FUZZY FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we consider fuzzy fractional partial differential equations under Caputo generalized Hukuhara differentiability. Some new results on the existence and uniqueness of two types of fuzzy solutions are studied via weakly contractive mapping in the partially ordered metric space. Some application examples are presented to illustrate our main results.

1. Introduction

Bede and Stefanini [5] have given a brilliant idea of the generalized Hukuhara derivatives (gH-derivatives) concepts for fuzzy-valued functions, for which the length of the diameter of gH-differentiable functions can be monotonically nonincreasing in time. Thus the longtime behavior of gH-differentiable solutions of fuzzy differential equations (DEs) approximates the crisp solutions. Since then, there has been a new trend in studying the behavior of solutions for fuzzy DEs in the abstract spaces. There exists a long list of references related to this topic, see for instance [2, 3, 4, 8, 10, 13, 18, 19].

Recently, the issue of fuzzy fractional DEs under gH-differentiability has emerged as the significant subject, this new theory turned out to be very attractive to many scientist [3, 9, 10, 14, 15, 16]. Fractional DEs provide an outstanding instrument to describe the complex phenomena in fields of viscoelasticity, electromagnetic waves, diffusion equations and so on [11]. Supporting for this subject, many different forms of fractional operators for fuzzy-valued functions were introduced such as the Grunwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives [3]. In which, Caputo's fractional derivatives for crisp functions originally introduced by Caputo [6] and afterwards adopted in the theory of linear viscoelasticity, satisfy the requirement on definition of fractional derivatives allowing the utilization of physically interpretable initial conditions of applied problems. The concepts of Caputo derivatives for fuzzy one-variable functions were proposed by Allahviranloo et al. [3] and followed by some authors [8, 10]. In previous paper [14], we established some new concepts on fractional integral, Caputo gH-derivatives of fuzzy-valued multivariable functions. As a consequence, the notions of fuzzy fractional partial differential equations (PDEs) were interpreted under the sense of two types of Caputo generalized differentiability.

Received: February 2016; Revised: July 2016; Accepted: October 2016

Key words and phrases: Fractional PDEs, Caputo gH-derivatives, Fuzzy weak solutions, Weakly contractive mapping, Partially ordered space.

In this paper, our model is embedded into partially order metric space of fuzzy-valued functions. For which some fixed point theorems for weakly contractive mappings in the partially ordered spaces with quasilinear structure can be employed. This approach was initiated by Nieto and Rodríguez-López in [17] with some applications to first-order initial value problems by using weak Lipschitz conditions. Hereabouts, Chalco-Cano et al. [18] used some more generalized fixed point results of weakly contractive mappings to analyze the existence and uniqueness of fuzzy solutions of the Cauchy problem for first order DEs in the setting of gH-derivatives. Following this direction, in this paper, we prove some results on the existence and uniqueness of two types of fuzzy solutions of the Darboux problem for fuzzy fractional wave equations without Lipschitz condition of the right-hand side. The uniqueness is understood in the sense that considered fuzzy solutions having no switching points.

The paper is organized as follows: In Section 2, we present a partially order in the fuzzy number space E and prove some properties in the partial metric space $(C(J,E),\lesssim)$. The notions of fractional gH-derivatives are revisited in Section 3 for fuzzy valued two-variable functions. In Section 4, we establish local boundary valued problems for hyperbolic equations with respect to two types of generalized Caputo derivatives. The well-posedness of these problem is proved in Section 5 with some illustrated examples.

2. **Fuzzy Partially Ordered Metric Spaces**

2.1. **Fuzzy Partially Ordered Metric Spaces.** We will recall some notions and preliminaries used throughout the paper, some of them were detailed in [4, 5, 11, 12].

Let $\mathcal{K}_c(\mathbb{R}^d)$ be the collection of all nonempty compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multiplication in $\mathcal{K}_c(\mathbb{R}^d)$ are defined as usual, i.e., $A, B \in \mathcal{K}_c(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, then we have $A+B = \{a+b \mid a \in A, b \in B\}$, $\lambda A = \{\lambda a \mid a \in A\}$. The Hausdorff-Pompeiu metric d_H in $\mathcal{K}_c(\mathbb{R}^d)$ is defined as

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\mathbb{R}^d}, \sup_{b\in B}\inf_{a\in A}\|a-b\|_{\mathbb{R}^d}\},$$

where $A, B \in \mathcal{K}_c(\mathbb{R}^d)$. It is well-known that $(\mathcal{K}_c(\mathbb{R}^d), d_H)$ is a complete metric space.

Denote by E the space of fuzzy sets on $\mathbb{R}$, which is a mapping $u : \mathbb{R} \to [0,1]$ that satisfies normal, fuzzy convex, upper semi-continuous and compactly supported. For $\alpha \in (0,1]$, the $\alpha-$level of u is defined by the set $[u]^\alpha = \{u \in \mathbb{R} \mid u(x) \geq \alpha\}$. For $\alpha = 0$, the support of u is defined as the set $[u]^0 = supp(u) = cl\{x \in \mathbb{R} \mid u(x) > 0\}$. It is clear that the $\alpha-$level set of a fuzzy set is a closed and bounded interval $[u^l_\alpha, u^r_\alpha]$, where u^l_α denotes the left-hand endpoint of $[u]^\alpha$ and u^r_α denotes the right-hand endpoint of $[u]^\alpha$. If u, v are two fuzzy sets, then $u = v$ if and only if $[u]^\alpha = [v]^\alpha$, for all $\alpha \in [0,1]$. The diameter of the $\alpha-$level set of u is defined by $len[u]^\alpha = u^r_\alpha - u^l_\alpha$. Supremum metric is the most commonly used metric on E is defined by

$$d_\infty(u,v) = \sup_{0\leq\alpha\leq 1} d_H\left([u]^\alpha, [v]^\alpha\right) = \sup_{0\leq\alpha\leq 1} \max\{|u^l_\alpha - v^l_\alpha|; |u^r_\alpha - v^r_\alpha|\},\ u, v \in E,$$

where $[u]^\alpha = [u^l_\alpha, u^r_\alpha], [v]^\alpha = [v^l_\alpha, v^r_\alpha]$. It is well-known that (E, d_∞) is a complete metric space.

For $x, y \in E$ with $[u]^\alpha = [u^l_\alpha, u^r_\alpha], [v]^\alpha = [v^l_\alpha, v^r_\alpha]$, we can define the partial order " $\lesssim$ " by

$$u \lesssim v \text{ if and only if } u^l_\alpha \leq v^l_\alpha \text{ and } u^r_\alpha \leq v^r_\alpha \text{ for all } \alpha \in [0,1].$$

We denote the converse of the partial order " $\lesssim$ " by " $\gtrsim$ ". The following properties of the partial order are proved concreted in [17].

Lemma 2.1. *On E the following properties hold:*

(1) *If $u \lesssim v$, then $u + w \lesssim v + w$ for $u, v, w \in E$.*
(2) *If $\{u_n\}_{n\in\mathbb{N}} \subset E$ is a nondecreasing sequence such that $u_n \to u$ in E, then $u_n \lesssim u$ for all $n \in \mathbb{N}$.*
(3) *Every pair of elements of E has an upper bound or a lower bound.*

For $J = [0, a] \times [0, b] \subset \mathbb{R}^2_+$, $C(J, E)$ is denoted by the space of continuous functions defined on J. We define the partial order $\lesssim$ on $C(J, E)$ as

$$x, y \in C(J, E), \quad x \lesssim y \text{ if and only if } x(t, s) \lesssim y(t, s), \quad \forall t, s \in J.$$

The metric d_r on $C(J, E)$ is defined by

$$d_r(x, y) = \sup_{(s,t)\in J} \left\{ t^{r_1} s^{r_2} d_\infty(x(t, s), y(t, s)) \right\},\ r = (r_1, r_2) \in (0, 1] \times (0, 1].$$

It holds that $(C(J, E), d_r)$ is a complete metric space.

Lemma 2.2. *Let $(E, \lesssim)$ be a partial ordered space of fuzzy sets, then we have*

(1) *$(C(J, E), \lesssim)$ is a partial ordered space;*
(2) *$(C(J, E), d_r)$ is a regular metric space, that is for every nondecreasing sequence $\{u_n\} \subset C(J, E)$, if $u_n \to u$ in $C(J, E)$, then $u_n \lesssim u$ for all $n \in \mathbb{N}$.*
(3) *Every pair of elements of $C(J, E)$ has an upper bound or a lower bound.*

Proof. The properties (1) and (3) are inherited from those in E. Hence, we shall only give the proof of property (2). Indeed, assume that $\{u_n\} \subset C(J, E)$ is a nondecreasing sequence and converge to u in $C(J, E)$, then $\{u_n(x, y)\}$ is a nondecreasing sequence in $C(J, E)$ for $(x, y) \in J$. Moreover, for each $(x, y) \in J$, $r := (r_1, r_2) \in (0, 1] \times (0, 1]$,

$$x^{r_1} y^{r_2} d_\infty\big(u_n(x, y), u(x, y)\big) \leq \sup_J \left\{ d_\infty(u_n(x, y), u(x, y)) x^{r_1} y^{r_2} \right\} = d_r(u_n, u).$$

Since $\lim_{n\to\infty} d_r(u_n, u) = 0$, we have $\lim_{n\to\infty} x^{r_1} y^{r_2} d_\infty\big(u_n(x, y), u(x, y)\big) = 0$. This implies $\lim_{n\to\infty} d_\infty\big(u_n(x, y), u(x, y)\big) = 0$ for fixed $(x, y) \in J$. Hence $u_n(x, y)$ converges to $u(x, y)$ in $C(J, E)$. From Lemma 2.1, we have $u_n(x, y) \lesssim u(x, y)$ for all $(x, y) \in J$ and for all $n \in \mathbb{N}$. This shows $u_n \lesssim u$ for all $n \in \mathbb{N}$. $\square$

2.2. **Generalized Fixed Point Theorems.** In this subsection, we recall some generalized fixed point theorems in partially ordered space that used throughout this paper.

Denote by $C(\mathbb{R}_+, \mathbb{R}_+)$ the space of all continuously nonnegative functions $\phi : [0, \infty) \to [0, \infty)$, for which $\phi(t) = 0$ if and only if $t = 0$. The indicator function of a set $A \subset \mathbb{R}$ is a mapping $1_A : A \to \mathbb{R}$ satisfied $1_A(t) = 1$ for all $t \in A$ and $1_A(t) = 0$ for all $t \notin A$. We denote $\mathcal{S}_0$ by the class of functions $\beta : [0, \infty) \to [0, 1)$, which satisfies the condition as follows: if $\beta(t_n) \to 1$, then it implies $t_n \to 0$. We denote $\mathcal{S} = \mathcal{S}_0 \cup \{1_{[0,\infty)}\}$.

Definition 2.3. [7] A nondecreasing function ψ in $C(\mathbb{R}_+, \mathbb{R}_+)$ is called an altering distance function on $[0, \infty)$.

Definition 2.4. [7] Let $(X, \leq)$ be a partially ordered set and $f : X \to X$. We say that f is monotone nondecreasing (or nonincreasing), if $x \leq y$ for $x, y \in X$, then $f(x) \leq f(y)$ (or $f(y) \geq f(x)$), respectively.

Theorem 2.5. [7] *Assume that $(X, \leq)$ is a partially ordered set such that every pair of elements of X has an upper bound or a lower bound. Moreover, there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $f : X \to X$ is a continuously monotone nondecreasing function satisfying*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y), \quad \text{for } x \geq y,$$

where $\beta(\cdot) \in \mathcal{S}$. If there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$, then f has a unique fixed point.

Theorem 2.6. [7] *Assume that $(X, \leq)$ is a partially ordered set such that every pair of elements of X has an upper bound or a lower bound. Moreover, there exists a metric d on X such that (X, d) is a complete metric space and X satisfies the following property: if a nonincreasing sequence $\{x_n\} \subset X$ converges to x in X, then $x \leq x_n$ for all $n \in \mathbb{N}$. Suppose that $f : X \to X$ is a monotone nondecreasing function satisfying the weakly contractive condition*

$$\psi(d(f(x), f(y))) \leq \psi(d(x, y)) - \phi(d(x, y)); \text{ for all } x \geq y,$$

for some altering distance functions ψ and ϕ. If there exists $x_0 \in X$ such that $x_0 \geq f(x_0)$, then f has a unique fixed point.

3. Fuzzy Fractional Integrability and gH-derivative

3.1. **Preliminaries on gH-derivative.** Let $u, v, w \in E$. An element w is called the H-difference of u and v, if it satisfies the equation $u = v + w$. If the H-difference exists, it will be denoted by $u \ominus v$ and $[u \ominus v]^\alpha = [u^l_\alpha - v^l_\alpha, u^r_\alpha - v^r_\alpha]$, for all $0 \leq \alpha \leq 1$.

Definition 3.1. [5] For $u, v \in E$, the gH-difference of u and v, denoted by $u \ominus_{gH} v$, is defined as the element $w \in E$ such that

$$u \ominus_{gH} v = w \Longleftrightarrow (i) \;\; u = v + w, \text{ or } \;\; (ii) \;\; v = u + (-1)w.$$

Notice that if $u \ominus v$ exists, then $u \ominus_{gH} v = u \ominus v$; if (i) and (ii) in Definition 3.1 are satisfied simultaneously, then w is a crisp number; also, $u \ominus_{gH} u = \hat{0}$, and if $u \ominus_{gH} v$ exists, then it is unique.

For a fuzzy mapping $f : J \subset \mathbb{R}^2_+ \to E$, we have the following notions of gH-partial derivative of a multivariable function.

Definition 3.2. [4] Let $(x_0, y_0) \in J$, then the gH-partial derivative in order 1 of a fuzzy mapping $f : J \to E$ at (x_0, y_0) with respect to variables x, y is the functions $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ given by

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) \ominus_{gH} f(x_0, y_0)}{h};$$
$$f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, y_0 + k) \ominus_{gH} f(x_0, y_0)}{k},$$

provided that the gH-differences $f(x_0 + h, y_0) \ominus_{gH} f(x_0, y_0)$, $f(x_0, y_0 + k) \ominus_{gH} f(x_0, y_0)$ exist and $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ in E.
The gH-partial derivative of higher order of f at the point $(x_0, y_0) \in I$ are defined similarly (see Definition 2.9 and 3.4 in [4]).

Denote $C^{i,j}_{gH}(J, E)$ $(i, j = 0, 1, 2)$ by a set of all functions $f : J \to E$ which have partial gH-derivative up to order i w.r.t x and up to j w.r.t y in J.

Definition 3.3. [4] Let $f : J \to E$ be partial gH-differentiable w.r.t x at $(x_0, y_0) \in J$. We say that f is (i)-gH differentiable w.r.t x at $(x_0, y_0) \in J$ if for all $\alpha \in [0, 1]$,

$$\Big[f_x(x_0, y_0)\Big]^\alpha = \Big[\frac{\partial f^l_\alpha}{\partial x}(x_0, y_0), \frac{\partial f^r_\alpha}{\partial x}(x_0, y_0)\Big]$$

and that f is (ii)-gH differentiable w.r.t x at $(x_0, y_0) \in J$ if for all $\alpha \in [0, 1]$,

$$\Big[f_y(x_0, y_0)\Big]^\alpha = \Big[\frac{\partial f^r_\alpha}{\partial x}(x_0, y_0), \frac{\partial f^l_\alpha}{\partial x}(x_0, y_0)\Big].$$

Types of (i)-gH and (ii)-gH derivatives of f w.r.t y at the point $(x_0, y_0) \in J$ are defined similarly.

Definition 3.4. [4] For any fixed x_0, we say that $(x_0, y) \in J$ is a switching point for the differentiability of f with respect to x, if in any neighborhood V of $(x_0, y) \in J$, there exist points $A(x_1, y), B(x_2, y)$ such that $x_1 < x_0 < x_2$ and
(type I) f is (i)-gH differentiable at A while f is (ii)-gH differentiable at B for all y, or
(type II) f is (i)-gH differentiable at B while f is (ii)-gH differentiable at A for all y.

Definition 3.5. Let $f \in C^{1,0}_{gH}(J, E)$ such that $f_x(\cdot, \cdot) \in C^{0,1}_{gH}(J, E)$ and they do not have any switching point on I. Denote f_{xy} by the mixed second order partial gH-derivative of f w.r.t x and y at $(x_0, y_0) \in J$. We say that

a) $f_{xy}(\cdot, \cdot)$ is a gH-derivative in type 1 at (x_0, y_0) (denoted by ${}_1D_{xy}f(x_0, y_0)$), if the type of gH-differentiability at (x_0, y_0) of both $f(\cdot, \cdot)$ (w.r.t x) and f_x (w.r.t y) are the same. Then,

$$\big[{}_1D_{xy}f(x_0, y_0)\big]^\alpha = \big[\partial_{xy} f^l_\alpha(x_0, y_0), \partial_{xy} f^r_\alpha(x_0, y_0)\big], \text{ for } \alpha \in [0, 1],$$

b) $f_{xy}(\cdot,\cdot)$ is a gH-derivative in type 2 at (x_0, y_0) (denoted by ${}_2D_{xy}f(x_0, y_0)$), if the type of gH-differentiability at (x_0, y_0) of both $f(\cdot,\cdot)$ (w.r.t x) and f_x (w.r.t y) are different. Then,

$$\left[{}_2D_{xy}f(x_0, y_0)\right]^\alpha = \left[\partial_{xy}f^r_\alpha(x_0, y_0), \partial_{xy}f^l_\alpha(x_0, y_0)\right], \text{ for } \alpha \in [0, 1].$$

3.2. **Preliminaries on Fuzzy Integral.** In this subsection, we present some definitions and properties of the fuzzy integral. The notion of integrability considered is the Aumann-integrability.

Definition 3.6. [12] A mapping $f : U \subset \mathbb{R}^m \to E$ is said to be strongly measurable if the set-valued mapping $f_\alpha : U \to K_c(\mathbb{R})$ given by $f_\alpha(\nu) = [f(\nu)]^\alpha, \nu \in U$ are Lebesgue-measurable for all $\alpha \in [0, 1]$.

A fuzzy mapping $f : U \subset \mathbb{R}^m \to E$ is called integrable bounded if there exists an integrable function $h : U \to [0; \infty)$, such that $d_\infty\big(f(\nu), \hat{0}\big) \le h(\nu)$, forall $\nu \in U$.

A strongly measurable and integrable bounded fuzzy function is called integrable. The fuzzy Aumann integral of $f : U \subset \mathbb{R}^m \to E$ is defined levelsetwise by the equation $[\int\limits_U f(\nu)d\nu]^\alpha = \Big[\int\limits_U f^l_\alpha(\nu)d\nu, \int\limits_U f^r_\alpha(\nu)d\nu\Big]$, where $[f(\nu)]^\alpha = [f^l_\alpha(\nu), f^r_\alpha(\nu)]$ for all $\alpha \in [0, 1]$. We will denote this integral by $\int\limits_U f\,(\nu)\,d\nu$.

It is well-known that if f is continuous on U, then f is integrable on U. Moreover. we have the following property.

Lemma 3.7. *Let U be a compact subset of $\mathbb{R}^2$, $u \lesssim v$ in $C(U, E)$ and $k : U \to \mathbb{R}_+$ Then,*

$$\int\limits_U k(x, y)u(x, y)dxdy \lesssim \int\limits_U k(x, y)v(x, y)dxdy.$$

Proof. From Definition 3.6, we have

$$[\int\limits_U k(x, y)u(x, y)dxdy]^\alpha = [\int\limits_U k(x, y)u^l_\alpha(x, y)dxdy, \int\limits_U k(x, y)u^r_\alpha(x, y)dxdy]$$

and

$$[\int\limits_U k(x, y)v(x, y)dxdy]^\alpha = [\int\limits_U k(x, y)v^l_\alpha(x, y)dxdy, \int\limits_U k(x, y)v^r_\alpha(x, y)dxdy].$$

As $u \lesssim v$ in $C(U, E)$, $u(x, y) \lesssim v(x, y)$ in E for all $(x, y) \in U$. This infers $u^l_\alpha(x, y) \le v^l_\alpha(x, y)$ and $u^r_\alpha(x, y) \le v^r_\alpha(x, y)$ for all $\alpha \in [0, 1]$. Since $k(x, y) \ge 0,\ \forall (x, y) \in U$, $k(x, y)u^l_\alpha(x, y) \le k(x, y)v^l_\alpha(x, y)$ and $k(x, y)u^r_\alpha(x, y) \le k(x, y)v^r_\alpha(x, y)$. Therefore, we get

$$\int\limits_U k(x, y)u^l_\alpha(x, y)dxdy \le \int\limits_U k(x, y)v^l_\alpha(x, y)dxdy$$

and

$$\int\limits_U k(x, y)u^r_\alpha(x, y)dxdy \le \int\limits_U k(x, y)v^r_\alpha(x, y)dxdy$$

for all $\alpha \in [0,1]$. From Definition 3.6, one has

$$\int\limits_U k(x,y)u(x,y)dxdy \lesssim \int\limits_U k(x,y)v(x,y)dxdy.$$

The proof is complete. □

3.3. **Fuzzy Fractional Integral and Derivative.** Let $q=(q_1,q_2)\in(0,1]\times(0,1]$ and $f\in L^1(J,\mathbb{R})$. In [1], the authors presented the mixed Riemann - Liouville fractional integral notion of order q for real-valued functions f as follows.

$$I^q f(x,y)=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}f(s,t)dtds,$$

provided that the expression on the right hand side is defined, for almost every $(x,y)\in J$.

Example 3.8. Let $u: J\to E$ be a fuzzy function, $u(x,y)=Cxy$, where C is a fuzzy number with $[C]^\alpha=[C^l_\alpha,C^r_\alpha]$ for all $\alpha\in[0,1]$. It shows $[u(x,y)]^\alpha=[C^l_\alpha xy, C^r_\alpha xy]$, for all $(x,y)\in J$ and $\alpha\in[0,1]$. Then, we get

$$\begin{aligned}I^q u^{l,r}_\alpha(x,y) &= \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}C^{l,r}_\alpha stdtds\\ &= \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^1(1-s_1)^{q_1-1}s_1x^{q_1+1}ds_1\int_0^1(1-t_1)^{q_2-1}y^{q_2+1}t_1C^{l,r}_\alpha dt\\ &= \frac{B(2,q_1)B(2,q_2)}{\Gamma(q_1)\Gamma(q_2)}x^{q_1+1}y^{q_2+1}C^{l,r}_\alpha\\ &= \frac{4}{\Gamma(q_1+2)\Gamma(q_2+2)}x^{q_1+1}y^{q_2+1}C^{l,r}_\alpha.\end{aligned}$$

The family of closed interval

$$\begin{aligned}[I^q u^l_\alpha(x,y), I^q u^r_\alpha(x,y)] &= \frac{B(2,q_1)B(2,q_2)}{\Gamma(q_1)\Gamma(q_2)}x^{q_1+1}y^{q_2+1}[C^l_\alpha,C^r_\alpha]\\ &= \frac{4}{\Gamma(q_1+2)\Gamma(q_2+2)}x^{q_1+1}y^{q_2+1}[C]^\alpha\end{aligned}$$

defines a fuzzy number $v\in E$. Now, we call it the fractional integral of fuzzy-valued function u.

Motivating by this example, we define the fractional integral of fuzzy-valued functions as follows.

Definition 3.9. Let $q=(q_1,q_2)\in(0,1]\times(0,1]$ and $u: J\to E$, $[u(\cdot,\cdot)]^\alpha=[u^l_\alpha(\cdot,\cdot),u^r_\alpha(\cdot,\cdot)]$ such that $u^l_\alpha,u^r_\alpha\in L^1(J,\mathbb{R})$ for all $\alpha\in[0,1]$. We define the left-sided mixed Riemann-Liouville fractional integral of order q for fuzzy-valued function u, by denoting

$$I^q u(x,y)=\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}u(s,t)dtds,$$

where

$$[I^q u(x,y)]^\alpha=[I^q u^\alpha_1(x,y), I^q u^\alpha_2(x,y)],\ (x,y)\in I.$$

In particular cases, we denote

$$I^0 u(x,y) = u(x,y), \text{ for } q = (0,0)$$
$$I^1 u(x,y) = \int_0^x \int_0^y u(s,t)dtds, \text{ for } q = (1,1), \ (x,y) \in I.$$

Adapting similarly the method in the proof of Remark 5 in [15] for multivariable case, we receive the following property of fuzzy fractional integral I^q.

Proposition 3.10. *Let* $p = (p_1, p_2), q = (q_1, q_2) \in (0,1] \times (0,1]$ *such that* $p + q$ *is still in* $(0,1] \times (0,1]$ *and* $u \in L^1(J,E)$ *then*

$$I^p I^q u = I^{p+q} u.$$

Definition 3.11. Let $q = (q_1, q_2) \in (0,1] \times (0,1]$ and $u \in C^{2,2}_{gH}(J,E)$. We say that

(1) The (1)-gH Caputo fractional derivative of order q with respect to x, y of the function u is defined as follows

$$^C\mathcal{D}^q_1 u(x,y) = I^{1-q}\Big({}_1D_{xy}u(x,y)\Big)$$

provided that the expression on the right hand side is defined.

(2) The (2)-gH Caputo fractional derivative of order q with respect to x, y of the function u is defined as follows

$$^C\mathcal{D}^q_2 u(x,y) = I^{1-q}\Big({}_2D_{xy}u(x,y)\Big)$$

provided that the expression on the right hand side is defined, where $1-q = (1-q_1, 1-q_2) \in (0,1] \times (0,1]$, $(x,y) \in J$.

Example 3.12. Consider the fuzzy function $u(\cdot,\cdot)$ given in Example 3.8. Then, the partial gH-derivatives of u with respect to x, y are calculated by $\dfrac{\partial u(x,y)}{\partial x} = Cy$, and ${}_1D_{xy}u(x,y) = C$. It is easy to see that

$$\begin{aligned}
[D^q_1 u(x,y)]^\alpha &= [I^{1-q}{}_1D_{xy}u(x,y)]^\alpha \\
&= \frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\Big[\textstyle\int_0^x\int_0^y (x-s)^{-q_1}(y-t)^{-q_2}C^l_\alpha dtds, \\
&\qquad\qquad \textstyle\int_0^x\int_0^y (x-s)^{-q_1}(y-t)^{-q_2}C^r_\alpha dtds\Big] \\
&= \frac{1}{\Gamma(1-q_1)\Gamma(1-q_2)}\frac{x^{1-q_1}y^{1-q_2}}{(1-q_1)(1-q_2)}[C^l_\alpha, C^r_\alpha] \\
&= \frac{1}{\Gamma(2-q_1)\Gamma(2-q_2)}x^{1-q_1}y^{1-q_2}[C]^\alpha.
\end{aligned}$$

Thus, $^C\mathcal{D}^q_1 u(x,y) = \dfrac{1}{\Gamma(2-q_1)\Gamma(2-q_2)}x^{-q_1}y^{-q_2}u(x,y).$

4. **State the Problem**

For arbitrary positive real numbers a and b, we denote $J_a = [0, a]$, $J_b = [0, b]$, $J = J_a \times J_b$. In this paper, we consider the fuzzy fractional PDEs as follows

$$^{C}\mathcal{D}_k^q u(x, y) = f(x, y, u(x, y)), \quad (x, y) \in J; \quad k = 1, 2, \tag{1}$$

with the initial conditions

$$\begin{cases} u(x, 0) = \eta_1(x), \; x \in [0, a], \\ u(0, y) = \eta_2(y), \; y \in [0, b], \end{cases} \tag{2}$$

where $q = (q_1, q_2) \in (0, 1] \times (0, 1]$, $\eta_1 \in C([0, a], E)$, $\eta_2 \in C([0, b], E)$ are given functions such that $\eta_2(y) \ominus \eta_1(0)$ exists for all $y \in [0, b]$ and $f : J \times C(J, E) \to E$ is a continuous function.

For $(x, y) \in J$, we denote

$$\psi(x, y) = \eta_2(y) + [\eta_1(x) \ominus \eta_1(0)]. \tag{3}$$

By adapting method used in the proof of Lemma 4.1 in [13] and using Proposition 3.10, we have the following result for fuzzy fractional PDEs.

Lemma 4.1. *Let $u \in C_{gH}^{2,2}(J, E)$ be a fuzzy function satisfying (1)-(2) in J. Moreover, the Hukuhara difference $\eta_2(y) \ominus \eta_1(0)$ exists for all $y \in [0, b]$.*

1) *If $k = 1$, then $u(x, y) = \psi(x, y) + I^q f\,(x, y, u(x, y))$ for $(x, y) \in J$.*
2) *If $k = 2$ and the Hukuhara difference $\psi(x, y) \ominus (-1) I^q f\,(x, y, u(x, y))$ exists for all $(x, y) \in J$, then $u(x, y) = \psi(x, y) \ominus (-1) I^q f\,(x, y, u(x, y))\,,\ (x, y) \in J$.*

Definition 4.2. A function $u \in C(J, E)$ is called

1) a (i)-weak solution of problem (1)-(2) if it satisfies the following fractional integral equation

$$u(x, y) = \psi(x, y) + I^q f(x, y, u(x, y)) \text{ for all } (x, y) \in J, \tag{4}$$

2) a (ii)-weak solution of problem (1)-(2) if it satisfies the fractional following integral equation

$$u(x, y) = \psi(x, y) \ominus (-1) I^q f(x, y, u(x, y)) \text{ for all } (x, y) \in J. \tag{5}$$

Definition 4.3. A fuzzy function $\mu \in C_{gH}^{2,2}(J, E)$ is called

1) a $(k)-$lower $(k = 1, 2)$ solution of problem (1)-(2) if

$$^{C}\mathcal{D}_k^q \mu(x, y) \lesssim f(x, y, \mu(x, y)); \; \mu(x, 0) \lesssim \eta_1(x); \; \mu(0, y) \lesssim \eta_2(y); \; \mu(0, 0) = \eta_1(0)$$

for $(x, y) \in J$ and

2) a $(k)-$upper $(k = 1, 2)$ solution of problem (1)-(2) if

$$^{C}\mathcal{D}_k^q \mu(x, y) \gtrsim f(x, y, \mu(x, y)); \; \mu(x, 0) \gtrsim \eta_1(x); \; \mu(0, y) \gtrsim \eta_2(y); \; \mu(0, 0) = \eta_1(0)$$

for $(x, y) \in J$.

5. **On the Solvability of the Problem**

Lemma 5.1. *For arbitrary increasing altering distance function ϕ and for all positive real numbers a, b, there exists $\lambda > 0$ such that the function*

$$\psi(t) = \phi(t) - \phi(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t), \quad t \in [0, \infty)$$

belongs to $C(\mathbb{R}_+, \mathbb{R}_+)$.

Proof. From the continuity of ϕ, we get that $\psi(\cdot)$ is a continuous function on $[0, \infty)$. Choose $\lambda > 0$ such that

$$\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b}) < 1.$$

Then for all $t \geq 0$ we have

$$\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t \leq t.$$

As ϕ is increasing, it follows

$$\phi(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t) \leq \phi(t)$$

for all $t \geq 0$. Hence, $\psi(t) \geq 0$ for all $t \geq 0$.

Now assume that $t > 0$. From $\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t < t$ and the increasing property of ϕ, it implies $\psi(t) > 0$. It follows that if $\psi(t) = 0$, then $t = 0$. The proof is complete. □

Theorem 5.2. *Let f be a function satisfied the following hypotheses.*
($\mathbf{h_1}$) *$f : J \times E \to E$ is nondecreasing in the third variable, i.e., if ν, $\xi \in E$ and $\nu \lesssim \xi$, then $f(x, y, \nu) \lesssim f(x, y, \xi)$, for all $(x, y) \in J$.*
($\mathbf{h_2}$) *f is a continuous function and satisfies Lipschitz condition, i.e. there exists a positive real number L such that*

$$d_\infty(f(x, y, \varphi_1), f(x, y, \varphi_2)) \leq L d_\infty(\varphi_1, \varphi_2), \text{ for all } \varphi_1 \lesssim \varphi_2, \ (x, y) \in J.$$

Suppose that there exists a (1)-lower (or (1)-upper) solution μ of the problem (1)-(2). Then, the problem (1)-(2) has a unique (i)-weak solution defined by (4) on $C(J, E)$.

Proof. We define the operator $T_1 : C(J, E) \to C(J, E)$ by

$$T_1 u(x, y) = \psi(x, y) + I^q f(x, y, u(x, y)), \quad (x, y) \in J. \tag{6}$$

Step 1: We will prove that T_1 is a nondecreasing operator in $C(J, E)$.

Assume that $u \lesssim v$ in $C(J, E)$ and from hypothesis of the nondecreasing property of f with respect to the third variable we have $f(x, y, u(x, y)) \lesssim f(x, y, v(x, y))$ for all $(x, y) \in J$. Then, from Lemma 3.7 we have

$$I^q f(x, y, u(x, y)) \lesssim I^q f(x, y, v(x, y)).$$

This infers that $T_1 u(x, y) \lesssim T_1 v(x, y)$ for all $(x, y) \in J$. Hence, $T_1 u \lesssim T_1 v$ in $C(J, E)$.

Step 2: The contractive-like property of the operator T_1.

For $u \lesssim v$ in $C(J,E)$, we have $u(x,y) \lesssim v(x,y)$ for all $(x,y) \in J$. It is implied from (h_2) that

$$d_\infty(f(x,y,u(x,y)), f(x,y,v(x,y))) \le L d_\infty(u(x,y), v(x,y)) \text{ for all } (x,y) \in J.$$

Thus,

$$\begin{aligned}
&d_\infty(T_1u(x,y), T_1v(x,y))\\
&= d_\infty\Big(\psi(x,y) + I^q f(x,y,u(x,y)), \psi(x,y) + I^q f(x,y,v(x,y))\Big)\\
&\le d_\infty\Big(I^q f(x,y,u(x,y)), I^q f(x,y,v(x,y))\Big)\\
&= \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} d_\infty\left(f(s,t,u(s,t)), f(s,t,v(s,t))\right) dtds\\
&\le \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} L d_\infty(u(s,t), v(s,t)) dtds\\
&\le \frac{L}{\Gamma(q_1)\Gamma(q_2)} d_{1-q}(u,v)\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} s^{q_1-1} t^{q_2-1} dtds\\
&\le \frac{L}{\Gamma(q_1)\Gamma(q_2)} d_{1-q}(u,v)\int_0^x (x-s)^{q_1-1} s^{q_1-1} ds \int_0^y (y-t)^{q_2-1} t^{q_2-1} dt\\
&\le \frac{L\Gamma(q_1)\Gamma(q_2)}{\Gamma(2q_1)\Gamma(2q_2)} x^{2q_1-1} y^{2q_2-1} d_{1-q}(u,v).
\end{aligned}$$

Therefore

$$d_{1-q}(T_1u, T_1v) \le \frac{L x^{q_1} y^{q_2}\Gamma(q_1)\Gamma(q_2)}{\Gamma(2q_1)\Gamma(2q_2)} d_{1-q}(u,v). \tag{7}$$

Now, we show that for $u, v \in C(J,E)$ and $u \lesssim v$, we get

$$d_{1-q}(T_1^n u, T_1^n v) \le \frac{L^n x^{nq_1} y^{nq_2}\Gamma(q_1)\Gamma(q_2)}{\Gamma((n+1)q_1)\Gamma((n+1)q_2)} d_{1-q}(u,v). \tag{8}$$

From the inequality (7), we can easily to infer that (8) is valid with $n = 1$. Assume that (8) is true for $n = k$. We need to prove that (8) is true for $n = k+1$. Indeed, we see that

$$\begin{aligned}
&d_{1-q}(T_1^{k+1}u, T_1^{k+1}v) = \sup_{(x,y)\in J} x^{1-q_1} y^{1-q_2} d_\infty(T_1(T_1^k u)(x,y), T_1(T_1^k v)(x,y))\\
&\le x^{1-q_1} y^{1-q_2} d_\infty(I^q f(x,y,T_1^k u(x,y)), I^q(f(x,y,T_1^k v(x,y))))\\
&\le \frac{x^{1-q_1} y^{1-q_2}}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} d_\infty\left(f(s,t,T_1^k u(s,t)), f(s,t,T_1^k v(s,t))\right) dtds.
\end{aligned}$$

Since f satisfies Lipschitz condition, we have

$$\begin{aligned}
&d_{1-q}(T_1^{k+1}u, T_1^{k+1}v)\\
&\le \frac{L x^{1-q_1} y^{1-q_2}}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} d_\infty\left(T_1^k u(s,t), T_1^k v(s,t)\right) dtds\\
&\le \frac{L^{k+1} x^{1-q_1} y^{1-q_2}}{\Gamma((k+1)q_1)\Gamma((k+1)q_2)} d_{1-q}(u,v)\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} s^{kq_1+q_1-1} t^{kq_2+q_2-1} dtds\\
&\le \frac{L^{k+1} x^{(k+1)q_1} y^{(k+1)q_2}}{\Gamma((k+1)q_1)\Gamma((k+1)q_2)} d_{1-q}(u,v) B(q_1,(k+1)q_1) B(q_2,(k+1)q_2)\\
&\le \frac{L^{k+1} x^{(k+1)q_1} y^{(k+1)q_2}\Gamma(q_1)\Gamma(q_2)}{\Gamma((k+2)q_1)\Gamma((k+2)q_2)} d_{1-q}(u,v)
\end{aligned}$$

and so, (8) is true for $n = k + 1$.

For arbitrary increasing altering distance function ϕ, from (8) we have:

$$\phi(d_{1-q}(T_1^n u, T_1^n v)) \leq \phi\Big(\frac{L^n x^{nq_1} y^{nq_2} \Gamma(q_1)\Gamma(q_2)}{\Gamma((n+1)q_1)\Gamma((n+1)q_2)} d_{1-q}(u,v)\Big)$$
$$= \phi(d_{1-q}(u,v)) - [\gamma(d_{1-q}(u,v)) - \gamma\Big(\frac{L^n x^{nq_1} y^{nq_2} \Gamma(q_1)\Gamma(q_2)}{\Gamma((n+1)q_1)\Gamma((n+1)q_2)} d_{1-q}(u,v)\Big)].$$

Denote $\psi(t) = \phi(t) - \phi\Big(\frac{L^n x^{nq_1} y^{nq_2} \Gamma(q_1)\Gamma(q_2)}{\Gamma((n+1)q_1)\Gamma((n+1)q_2)} t\Big)$, $t \in [0, \infty)$. Then, from Lemma 5.1, ψ belongs to $C(\mathbb{R}_+, \mathbb{R}_+)$ and

$$\phi(d_{1-q}(T_1^n u, T_1^n v)) \leq \phi(d_{1-q}(u,v)) - \psi(d_{1-q}(u,v)) \text{ for all } u \lesssim v.$$

Therefore, the operator T_1^n satisfies the contractive-like property.

Step 3: If $\mu \in C_{gH}^{2,2}(J, E)$ is (1)-lower solution for problem (1)-(2), then $\mu \lesssim T_1\mu$. Indeed

$$\begin{aligned}\mu(x,y) &\lesssim \mu(x,0) + \mu(0,y) \ominus \mu(0,0) + I^q(x,y,\mu(x,y)) \\ &\lesssim \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I^q f(x,y,\mu(x,y)) \\ &= (T_1\mu)(x,y)\end{aligned}$$

for all $(x,y) \in J$.

From Step 1 to Step 3 we see that the operator T_1^n verifies all hypotheses of Theorem 2.6 for a sufficiently large number n. In consequence, T_1^n has a fixed point in $C(J,E)$. Noting that $C(J,E)$ satisfies that every pair of elements of $C(J,E)$ has an upper bound. It follows that the operator T_1^n has a unique fixed point, so does T_1. That is the unique (i)-weak solution of problem (1)-(2). □

Remark 5.3. The existence of (i)-weak solution of problem (1)-(2) w.r.t the (1)-gH Caputo fractional derivative ${}^C\mathcal{D}_1^q u(x,y)$ is guaranteed by the weakly nondecreasing property of function f in the hypothesis (h_1) and the weak Lipschitz condition of function f in hypothesis (h_2). The existence of (ii)-weak solution w.r.t the (2)-gH Caputo fractional derivative ${}^C\mathcal{D}_2^q u(x,y)$ is more difficult due to whenever H-differences exists. The following results give some necessary conditions for gaining the (ii)-weak solution of problem (1)-(2).

For all $(x,y) \in J$, denote

$$C_f(J,E) = \big\{u \in C(J,E) : \psi(x,y) \ominus (-1) I^q f(x,y,u(x,y)) \text{ exists}\big\},$$

where $\psi(x,y)$ is defined on (2).

Lemma 5.4. [14] *If $f : J \times E \to E$ is a continuous function, then $(C_f(J,E), d_r)$ is a complete metric space.*

Theorem 5.5. *Assume that f satisfies the hypotheses (h_1)-(h_2) in Theorem 5.2. Moreover, the following hypotheses are fulfilled*

$(\mathbf{h_3})$ *$\eta_1(x) + \eta_2(y)$ is not a crisp number and the space $C_f(J,E) \neq \emptyset$.*

$(\mathbf{h_4})$ *For all $u \in C_f(J,E)$, for all $(x,y) \in J$, $T_2u(x,y) \in C_f(J,E)$, where $T_2u(x,y) = \psi(x,y) \ominus\ (-1) I^q f(x,y,u(x,y))$.*

Suppose that there exists a (2)-lower solution (or (2)-upper solution) μ for problem (1) and (2). Then, the problem (1)-(2) has a unique (ii)-weak solution on $(C_f(J,E),d_r)$.

Proof. We define operator $T_2 : C_f(J,E) \to C_f(J,E)$ by

$$T_2u(x,y) = \psi(x,y) \ominus \ (-1)I^q f(x,y,u(x,y)), (x,y) \in J. \tag{9}$$

By hypothesis (h_4) the operator T_2 is well-defined. For $u \lesssim v$ in $C_f(J,E)$, using analogous arguments as in the proof of Theorem 5.2, we have

$$\begin{aligned} &d_\infty(T_2u(x,y), T_2v(x,y)) \\ &\quad = d_\infty\Big(\psi(x,y) \ominus (-1)I^q f(x,y,u(x,y)), \psi(x,y) \ominus (-1)I^q f(x,y,v(x,y))\Big) \\ &\quad \leq d_\infty(I^q f(x,y,u(x,y)), I^q f(x,y,v(x,y))) \\ &\quad \leq \frac{L\Gamma(q_1)\Gamma(q_2)}{\Gamma(2q_1)\Gamma(2q_2)} x^{2q_1-1}y^{2q_2-1}d_{1-q}(u,v). \end{aligned}$$

It follows

$$d_{1-q}(T_2^n u, T_2^n v) \leq \frac{L^n x^{nq_1}y^{nq_2}\Gamma(q_1)\Gamma(q_2)}{\Gamma((n+1)q_1)\Gamma((n+1)q_2)} d_{1-q}(u,v),$$

for all $n \in \mathbb{N}$. Similarly T_2 is a contractive-like mapping in $C_f(J,E)$.

Now assume that $u \lesssim v$ in $C_f(J,E)$. We need to indicate the nondecreasing property of the operator T_2, that means $T_2u \lesssim T_2v$. Assume that T_2u is not less than or equal to T_2v, then there exists $(x,y) \in J$ such that $T_2u(x,y)$ is "greater than" $T_2v(x,y)$, denoted by $T_2u(x,y) \succ T_2v(x,y)$.
Since $u(x,y) \lesssim v(x,y)$ for all $(s,t) \in J$ and hypothesis of nondecreasing property of function f, we have $f(x,y,u(x,y)) \lesssim f(x,y,v(x,y))$ for all $(s,t) \in J$. From Lemma 5.1, we have

$$\int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} f(s,t,u(s,t)) \lesssim \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} f(s,t,v(s,t))$$

hold for all $(x,y) \in J$. Hence,

$$\begin{aligned} \psi(x,y) &= T_2(v(x,y)) + (-1)\frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} f(s,t,v(s,t)) \\ &\prec T_2(u(x,y)) + (-1)\frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} f(s,t,v(s,t)) \\ &\lesssim T_2(u(x,y)) + (-1)\frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} f(s,t,u(s,t)) \\ &= \psi(x,y). \end{aligned}$$

It implies the contraction.

From $T_2u(x,y) \lesssim T_2v(x,y)$ for all $(x,y) \in J$, for some integers n we obtain $T_2^n u(x,y) \lesssim T_2^n v(x,y)$, $\forall (x,y) \in J$. It shows $T_2^n u \lesssim T_2^n v$. This infers that T_2^n is a nondecreasing operator in $C_f(J,E)$.

Because there exists a (2)-lower solution $\mu \in C^{2,2}_{gH}(J,E)$ for problem (1)-(2), then

$$\mu(x,y) \lesssim \mu(x,0) + \mu(0,y) \ominus \mu(0,0) \ominus (-1)I^q(x,y,\mu(x,y))$$

or

$$\mu(x,y) + (-1)I^q f(x,y,\mu(x,y)) \lesssim \mu(x,0) + \mu(0,y) \ominus \eta_1(0),$$

for all $(x,y) \in J$.

Assume that there exists $(x,y) \in J$ such that $\mu(x,y)$ does not "$\lesssim$" $T_2\mu(x,y)$. From ${}^C\mathcal{D}^q_2\mu(x,y) \lesssim f(x,y,\mu(x,y))$, we have

$$(-1)I^{q\,C}\mathcal{D}^q_2\mu(s,t)dsdt \gtrsim (-1)\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}f(s,t,\mu(s,t))dsdt.$$

Hence, we get

$$\begin{aligned}
&\mu(x,0) + \mu(0,y) \ominus \mu(0,0)\\
&\quad = \mu(x,y) + (-1)\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}f(s,t,\mu(s,t))dsdt\\
&\quad \succ T_2\mu(x,y) + (-1)\frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_0^x\int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1}f(s,t,\mu(s,t))dsdt\\
&\quad = \psi(x,y)
\end{aligned}$$

That follows $\mu(x,0) + \mu(0,y) \ominus \mu(0,0) \succ \psi(x,y)$ and this is contractive with hypothesis $\mu(x,0) \lesssim \eta_1(x)$ and $\mu(0,y) \lesssim \eta_2(y)$. Therefore, $\mu \lesssim T_2\mu$ in $C_f(J,E)$. It follows $\mu \lesssim T_2^n\mu$ in $C_f(J,E), n \in \mathbb{N}$.

If μ is an (2)-upper solution to the problem (1)-(2), then by using analogous arguments we receive $\mu \gtrsim T_2\mu$ in $C_f(J,E)$. Then $\mu \gtrsim T_2^n\mu$ in $C_f(J,E), n \in \mathbb{N}$. Therefore, the operator T_2^n verifies all hypotheses of Theorem 2.6 for some large numbers n, and therefore, T_2^n has a fixed point in $C_f(J,E)$. The uniqueness comes from the validity of Lemma 2.2. $\square$

Now we prove the existence of solution for problem (1)-(2) by applying the results of Theorem 2.5 in case $\beta \in \mathcal{S}_0$.

In the space $C(J,E)$ we consider metric

$$d(u,v) = \sup_J\{d_\infty(u(x,y),v(x,y))\}.$$

Because of the compactness of J in $\mathbb{R}^2_+$, it is easy to see that $(C(J,E),d)$ is a complete metric space.

For an arbitrary altering distance function η, let $\mathcal{B}_\eta$ be the class of functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfies the following conditions

i) φ is increasing,

ii) $\beta(t) \in \mathcal{S}_0$, where $\beta(t) = \dfrac{\varphi(\eta(t))}{\eta(t)}$.

Theorem 5.6. *Consider function f satisfied the hypothesis (h_1) in Theorem 5.2. Assume that for some altering distance functions ψ, $\psi(t) < t$ for all $t > 0$, the following inequality hold*

$$d_\infty(f(x,y,u(x,y)),f(x,y,v(x,y))) \leq \frac{1}{A}\varphi\big(\psi(d_\infty(u(x,y),v(x,y)))\big) \tag{10}$$

for $u \lesssim v$ *in* $C(J,E)$, $(x,y) \in J$, $\varphi \in \mathcal{B}_\psi$ *and* $A = \dfrac{1}{\Gamma(q_1)\Gamma(q_2)} \dfrac{a^{q_1}}{q_1} \dfrac{b^{q_2}}{q_2}$. *Then, the existence of an (1)-lower solution (or an (1)-upper solution)* μ *of problem (1)-(2) provides the existence of a unique (i)-weak fuzzy solution of problem (1)-(2).*

Proof. Consider the operator $T_1 : (C(J,E), d) \to (C(J,E), d)$ defined by (6). Assume that $u \lesssim v$ in $C(J,E)$ and from the nondecreasing property of function f with respect to the third variable we have $f(x,y,u(x,y)) \lesssim f(x,y,v(x,y))$ for all $(x,y) \in J$. Then, from Lemma 3.7 we have

$$I^q f(x,y,u(x,y)) \lesssim I^q f(x,y,v(x,y)).$$

This implies $T_1 u(x,y) \lesssim T_1 v(x,y)$ for all $(x,y) \in J$ or $T_1 u \lesssim T_1 v$. Hence, T_1 is a nondecreasing operator in $C(J,E)$.
For $u \lesssim v$ in $C(J,E)$ we have

$$\begin{aligned} d_\infty(T_1 u(x,y), T_1 v(x,y)) &\le d_\infty(I^q f(x,y,u(x,y)), I^q f(x,y,v(x,y))) \\ &\le \int_0^x \int_0^y d_\infty(f(s,t,u(s,t)), f(s,t,v(s,t))) ds dt \\ &\le \frac{1}{A} \frac{1}{\Gamma(q_1)\Gamma(q_2)} \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} \varphi\big(\psi(d_\infty(u(x,y), v(x,y)))\big) ds dt. \end{aligned}$$

Since $d_\infty(u(x,y), v(x,y)) \le d(u,v)$ for all $(x,y) \in J$, by using hypothesis of the nondecreasing property of functions ψ and φ we get

$$\psi(d_\infty(u(x,y), v(x,y))) \le \psi(d(u,v))$$

and

$$\varphi(\psi(d_\infty(u(x,y), v(x,y)))) \le \varphi(\psi(d(u,v)))$$

for all $(x,y) \in J$. It follows for all $(x,y) \in J$,

$$\begin{aligned} d_\infty(T_1 u(x,y), T_1 v(x,y)) &\le \frac{1}{A} \frac{1}{\Gamma(q_1)\Gamma(q_2)} \varphi\big(\psi(d(u,v))\big) \int_0^x \int_0^y (x-s)^{q_1-1}(y-t)^{q_2-1} ds dt \\ &= \frac{1}{A} \frac{1}{\Gamma(q_1)\Gamma(q_2)} \varphi\big(\psi(d(u,v))\big) \tfrac{x^{q_1}}{q_1} \tfrac{y^{q_2}}{q_2} \\ &\le \frac{1}{A} A \varphi\big(\psi(d(u,v))\big) \\ &= \varphi\big(\psi(d(u,v))\big). \end{aligned}$$

Thus, $d(T_1 u, T_1 v) \le \varphi\big(\psi(d(u,v))\big)$ for $u \lesssim v$ in $C(J,E)$. From nondecreasing property of ψ we get

$$\begin{aligned} \psi\big(d(T_1 u, T_1 v)\big) &\le \psi\Big(\varphi\big(\psi(d(u,v))\big)\Big) \\ &\le \varphi\big(\psi(d(u,v))\big) \\ &= \frac{\varphi\big(\psi(d(u,v))\big)}{\psi(d(u,v))} \psi(d(u,v)) \\ &= \beta(d(u,v))\psi(d(u,v)), \end{aligned}$$

where $\beta(t) = \dfrac{\varphi(\psi(t))}{\psi(t)} \in \mathcal{S}_0$.

Finally, let μ be a (1)-lower solution for problem (1)-(2). We will show that $\mu \lesssim T_1\mu$. Indeed, we have

$$\begin{aligned}\mu(x,y) &\lesssim \mu(x,0) + \mu(0,y) \ominus \mu(0,0) + I^q(x,y,\mu(x,y))\\ &\lesssim \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I^q f(x,y,\mu(x,y))\\ &= T_1(\mu)(x,y)\end{aligned}$$

for all $(x,y) \in J$. That follows $\mu \lesssim T_1\mu$.

Overall, we can see that the operator T_1 verifies all hypotheses of Theorem 5.2 in case $\beta \in \mathcal{S}_0$. Consequently, T_1 has a fixed point in $C(J,E)$. Noting that $C(J,E)$ satisfies that every pair of elements of $C(J,E)$ has an (1)-upper bound. It follows that the operator T_1 has a unique fixed point.

It repeats similarly if there exists a (1)-upper solution μ for problem (1)-(2), then one has $\mu \gtrsim T_1\mu$. □

Theorem 5.7. *Suppose that f satisfies the hypotheses (h_1), (h_3) and (h_4). Then the existence of a (2)-lower solution (an (2)-upper solution) μ for problem (1)-(2) provides the existence of a unique (ii)-weak fuzzy solution of problem (1)-(2).*

Proof. Analogous arguments are used for the operator T_2 in Theorem 5.5, we consider the operator T_2 defined by (9) and receive the existence of unique (ii)-weak solution for problem (1)-(2). □

Example 5.8. Denote $E_+ = \{z \in E, \hat{0} \lesssim z\}$, where $\hat{0}$ is defined by $\hat{0}(t) = 1$ if $t = 0$ and $\hat{0}(t) = 0$ in other cases. In this example, we consider the following fuzzy partial hyperbolic equation under Caputo gH-differentiability

$$\begin{cases} {}^C\mathcal{D}^q_k u(xy) &= f(x,y,u(x,y)), \quad (x,y) \in J = [0,a] \times [0,b],\ k = 1,2,\\ u(x,0) &= 0, \quad x \in J_a\\ u(0,y) &= 0, \quad y \in J_b, \end{cases} \tag{11}$$

where $f : \ J \times C(J,E) \to E_+$ is continuous and nondecreasing with respect to the third variable and suppose that

$$d_\infty(f(x,y,u(x,y)), f(x,y,v(x,y))) \leq \frac{1}{ab} \ln\left(1 + d^2_\infty(u(x,y), v(x,y))\right)$$

if $u \lesssim v$ in $C(J,E)$ for all $(x,y) \in J$. Then, problem (11) has a unique nonnegative (1)-weak fuzzy solution.

In addition to the hypothesis (h_3) satisfied the problem (11) has a unique nonnegative (ii)-weak fuzzy solution.

Proof. Consider the cone $P = \{u \in C(J,E) : u \gtrsim \hat{0}\}$. Obviously (P,d) is a complete metric space. The operator $T_1(u)(x,y) = I^q f(x,y,u(x,y))$ applies P into itself since $f(x,y,u)$ is a nonnegative continuous function. $T_1(\hat{0}) \gtrsim \hat{0}$. Then, from Theorem 5.5 and Theorem 5.7 with $\varphi(t) = \ln(1+t)$ and $\psi(t) = t^2$, we receive the conclusion. □

Definition 5.9. [5]*(Zadeh's Extension Principle)* Give a crisp function $f:\mathbb{R}\to\mathbb{R}$, Zadeh's extension of f is a function $F:E\to E$, $v=F(u)$ defined by

$$v(y)=\begin{cases}\sup\{u(x):x\in X, f(x)=y\} & \textit{when } f^{-1}(y)\neq\emptyset\\ 0 & \textit{otherwise}\end{cases}$$

Lemma 5.10. [5] *Given a continuous function $f:\mathbb{R}\to\mathbb{R}$, it can be extended to a fuzzy function $F:E\to E$ and given $u\in E$ we can determine $v=F(u)\in E$ by its level set $[v]^\alpha=F([u]^\alpha)$, $\forall\alpha\in[0;1]$, i.e., we have $[v]^\alpha=[v^l_\alpha,v^r_\alpha]$, where*

$$v^l_\alpha=\inf\{f(x)|x\in[u]^\alpha\},\ v^r_\alpha=\sup\{f(x)|x\in[u]^\alpha\}.$$

Example 5.11. Given a continuous real-valued function $f(x)=\ln^2 x$. According to Lemma 5.10 and Definition 5.9, it can be extended to a fuzzy function $F:\ E\to E$, $v=F(u)=\ln^2 u$, where

$$\begin{aligned}v(y)&=\begin{cases}\sup\{u(x):y=\ln^2 x\} & \textit{when } f^{-1}(y)\neq\emptyset\\ 0 & \textit{otherwise}\end{cases}\\ &=\begin{cases}\sup\{u(x):x=e^{\sqrt{y}}\text{ or }x=e^{-\sqrt{y}}\} & \textit{when } y\in[0,+\infty)\\ 0 & \textit{otherwise}\end{cases}\\ &=\begin{cases}\sup\{u(e^{\sqrt{y}},u(e^{-\sqrt{y}})\} & \textit{when } y\in[0,+\infty)\\ 0 & \textit{otherwise}\end{cases}\end{aligned}$$

Consider a triangular fuzzy number $[u]^\alpha=[\alpha,2-\alpha],\ \alpha\in[0,1]$. Then

$$v\left(\frac{1}{4}\right)=\sup\left\{u\left(\sqrt{e}\right),u\left(\frac{1}{\sqrt{e}}\right)\right\}=\frac{1}{\sqrt{e}}.$$

For $y=3$, we have

$$v(3)=\sup\{u(e^{\sqrt{3}}),u(e^{-\sqrt{3}})\}=e^{-\sqrt{3}}.$$

For $y=4$, then

$$v(4)=\sup\{u(e^2),u(e^{-2})\}=e^{-2}.$$

Next, for $\alpha\in(0,1]$ we have

$$\begin{aligned}[v]^\alpha&=[\ln^2 u]^\alpha=\{y\in\mathbb{R}:v(y)\geq\alpha\}\\ &=\{y\in[0,+\infty):u(e^{\sqrt{y}})\geq\alpha\}\\ &=\{\ln^2 t:u(t)\geq\alpha\}\\ &=\{\ln^2 t:t\in[u]^\alpha\}\\ &=(\ln[u]^\alpha)^2.\end{aligned}$$

Example 5.12. In this example, we extend a continuous real-valued function $f(x)=x^2$ to a fuzzy function $F:\ E\to E$, $v=F(u)=u^2$, where

$$v(y)=F(u)(y)=(u^2)(y)=\begin{cases}\sup\{u(\sqrt{y})\} & \textit{when } y\in[0,+\infty)\\ 0 & \textit{otherwise.}\end{cases}$$

Next, for $\alpha \in (0,1]$ we have

$$\begin{aligned}[v]^\alpha = [u^2]^\alpha &= \{y \in \mathbb{R} : v(y) \ge \alpha\} = \{y \in [0,+\infty) : u(\sqrt{y}) \ge \alpha\} \\ &= \{t^2 : u(t) \ge \alpha\} = \{t^2 : t \in [u]^\alpha\} = \left([u]^\alpha\right)^2.\end{aligned}$$

Let $[u]^0 = [a,b];\ [v]^0 = [c,d]$. For $t \in [u]^\alpha \subset [a,b]$, $t' \in [\overline{u}]^\alpha \subset [c,d]$, we have

$$|t + t'| \le |t| + |t'| \le \max\{|a|,|b|\} + \max\{|c|,|d|\} = M_0.$$

Additionally, for arbitrary $u,\, \overline{u} \in E$, $\alpha \in [0,1]$ we have

$$d(u^2, \overline{u}^2) = \sup_{\alpha\in[0,1]} d_H\left([u^2]^\alpha, [\overline{u}^2]^\alpha\right) = \sup_{\alpha\in[0,1]} d_H\left(\left([u]^\alpha\right)^2, \left([\overline{u}]^\alpha\right)^2\right),$$

where

$$\begin{aligned}d_H\left(\left([u]^\alpha\right)^2, \left([\overline{u}]^\alpha\right)^2\right) &= \max\left\{\sup_{\lambda\in([u]^\alpha)^2} \inf_{\mu\in([\overline{u}]^\alpha)^2} |\lambda - \mu|;\ \sup_{\mu\in([\overline{u}]^\alpha)^2} \inf_{\lambda\in([u]^\alpha)^2} |\lambda - \mu|\right\} \\ &= \max\left\{\sup_{t\in[u]^\alpha} \inf_{\overline{t}\in[\overline{u}]^\alpha} |t^2 - \overline{t}^2|;\ \sup_{t\in[\overline{u}]^\alpha} \inf_{\overline{t}\in[u]^\alpha} |t^2 - \overline{t}^2|\right\} \\ &\le M_0 \max\left\{\sup_{t\in[u]^\alpha} \inf_{\overline{t}\in[\overline{u}]^\alpha} |t - \overline{t}|;\ \sup_{t\in[\overline{u}]^\alpha} \inf_{\overline{t}\in[u]^\alpha} |t - \overline{t}|\right\} \\ &= M_0\, d_H\left([u]^\alpha, [\overline{u}]^\alpha\right).\end{aligned}$$

Thus

$$\begin{aligned}d(u^2, \overline{u}^2) &= \sup_{\alpha\in[0,1]} d_H\left(([u]^\alpha)^2, ([\overline{u}]^\alpha)^2\right) \\ &\le M_0 \sup_{\alpha\in[0,1]} d_H\left([u]^\alpha, [\overline{u}]^\alpha\right) \\ &= M_0\, d(u, \overline{u}).\end{aligned}$$

Example 5.13. Now we consider the following fractional hyperbolic PDEs in the fuzzy number space E_+

$${}^C\mathcal{D}_1^{\frac{1}{2}} u(x,y) = f(x,y,u(x,y)), \quad (x,y) \in J := [0,1] \times [0,1], \tag{12}$$

with the initial conditions

$$u(x,0) = \eta_1(x) = C,\ u(0,y) = \eta_2(y) = C,\ u(0,0) = C \tag{13}$$

for all $x \in [0,1], y \in [0,1]$ and $C = (1,2,3)$ is a triangle fuzzy number and

$$f(x,y,\varphi) = \begin{cases} x^2y^2\varphi^2 & \text{if } (x,y,\varphi) \in J \times E_{\lesssim} \\ x^2y^2\ln^2\varphi & \text{others}\end{cases}$$

where

$$E_{\lesssim} = \underset{\phi\in E}{\cup} E_{\lesssim\phi} = \underset{\phi\in E}{\cup} \{\varphi \in E | \varphi \lesssim \phi\}.$$

It is easy to see that if $\varphi_1 \lesssim \varphi_2$ then $\varphi_1, \varphi_2 \in E_{\lesssim \varphi_2}$ and $\varphi_1, \varphi_2 \in E_{\lesssim}$. Therefore, we have

$$\begin{aligned} d(f(x,y,\varphi_1), f(x,y,\varphi_2)) &= d(x^2y^2\varphi_1^2, x^2y^2\varphi_2^2) \\ &= x^2y^2 d(\varphi_1^2, \varphi_2^2) \\ &\le x^2y^2 M_0\, d(\varphi_1, \varphi_2) \\ &\le M_0\, d(\varphi_1, \varphi_2). \end{aligned}$$

Thus f satisfies Lipschitz condition in the subspace $E_{\lesssim} \subset E$. Moreover f is non-decreasing with respect to the third variable in E_+.

Consider $\mu(x,y) = C \in E_+$, we have ${}^C\mathcal{D}_1^{\frac{1}{2}}\mu(x,y) = \hat{0} \lesssim f(x,y,\mu(x,y))$ and $\mu(x,0) \lesssim \eta_1(x)$, $\mu(0,y) \lesssim \eta_2(y)$ for all $(x,y) \in J$. It means that $\mu(x,y)$ is a (1)-lower solution of (15) - (16). Applying Theorem 5.2, the problem (12) - (13) has a unique (i)-weak solution on $C(J,E)$.

Now we consider

$$g(x,y,\varphi) = \begin{cases} xy\varphi & \text{if } (x,y,\varphi) \in J \times E_{\lesssim} \\ \Phi(x,y,\varphi) & \text{others} \end{cases}$$

such that $g \in C(J,E)$. In addition, we assume that

$$C_g(J,E) = \left\{u \in C(J,E) : C \ominus (-1)I^{\frac{1}{2}} g(x,y,u(x,y)) \text{ exists}\right\} \neq \emptyset.$$

Then from Theorem 5.5, the existence of (2)-lower solution $\mu(x,y) = C \in E$ of the problem

$$\begin{aligned} {}^C\mathcal{D}_2^{\frac{1}{2}} u(x,y) = g(x,y,u(x,y)), \quad (x,y) \in J := [0,1] \times [0,1] \\ u(x,0) = \eta_1(x) = C,\ u(0,y) = \eta_2(y) = C,\ u(0,0) = C \end{aligned}$$

for all $x \in [0,1], y \in [0,1]$ and C an arbitrary triangle fuzzy number, implies the existence of its unique (ii)-weak solution.

Acknowledgements. The authors would like to express their gratitude to Editor-in Chief and the anonymous referees for their helpful comments and suggestions, which have greatly improved the manuscript. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2015.08.

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