where $\gamma_1, \gamma_2, \eta_1, \eta_2 \in L, \gamma_1 \wedge \gamma_2 >_L \alpha, \eta_1 \vee \eta_2 <_L \beta$ and $\gamma_1 \leq_L N(\eta_1), \gamma_2 \leq_L N(\eta_2)$.

Theorem 3.2. *For any (α, β)-language $\mathcal{L} \subseteq X^*$, the following properties are equivalent:*

(1) *$Q_{\mathcal{L}}$ is finite,*
(2) *$\mathcal{L}$ is a recognizable (α, β)-language.*

Proof. $1 \Rightarrow 2$. First, let $Q_{\mathcal{L}}$ be finite. Then $\tilde{F}^*_{\mathcal{L}} = (Q_{\mathcal{L}}, X, \mathcal{L}_\Lambda, \{b\}, \tilde{\delta}^*_{\mathcal{L}}, \tilde{\omega}_{\mathcal{L}}, F_1, F_2, F_3, F_4)$ is a IGLFA. We show that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*_{\mathcal{L}}) = \mathcal{L}$. If $w \in \mathcal{L}$, then $\omega_{\mathcal{L}1}(\mathcal{L}_w, b) >_L \alpha$ and $\omega_{\mathcal{L}2}(\mathcal{L}_w, b) <_L \beta$. Also, we have

$$\tilde{\delta}^*_{\mathcal{L}1}((\mathcal{L}_\Lambda, \mu^{t_0}(\mathcal{L}_\Lambda), \nu^{t_0}(\mathcal{L}_\Lambda)), w, \mathcal{L}_w) >_L \alpha,$$

and

$$\tilde{\delta}^*_{\mathcal{L}2}((\mathcal{L}_\Lambda, \mu^{t_0}(\mathcal{L}_\Lambda), \nu^{t_0}(\mathcal{L}_\Lambda)), w, \mathcal{L}_w) <_L \beta.$$

Hence, $w \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_{\mathcal{L}})$. Now, let $w \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_{\mathcal{L}})$. Then there exists $\mathcal{L}_z \in Q_{\mathcal{L}}$ such that

$$\tilde{\delta}^*_{\mathcal{L}1}((\mathcal{L}_\Lambda, \mu^{t_0}(\mathcal{L}_\Lambda), \nu^{t_0}(\mathcal{L}_\Lambda)), w, \mathcal{L}_z) \wedge \tilde{\omega}_{\mathcal{L}1}((\mathcal{L}_z, \mu^{t_0+|w|}(\mathcal{L}_z), \nu^{t_0+|w|}(\mathcal{L}_z)), b) >_L \alpha, \tag{9}$$

and

$$\tilde{\delta}^*_{\mathcal{L}2}((\mathcal{L}_\Lambda, \mu^{t_0}(\mathcal{L}_\Lambda), \nu^{t_0}(\mathcal{L}_\Lambda)), w, \mathcal{L}_z) \vee \tilde{\omega}_{\mathcal{L}2}((\mathcal{L}_z, \mu^{t_0+|w|}(\mathcal{L}_z), \nu^{t_0+|w|}(\mathcal{L}_z)), b) <_L \beta. \tag{10}$$

By considering (3.1) and (3.1), we have $\mathcal{L}_z = \mathcal{L}_w$. Also, (3.2) and (3.2) imply that $\omega_{\mathcal{L}1}(\mathcal{L}_w, b) >_L \alpha$ and $\omega_{\mathcal{L}2}(\mathcal{L}_w, b) <_L \beta$. Therefore, $w \in \mathcal{L}$. Hence, $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*_{\mathcal{L}}) = \mathcal{L}$.

$2 \Rightarrow 1$. Suppose that $\mathcal{L}$ be a recognizable (α, β)-language for some (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Define a map $f : Q \to Q_{\mathcal{L}}$ by $f(q) = \mathcal{L}_u$, where

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

It is clear that the map f is well defined and surjective. So, $|Q_{\mathcal{L}}| \leq |Q|$ and hence, $Q_{\mathcal{L}}$ is finite. $\square$

Example 3.3. Let the bounded lattice $L = (L, \leq_L, T, S, 0, 1)$ as in Figure 1, where $L = \{0, a, b, c, d, 1\}$ and $N(0) = 1, N(1) = 0, N(a) = b, N(b) = a, N(c) = d, N(d) = c$. Suppose that $\mathcal{L} = \{u, v\}^* uv \{u, v\}^* \cup \{u, v\}^* vu \{u, v\}^*$ be a (a, b)-language. Then

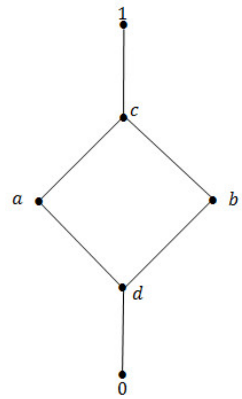

Figure 1. The Bounded Lattice L of Example 3.3

by Definition 3.1, $\mathcal{L}_\Lambda = \mathcal{L}, \mathcal{L}_u = v\{u, v\}^* \cup \mathcal{L}, \mathcal{L}_v = u\{u, v\}^* \cup \mathcal{L}, \mathcal{L}_{u^2} = \mathcal{L}_u, \mathcal{L}_{uv} = \mathcal{L}_{vu} = X^* = \mathcal{L}_{uvu} = \mathcal{L}_{uv^2}, \mathcal{L}_{v^2} = \mathcal{L}_v$. Therefore, we have $\tilde{F}^*_{\mathcal{L}}$ as in Figure 2.

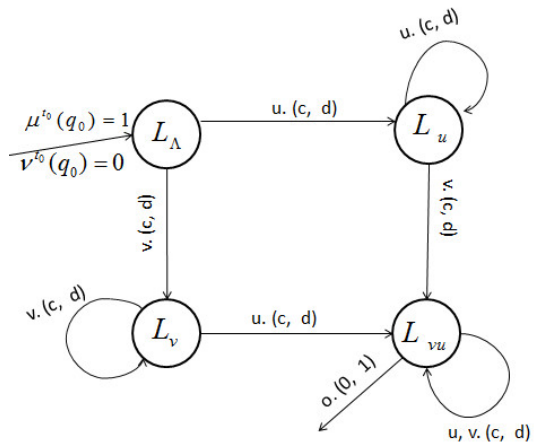

Figure 2. The Max-Min IGLFA of Example 3.3

The following algorithm determines the states of minimal automaton.

1. Algorithm (states of minimal automaton)

Step 1: input: (α, β)-accessible max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ and $\alpha, \beta \in L, \alpha \leq_L N(\beta)$,

Step 2: $A_l = \{x \in X^* \big| |x| = l\}$, let $l = 0$,

Step 3: $q_1 \iota_0^{\alpha,\beta} q_2$ if and only if

$$\tilde{\omega}_1((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), b_1) >_L \alpha \ \& \ \tilde{\omega}_2((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), b'_1) <_L \beta \Longleftrightarrow$$
$$\tilde{\omega}_1((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), b_2) >_L \alpha \ \& \ \tilde{\omega}_2((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), b'_2) <_L \beta$$

for some $b_1, b'_1, b_2, b'_2 \in Z$, where $q_1 \in Q_{act}(t_i)$ and $q_2 \in Q_{act}(t_j)$,

Step 4: $l = l + 1$,

Step 5: $q_1 \iota_l^{\alpha,\beta} q_2$ if and only if $q_1 \iota_{l-1}^{\alpha,\beta} q_2$ and

$$\{w \in X^* | \tilde{\delta}_1^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b_1) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b'_1) <_L \beta, |w| = l\} =$$
$$\{w \in X^* | \tilde{\delta}_1^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b_1) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b'_2) <_L \beta, |w| = l\},$$

for some $b_1, b'_1, b_2, b'_2 \in Z$, where $q_1 \in Q_{act}(t_i), q_2 \in Q_{act}(t_j)$,

Step 6: if $\iota_{l-1}^{\alpha,\beta} = \iota_l^{\alpha,\beta}$ go to next step, else go to Step 5,

Step 7: $\iota^{\alpha,\beta} = \iota_l^{\alpha,\beta}$,

Step 8: output: $\iota^{\alpha,\beta}$.

Now, consider the (α, β)-accessible max-min IGLFA $\tilde{F}^*$. Suppose that $q_1 \iota^{\alpha,\beta} q_2$. Since $\tilde{F}^*$ is (α, β)-accessible. Then there exist $u, v \in X^*$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) >_L \alpha, \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) <_L \beta,$$

and

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), v, q_2) >_L \alpha, \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), v, q_2) <_L \beta.$$

Therefore, by Algorithm 1, $\mathcal{L}_u = \mathcal{L}_v$.

Steps 4 to 6 of the Algorithm 1, are a loop. The loop must be repeated at most $O(|Q|)$ times. The order of time complexity to calculate $l = 1$ is $O(|Q|^3|X||Z|)$, to calculate $l = 2$ is $O(|Q|^4|X||Z|)$. If we continue this process, then the time complexity of calculating this algorithm is $O(|Q|^{|Q|+3}|X|^{|Q|+1}|Z|)$.

Example 3.4. Consider the bounded lattice $L = (L, \leq_L, T, S, 0, 1)$ as in Figure 1, and let $\mathcal{L} = \{u, v\}^* uv \{u, v\}^* \cup \{u, v\}^* vu \{u, v\}^*$ be a (a, b)-language. Suppose that $\tilde{F}^*$ as in Figure 3, and $\alpha = a$ and $\beta = b$. It is clear that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}$. Then by Algorithm 1, we have

Stage 1: $q_2 \iota_0^{\alpha,\beta} q_4, q_0 \iota_0^{\alpha,\beta} q_1 \iota_0^{\alpha,\beta} q_3$,
Stage 2: $q_2 \iota_1^{\alpha,\beta} q_4$,
Stage 3: $q_2 \iota_2^{\alpha,\beta} q_4$.

Therefore, we have $\iota^{\alpha,\beta} = \{[q_0], [q_1], [q_2], [q_4]\}$. Then by Example 3.3, we get that the number of states of $\tilde{F}_m^*$ is equal to the number of the member of $\iota^{\alpha,\beta}$.

Theorem 3.5. *Let $\mathcal{L}$ be a recognizable (α, β)-language. Then* $\tilde{F}_{\mathcal{L}}^* = (Q_{\mathcal{L}}, X, \mathcal{L}_\Lambda, \{b\}, \tilde{\delta}_{\mathcal{L}}^*, \tilde{\omega}_{\mathcal{L}}, F_1, F_2, F_3, F_4)$ *is a minimal (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA, which* $\mathcal{L}(\tilde{F}_{\mathcal{L}}^*) = \mathcal{L}$.

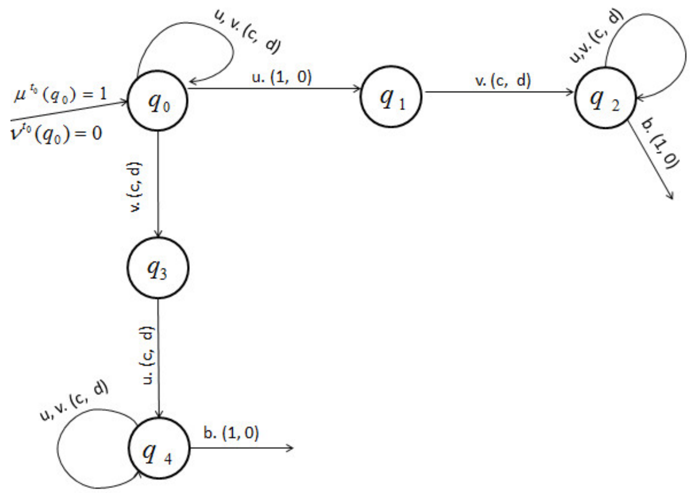

FIGURE 3. The Max-Min IGLFA of Example 3.4

Proof. It is clear by the proof of Theorem 3.2. □

Theorem 3.6. *For every recognizable* $(\alpha,\beta)-$*language* $\mathcal{L}$*, the minimal max-min IGLFA* $\tilde{F}^*_{\mathcal{L}}$ *defined in Definition 3.1, is* (α,β)*-reduced.*

Proof. Let $\tilde{F}^*$ be an (α,β)-accessible, (α,β)-complete, deterministic max-min IGLFA. Suppose that $q_1 = \mathcal{L}_u, q_2 = \mathcal{L}_v$. Now, let $q_1\rho^{\alpha,\beta}q_2$. Then

$$\begin{aligned} A =&\{w \in X^* \big| \tilde{\delta}^*_{\mathcal{L}1}((\mathcal{L}_u, \mu^{t_i}(\mathcal{L}_u), \nu^{t_i}(\mathcal{L}_u)), w, q) \wedge \tilde{\omega}_{\mathcal{L}1}((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,\\ &\tilde{\delta}^*_{\mathcal{L}2}((\mathcal{L}_u, \mu^{t_i}(\mathcal{L}_u), \nu^{t_i}(\mathcal{L}_u)), w, q) \vee \tilde{\omega}_{\mathcal{L}2}((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,\\ &\text{for some } b, b' \in Z, q \in Q_{\mathcal{L}}\} =\\ B =&\{w \in X^* \big| \tilde{\delta}^*_{\mathcal{L}1}((\mathcal{L}_v, \mu^{t_j}(\mathcal{L}_v), \nu^{t_j}(\mathcal{L}_v)), w, q) \wedge \tilde{\omega}_{\mathcal{L}1}((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,\\ &\tilde{\delta}^*_{\mathcal{L}2}((\mathcal{L}_v, \mu^{t_j}(\mathcal{L}_v), \nu^{t_j}(\mathcal{L}_v)), w, q) \vee \tilde{\omega}_{\mathcal{L}2}((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,\\ &\text{for some } b, b' \in Z, q \in Q_{\mathcal{L}}\}. \end{aligned}$$

So, $w \in A$ if and only if $w \in B$, for every $w \in X^*$. This implies that

$$\omega_1(\mathcal{L}_{uw}, b) >_L \alpha, \omega_2(\mathcal{L}_{uw}, b) <_L \beta \Longleftrightarrow \omega_1(\mathcal{L}_{vw}, b) >_L \alpha, \omega_2(\mathcal{L}_{vw}, b) <_L \beta,$$

for every $w \in X^*$. Therefore, $uw \in \mathcal{L}$ if and only if $vw \in \mathcal{L}$, for every $w \in X^*$. So, $\mathcal{L}_u = \mathcal{L}_v$. Hence, $\tilde{F}^*_{\mathcal{L}}$ is (α,β)-reduced. □

Theorem 3.7. *Let* $\mathcal{L}$ *be a recognizable* (α,β)*-language. Suppose that* $\tilde{F}^*_{\mathcal{L}}$ *be the max-min IGLFA defined in Definition 3.1, and* $\tilde{F}^*$ *be an* (α,β)*-complete,* (α,β)*-accessible, deterministic* (α,β)*-reduced max-min IGLFA. Then* $\tilde{F}^*_{\mathcal{L}}$ *and* $\tilde{F}^*$ *are isomorphism with threshold* (α,β)*.*

Proof. Let $\tilde{F}^*_{\mathcal{L}} = (Q_{\mathcal{L}}, X, \{\mathcal{L}_\Lambda\}, \{b\}, \tilde{\delta}^*_{\mathcal{L}}, \tilde{\omega}_{\mathcal{L}}, F_1, F_2, F_3, F_4)$ and $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Define $\xi : Q \to Q_{\mathcal{L}}$ by $\xi(q) = \mathcal{L}_u$, where

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

By (α, β)-accessibility property of $\tilde{F}^*$, $\mu^{t_0}_Q(q_0) >_L \alpha$ and $\nu^{t_0}_Q(q_0) <_L \beta$ also, we have $\mu^{t_0}_{Q_{\mathcal{L}}}(\mathcal{L}_\Lambda) >_L \alpha$ and $\nu^{t_0}_{Q_{\mathcal{L}}}(\mathcal{L}_\Lambda) <_L \beta$.

Let $q_1, q_2 \in Q$ and $q_1 = q_2$. Then $q_1 \rho^{\alpha,\beta} q_2$. So, $\mathcal{L}_u = \mathcal{L}_v$, i.e., $\xi(q_1) = \xi(q_2)$. Hence, ξ is well defined.

Let $\mathcal{L}_u \in Q_{\mathcal{L}}$. By the (α, β)-complete property of $\tilde{F}^*$, there exists $q \in Q$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Then $\xi(q) = \mathcal{L}_u$. Therefore, ξ is surjective.

Now, let $\delta_1(q, a, q') >_L \alpha$ and $\delta_2(q, a, q') <_L \beta$. Since $\tilde{F}^*$ is (α, β)-accessible, then there exists $u \in X^*$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Therefore,

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), ua, q') >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), ua, q') <_L \beta.$$

Hence, $\delta_{\mathcal{L}1}(\xi(q), a, \xi(q')) >_L \alpha$ and $\delta_{\mathcal{L}2}(\xi(q), a, \xi(q')) <_L \beta$, where $\xi(q) = \mathcal{L}_u$ and $\xi(q') = \mathcal{L}_{ua}$. Let $\delta_{\mathcal{L}1}(\xi(q), a, \xi(q')) >_L \alpha$ and $\delta_{\mathcal{L}2}(\xi(q), a, \xi(q')) <_L \beta$, where $\xi(q) = \mathcal{L}_u$ and $\xi(q') = \mathcal{L}_v$. Then $\mathcal{L}_{ua} = \mathcal{L}_v$. We have $w \in \mathcal{L}_{ua}$ if and only if $uaw \in \mathcal{L}$ if and only if $aw \in \mathcal{L}_u$, for every $w \in X^*$. So,

$$\begin{aligned} &\tilde{\delta}^*_1((q, \mu^{t_i}(q), \nu^{t_i}(q)), aw, p) \wedge \tilde{\omega}_1((p, \mu^{t_i+|aw|}(p), \nu^{t_i+|aw|}(p)), b) >_L \alpha, \\ &\tilde{\delta}^*_2((q, \mu^{t_i}(q), \nu^{t_i}(q)), aw, p) \vee \tilde{\omega}_2((p, \mu^{t_i+|aw|}(p), \nu^{t_i+|aw|}(p)), b') <_L \beta, \end{aligned} \tag{11}$$

if and only if

$$\begin{aligned} &\tilde{\delta}^*_1((q', \mu^{t_{i+1}}(q'), \nu^{t_{i+1}}(q')), w, p) \wedge \tilde{\omega}_1((p, \mu^{t_{i+1}+|w|}(p), \nu^{t_{i+1}+|w|}(p)), b) >_L \alpha, \\ &\tilde{\delta}^*_2((q', \mu^{t_{i+1}}(q'), \nu^{t_{i+1}}(q')), w, p) \vee \tilde{\omega}_2((p, \mu^{t_{i+1}+|w|}(p), \nu^{t_{i+1}+|w|}(p)), b') <_L \beta, \end{aligned} \tag{12}$$

where $|u| = i$ and $b, b' \in Z$. By (α, β)-complete and deterministic properties of $\tilde{F}^*$ and (3.7) and (3.7) we have that $\delta_1(q, a, q') >_L \alpha$ and $\delta_2(q, a, q') <_L \beta$.

Let $q \in Q$ and $\omega_1(q, b) >_L \alpha$ and $\omega_2(q, b') <_L \beta$, for some $b, b' \in Z$. By the (α, β)-accessibility property of $\tilde{F}^*$

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Thus $u \in \mathcal{L}$. Therefore, $\omega_{\mathcal{L}1}(\mathcal{L}_u, b) >_L \alpha$ and $\omega_{\mathcal{L}2}(\mathcal{L}_u, b') <_L \beta$, for some $b, b' \in Z$. Hence, $\omega_{\mathcal{L}1}(\xi(q), b) >_L \alpha$ and $\omega_{\mathcal{L}2}(\xi(q), b') <_L \beta$, for some $b, b' \in Z$.

So, $\tilde{F}^*_{\mathcal{L}}$ and $\tilde{F}^*$ are homomorphism with threshold (α, β). Now, let $q_1, q_2 \in Q$ and $\xi(q_1) = \xi(q_2)$. Then there exist $x, y \in X^*$ such that $\mathcal{L}_x = \xi(q_1) = \xi(q_2) = \mathcal{L}_y$. Therefore, $q_1 \rho^{\alpha,\beta} q_2$. By the (α, β)-reduced property of $\tilde{F}^*$, $q_1 = q_2$. Thus, ξ is one-one.

Now, let $\omega_{\mathcal{L}1}(\xi(q), b) >_L \alpha$ and $\omega_{\mathcal{L}2}(\xi(q), b') <_L \beta$ for some $b, b' \in Z$, where $\xi(q) = \mathcal{L}_u$, i.e., $\omega_{\mathcal{L}1}(\mathcal{L}_u, b) >_L \alpha$ and $\omega_{\mathcal{L}2}(\mathcal{L}_u, b') <_L \beta$, for some $b, b' \in Z$. This implies that $u \in \mathcal{L}$. So,

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q') \wedge \tilde{\omega}_1((q', \mu^{t_0+|u|}(q'), \nu^{t_0+|u|}(q')), b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q') \vee \tilde{\omega}_2((q', \mu^{t_0+|u|}(q'), \nu^{t_0+|u|}(q')), b') <_L \beta,$$

for some $b, b' \in Z$. Also, we have

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Since $\tilde{F}^*$ is deterministic, then $q = q'$. Therefore, $\omega_1(q, b) >_L \alpha$ and $\omega_2(q, b') <_L \beta$, for some $b, b' \in Z$. Hence, $\tilde{F}^*_{\mathcal{L}}$ and $\tilde{F}^*$ are isomorphic with threshold (α, β). □

Theorem 3.8. *Let $\mathcal{L}$ be a recognizable (α, β)-language. Suppose that $\tilde{F}^*_{\mathcal{L}}$ be the max-min IGLFA defined in Definition 3.1, and let $\tilde{F}^*_m$ be the (α, β)-complete, deterministic max-min IGLFA defined in Theorem 4.23, in* [22]*. Then $\tilde{F}^*_{\mathcal{L}}$ and $\tilde{F}^*_m$ are isomorphism with threshold (α, β).*

Proof. The proof follows from Theorem 3.7, and Theorem 4.26 in [22]. □

4. Statewise (α, β)-Minimal Max-Min Intuitionistic General L-Fuzzy Automata

The minimization of IGLFA is a consequence of the theory of equivalence of IGLFA. In this section, we give a statewise (α, β)-minimal max-min IGLFA. Finally, we comparison minimal intuitionistic general L-fuzzy automaton and statewise (,)-minimal intuitionistic general L-fuzzy automaton together.

Definition 4.1. We say that max-min IGLFA $\tilde{F}^*$ is strong deterministic if for any $q \in Q, a \in X$ there exists at most one $p \in Q$ such that $\delta_2(q, a, p) <_L 1$.

Theorem 4.2. *Let $\tilde{F}^*$ be a max-min IGLFA. Then there exists a strong deterministic max-min IGLFA $\tilde{F}^*_{sd}$ recognizing $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$.*

Proof. The proof follows from the proof of Theorem 4.11, in [22]. □

The following algorithm determines (α, β)-equivalence classes of max-min IGLFA $\tilde{F}^*$.

2. (α, β)**-equivalence Algorithm**

Step 1: input: max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ and $\alpha, \beta \in L, \alpha \leq_L N(\beta)$,

Step 2: $\tilde{R}'^{\alpha,\beta} = \{(q, \mu^{t_0}(q), \nu^{t_0}(q)) \big| \mu^{t_0}(q) >_L \alpha, \nu^{t_0}(q) <_L \beta\}$, $R = Domain$ $(\tilde{R}'^{\alpha,\beta})$, $i = 0$,

Step 3: $\rho_0^{\alpha,\beta} = \{R, Q - R\}$, as the equivalent classes

Step 4: $i = i + 1$,

Step 5: $q_1\rho_i^{\alpha,\beta}q_2$ if and only if $q_1\rho_{i-1}^{\alpha,\beta}q_2$ and if there exists $p_1 \in Q$ such that $\delta_1(p_1,a,q_1) >_L \alpha, \delta_2(p_1,a,q_1) <_L \beta$, then there exists $p_2 \in Q$ such that $\delta_1(p_2,a,q_2) >_L \alpha, \delta_2(p_2,a,q_2) <_L \beta$ and $p_1\rho_{i-1}^{\alpha,\beta}p_2$ and vice versa,

Step 6: if $\rho_{i-1}^{\alpha,\beta} = \rho_i^{\alpha,\beta}$ go to next step, else go to Step 4,

Step 7: $\rho^{\alpha,\beta} = \rho_i^{\alpha,\beta}$,

Step 8: output: $\rho^{\alpha,\beta}$.

Steps 4 to 6 of (α,β)-equivalence Algorithm, are a loop. The loop must repeat at most $|Q|+1$ times. Also, the time complexity to calculate Step 5, is $O(|X||Q|^4)$. Then the order of time complexity is at most $O(|X||Q|^5)$.

Theorem 4.3. *For every $i \geq 0$, $\rho_i^{\alpha,\beta}$ is an equivalence relation on Q.*

Proof. We prove by induction on i. Clearly, $\rho_0^{\alpha,\beta}$ is an equivalence relation on Q. Let $i=1$ and $q_1,q_2,q_3 \in Q$. It is obvious that $q_1\rho_1^{\alpha,\beta}q_1$ and $q_1\rho_1^{\alpha,\beta}q_2$ if and only if $q_2\rho_1^{\alpha,\beta}q_1$. Now, let $q_1\rho_1^{\alpha,\beta}q_2$ and $q_2\rho_1^{\alpha,\beta}q_3$. Then

$$\exists p_1 \in Q \ s.t \ \delta_1(p_1,a,q_1) >_L \alpha, \delta_2(p_1,a,q_1) <_L \beta \Longleftrightarrow$$
$$\exists p_2 \in Q \ s.t \ \delta_1(p_2,a,q_2) >_L \alpha, \delta_2(p_2,a,q_2) <_L \beta, \ \& \ p_1\rho_0^{\alpha,\beta}p_2 \Longleftrightarrow$$
$$\exists p_3 \in Q \ s.t \ \delta_1(p_3,a,q_3) >_L \alpha, \delta_2(p_3,a,q_3) <_L \beta, \ \& \ p_2\rho_0^{\alpha,\beta}p_3.$$

Therefore, there exists $p_1 \in Q$ such that $\delta_1(p_1,a,q_1) >_L \alpha, \delta_2(p_1,a,q_1) <_L \beta$ if and only if there exists $p_3 \in Q$ such that $\delta_1(p_3,a,q_3) >_L \alpha, \delta_2(p_3,a,q_3) <_L \beta$ and $p_1\rho_0^{\alpha,\beta}p_3$. Then $\rho_1^{\alpha,\beta}$ is an equivalence relation. Now, let the claim holds, for every $i \leq n$. Suppose that $i = n$.

$$\exists p_1 \in Q \ s.t \ \delta_1(p_1,a,q_1) >_L \alpha, \delta_2(p_1,a,q_1) <_L \beta \Longleftrightarrow$$
$$\exists p_2 \in Q \ s.t \ \delta_1(p_2,a,q_2) >_L \alpha, \delta_2(p_2,a,q_2) <_L \beta \ \& \ p_1\rho_{n-1}^{\alpha,\beta}p_2 \Longleftrightarrow$$
$$\exists p_3 \in Q \ s.t \ \delta_1(p_3,a,q_3) >_L \alpha, \delta_2(p_3,a,q_3) <_L \beta \ \& \ p_2\rho_{n-1}^{\alpha,\beta}p_3.$$

Then there exists $p_1 \in Q$ such that $\delta_1(p_1,a,q_1) >_L \alpha, \delta_2(p_1,a,q_1) <_L \beta$ if and only if there exists $p_3 \in Q$ such that $\delta_1(p_3,a,q_3) >_L \alpha, \delta_2(p_3,a,q_3) <_L \beta$ and $p_1\rho_{n-1}^{\alpha,\beta}p_3$. Then $\rho_n^{\alpha,\beta}$ is an equivalence relation. Hence, $\rho^{\alpha,\beta}$ is an equivalence relation. □

We show $q_1 \approx^{\alpha,\beta} q_2$ if and only if $q_1\rho^{\alpha,\beta}q_2$, for every $q_1,q_2 \in Q$.

The following algorithm process the (α,β)-equivalence of two max-min IGLFAs $\tilde{F}_1^*$ and $\tilde{F}_1^*$.

3. (α,β)-equivalence Algorithm of Two Max-Min IGLFAs

Step 1: input: max-min IGLFAs $\tilde{F}_1^* = (Q_1, X, \tilde{R}_1, Z_1, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$, $\tilde{F}_2 = (Q_2, X, \tilde{R}_2,$
$Z_2, \tilde{\delta}^{*'}, \tilde{\omega}', F_1, F_2, F_3, F_4)$, $\alpha,\beta \in L, \alpha \leq_L N(\beta)$ and $i=0$, let $q_1 \in Q_1, q_2 \in Q_2$,

Step 2: $q_1\tau_i^{\alpha,\beta}q_2$ if and only if $\mu_{Q_1}^{t_0}(q_1) >_L \alpha, \nu_{Q_1}^{t_0}(q_1) <_L \beta \Leftrightarrow \mu_{Q_2}^{t_0}(q_2) >_L \alpha, \nu_{Q_2}^{t_0}(q_2) <_L \beta$,

Step 3: $i = i + 1$,

Step 4: $q_1\tau_i^{\alpha,\beta}q_2$ if and only if $q_1\tau_{i-1}^{\alpha,\beta}q_2$ and there exists $p_1 \in Q_1$ such that $\delta_1(p_1, a, q_1) >_L \alpha$ and $\delta_2(p_1, a, q_1) <_L \beta$, then there exists $p_2 \in Q_2$ such that $\delta_1'(p_2, a, q_2) >_L \alpha$ and $\delta_2'(p_2, a, q_2) <_L \beta$ and $p_1\tau_{i-1}^{\alpha,\beta}p_2$ and vice versa,

Step 5: if $\tau_{i-1}^{\alpha,\beta} = \tau_i^{\alpha,\beta}$ go to next step, else go to Step 3,

Step 6: $\tau^{\alpha,\beta} = \tau_i^{\alpha,\beta}$,

Step 7: **output:** $\tau^{\alpha,\beta}$.

Notation 2. Steps 3 to 5 of (α, β)-equivalence Algorithm of Two Max-Min IGLFAs, are a loop and this loop must be repeated at most $\max\{|Q_1|, |Q_2|\} + 1$ times. The time complexity to calculate Step 4 is $O(|X||Q_1||Q_2|(\max\{|Q_1|, |Q_2|\}))$. Then the total time complexity is at most $O(|X||Q_1||Q_2|(\max\{|Q_1|, |Q_2|\})^2)$.

We show $q_1 \equiv^{\alpha,\beta} q_2$ if and only if $q_1\tau^{\alpha,\beta}q_2$, for every $q_1 \in Q_1, q_2 \in Q_2$. It is clear that $\equiv^{\alpha,\beta}$ is equivalence relations.

Lemma 4.4. *Let $\tilde{F}_i^*$ be a max-min IGLFA and $q_i, p_i \in Q_i$, where $i = 1, 2, 3$.*

(1) *Let $q_1 \equiv^{\alpha,\beta} q_2$ and $q_2 \equiv^{\alpha,\beta} q_3$. Then $q_1 \equiv^{\alpha,\beta} q_3$.*
(2) *Let $p_1 \approx^{\alpha,\beta} q_1$ and $q_1 \equiv^{\alpha,\beta} q_2$. Then $p_1 \equiv^{\alpha,\beta} q_2$.*
(3) *Let $q_1 \equiv^{\alpha,\beta} q_2$ and $p_1 \equiv^{\alpha,\beta} q_2$. Then $q_1 \approx^{\alpha,\beta} p_1$.*

Proof. 1. Let $q_1\tau^{\alpha,\beta}q_2$ and $q_2\tau^{\alpha,\beta}q_3$ and $\tau^{\alpha,\beta} = \tau_i^{\alpha,\beta}$. We prove the claim by induction on i. Let $i = 0$. Then $\mu_{Q_1}^{t_0}(q_1) >_L \alpha, \nu_{Q_1}^{t_0}(q_1) <_L \beta$ if and only if $\mu_{Q_2}^{t_0}(q_2) >_L \alpha, \nu_{Q_2}^{t_0}(q_2) <_L \beta$ if and only if $\mu_{Q_3}^{t_0}(q_3) >_L \alpha, \nu_{Q_3}^{t_0}(q_3) <_L \beta$. Therefore, $q_1\tau_0^{\alpha,\beta}q_3$. Suppose that the claim holds, for any positive integer $i-1$. If there exists $p_1 \in Q_1$ such that $\delta_1^1(p_1, a, q_1) >_L \alpha, \delta_2^1(p_1, a, q_1) <_L \beta$, then there exists $p_2 \in Q_2$ such that $\delta_1^2(p_2, a, q_2) >_L \alpha, \delta_2^2(p_2, a, q_2) <_L \beta$ and $p_1\tau_{i-1}^{\alpha,\beta}p_2$ and also, there exists $p_3 \in Q_3$ such that $\delta_1^3(p_3, a, q_3) >_L \alpha, \delta_2^3(p_3, a, q_3) <_L \beta$ and $p_2\tau_{i-1}^{\alpha,\beta}p_3$. In a similar manner if there exists $p_3 \in Q_3$ such that $\delta_1^3(p_3, a, q_3) >_L \alpha, \delta_1^3(p_3, a, q_3) <_L \beta$, then there exists $p_1 \in Q_1$ such that $\delta_1^1(p_1, a, q_1) >_L \alpha, \delta_1^1(p_1, a, q_1) <_L \beta$ and $p_3\tau_{i-1}^{\alpha,\beta}p_1$. Hence, $q_1\tau_{i-1}^{\alpha,\beta}q_3$.
2. Let $p_1\rho^{\alpha,\beta}q_1$ and $q_1\tau^{\alpha,\beta}q_2$, where $\rho^{\alpha,\beta} = \rho_i^{\alpha,\beta}, \tau^{\alpha,\beta} = \tau_j^{\alpha,\beta}$. We prove the claim by induction on $n = \max\{i, j\}$. Let $n = 0$. Then $\mu_{Q_1}^{t_0}(p_1) >_L \alpha, \nu_{Q_1}^{t_0}(p_1) <_L \beta$ if and only if $\mu_{Q_1}^{t_0}(q_1) >_L \alpha, \nu_{Q_1}^{t_0}(q_1) <_L \beta$ if and only if $\mu_{Q_2}^{t_0}(q_2) >_L \alpha, \nu_{Q_2}^{t_0}(q_2) <_L \beta$. Let the claim holds, for any positive integer $n-1$. Now, Suppose that there exists $p_1' \in Q_1$ such that $\delta_1^1(p_1', a, p_1) >_L \alpha$ and $\delta_2^1(p_1', a, p_1) <_L \beta$. Then there exists $q_1' \in Q_1$ such that $\delta_1^1(q_1', a, q_1) >_L \alpha, \delta_2^1(q_1', a, q_1) <_L \beta$ and $p_1'\rho_{n-1}^{\alpha,\beta}q_1'$ and also, there exists $p_2' \in Q_2$ such that $\delta_1^2(p_2', a, q_2) >_L \alpha, \delta_2^2(p_2', a, q_2) <_L \beta$ and $q_1'\tau_{n-1}^{\alpha,\beta}p_2'$. In the similar way if there exists $p_2' \in Q_2$ such that $\delta_1^2(p_2', a, q_2) >_L \alpha, \delta_2^2(p_2', a, q_2) <_L \beta$, then there exists $p_1' \in Q_1$ such that $\delta_1^1(p_1', a, p_1) >_L \alpha, \delta_2^1(p_1', a, p_1) <_L \beta$, where

$p'_1 \tau^{\alpha,\beta}_{n-1} p'_2$. Hence, $p_1 \equiv^{\alpha,\beta} q_2$.
3. Can be proved in a similar manner. □

Definition 4.5. Let $\tilde{F}^*_1$ and $\tilde{F}^*_2$ be two max-min IGLFAs. Then we say that $\tilde{F}^*_1$ and $\tilde{F}^*_2$ are statewise (α, β)-equivalent, written $\tilde{F}^*_1 \equiv^{\alpha,\beta} \tilde{F}^*_2$, if for any $q_1 \in Q_1$ there exists $q_2 \in Q_2$ such that $q_1 \equiv^{\alpha,\beta} q_2$ and vice versa.

Definition 4.6. Let $\tilde{F}^*$ be a strong deterministic max-min IGLFA. Then we say that $\tilde{F}^*$ is statewise (α, β)-minimal if $\tilde{F}^*$ is not statewise (α, β)-equivalent to any max-min IGLFA with a fewer number of states that preserve (α, β)-language.

Theorem 4.7. *For every strong deterministic max-min IGLFA $\tilde{F}^*$ there exists a statewise (α, β)-minimal max-min IGLFA $\tilde{F}^*_m$.*

Proof. Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Consider $Q_m = \{[q] \big| q \in Q\}$, where $[q] = \{p \in Q \big| q \approx^{\alpha,\beta} p\}$, $\tilde{R}_m = \{[q] \big| q \in \tilde{R}, \mu^{t_0}_Q(q) >_L \alpha, \nu^{t_0}_Q(q) <_L \beta\}, Z_m = \{e\}$, $\mu^{t_0}_{Q_m}([q]) = \vee\{\mu^{t_0}_Q(s) \big| s \approx^{\alpha,\beta} q\}$ and $\nu^{t_0}_{Q_m}([q]) = \wedge\{\nu^{t_0}_Q(s) \big| s \approx^{\alpha,\beta} q\}$. Now, define $\delta_m : Q_m \times X \times Q_m \to L \times L$, where

$$\delta_{m1}([q], a, [p]) = \begin{cases} \gamma_1 & if\ \delta_1(s, a, t) >_L \alpha, \delta_2(s, a, t) <_L \beta, \text{where } s \approx^{\alpha,\beta} q, t \approx^{\alpha,\beta} p \\ 0 & otherwise \end{cases}, \tag{13}$$

$$\delta_{m2}([q], a, [p]) = \begin{cases} \eta_1 & if\ \delta_1(s, a, t) >_L \alpha, \delta_2(s, a, t) <_L \beta, \text{where } s \approx^{\alpha,\beta} q, t \approx^{\alpha,\beta} p \\ 1 & otherwise \end{cases}, \tag{14}$$

$\gamma_1, \eta_1 \in L$ and $\gamma_1 \leq_L N(\eta_1)$. Also, define $\omega_m : Q_m \times Z_m \to L \times L$, where

$$\omega_{m1}([q], e) = \begin{cases} \gamma_2 & if\ \omega_1(s, b) >_L \alpha, \omega_2(s, b') <_L \beta, \text{where } s \approx^{\alpha,\beta} q,\ b, b' \in Z, \\ 0 & otherwise \end{cases}, \tag{15}$$

$$\omega_{m2}([q], e) = \begin{cases} \eta_2 & if\ \omega_1(s, b) >_L \alpha, \omega_2(s, b') <_L \beta, \text{where } s \approx^{\alpha,\beta} q,\ b, b' \in Z, \\ 1 & otherwise \end{cases}, \tag{16}$$

$\gamma_2, \eta_2 \in L$ and $\gamma_2 \leq_L N(\eta_2)$.

Now, we show that δ_m is well defined. Let $[q_1] = [q_2]$, $[p_1] = [p_2]$ and $\delta_{m1}([q_1], a, [p_1]) = \gamma_1, \delta_{m2}([q_1], a, [p_1]) = \eta_1$. Then there exist $r_1, s_1 \in Q$ such that $\delta_1(r_1, a, s_1) >_L \alpha$ and $\delta_2(r_1, a, s_1) <_L \beta$, where $r_1 \approx^{\alpha,\beta} q_1 \approx^{\alpha,\beta} q_2, s_1 \approx^{\alpha,\beta} p_1 \approx^{\alpha,\beta} p_2$. Therefore, $\delta_{m1}([q_2], a, [p_2]) = \gamma_1, \delta_{m2}([q_2], a, [p_2]) = \eta_1$. Clearly, ω_m is well defined.
Now, we show that $\tilde{F}^* \equiv^{\alpha,\beta} \tilde{F}^*_m$. We prove for any $q \in Q$, $q\tau^{\alpha,\beta}_i[q]$. If $i = 0$, then the claim holds. So, for $i = 0$, the theorem holds. Now, we continue the proof by induction on i. Let $i = 1$. If there exists $p_1 \in Q$ such that $\delta_1(p_1, a, q_1) >_L \alpha, \delta_2(p_1, a, q_1) <_L \beta$, then $\delta_{m1}([p_1], a, [q_1]) >_L \alpha, \delta_{m2}([p_1], a, [q_1]) <_L \beta$ and $p_1\tau^{\alpha,\beta}_0[p_1]$. Also, if there exists $[p_1] \in Q_m$ such that $\delta_{m1}([p_1], a, [q_1]) >_L \alpha, \delta_{m2}([p_1], a, [q_1]) <_L \beta$, then there exist $s, t \in Q$ such that $s \approx^{\alpha,\beta} p_1, t \approx^{\alpha,\beta} q_1$ and $\delta_1(s, a, t) >_L \alpha, \delta_2(s, a, t) <_L \beta$. Since $t \approx^{\alpha} q_1$ and $\delta_1(s, a, t) >_L \alpha, \delta_2(s, a, t) <_L \beta$, then there exists $p_2 \in Q$ such that $\delta_1(p_2, a, q_1) >_L \alpha, \delta_2(p_2, a, q_1) <_L \beta$ and $p_2 \approx^{\alpha,\beta} s \approx^{\alpha,\beta} p_1$. Then $p_2 \approx^{\alpha,\beta} p_1\tau^{\alpha,\beta}_0[p_1]$. So, $q_1\tau^{\alpha,\beta}_1[q_1]$. Suppose that the claim holds for any $i \leq l$, where l is a positive integer. Also, we show that it holds,

for $i = l$. If there exists $p_1 \in Q$ such that $\delta_1(p_1, a, q_1) >_L \alpha, \delta_2(p_1, a, q_1) <_L \beta$, then $\delta_{m1}([p_1], a, [q_1]) >_L \alpha, \delta_{m2}([p_1], a, [q_1]) <_L \beta$ and $p_1 \tau_{l-1}^{\alpha,\beta} [p_1]$. If there exists $[p_1] \in Q_m$ such that $\delta_{m1}([p_1], a, [q_1]) >_L \alpha, \delta_{m2}([p_1], a, [q_1]) <_L \beta$, then there is $t \approx^{\alpha,\beta} p_1$ such that $\delta_1(t, a, q_1) >_L \alpha, \delta_2(t, a, q_1) <_L \beta$, where $t \approx^{\alpha,\beta} p_1 \tau_{l-1}^{\alpha,\beta} [p_1]$. So, $q_1 \tau_l^{\alpha,\beta} [q_1]$. Therefore, $q \equiv^{\alpha,\beta} [q]$. Hence, $\tilde{F}^* \equiv^{\alpha,\beta} \tilde{F}^*_m$.

Now, we show that for any max-min IGLFA $\tilde{F}^*_1 = (Q_1, X, \tilde{R}_1, Z_1, \tilde{\delta}^*_1, \tilde{\omega}_1, F_1, F_2, F_3, F_4)$ such that $\tilde{F}^*_1 \equiv^{\alpha,\beta} \tilde{F}^*$we have $|\tilde{F}^*_m| \le |\tilde{F}^*_1|$. Since $\tilde{F}^*_m \equiv^{\alpha,\beta} \tilde{F}^*$, then $\tilde{F}^*_m \equiv^{\alpha,\beta} \tilde{F}^*_1$. Therefore, for any $q, p \in Q_1$, where $q \neq p$ there exist $q_1, p_1 \in Q$ such that $[q_1], [p_1] \in Q_m$ and $[q_1] \equiv^{\alpha,\beta} q, [p_1] \equiv^{\alpha,\beta} p$. Now, let $[q_1], [p_1] \in Q_m$, where $[q_1] \neq [p_1]$ and $q_1, p_1 \in Q$. Then there exist $p, q \in Q_1$ such that $[q_1] \equiv^{\alpha,\beta} q, [p_1] \equiv^{\alpha,\beta} p$, but $p \neq q$. Since if $p = q$, then $[q_1] \approx^{\alpha,\beta} [p_1]$ and $[q_1] \equiv^{\alpha,\beta} q_1, [p_1] \equiv^{\alpha,\beta} p_1$. So, $q_1 \equiv^{\alpha,\beta} [q_1] \approx^{\alpha,\beta} [p_1] \equiv^{\alpha,\beta} p_1$. These imply that $q_1 \approx^{\alpha,\beta} p_1$, then $[q_1] = [p_1]$ which is a contradiction. Then $|\tilde{F}^*_m| \le |\tilde{F}^*_1|$.

Now, we prove $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m)$. Let $u_1 u_2 ... u_{k+1} = x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Then there exist $q \in \tilde{R}, p \in Q$ and $b, b' \in Z$ such that

$$\tilde{\delta}^*_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

$$\tilde{\delta}^*_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Therefore, there exist $p_1, p_2, ..., p_k, p'_1, p'_2, ..., p'_k \in Q$ such that $\mu^{t_0}(q) \wedge \delta_1(q, u_1, p_1) \wedge \delta_1(p_1, u_2, p_2) \wedge ... \wedge \delta_1(p_k, u_{k+1}, p) >_L \alpha$ and $\nu^{t_0}(q) \vee \delta_2(q, u_1, p_1) \vee \delta_2(p_1, u_2, p_2) \vee ... \vee \delta_2(p_k, u_{k+1}, p) <_L \beta$. Since $\tilde{F}^*$ is strong deterministic, then $p_1 = p'_1, p_2 = p'_2, ..., p_k = p'_k$. Then $\delta_{m1}([q], u_1, [p_1]) \wedge \delta_{m1}([p_1], u_2, [p_2]) \wedge ... \wedge \delta_{m1}([p_k], u_{k+1}, [p]) >_L \alpha, \delta_{m2}([q], u_1, [p_1]) \vee \delta_{m2}([p_1], u_2, [p_2]) \vee ... \vee \delta_{m2}([p_k], u_{k+1}, [p]) <_L \beta$ and $\omega_{m1}([p], e) >_L \alpha, \omega_{m2}([p], e) <_L \beta$. Also, we have $\mu^{t_0}([q]) \ge \mu^{t_0}(q), \nu^{t_0}([q]) \le \nu^{t_0}(q)$. Hence, $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m)$. Now, let $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m)$. Then there exist $[q] \in \tilde{R}_m, [p] \in Q_m$ and $e \in Z$ such that

$$\tilde{\delta}^*_{m1}(([q], \mu^{t_0}([q]), \nu^{t_0}([q])), x, [p]) \wedge \tilde{\omega}_{m1}(([p], \mu^{t_0+|x|}([p]), \nu^{t_0+|x|}([p])), e) >_L \alpha,$$

$$\tilde{\delta}^*_{m2}(([q], \mu^{t_0}([q]), \nu^{t_0}([q])), x, [p]) \vee \tilde{\omega}_{m2}(([p], \mu^{t_0+|x|}([p]), \nu^{t_0+|x|}([p])), e) <_L \beta.$$

These imply that $\omega_{m1}([p], e) >_L \alpha, \omega_{m2}([p], e) <_L \beta$. Therefore, there exists $t \in Q$ and $b, b' \in Z$ such that $t \approx^{\alpha,\beta} p$ and $\omega_1(t, b) >_L \alpha, \omega_2(t, b') <_L \beta$. Let $x = u_1 ... u_k$. Then there exist $[p_1], [p_2], ..., [p_{k-1}] \in Q_m$ such that $\delta_{m1}([q], u_1, [p_1]) \wedge \delta_{m1}([p_1], u_2, [p_2]) \wedge ... \wedge \delta_{m1}([p_{k-1}], u_k, [p]) >_L \alpha$. By (4.7) and (4.7), we have $\delta_{m2}([q], u_1, [p_1]) \vee \delta_{m2}([p_1], u_2, [p_2]) \vee ... \vee \delta_{m2}([p_{k-1}], u_k, [p]) <_L \beta$. Since $\delta_{m1}([p_{k-1}], u_k, [p]) >_L \alpha, \delta_{m2}([p_{k-1}], u_k, [p]) <_L \beta$, then there exist $r', t' \in Q$, where $r' \approx^{\alpha,\beta} p_{k-1}$, $t' \approx^{\alpha,\beta} p$ and $\delta_1(r', u_k, t') >_L \alpha, \delta_2(r', u_k, t') <_L \beta$. We have $t' \approx^{\alpha,\beta} p \approx^{\alpha,\beta} t$ so, there exists $r_{k-1} \in Q$ such that $\delta_1(r_{k-1}, u_k, t) >_L \alpha, \delta_2(r_{k-1}, u_k, t) <_L \beta$. Therefore, $r_{k-1} \approx^{\alpha,\beta} r' \approx^{\alpha,\beta} p_{k-1} \equiv^{\alpha,\beta} [p_{k-1}]$. Also, $\delta_{m1}([p_{k-2}], u_{k-1}, [p_{k-1}]) >_L \alpha, \delta_{m2}([p_{k-2}], u_{k-1}, [p_{k-1}]) <_L \beta$ and $r_{k-1} \equiv^{\alpha,\beta} [p_{k-1}]$. Then there exists $r_{k-2} \in Q$, where $r_{k-2} \approx^{\alpha,\beta} p_{k-2}$ and $\delta_1(r_{k-2}, u_{k-1}, r_{k-1}) >_L \alpha, \delta_2(r_{k-2}, u_{k-1}, r_{k-1}) <_L \beta$. So, if we continue this process, then by some manipulation we get that $r_{k-2}, ..., r_1, r \in Q$ such that $r_{k-2} \equiv^{\alpha,\beta} [p_{k-2}], ..., r_1 \equiv^{\alpha,\beta} [p_1], r \equiv^{\alpha,\beta} [q]$ and $\delta_1(r_{k-3}, u_{k-2}, r_{k-2}) \wedge ... \wedge \delta_1(r_1, u_2, r_2) \wedge \delta_1(r, u_1, r_1) >_L \alpha$, $\delta_2(r_{k-3}, u_{k-2}, r_{k-2}) \vee ... \vee \delta_2(r_1, u_2, r_2) \vee \delta_2(r, u_1, r_1) <_L \beta$. Clearly, $r \approx^{\alpha,\beta} q$. Therefore, this implies that $\mu^{t_0}(r) >_L$

$\alpha, \nu^{t_0}(r) <_L \beta$. So, $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Hence, $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m)$. Then the claim holds. □

Theorem 4.8. *Let $\tilde{F}^*$ be an (α,β)-complete, (α,β)-accessible, deterministic max-min IGLFA recognizing (α,β)-language $\mathcal{L}$ and $\tilde{F}^*_{\mathcal{L}}$ be the max-min IGLFA defined in Definition 3.1. Suppose that $\tilde{F}^*_m$ be statewise (α,β)-minimal to $\tilde{F}^*$. Then $\tilde{F}^*_m$ and $\tilde{F}^*_{\mathcal{L}}$ are homomorphism with threshold (α,β).*

Proof. Let $\tilde{F}^*_{\mathcal{L}} = (Q_{\mathcal{L}}, X, \{\mathcal{L}_\Lambda\}, \{b\}, \tilde{\delta}^*_{\mathcal{L}}, \tilde{\omega}_{\mathcal{L}}, F_1, F_2, F_3, F_4)$ and $\tilde{F}_m = (Q_m, X, \tilde{R}_m, Z_m, \tilde{\delta}, \tilde{\omega}_m, F_1, F_2, F_3, F_4)$. Define $\xi : Q_m \to Q_{\mathcal{L}}$ by $\xi([q]) = \mathcal{L}_u$, where

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Since $\tilde{F}^*$ is (α,β)-accessible, then $\mu^{t_0}_Q(q_0) >_L \alpha$ and $\nu^{t_0}_Q(q_0) <_L \beta$ also, we have $\mu^{t_0}_{Q_{\mathcal{L}}}(\mathcal{L}_\Lambda) >_L \alpha$ and $\nu^{t_0}_{Q_{\mathcal{L}}}(\mathcal{L}_\Lambda) <_L \beta$.

Let $[q_1], [q_2] \in Q_m$ and $[q_1] = [q_2]$. Then $q_1 \approx^{\alpha,\beta} q_2$. Therefore, there exist $u = u_1 u_2 ... u_{k+1} \in X^*$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) <_L \beta.$$

So, there exist $p_1, p_2, ..., p_k \in Q$ such that $\mu^{t_0}(q_0) \wedge \delta_1(q_0, u_1, p_1) \wedge \delta_1(p_1, u_2, p_2) \wedge ... \wedge \delta_1(p_k, u_{k+1}, q_1) >_L \alpha$ and $\nu^{t_0}(q_0) \vee \delta_2(q_0, u_1, p_1) \vee \delta_2(p_1, u_2, p_2) \vee ... \vee \delta_2(p_k, u_{k+1}, q_1) <_L \beta$. Since $q_1 \approx^{\alpha,\beta} q_2$ and $\delta_1(p_k, u_{k+1}, q_1) >_L \alpha, \delta_2(p_k, u_{k+1}, q_1) <_L \beta$, then there exists $p'_k \in Q$ such that $\delta_1(p'_k, u_{k+1}, q_2) >_L \alpha, \delta_2(p'_k, u_{k+1}, q_2) <_L \beta$ and $p'_k \approx^{\alpha,\beta} p_k$. Also, $\delta_1(p_{k-1}, u_k, p_k) >_L \alpha, \delta_2(p_{k-1}, u_k, p_k) <_L \beta$. Then there exists $p'_{k-1} \in Q$ such that $\delta_1(p'_{k-1}, u_k, p'_k) >_L \alpha, \delta_2(p'_{k-1}, u_k, p'_k) <_L \beta$ and $p'_{k-1} \approx^{\alpha,\beta} p_{k-1}$. By continuing this process we obtain $p'_{k-2}, ..., p'_1, q'_0 \in Q$ such that $p'_{k-2} \approx^{\alpha,\beta} p_{k-2}, ..., p'_1 \approx^{\alpha,\beta} p_1, q'_0 \approx^{\alpha,\beta} q_0$ and

$$\tilde{\delta}^*_1((q'_0, \mu^{t_0}(q'_0), \nu^{t_0}(q'_0)), u, q_2) >_L \alpha \text{ and } \tilde{\delta}^*_2((q'_0, \mu^{t_0}(q'_0), \nu^{t_0}(q'_0)), u, q_2) <_L \beta.$$

Therefore, $\xi([q_1]) = \xi([q_2])$. So, ξ is well defined.

Now, let $\delta_{m1}([q], a, [p]) >_L \alpha$ and $\delta_{m2}([q], a, [p]) <_L \beta$. Then there exist $q_1, p_1 \in Q$ such that $q_1 \approx^{\alpha,\beta} q, p_1 \approx^{\alpha,\beta} p$ and $\delta_1(q_1, a, p_1) >_L \alpha$, $\delta_2(q_1, a, p_1) <_L \beta$. By (α,β)-accessibility property there exists $q_0 \in \tilde{R}$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) <_L \beta.$$

Therefore,

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), ua, p_2) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), ua, p_2) <_L \beta.$$

Since $q_1 \approx^{\alpha,\beta} q$ and $p_1 \approx^{\alpha,\beta} p$, then $\xi([q]) = \mathcal{L}_u$ and $\xi([p]) = \mathcal{L}_{ua}$. Therefore, $\delta_{\mathcal{L}1}(\xi([q]), a, \xi([p])) >_L \alpha$ and $\delta_{\mathcal{L}2}(\xi([q]), a, \xi([p])) <_L \beta$.

Suppose that $\delta_{\mathcal{L}1}(\xi([q]), a, \xi([p])) >_L \alpha$ and $\delta_{\mathcal{L}2}(\xi([q]), a, \xi([p])) <_L \beta$, where $\xi([q]) = \mathcal{L}_u$ and $\xi([p]) = \mathcal{L}_v$. Then $\mathcal{L}_{ua} = \mathcal{L}_v$. So, there exists $q_0 \in \tilde{R}$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha, \ \& \ \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta, \tag{17}$$

and

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), ua, p) >_L \alpha, \ \& \ \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), ua, p) <_L \beta. \tag{18}$$

Then there exists $p' \in Q$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, p') >_L \alpha \ \&\ \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, p') <_L \beta. \quad (19)$$

by considering (4.8) and (4.8) and the deterministic property of $\tilde{F}^*$, we have $\delta_1(q, a, p) >_L \alpha$ and $\delta_2(q, a, p) <_L \beta$. Therefore, $\delta_{m1}([q], a, [p]) >_L \alpha$ and $\delta_{m2}([q], a, [p]) <_L \beta$.

Now, let $[q] \in Q$ and $\omega_{m1}([q], b_1) >_L \alpha$, $\omega_{m2}([q], b_2) <_L \beta$, where $b_1, b_2 \in Z_m$. Then there exists $p \in Q$ such that $p \approx^{\alpha,\beta} q$ and $\omega_1(p, b) >_L \alpha$, $\omega_2(p, b') <_L \beta$, for some $b, b' \in Z$. Since $\tilde{F}^*$ is (α, β)-accessible, then there exist $q_0 \in Q, u \in X^*$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, p) >_L \alpha, \ \&\ \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, p) <_L \beta.$$

These imply that $u \in \mathcal{L}$. Therefore, $\omega_{\mathcal{L}1}(\mathcal{L}_u, b_1) >_L \alpha$ and $\omega_{\mathcal{L}2}(\mathcal{L}_u, b_2) <_L \beta$, where $b_1, b_2 \in Z_m$, i.e., $\omega_{\mathcal{L}1}(\xi(q), b_1) >_L \alpha$ and $\omega_{\mathcal{L}2}(\xi(q), b_2) <_L \beta$.

Let $\mathcal{L}_u \in Q_{\mathcal{L}}$. By the (α, β)-complete property of $\tilde{F}^*$, there exists $q \in Q$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha, \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Then $\xi(q) = \mathcal{L}_u$. Therefore, ξ is surjective. Hence, $\tilde{F}_{\mathcal{L}}^*$ and $\tilde{F}_m^*$ are homomorphism with threshold (α, β). □

Now, we give some examples for comparison two minimal IGLFA and statewise (α, β)-minimal IGLFA.

Example 4.9. Consider the bounded lattice $L = (L, \leq_L, T, S, 0, 1)$ in Example 3.3, $\alpha = a, \beta = b$ and max-min IGLFA as in Figure 4.

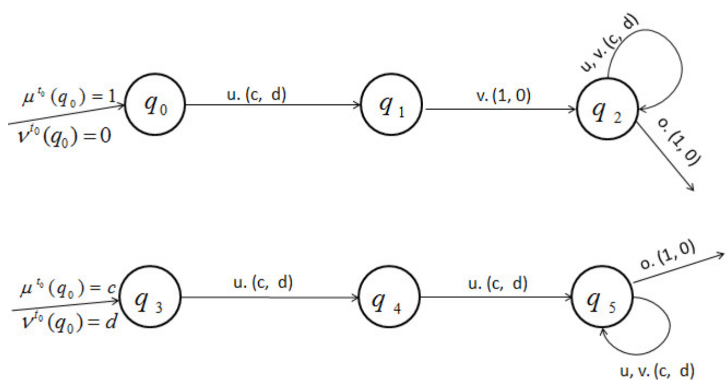

FIGURE 4. The Max-Min IGLFA of Example 4.9

By (α, β)-equivalence classes Algorithm, we have

(1) $\rho_0^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_2, q_4, q_5\}\}$,
(2) $\rho_1^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2, q_5\}\}$,
(3) $\rho_2^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2\}, \{q_5\}\}$,
(4) $\rho_3^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2\}, \{q_5\}\}$.

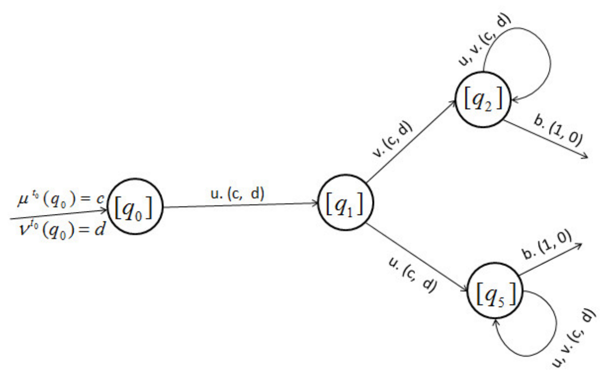

FIGURE 5. The Statewise (α, β)-minimal $\tilde{F}^*_m$ of Example 4.9

Then by the proof of Theorem 4.7, $Q_m = \{[q_0], [q_1], [q_2], [q_5]\}$. Therefore, we have statewise (a, b)-minimal $\tilde{F}^*_m$ as in Figure 5. Also, we have $\mathcal{L}^{a,b}(\tilde{F}^*) = uv\{u, v\}^* \cup u^2\{u, v\}^*$. Then by Definition 3.1, we obtain $\mathcal{L}_\Lambda = \mathcal{L}, \mathcal{L}_u = v\{u, v\}^* \cup u\{u, v\}^*, \mathcal{L}_v = \emptyset = \mathcal{L}_{vu} = \mathcal{L}_{v^2}, \mathcal{L}_{uv} = \{u, v\}^* = \mathcal{L}_{u^2} = \mathcal{L}_{uvu} = \mathcal{L}_{uv^2}$. So, $Q_\mathcal{L} = \{\mathcal{L}_\Lambda, \mathcal{L}_u, \mathcal{L}_v, \mathcal{L}_{uv}\}$. Hence, $\tilde{F}^*_\mathcal{L}$ is as in Figure 10. In this example, the number of states $\tilde{F}^*_m$ is equal to

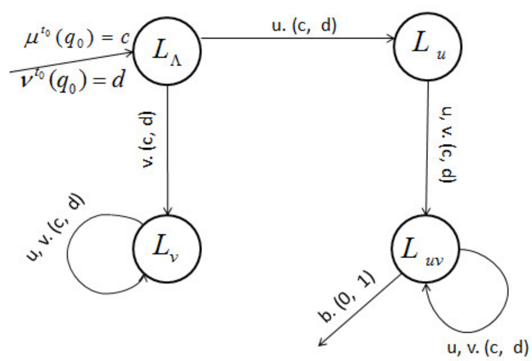

FIGURE 6. The Minimal $\tilde{F}^*_\mathcal{L}$ of Example 4.9

the number of states $\tilde{F}^*_\mathcal{L}$.

Example 4.10. Let the bounded lattice $L = (L, \leq_L, T, S, 0, 1)$ in Example 3.3, $\alpha = a, \beta = b$ and max-min IGLFA as in Figure 7. By considering (α, β)-equivalence Algorithm, we have

(1) $\rho_0^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_2, q_4\}\}$,
(2) $\rho_1^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2\}\}\}$,
(3) $\rho_2^{\alpha,\beta} = \{\{q_0, q_3\}, \{q_1, q_4\}, \{q_2\}\}\}$.

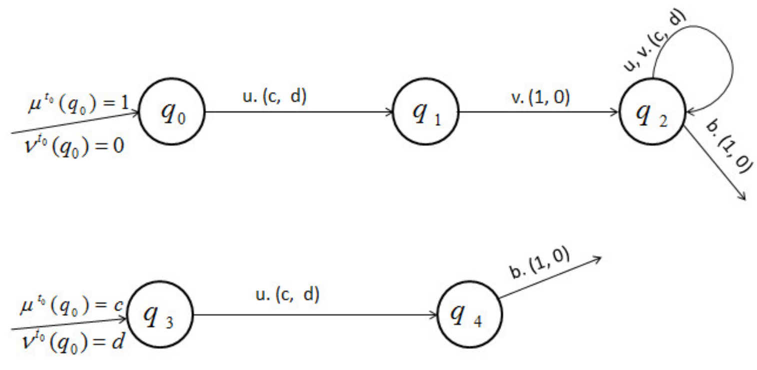

FIGURE 7. The Max-Min IGLFA of Example 4.10

Then by the proof of Theorem 4.7, $Q_m = \{[q_0], [q_1], [q_2]\}$. Therefore, we have statewise (a, b)-minimal $\tilde{F}^*_m$ as in Figure 8.

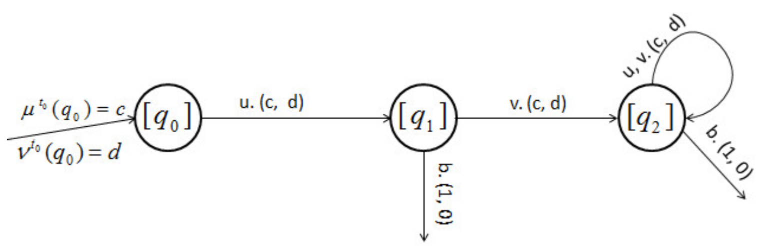

FIGURE 8. The Statewise (a, b)-minimal $\tilde{F}^*_m$ of Example 4.10

Also, we have $\mathcal{L}^{a,b}(\tilde{F}^*) = uv\{u, v\}^* \cup u$. Then by Definition 3.1, we obtain $\tilde{F}^*_{\mathcal{L}}$ as in Figure 9.

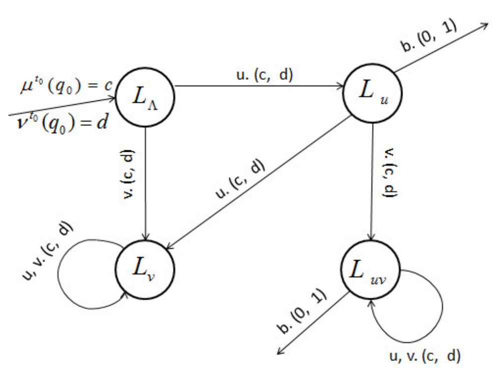

FIGURE 9. The Minimal $\tilde{F}^*_{\mathcal{L}}$ of Example 4.10

In this example, the number of states $\tilde{F}^*_m$ is less than the number of states $\tilde{F}^*_{\mathcal{L}}$.

Example 4.11. Consider the bounded lattice $L = (L, \leq_L, T, S, 0, 1)$ in Example 3.3, $\alpha = a, \beta = b$ and max-min IGLFA as in Figure 10.

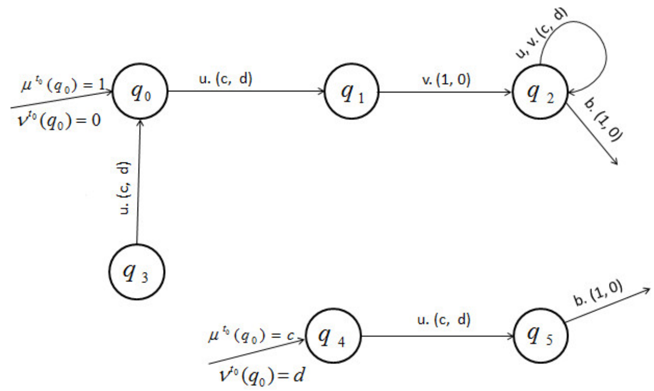

FIGURE 10. The Max-Min IGLFA of Example 4.11

By considering (α, β)-equivalence Algorithm, we have

(1) $\rho_0^{\alpha,\beta} = \{\{q_0, q_4\}, \{q_1, q_2, q_3, q_5\}\}$,
(2) $\rho_1^{\alpha,\beta} = \{\{q_0\}, \{q_4\}, \{q_1, q_5\}, \{q_2\}, \{q_3\}\}$,
(3) $\rho_2^{\alpha,\beta} = \{\{q_0\}, \{q_4\}, \{q_1\}, \{q_5\}, \{q_2\}, \{q_3\}\}$,
(4) $\rho_3^{\alpha,\beta} = \{\{q_0\}, \{q_4\}, \{q_1\}, \{q_5\}, \{q_2\}, \{q_3\}\}$.

Then by the proof of Theorem 4.7, $Q_m = \{[q_0], [q_1], [q_2], [q_3], [q_4], [q_5]\}$. Therefore, we have statewise (a, b)-minimal $\tilde{F}^*_m$ as in Figure 11.

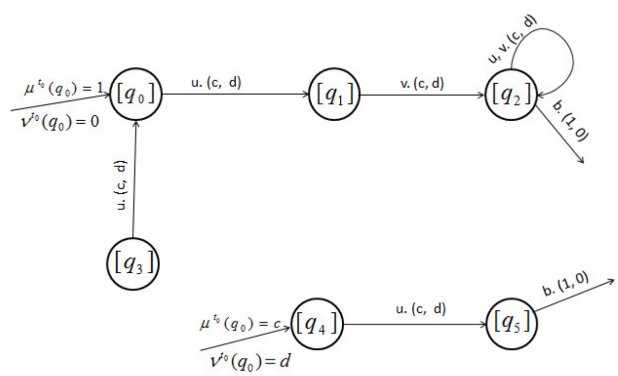

FIGURE 11. The Statewise (a, b)-minimal $\tilde{F}^*_m$ of Example 4.11

Also, we have $\mathcal{L}^{a,b}(\tilde{F}^*) = uv\{u, v\}^* \cup u$. Then by Definition 3.1, we obtain $\tilde{F}^*_{\mathcal{L}}$ as in Figure 9. In this example, the number of states $\tilde{F}^*_m$ is more than the number of states $\tilde{F}^*_{\mathcal{L}}$.

5. **Conclusion**

In this note, by considering the notion of intuitionistic general L-fuzzy automaton and (α, β)-language, we have proved that for any (α, β)-language $\mathcal{L}$, there exists a minimal max-min IGLFA recognizing $\mathcal{L}$. Also, we have shown that the minimal max-min IGLFA is isomorphic with threshold (α, β) to any (α, β)-reduced (α, β)-complete, (α, β)-accessible, deterministic max-min IGLFA. After that, for any strong deterministic max-min IGLFA, we have obtained the statewise (α, β)-minimal max-min IGLFA. Also, we have proved that if $\tilde{F}^*$ is an (α, β)-complete, (α, β)-accessible, deterministic max-min IGLFA and it is recognizing (α, β)-language $\mathcal{L}$, then the minimal $\tilde{F}^*_{\mathcal{L}}$ is homomorphism with threshold (α, β) to statewise (α, β)-minimal $\tilde{F}^*_m$, where $\tilde{F}^*_m$ is statewise (α, β)-equivalent to $\tilde{F}^*$.

In future research, we plan to find some algorithms which can calculate minimal automaton and statewise minimal automaton with less complexity than presented algorithms?

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