IMPLICATIONS, COIMPLICATIONS AND LEFT SEMI-UNINORMS ON A COMPLETE LATTICE

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Abstract. In this paper, we firstly show that the N-dual operation of the right residual implication, which is induced by a left-conjunctive right arbitrary $\vee$-distributive left semi-uninorm, is the right residual coimplication induced by its N-dual operation. As a dual result, the N-dual operation of the right residual coimplication, which is induced by a left-disjunctive right arbitrary $\wedge$-distributive left semi-uninorm, is the right residual implication induced by its N-dual operation. Then, we demonstrate that the N-dual operations of the left semi-uninorms induced by an implication and a coimplication, which satisfy the neutrality principle, are the left semi-uninorms. Finally, we reveal the relationships between conjunctive right arbitrary $\vee$-distributive left semi-uninorms induced by implications and disjunctive right arbitrary $\wedge$-distributive left semi-uninorms induced by coimplications, where both implications and coimplications satisfy the neutrality principle.

1. Introduction

Uninorms, introduced by Yager and Rybalov [27], and studied by Fodor et al. [9], are special aggregation operators that have been proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling (see [10, 25, 26]). This kind of operation is an important generalization of both t-norms and t-conorms and a special combination of t-norms and t-conorms. But, there are real-life situations when truth functions cannot be associative or commutative (see [6, 7]). By throwing away the commutativity from the axioms of uninorms, Mas et al. introduced the concepts of left and right uninorms in [15, 16], and Wang and Fang [23, 24] studied the left and right uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [13] introduced the concept of semi-uninorms, and Su et al. [22] discussed the notion of left and right semi-uninorms, on a complete lattice. On the other hand, it is well known that a uninorm (semi-uninorm, left and right uninorms) U is conjunctive or disjunctive whenever $U(0,1) = 0$ or 1, respectively. This fact allows us to use uninorms (semi-uninorm, left and right uninorms and so on) in defining fuzzy implications and coimplications (see [4, 5, 13, 20]).

Constructing fuzzy connecives is an interesting topic. Recently, Jenei and Montagna [12] introduced several new types of constructions of left-continuous t-norms,

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Mas et al. [17] derived two types of implications from uninorms, Ruiz and Torrens [18] investigated the residual implications and coimplications from idempotent uninorms, Su and Liu [19] studied the characterizations of residual coimplications of pseudo-uninorms on a complete lattice, and Su and Wang [21] discussed constructions of implications and coimplications on a complete lattice. In this paper, motivated by these works, we will further focus on this issue and investigate constructions of implications, coimplications and left semi-uninorms on a complete lattice.

The organization of this study is as follows. Section 2 recalls some necessary concepts examples about implications, coimplications, left semi-uninorms and N-dual operations. In Section 3, we show that the N-dual operation of the right residual implication, which is induced by a left-conjunctive right arbitrary $\vee$-distributive left semi-uninorm, is the right residual coimplication, which is induced by its N-dual operation. As a dual result, the N-dual operation of the right residual coimplication, which is induced by a left-disjunctive right arbitrary $\wedge$-distributive left semi-uninorm, is the right residual implication, which is induced by its N-dual operation. Then, we demonstrate that the N-dual operations of the left semi-uninorms induced by an implication and a coimplication, which satisfy the neutrality principle, are the left semi-uninorms. In Section 4, we reveal the relationships between conjunctive right arbitrary $\vee$-distributive left semi-uninorms induced by implications and disjunctive right arbitrary $\wedge$-distributive left semi-uninorms induced by coimplications, where both implications and coimplications satisfy the neutrality principle.

The knowledge about lattices required in this paper can be found in [11].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. Implications, Coimplications, Left Semi-uniorms and N-dual Operations

In this section, we briefly recall some concepts and examples which will be used in the paper.

Definition 2.1. (Baczyński and Jayaram [1], De Baets [3], De Baets and Fodor [4], Fodor and Roubens [8]) An implication I on L is a hybrid monotonous (with non-increasing first and non-decreasing second partial mappings) binary operation that satisfies the boundary conditions $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$. A coimplication C on L is a hybrid monotonous binary operation that satisfies the corner conditions $C(0,0) = C(1,1) = 0$ and $C(0,1) = 1$.

An implication I (a coimplication C) is said to satisfy the neutrality principle with respect to e (w.r.t. e, for short) if $I(e,y) = y$ ($C(e,y) = y$) for any $y \in L$.

Note that for any implication I and coimplication C on L, due to the monotonicity, the absorption principle holds, i.e., $I(0,x) = I(x,1) = 1$ and $C(x,0) = C(1,x) = 0$ for any $x \in L$.

Definition 2.2. (Wang and Fang [23, 24]) A binary operation U on L is called left (right) arbitrary $\vee$-distributive if

$$U(\bigvee_{j\in J} x_j, y) = \bigvee_{j\in J} U(x_j, y) \ \Big(U\big(x, \bigvee_{j\in J} y_j\big) = \bigvee_{j\in J} U(x, y_j)\Big) \ \forall x, y, x_j, y_j \in L;$$

left (right) arbitrary $\wedge$-distributive if

$$U(\bigwedge_{j\in J} x_j, y) = \bigwedge_{j\in J} U(x_j, y) \ \Big(U\big(x, \bigwedge_{j\in J} y_j\big) = \bigwedge_{j\in J} U(x, y_j)\Big) \ \forall x, y, x_j, y_j \in L.$$

If a binary operation U is left arbitrary $\vee$-distributive ($\wedge$-distributive) and also right arbitrary $\vee$-distributive ($\wedge$-distributive), then U is said to be arbitrary $\vee$-distributive ($\wedge$-distributive).

Left (right) arbitrary $\vee$-distributivity and left (right) arbitrary $\wedge$-distributivity are, respectively, called left (right) infinitely $\vee$-distributivity and left (right) infinitely $\wedge$-distributivity in [23, 24]. But, speaking of "infinitely" distributivity is not appropriate, since the index set J may be a finite or empty set.

Noting that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1 (see [2, 11]), we have that

$$U(0, y) = U(\bigvee_{j\in\emptyset} x_j, y) = \bigvee_{j\in\emptyset} U(x_j, y) = 0\Big(U(x, 0) = U\big(x, \bigvee_{j\in\emptyset} y_j\big) = \bigvee_{j\in\emptyset} U(x, y_j) = 0\Big)$$

for any $x, y \in L$ when U is left (right) arbitrary $\vee$-distributive and

$$U(1, y) = U(\bigwedge_{j\in\emptyset} x_j, y) = \bigwedge_{j\in\emptyset} U(x_j, y) = 1 \ \Big(U(x, 1) = U\big(x, \bigwedge_{j\in\emptyset} y_j\big) = \bigwedge_{j\in\emptyset} U(x, y_j) = 1\Big)$$

for any $x, y \in L$ when U is left (right) arbitrary $\wedge$-distributive.

For the sake of convenience, we introduce the following symbols:

$\mathcal{I}(L)$ ($\mathcal{C}(L)$): the set of all implications (coimplications) on L;

$\mathcal{I}_\wedge(L)$ ($\mathcal{C}_\vee(L)$): the set of all right arbitrary $\wedge$-distributive ($\vee$-distributive) implications (coimplications) on L;

$\mathcal{I}^{npe}(L)$ ($\mathcal{C}^{npe}(L)$): the set of all implications (coimplications) which satisfy the neutrality principle w.r.t. e on L;

$\mathcal{I}^{npe}_\wedge(L)$ ($\mathcal{C}^{npe}_\vee(L)$): the set of all right arbitrary $\wedge$-distributive ($\vee$-distributive) implications (coimplications) which satisfy the neutrality principle w.r.t. e on L.

Example 2.3. (Su and Wang [21]) Let

$$I_W(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad I_M(x, y) = \begin{cases} 0 & \text{if } (x, y) = (1, 0), \\ 1 & \text{otherwise,} \end{cases}$$

$$C_W(x, y) = \begin{cases} 1 & \text{if } (x, y) = (0, 1), \\ 0 & \text{otherwise,} \end{cases} \qquad C_M(x, y) = \begin{cases} 0 & \text{if } x = 1 \text{ or } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

where x and y are elements of L. It is easy to see that I_W and I_M are, respectively, the smallest and greatest elements of $\mathcal{I}(L)$ and I_W is also the smallest element of $\mathcal{I}_\wedge(L)$. C_W and C_M are, respectively, the smallest and greatest elements of $\mathcal{C}(L)$ and C_M is also the largest element of $\mathcal{C}_\vee(L)$.

Example 2.4. (Su and Wang [21]) Let $L = \{0, a, b, 1\}$ be a lattice, where $0 < a < 1$, $0 < b < 1$, $a \wedge b = 0$ and $a \vee b = 1$. Define two implications I_1, I_2 and two coimplications C_1, C_2 as follows:

I_1	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	0	a	b	1

I_2	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	0	b	a	1

C_1	0	a	b	1
0	0	a	b	1
a	0	0	0	0
b	0	0	0	0
1	0	0	0	0

C_2	0	a	b	1
0	0	b	a	1
a	0	0	0	0
b	0	0	0	0
1	0	0	0	0

It is straightforward to verify that I_1 and I_2 are two right arbitrary $\wedge$-distributive implications, C_1 and C_2 are two right arbitrary $\vee$-distributive coimplications, $I_1 \vee I_2 = I_M$ and $C_1 \wedge C_2 = C_W$. But I_M is not right arbitrary $\wedge$-distributive and C_W is not right arbitrary $\vee$-distributive.

This example shows that $\mathcal{I}_\wedge(L)$ is not a join-semilattice and $\mathcal{C}_\vee(L)$ is not a meet-semilattice.

Definition 2.5. (Su et al. [22]) A binary operation U on L is called a left (right) semi-uninorm if it satisfies the following two conditions:

(U1) there exists a left (right) neutral element, i. e., an element $e_L \in L$ ($e_R \in L$) satisfying $U(e_L, x) = x$ ($U(x, e_R) = x$) for all $x \in L$,

(U2) U is non-decreasing in each variable.

In the sequel, we only discuss left semi-uninorms. Similar results hold for right semi-uninorms.

For any left semi-uninorm U on L, U is said to be left-conjunctive and right-conjunctive if $U(0, 1) = 0$ and $U(1, 0) = 0$, respectively. U is called conjunctive if both $U(0, 1) = 0$ and $U(1, 0) = 0$ since it satisfies the classical boundary conditions of AND. U is said to be left-disjunctive and right-disjunctive if $U(1, 0) = 1$ and $U(0, 1) = 1$, respectively. We call U disjunctive if both $U(1, 0) = 1$ and $U(0, 1) = 1$ by a similar reason.

If a left semi-uninorm U is associative, then U is the left uninorm in [23, 24]. If a left semi-uninorm U with the left neutral element e_L has a right neutral element e_R, then $e_L = U(e_L, e_R) = e_R$. Let $e = e_L = e_R$. Here, U is the semi-uninorm in [13].

Now, for the sake of convenience, we list the following symbols:

$\mathcal{U}_s^{e_L}(L)$: the set of all left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{s\vee}^{e_L}(L)$: the set of all right arbitrary $\vee$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{s\wedge}^{e_L}(L)$: the set of all right arbitrary $\wedge$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{cs}^{e_L}(L)$: the set of all conjunctive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{cs\vee}^{e_L}(L)$: the set of all conjunctive right arbitrary $\vee$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{ds}^{e_L}(L)$: the set of all disjunctive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{ds\wedge}^{e_L}(L)$: the set of all disjunctive right arbitrary $\wedge$-distributive left semi-uninorms with left neutral element e_L on L.

Example 2.6. Let $e_L \in L$,

$$U_s^W(x,y) = \begin{cases} y & \text{if } x \geq e_L, \\ 0 & \text{otherwise,} \end{cases} \quad U_s^M(x,y) = \begin{cases} y & \text{if } x \leq e_L, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{cs}^M(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ y & \text{if } 0 < x \leq e_L, y \neq 0, \\ 1 & \text{otherwise,} \end{cases} \quad U_{ds}^W(x,y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1, \\ y & \text{if } e_L \leq x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where x and y are elements of L. By virtue of Example 2.5 in [22], we know that U_s^W and U_s^M are, respectively, the smallest and greatest elements of $\mathcal{U}_s^{e_L}(L)$; U_s^W is the smallest element of $\mathcal{U}_{s\vee}^{e_L}(L)$; and U_s^M is the greatest element of $\mathcal{U}_{s\wedge}^{e_L}(L)$. Moreover, it is easy to see that U_{cs}^M is the greatest element of $\mathcal{U}_{cs\vee}^{e_L}(L)$; U_{ds}^W is the smallest element of $\mathcal{U}_{ds\wedge}^{e_L}(L)$; U_s^W is the smallest element of $\mathcal{U}_{cs\vee}^{e_L}(L)$ when $e_L \neq 0$; and U_s^M is the greatest element of $\mathcal{U}_{ds\wedge}^{e_L}(L)$ when $e_L \neq 1$.

Definition 2.7. (Ma and Wu [14]) A mapping $N : L \to L$ is called a negation if

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $x \leq y,\ x, y \in L \Rightarrow N(y) \leq N(x)$.

A negation N is called strong if it is an involution, i. e., $N(N(x)) = x$ for any $x \in L$.

Definition 2.8. (De Baets [3]) Consider a strong negation N on L. The N-dual operation of a binary operation A on L is the binary operation A_N on L defined by

$$A_N(x,y) = N^{-1}\Big(A\big(N(x), N(y)\big)\Big) \ \forall x, y \in L.$$

Note that $(A_N)_{N^{-1}} = (A_N)_N = A$ for any binary operation A on L.

Moreover, for any nonempty subfamily $\{A_j \mid j \in J\}$ of $L^{L\times L}$, the least upper bound $\vee_{j\in J} A_j$ and the greatest lower bound $\wedge_{j\in J} A_j$ of $A_j^{'}s$ are, respectively, defined by

$$(\bigvee_{j\in J} A_j)(x,y) = \bigvee_{j\in J} A_j(x,y) \text{ and } (\bigwedge_{j\in J} A_j)(x,y) = \bigwedge_{j\in J} A_j(x,y) \ \forall x, y \in L.$$

3. The Residual Implications and Coimplicatons Induced by Left Semi-uninorms and the Left Semi-uninorms Induced by Implications and Coimplications

Recently, De Baets and Fodor [4] investigated the residual operators of uninorms on $[0,1]$, Torrens et al. [17, 18] studied the implications and coimplications derived from uninorms on $[0,1]$. Now, we consider the residual implications and coimplications induced by left semi-uninorms on a complete lattice.

For a binary operation U on L, let

$$I_U^L(x,y)=\bigvee\{z\in L\mid U(z,x)\le y\},\ I_U^R(x,y)=\bigvee\{z\in L\mid U(x,z)\le y\}\ \forall x,y\in L.$$

Here, I_U^L and I_U^R are, respectively, called the left and right residuum of U.

When U is a left semi-uninorm on L, it is easy to see that I_U^L and I_U^R are all non-increasing in the first variable and non-decreasing in the second one.

For any operation U on L and $x,y\in L$, it follows from Theorems 4.1 and 4.2 in [23] that

(1) $I_U^L(x,1)=I_U^R(x,1)=1$.

(2) $x\le I_U^L\big(y,U(x,y)\big)$ and $y\le I_U^R\big(x,U(x,y)\big)$.

(3) If $U(1,0)=0$, then $I_U^L(0,y)=1$ and if $U(0,1)=0$, then $I_U^R(0,y)=1$.

(4) If U is a left semi-uninorm with the left neutral element e_L, then $I_U^R(e_L,y)=y$ for any $y\in L$.

By virtue of Theorems 3.1, 3.3 and 3.4 in [13], we see that if U is a left-conjunctive left semi-uninorm with the left neutral element e_L, then I_U^R is an implication which satisfies the neutrality principle w.r.t. e_L; if U is a left-conjunctive right arbitrary $\vee$-distributive left semi-uninorm with the left neutral element e_L, then I_U^R is a right arbitrary $\wedge$-distributive implication and

$$I_U^R(x,y)=\max\{z\in L\mid U(x,z)\le y\}.$$

Here, I_U^R is called the right residual implication induced by the left semi-uninorm U.

By Theorems 4.4 and 4.5 in [23] or Theorems 3.3 and 3.4 in [13], we know that if a binary operation U is right arbitrary $\vee$-distributive, then U and I_U^R satisfy the generalized modus ponens (GMP) rule (see [4]) $U\big(x,I_U^R(x,y)\big)\le y$ and the following right residual (implication) principle:

$$U(x,z)\le y\Leftrightarrow z\le I_U^R(x,y)\ \forall x,y,z\in L;$$

if U is left arbitrary $\vee$-distributive, then U and I_U^L satisfy GMP rule in the form $U(I_U^L(x,y),x)\le y$ and the following left residual (implication) principle:

$$U(z,x)\le y\Leftrightarrow z\le I_U^L(x,y)\ \forall x,y,z\in L.$$

Example 3.1. For some left semi-uninorms in Example 2.6, a simple computation shows that

$$I^R_{U_s^W}(x,y)=\begin{cases} y & \text{if } x\ge e_L,\\ 1 & \text{otherwise,}\end{cases}\quad I^L_{U_{cs}^M}(x,y)=\begin{cases} 1 & \text{if } x=0 \text{ or } y=1,\\ e_L & \text{if } 0<x\le y<1,\\ 0 & \text{otherwise,}\end{cases}$$

$$I^R_{U_{cs}^M}(x,y)=\begin{cases} 1 & \text{if } x=0 \text{ or } y=1,\\ y & \text{if } 0<x\le e_L, y\ne 1,\\ 0 & \text{otherwise,}\end{cases}$$

When $e_L\in L\setminus\{0\}$, we see that $I^L_{U_{cs}^M}$ is an implication; $I^R_{U_{cs}^M}$ is the smallest element of $\mathcal{I}_\wedge^{npe_L}(L)$; and $I^R_{U_s^W}$ is the greatest element of $\mathcal{I}_\wedge^{npe_L}(L)$.

For a binary operation U on L, let

$$C_U^L(x,y) = \bigwedge\{z \in L \mid y \le U(z,x)\},\ C_U^R(x,y) = \bigwedge\{z \in L \mid y \le U(x,z)\}\ \forall x,y \in L.$$

Here, C_U^L and C_U^R are, respectively, called the left and right deresiduum of U.

For any operation U on L, it follows from Theorems 3.1 and 3.2 in [24] that

(1) $C_U^L(x,0) = C_U^R(x,0) = 0$ for any $x \in L$.

(2) For any $x,y \in L$, $C_U^L\big(y,U(x,y)\big) \le x$ and $C_U^R\big(x,U(x,y)\big) \le y$.

(3) If U is right-disjunctive, then $C_U^L(1,y) = 0$ and if U is left-disjunctive, then $C_U^R(1,y) = 0$.

(4) If U is a left semi-uninorm with the left neutral element e_L, then $C_U^R(e_L,y) = y$ for any $y \in L$.

It is easy to see that C_U^L and C_U^R are all non-increasing in the first variable and non-decreasing in the second one when U is a left semi-uninorm; $C_U^L(e,x) = C_U^R(e,x) = x$ for any $x \in L$ when U is a semi-uninorm with the neutral element e.

Example 3.2. For some left semi-uninorms in Example 2.6, a simple computation shows that

$$C^R_{U_s^M}(x,y) = \begin{cases} y & \text{if } x \le e_L, \\ 0 & \text{otherwise,} \end{cases} \quad C^L_{U_{ds}^W}(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ or } y = 0, \\ e_L & \text{if } 0 < y \le x < 1, \\ 1 & \text{otherwise,} \end{cases}$$

$$C^R_{U_{ds}^W}(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ or } y = 0, \\ y & \text{if } e_L \le x < 1, \\ 1 & \text{otherwise,} \end{cases}$$

When $e_L \in L\backslash\{1\}$, we see that $C^L_{U_{ds}^W}$ is a coimplication, $C^R_{U_s^M}$ is the smallest element of $\mathcal{C}_\vee^{npe_L}(L)$; and $C^R_{U_{ds}^W}$ is the greatest element of $\mathcal{C}_\vee^{npe_L}(L)$.

Theorem 3.3. *If $U \in \mathcal{U}_s^{e_L}(L)$ is left-disjunctive, then $C_U^R \in \mathcal{C}(L)$.*

Proof. If U is a left-disjunctive left semi-uninorm with the left neutral element e_L, then C_U^R is non-increasing in its first and non-decreasing in its second variable and $C_U^R(1,1) = 0$. Moreover,

$$C_U^R(0,0) = \bigwedge\{z \in L \mid 0 \le U(0,z)\} = 0.$$

By the non-decreasingness of U, we see that

$$C_U^R(0,1) = \bigwedge\{z \in L \mid U(0,z) = 1\} \ge \bigwedge\{z \in L \mid z = U(e_L,z) \ge U(0,z) = 1\} = 1.$$

Thus, C_U^R is a coimplication on L. □

Moreover, if $U \in \mathcal{U}_{s\wedge}^{e_L}(L)$ is left-disjunctive, then it follows from Theorems 3.1 and 3.2 in [13], Theorem 3.5 in [24] and Theorem 3.3 that $C_U^R \in \mathcal{C}_\vee(L)$ and

$$C_U^R(x,y) = \min\{z \in L \mid y \le U(x,z)\}.$$

Here, C_U^R is called the right residual coimplication induced by the left semi-uninorm U.

If P and Q are two propositions, then the property $U(x, C_U^R(x,y)) \ge y$ is a generalization of the following tautology $Q \Rightarrow (P \vee (P \not\Leftarrow Q))$ in classical logic and is in some sense dual to the modus ponens (see [3]). By Theorems 3.3 and 3.4 in [24], we know that U and C_U^R satisfy the generalized dual modus ponens rule and the following right residual (coimplication) principle:

$$y \le U(x,z) \Leftrightarrow C_U^R(x,y) \le z \ \forall x,y,z \in L$$

when U is a right arbitrary $\wedge$-distributive left semi-uninorm on L.

The following theorem reveals the relationships between the residual implications and the residual coimplications.

Theorem 3.4. *Let U be a binary operation and N strong negation on L. Then*
(1) $(I_U^L)_N = C_{U_N}^L$ and $(C_U^L)_N = I_{U_N}^L$.
(2) $(I_U^R)_N = C_{U_N}^R$ and $(C_U^R)_N = I_{U_N}^R$.

Proof. We only prove that statement (1) holds.

Noting that the strong negation N is a bijection, by Definition 2.8, we have that

$$\begin{aligned}
(I_U^L)_N(x,y) &= N\Big(I_U^L\big(N(x),N(y)\big)\Big) \\
&= N\Big(\bigvee\big\{z \in L \mid U(z,N(x)) \le N(y)\big\}\Big) \\
&= \bigwedge\big\{N(z) \in L \mid N\Big(U\big(N(N(z)),N(x)\big)\Big) \ge y\big\} \\
&= \bigwedge\big\{N(z) \in L \mid y \le U_N(N(z),x)\big\} \\
&= \bigwedge\big\{u \in L \mid y \le U_N(u,x)\big\} = C_{U_N}^L(x,y) \ \ \forall x,y \in L.
\end{aligned}$$

Thus, $(I_U^L)_N = C_{U_N}^L$. Moreover, $(I_{U_N}^L)_N = C_{(U_N)_N}^L = C_U^L$ and so $(C_U^L)_N = I_{U_N}^L$. □

By virtue of Theorem 3.4, we see that the N-dual operation of the right residual implication, which is induced by a left-conjunctive right arbitrary $\vee$-distributive left semi-uninorm, is the right residual coimplication induced by its N-dual operation and the N-dual operation of the right residual coimplication, which is induced by a left-disjunctive right arbitrary $\wedge$-distributive left semi-uninorm, is the right residual implication induced by its N-dual operation.

Liu [13] discussed the semi-uninorms induced by implications, and Su and Wang [20] studied the pseudo-uninorms induced by coimplications. Below, we investigate the left semi-uninorms induced by implications and coimplications on a complete lattice.

For an implication I on L, let

$$U_I^L(x,y) = \bigwedge\big\{z \in L \mid x \le I(y,z)\big\},\ U_I^R(x,y) = \bigwedge\big\{z \in L \mid y \le I(x,z)\big\}\ \forall x,y \in L.$$

Clearly, $U_I^R = C_I^R$, $U_I^L(0,x) = U_I^R(x,0) = 0$, $U_I^L(1,x) = U_I^R(x,1)$ for any $x \in L$. It is easy to see that U_I^L and U_I^R are all non-decreasing in its each variable and

$$U_I^L\big(I(x,y),x\big) \le y,\ U_I^R\big(x,I(x,y)\big) \le y\ \forall x,y \in L,$$

i.e., U_I^L and I, U_I^R and I satisfy the GMP rule.

For a coimplication C on L, let

$$U_C^L(x,y) = \bigvee\big\{z \in L \mid C(y,z) \le x\big\},\ U_C^R(x,y) = \bigvee\big\{z \in L \mid C(x,z) \le y\big\}\ \forall x,y \in L.$$

Obviously, $U_C^R = I_C^R$, $U_C^L(1,x) = U_C^R(x,1) = 1$; $U_C^L(0,x) = U_C^R(x,0) = \vee\{z \in L \mid C(x,z) = 0\}$ for any $x \in L$. It is also easy to see that U_C^L and U_C^R are all non-decreasing in its each variable and

$$y \leq U_C^L\big(C(x,y),x\big),\ y \leq U_C^R\big(x, C(x,y)\big)\ \forall x,y \in L.$$

These explain that U_C^L and C, U_C^R and C satisfy the generalized dual modus ponens rule.

Example 3.5. For I_W, I_M, C_W and C_M in Example 2.3, we have that

$$U_{I_W}^L(x,y) = U_{I_W}^R(x,y) = \begin{cases} 0 & \text{if } x=0 \text{ or } y=0, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{I_M}^L(x,y) = \begin{cases} \wedge_{a\in L\setminus\{0\}} a & \text{if } x>0, y=1, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{I_M}^R(x,y) = \begin{cases} \wedge_{a\in L\setminus\{0\}} a & \text{if } x=1, y>0, \\ 0 & \text{otherwise.} \end{cases}$$

$$U_{C_M}^L(x,y) = U_{C_M}^R(x,y) = \begin{cases} 1 & \text{if } x=1 \text{ or } y=1, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{C_W}^L(x,y) = \begin{cases} \vee_{a\in L\setminus\{1\}} a & \text{if } x<1, y=0, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{C_W}^R(x,y) = \begin{cases} \vee_{a\in L\setminus\{1\}} a & \text{if } x=0, y<1, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, these operations induced by implications I_W and I_M and coimplications C_W and C_M are neither left semi-uninorms nor right semi-uninorms on L.

Now, we find some conditions such that these operations induced by implications and coimplications are left semi-uninorms.

Theorem 3.6. *Let $I \in \mathcal{I}(L)$ and $C \in \mathcal{C}(L)$. If I and C satisfies the neutrality principle w.r.t. e_L, then U_I^R, $U_C^R \in \mathcal{U}_s^{e_L}(L)$. Moreover, if $I \in \mathcal{I}_\wedge(L)$ and $C \in \mathcal{C}_\vee(L)$, then $U_I^R \in \mathcal{U}_{s\vee}^{e_L}(L)$ and $U_C^R \in \mathcal{U}_{s\wedge}^{e_L}(L)$.*

Here, U_I^R and U_C^R are called the left semi-uninorms induced by the implication I and the coimplication C, respectively.

Proof. Assume that $C \in \mathcal{C}(L)$. Then U_C^R is non-decreasing in each variable. If C satisfies the neutrality principle w.r.t. e_L, then

$$U_C^R(e_L,y) = \bigvee\big\{z \in L \mid C(e_L,z) \leq\ y\big\} = \bigvee\big\{z \in L \mid z \leq y\big\} = y\ \forall y \in L.$$

So, $U_C^R \in \mathcal{U}_s^{e_L}(L)$. Moreover, if C is a right arbitrary $\vee$-distributive, then it follows from Theorem 5.3 in [20] that U_C^R is right arbitrary $\wedge$-distributive. Thus, $U_C^R \in \mathcal{U}_{s\wedge}^{e_L}(L)$.

Similarly, we can show that $U_I^R \in \mathcal{U}_s^{e_L}(L)$ when I satisfies the neutrality principle w.r.t. e_L and $U_I^R \in \mathcal{U}_{s\vee}^{e_L}(L)$ when $I \in \mathcal{I}_\wedge(L)$ satisfies the neutrality principle w.r.t. e_L. □

When $I \in \mathcal{I}(L)$, $I(0,x) = 1$ for any $x \in L$ and hence $U_I^L(1,0) = U_I^R(0,1) = 0$. Thus, U_I^R in Theorem 3.6 is the conjunctive left semi-uninorms induced by the implication I.

When $C \in \mathcal{C}(L)$, $C(1,x) = 0$ for any $x \in L$ and hence $U_C^L(0,1) = U_C^R(1,0) = 1$. Thus, U_C^R in Theorem 3.6 is the disjunctive left semi-uninorms induced by the coimplication C.

By virtue of Theorems 4.2 and 4.3 in [13] and Theorems 5.1 and 5.2 in [20], we know that I, U_I^L and U_I^R satisfy the following adjunction conditions:

$$x \le I(y,z) \Leftrightarrow U_I^L(x,y) \le z,\ y \le I(x,z) \Leftrightarrow U_I^R(x,y) \le z\ \forall x,y,z \in L$$

when I is a right arbitrary $\wedge$-distributive implication on L; C, U_C^L and U_C^R satisfy the following adjunction conditions:

$$C(y,z) \le x \Leftrightarrow z \le U_C^L(x,y),\ C(x,z) \le y \Leftrightarrow z \le U_C^R(x,y)\ \forall x,y,z \in L$$

when C is a right arbitrary $\vee$-distributive coimplication on L.

The following theorem reveals the relationships between the left semi-uninorms induced by implications and coimplications.

Theorem 3.7. *Let I be an implication, C a coimplication and N a strong negation on L. Then*
(1) $(U_C^L)_N = U_{C_N}^L$ and $(U_I^L)_N = U_{I_N}^L$.
(2) $(U_C^R)_N = U_{C_N}^R$ and $(U_I^R)_N = U_{I_N}^R$.

Proof. We only prove that statement (1) holds.

If I is an implication and C a coimplication, then it is easy to see that I_N is a coimplication and C_N an implication. By Definition 2.8, we see that

$$\begin{aligned}
(U_C^L)_N(x,y) &= N\Big(U_C^L\big(N(x),N(y)\big)\Big)\\
&= N\Big(\bigvee\big\{z \in L \mid C\big(N(y),z\big) \le N(x)\big\}\Big)\\
&= \bigwedge\big\{N(z) \in L \mid C(N(y),z) \le N(x)\big\}\\
&= \bigwedge\big\{N(z) \in L \mid N\big(C(N(y),N(N(z)))\big) \ge x\big\}\\
&= \bigwedge\big\{N(z) \in L \mid C_N(y,N(z)) \ge x\big\}\\
&= \bigwedge\big\{u \in L \mid C_N(y,u) \ge x\big\}\\
&= (U_{C_N}^L)(x,y)\ \forall x,y \in L.
\end{aligned}$$

Thus, $(U_C^L)_N = U_{C_N}^L$.

We can prove in an analogous way that $(U_I^L)_N = U_{I_N}^L$. □

By Theorems 3.6 and 3.7, we know that the N-dual operation of the left semi-uninorm induced by an implication, which satisfies the neutrality principle w.r.t. e_L, is the left semi-uninorm induced by its N-dual operation. As a dual result, the N-dual operation of the left semi-uninorm induced by a coimplication, which satisfies the neutrality principle w.r.t. e_L, is the left semi-uninorm induced by its N-dual operation.

4. The Relations Between Conjunctive Left Semi-uninorms Induced by Implications and Disjunctive Left Semi-uninorms Induced by Coimplications

We know that the N-dual operations of an implication and a coimplication are, respectively, a coimplication and an implication and the N-dual operation of a left semi-uninorm is a left semi-uninorm. By virtue of Theorem 3.4, we see that the N-dual operations of the right residual implication and coimplication, which are induced by a left semi-uninorm, are, respectively, the right residual coimplication and implication, which are induced by its N-dual operation. By Theorem 3.7, we know that the N-dual operations of the left semi-uninorms induced by an implication and a coimplication, which satisfy the neutrality principle, are the left semi-uninorms.

In the final section, we reveal the relationships between conjunctive right arbitrary $\vee$-distributive left semi-uninorms induced by implications and disjunctive right arbitrary $\wedge$-distributive left semi-uninorms induced by coimplications on a complete lattice.

Theorem 4.1. *(1) If $U \in \mathcal{U}_{s\vee}^{e_L}(L)$ is left-conjunctive, then $I_U^R \in \mathcal{I}_\wedge(L)$ satisfies the neutrality principle w.r.t. e_L and $U_{I_U^R}^R = U$.*

(2) If $U \in \mathcal{U}_{s\wedge}^{e_L}(L)$ is left-disjunctive, then $C_U^R \in \mathcal{C}_\vee(L)$ satisfies the neutrality principle w.r.t. e_L and $U_{C_U^R}^R = U$.

(3) If $I \in \mathcal{I}_\wedge(L)$ satisfies the neutrality principle w.r.t. e_L, then $U_I^R \in \mathcal{U}_{s\vee}^{e_L}(L)$ is conjunctive and $I_{U_I^R}^R = I$.

(4) If $C \in \mathcal{C}_\vee(L)$ satisfies the neutrality principle w.r.t. e_L, then $U_C^R \in \mathcal{U}_{s\wedge}^{e_L}(L)$ is disjunctive and $C_{U_C^R}^R = C$.

Proof. We only prove that statements (1) and (3) hold.

(1) If U is a left-conjunctive right arbitrary $\vee$-distributive left semi-uninorm, then $I_U^R \in \mathcal{I}_\wedge(L)$ satisfies the neutrality principle w.r.t. e_L by Theorem 3.1 in [13] and Theorem 4.6 in [23]. Moreover, it follows from the right residual (implication) principle that

$$U_{I_U^R}^R(x,y) = \bigwedge\{z \in L \mid y \le I_U^R(x,z)\} = \bigwedge\{z \in L \mid U(x,y) \le z\} = U(x,y)\ \forall x,y \in L.$$

Thus, $U_{I_U^R}^R = U$.

(3) If $I \in \mathcal{I}_\wedge(L)$ satisfies the neutrality principle w.r.t. e_L, then U_I^R is a conjunctive right arbitrary $\vee$-distributive left semi-uninorm by Theorem 3.6. Moreover, it follows from the adjunction condition that

$$I_{U_I^R}^R(x,y) = \bigvee\{z \in L \mid U_I^R(x,z) \le y\} = \bigvee\{z \in L \mid z \le I(x,y)\} = I(x,y)\ \forall x,y \in L.$$

Therefore, $I_{U_I^R}^R = I$. □

Example 4.2. Let $L = [0,1]$,

$$U(x,y) = \begin{cases} \frac{1}{4}xy & \text{if } y = 0 \text{ or } x < \frac{1}{2}, \\ y & \text{if } x = \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, $U \in \mathcal{U}_{s\vee}^{\frac{1}{2}}([0,1])$ is left-conjunctive and

$$I_U^R(x,y) = \sup\{z \in [0,1] \mid U(x,z) \le y\} = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ \min\{1, \frac{4y}{x}\} & \text{if } 0 < x < \frac{1}{2}, \\ y & \text{if } x = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $I_U^R \in \mathcal{I}_\wedge([0,1])$ satisfies the neutrality principle w.r.t. $\frac{1}{2}$ and

$$U_{I_U^R}^R(x,y) = \inf\{z \in [0,1] \mid y \le I_U^R(x,z)\} = \begin{cases} \frac{1}{4}xy & \text{if } y = 0 \text{ or } x < \frac{1}{2}, \\ y & \text{if } x = \frac{1}{2}, \\ 1 & \text{otherwise,} \end{cases}$$

i.e., $U_{I_U^R}^R = U$.

Theorem 4.3. *(1) If $e_L \neq 0$, then $\mathcal{U}_{cs\vee}^{e_L}(L)$ is a complete lattice with the smallest element U_s^W and greatest element U_{cs}^M.*

(2) If $e_L \neq 1$, then $\mathcal{U}_{ds\wedge}^{e_L}(L)$ is a complete lattice with the smallest element U_{ds}^W and greatest element U_s^M.

(3) If $e_L \neq 0$, then $\mathcal{I}_\wedge^{npe_L}(L)$ is a complete lattice with the smallest element $I_{U_{cs}^M}^R$ and greatest element $I_{U_s^W}^R$.

(4) If $e_L \neq 1$, then $\mathcal{C}_\vee^{npe_L}(L)$ is a complete lattice with the smallest element $C_{U_s^M}^R$ and greatest element $C_{U_{ds}^W}^R$.

Proof. We only prove that statements (1) and (3) hold.

(1) Suppose that $U_j \in \mathcal{U}_{cs\vee}^{e_L}(L)$ $(j \in J)$ and $J \neq \emptyset$. Clearly, $\bigvee_{j\in J} U_j \in \mathcal{U}_{cs}^{e_L}(L)$. Moreover, for any index set K and any $x, y_k \in L$ $(k \in K)$, we have that

$$\begin{aligned} \big(\bigvee_{j\in J} U_j\big)\big(x, \bigvee_{k\in K} y_k\big) &= \bigvee_{j\in J} U_j\big(x, \bigvee_{k\in K} y_k\big) = \bigvee_{j\in J}\bigvee_{k\in K} U_j(x,y_k) \\ = \bigvee_{k\in K}\bigvee_{j\in J} U_j(x,y_k) &= \bigvee_{k\in K}\big(\bigvee_{j\in J} U_j(x,y_k)\big) = \bigvee_{k\in K}\big((\bigvee_{j\in J} U_j)(x,y_k)\big). \end{aligned}$$

Hence, $\bigvee_{j\in J} U_j \in \mathcal{U}_{cs\vee}^{e_L}(L)$. By virtue of Theorem 4.2 in [2] and Example 2.6, we see that $\mathcal{U}_{cs\vee}^{e_L}(L)$ is a complete lattice with the smallest element U_s^W and greatest element U_{cs}^M when $e_L \neq 0$.

(3) Assume that $e_L \neq 0$, $I_j \in \mathcal{I}_\wedge^{npe_L}(L)$ $(j \in J)$, and $J \neq \emptyset$. Clearly, $\bigwedge_{j\in J} I_j \in \mathcal{I}^{npe_L}(L)$. Moreover, for any index set K and any $x, y_k \in L$ $(k \in K)$, we see that

$$\begin{aligned} \big(\bigwedge_{j\in J} I_j\big)\big(x, \bigwedge_{k\in K} y_k\big) &= \bigwedge_{j\in J} I_j\big(x, \bigwedge_{k\in K} y_k\big) = \bigwedge_{j\in J}\bigwedge_{k\in K} I_j(x,y_k) \\ = \bigwedge_{k\in K}\bigwedge_{j\in J} I_j(x,y_k) &= \bigwedge_{k\in K}\big(\bigwedge_{j\in J} I_j(x,y_k)\big) = \bigwedge_{k\in K}\big((\bigwedge_{j\in J} I_j)(x,y_k)\big). \end{aligned}$$

Hence, $\bigwedge_{j\in J} I_j \in \mathcal{I}_\wedge^{npe_L}(L)$. By virtue of Theorem 4.2 in [2] and Example 3.1, we know that $\mathcal{I}_\wedge^{npe_L}(L)$ is a complete lattice with the smallest element $I_{U_{cs}^M}^R$ and greatest element $I_{U_s^W}^R$. □

Define two mappings $\varphi_1 : \mathcal{U}^{e_L}_{cs\vee}(L) \to \mathcal{I}^{npe_L}_{\wedge}(L)$ and $\varphi_2 : \mathcal{U}^{e_L}_{ds\wedge}(L) \to \mathcal{C}^{npe_L}_{\vee}(L)$ as follows:

$$\varphi_1(U) = I^R_U \ \forall U \in \mathcal{U}^{e_L}_{cs\vee}(L),\ \varphi_2(U) = C^R_U \ \forall U \in \mathcal{U}^{e_L}_{ds\wedge}(L).$$

Then it follows from Theorem 4.1 that φ_1 and φ_2 are all invertible,

$$\varphi_1^{-1}(I) = U^R_I \ \forall I \in \mathcal{I}^{npe_L}_{\wedge}(L),\ \varphi_2^{-1}(C) = U^R_C \ \forall C \in \mathcal{C}^{npe_L}_{\vee}(L).$$

Moreover, we have the following theorem.

Theorem 4.4. *(1)* $(\mathcal{U}^{e_L}_{cs\vee}(L), \vee)$ *is order-reversing isomorphic to* $(\mathcal{I}^{npe_L}_{\wedge}(L), \wedge)$.

(2) $(\mathcal{U}^{e_L}_{ds\wedge}(L), \wedge)$ *is order-reversing isomorphic to* $(\mathcal{C}^{npe_L}_{\vee}(L), \vee)$.

(3) $(\mathcal{U}^{e_L}_{cs\vee}(L), \vee)$ *is order-reversing isomorphic to* $(\mathcal{U}^{N(e_L)}_{ds\wedge}(L), \wedge)$.

(4) $(\mathcal{I}^{npe_L}_{\wedge}(L), \wedge)$ *is order-reversing isomorphic to* $(\mathcal{C}^{npN(e_L)}_{\vee}(L), \vee)$.

Proof. (1) If $U_1,\ U_2 \in \mathcal{U}^{e_L}_{cs\vee}(L)$, then it is easy to see that $U_1 \vee U_2 \in \mathcal{U}^{e_L}_{cs\vee}(L)$. Moreover, it follows from the right residual (implication) principle that

$$\begin{aligned}
I^R_{(U_1\vee U_2)}(x,y) &= \bigvee\{z \in L \mid (U_1 \vee U_2)(x,z) \le y\} \\
&= \bigvee\{z \in L \mid U_1(x,z) \vee U_2(x,z) \le y\} \\
&= \bigvee\{z \in L \mid U_1(x,z) \le y,\ U_2(x,z) \le y\} \\
&= \bigvee\{z \in L \mid z \le I^R_{U_1}(x,y),\ z \le I^R_{U_2}(x,y)\} \\
&= \bigvee\{z \in L \mid z \le I^R_{U_1}(x,y) \wedge I^R_{U_2}(x,y)\} \\
&= \big(I^R_{U_1} \wedge I^R_{U_2}\big)(x,y) \ \forall x,y \in L,
\end{aligned}$$

i.e., $\varphi_1(U_1 \vee U_2) = \varphi_1(U_1) \wedge \varphi_1(U_2)$. Thus, φ_1 is an order-reversing isomorphism of $(\mathcal{U}^{e_L}_{cs\vee}(L), \vee)$ onto $(\mathcal{I}^{npe_L}_{\wedge}(L), \wedge)$.

(2) If $U_1, U_2 \in \mathcal{U}^{e_L}_{ds\wedge}(L)$, then $U_1 \wedge U_2 \in \mathcal{U}^{e_L}_{ds\wedge}(L)$. Moreover, it follows from the right residual (coimplication) principle that

$$\begin{aligned}
C^R_{(U_1\wedge U_2)}(x,y) &= \bigwedge\{z \in L \mid y \le (U_1 \wedge U_2)(x,z)\} \\
&= \bigwedge\{z \in L \mid y \le U_1(x,z) \wedge U_2(x,z)\} \\
&= \bigwedge\{z \in L \mid y \le U_1(x,z),\ y \le U_2(x,z)\} \\
&= \bigwedge\{z \in L \mid C^R_{U_1}(x,y) \le z,\ C^R_{U_2}(x,y) \le z\} \\
&= \bigwedge\{z \in L \mid C^R_{U_1}(x,y) \vee C^R_{U_2}(x,y) \le z\} \\
&= \big(C^R_{U_1} \vee C^R_{U_2}\big)(x,y) \ \forall x,y \in L,
\end{aligned}$$

i.e., $\varphi_2(U_1 \wedge U_2) = \varphi_2(U_1) \vee \varphi_2(U_2)$. So, φ_2 is an order-reversing isomorphism of $(\mathcal{U}^{e_L}_{ds\wedge}(L), \wedge)$ onto $(\mathcal{C}^{npe_L}_{\vee}(L), \vee)$.

(3) Define $f : \mathcal{U}^{e_L}_{cs\vee}(L) \to \mathcal{U}^{N(e_L)}_{ds\wedge}(L)$ as follows: $f(U) = U_N \ \forall U \in \mathcal{U}^{e_L}_{cs\vee}(L)$.

(i) If $U \in \mathcal{U}^{e_L}_{cs\vee}(L)$, then U_N is a right arbitrary $\wedge$-distributive left semi-uninorm with the left neutral element $N(e_L)$. Noting that U is a conjunctive left semi-uninorm, we have that

$$U_N(1,0) = N^{-1}\big(U(N(1),(N(0))\big) = N^{-1}\big(U(0,1)\big) = N^{-1}(0) = 1,$$

$$U_N(0,1) = N^{-1}\big(U(N(0),(N(1))\big) = N^{-1}\big(U(1,0)\big) = N^{-1}(0) = 1.$$

Thus, $U_N \in \mathcal{U}^{N(e_L)}_{ds\wedge}(L)$ and so f is a morphism of $\mathcal{U}^{e_L}_{cs\vee}(L)$ into $\mathcal{U}^{N(e_L)}_{ds\wedge}(L)$.

(ii) If $U_1, U_2 \in \mathcal{U}^{e_L}_{cs\vee}(L)$ and $f(U_1) = f(U_2)$, then

$$(U_1)_N = (U_2)_N,\ U_1 = ((U_1)_N)_N = ((U_2)_N)_N = U_2.$$

Moreover, for any $U \in \mathcal{U}^{N(e_L)}_{ds\wedge}(L)$, we have that $U_N \in \mathcal{U}^{e_L}_{cs\vee}(L)$ and $f(U_N) = (U_N)_N = U$. Thus, f is a bijection.

(iii) If $U_1, U_2 \in \mathcal{U}^{e_L}_{cs\vee}(L)$, then

$$f(U_1 \vee U_2) = (U_1 \vee U_2)_N = (U_1)_N \wedge (U_2)_N = f(U_1) \wedge f(U_2).$$

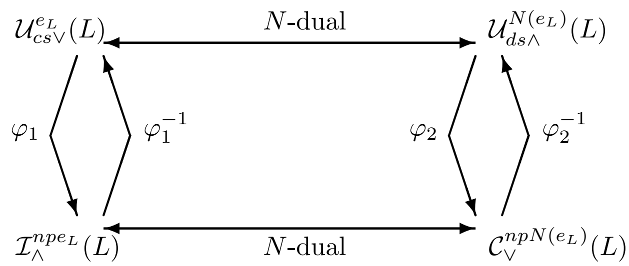
Therefore, f is an order-inversing isomorphism of $(\mathcal{U}^{e_L}_{cs\vee}(L), \vee)$ onto $(\mathcal{U}^{N(e_L)}_{ds\wedge}(L), \wedge)$.

(4) Define $g : \mathcal{I}^{npe_L}_{\wedge}(L) \to \mathcal{C}^{npN(e_L)}_{\vee}(L)$ as follows: $g(I) = I_N\ \forall I \in \mathcal{I}^{npe_L}_{\wedge}(L)$. If $I \in \mathcal{I}^{npe_L}_{\wedge}(L)$, then $I_N \in \mathcal{C}_{\vee}(L)$ and

$$\begin{aligned} I_N\big(N(e_L), x\big) &= N^{-1}\Big(I\big(N(N(e_L)), N(x)\big)\Big) \\ &= N^{-1}\big(I(e_L, N(x)\big) = N^{-1}\big(N(x)\big) = x\ \forall x \in L. \end{aligned}$$

Thus, $I_N \in \mathcal{C}^{npN(e_L)}_{\vee}(L)$ and g is a morphism of $\mathcal{I}^{npe_L}_{\wedge}(L)$ into $\mathcal{C}^{npN(e_L)}_{\vee}(L)$. Moreover, by the proof of statement (3), we see that g is an order-inversing isomorphism of $(\mathcal{I}^{npe_L}_{\wedge}(L), \wedge)$ onto $(\mathcal{C}^{npN(e_L)}_{\vee}(L), \vee)$. □

By Theorems 4.1, 4.3 and 4.4, we can get the relational graph as follows:



5. Conclusions and Future Works

In this paper, we have discussed the residual implications and coimplications induced by left semi-uninorms and the left semi-uninorms induced by implications and coimplications. We have shown that the N-dual operations of the right residual implication and coimplication, which are induced by a left semi-uninorm, are, respectively, the right residual coimplication and implication, which are induced by its N-dual operation; demonstrated that the N-dual operations of the left semi-uninorms induced by an implication and a coimplication, which satisfy the neutrality principle, are all left semi-uninorms; and revealed the relationships between conjunctive right arbitrary $\vee$-distributive left semi-uninorms induced by implication and disjunctive right arbitrary $\wedge$-distributive left semi-uninorms induced by coimplication, where both implications and coimplications satisfy the neutrality principle.

In forthcoming papers, we will further investigate the constructions of left (right) semi-uninorms, implications and coimplications on a complete lattice.

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