

## GENERALIZATION OF $(\in, \in \vee q)$ -FUZZY SUBNEAR-RINGS AND IDEALS

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**ABSTRACT.** In this paper, we introduce the notion of  $(\in, \in \vee q_k)$ -fuzzy subnear-ring which is a generalization of  $(\in, \in \vee q)$ -fuzzy subnear-ring. We have given examples which are  $(\in, \in \vee q_k)$ -fuzzy ideals but they are not  $(\in, \in \vee q)$ -fuzzy ideals. We have also introduced the notions of  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals and  $(\in, \in \vee q_k)$ -fuzzy bi-ideals of near-ring. We have characterized  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals and  $(\in, \in \vee q_k)$ -fuzzy bi-ideals of near-rings.

### 1. Introduction

In 1965 Zadeh [24] introduced the concept of fuzzy subsets and studied their properties on the lines parallel to set theory. In 1971, Rosenfeld [17] defined a fuzzy subgroup and gave some of its properties. Rosenfeld's definition of a fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector spaces and so on. In 1981 Das [6] explained the inter-relationship between a fuzzy subgroups and its  $t$ -level subsets. Fuzzy subrings and ideals were first introduced by Wang-jin Liu [12] in 1982. Subsequently, Mukherjee and Sen [14], Swamy and Swamy [20], Yue [23], Dixit et al [7] and Rajesh Kumar [10] applied some basic concepts pertaining to ideals from classical ring theory and developed a theory of fuzzy rings. The notions of fuzzy subnear-ring and ideal were first introduced by Abou-Zaid [1] in 1991. The concept of quasi-coincidence of a fuzzy point with a fuzzy subset was introduced by Pu Pao-Ming and Liu Ying-Ming [13] in 1980. The idea of quasi-coincidence of a fuzzy point with a fuzzy set was introduced by Bhakat and Das [2] in 1992. In particular, the  $(\in, \in \vee q)$ -fuzzy subgroup is an important and useful generalization of a fuzzy subgroup. In [4], Bhakat and Das have extended the notion of  $(\in, \in \vee q)$ -fuzzy subgroups to the notion of  $(\in, \in \vee q)$ -fuzzy subrings. Narayanan and Manikantan [15] have extended these results to near-rings. We introduce the notion of an  $(\in, \in \vee q_k)$ -fuzzy subnear-ring which is a generalization of an  $(\in, \in \vee q)$ -fuzzy subnear-ring. We give examples which are  $(\in, \in \vee q_k)$ -fuzzy ideals but not  $(\in, \in \vee q)$ -fuzzy ideals. Finally, we

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*Key words:* Near ring, Fuzzy subnear-ring, Fuzzy ideal, Fuzzy quasi-ideal, Fuzzy bi-ideal  $(\in, \in \vee q)$ -fuzzy subnear-ring,  $(\in, \in \vee q)$ -fuzzy ideal,  $(\in, \in \vee q)$ -fuzzy quasi-ideal,  $(\in, \in \vee q)$ -fuzzy bi-ideal,  $(\in, \in \vee q_k)$ -fuzzy subnear-ring,  $(\in, \in \vee q_k)$ -fuzzy ideal,  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal,  $(\in, \in \vee q_k)$ -fuzzy bi-ideal.

introduce the notions of  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals and  $(\in, \in \vee q_k)$ -fuzzy bi-ideals of near-rings which are the generalization of fuzzy bi-ideals, fuzzy quasi-ideals,  $(\in, \in \vee q)$ -fuzzy quasi-ideals,  $(\in, \in \vee q)$ -fuzzy bi-ideals of near-ring  $N$ . We also characterize  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals and  $(\in, \in \vee q_k)$ -fuzzy bi-ideals of near-rings.

## 2. Preliminaries

For the sake of completeness we first recall some definitions and results proposed by the early pioneers.

**Definition 2.1.** A *near-ring*  $N$  is a system with two binary operations  $+$  and  $\cdot$  such that:

- (1)  $(N, +)$  is a group, not necessarily abelian,
- (2)  $(N, \cdot)$  is a semigroup,
- (3)  $(x + y)z = xz + yz$ , for all  $x, y, z \in N$ .

We will use the word "near-ring" to mean "right distributive near-ring" and write  $xy$  instead of  $x \cdot y$ . Note that  $0 \cdot x = 0$  and  $(-x)y = -xy$  but in general  $x \cdot 0 \neq 0$  for some  $x \in N$ .

**Definition 2.2.** Let  $(N, +, \cdot)$  be a near-ring. A subset  $I$  of  $N$  is said to be an *ideal* of  $N$  if:

- (1)  $(I, +)$  is a normal subgroup of  $(N, +)$ ,
- (2)  $IN \subseteq I$ ,
- (3)  $n_1(n_2 + i) - n_1n_2 \in I$ , for all  $i \in I$  and  $n_1, n_2 \in N$ .

If  $I$  satisfies (1) and (2), then it is called a *right ideal* of  $N$ . If  $I$  satisfies (1) and (3), then it is called a *left ideal* of  $N$ .

Let  $N$  be a near-ring. Given two subsets  $A$  and  $B$  of  $N$ , the product  $AB = \{ab \mid a \in A, b \in B\}$  and  $A * B = \{a(a' + b) - aa' \mid a, a' \in A, b \in B\}$ . In what follows,  $N$  will denote right distributive near-ring, unless otherwise specified. For the basic terminology and notation we refer to Pilz [16] and Abou-Zaid [1].

**Definition 2.3.** Let  $S$  be any set. A mapping  $\mu : S \rightarrow [0, 1]$  is called a *fuzzy subset* of  $S$ .

A fuzzy subset  $\mu : S \rightarrow [0, 1]$  is nonempty if  $\mu$  is not the constant map with value 0. For any two fuzzy subsets  $\lambda$  and  $\mu$  of  $S$ ,  $\lambda \leq \mu$  means that  $\lambda(a) \leq \mu(a)$  for all  $a \in S$ . The characteristic function of  $N$  is denoted by  $\mathbf{N}$  and the characteristic function of a subset  $A$  is denoted by  $f_A$ . The image of a fuzzy subset  $\mu$  is denoted by  $Im(\mu) = \{\mu(n) \mid n \in N\}$  and  $|Im(\mu)|$  denotes the cardinality of  $Im(\mu)$ . Hereafter, we consider only nonempty fuzzy subsets of  $N$ .

**Definition 2.4.** [12] Let  $\mu$  be a fuzzy subset of a ring  $R$ . Then  $\mu$  is called a *fuzzy subring* (resp. *ideal*) of  $R$  if for all  $x, y \in R$ :

- (1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ ,
- (2)  $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$  (resp.  $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ ).

**Definition 2.5.** Let  $\mu$  be a fuzzy subset of  $N$ . Then  $\mu$  is called a *fuzzy left (right)  $N$ -subgroup* of  $N$  if for all  $x, y \in N$ :

- (1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ ,
- (2)  $\mu(xy) \geq \mu(y)(\mu(xy) \geq \mu(x))$ .

If  $\mu$  is both left and right fuzzy  $N$ -subgroup of  $N$ , then it is called a *fuzzy  $N$ -subgroup* of  $N$ .

**Definition 2.6.** Let  $\mu$  be any fuzzy subset of  $N$ . For  $t \in [0, 1]$ , the set  $\mu_t = \{x \in N \mid \mu(x) \geq t\}$  is called a *level subset* of  $\mu$ .

**Definition 2.7.** Let  $f$  and  $g$  be any two fuzzy subsets of  $N$ . Then  $f \cap g$ ,  $f \cup g$ ,  $f + g$ ,  $f g$  and  $f * g$  are fuzzy subsets of  $N$  defined by:

$$(f \cap g)(x) = \min\{f(x), g(x)\}$$

$$(f \cup g)(x) = \max\{f(x), g(x)\}$$

$$(f + g)(x) = \begin{cases} \sup_{x=y+z} \{\min\{f(y), g(z)\}\}, & \text{if } x \text{ is expressed as } x = y + z, \\ 0 & \text{, otherwise,} \end{cases}$$

$$(f g)(x) = \begin{cases} \sup_{x=yz} \{\min\{f(y), g(z)\}\}, & \text{if } x \text{ is expressed as } x = yz, \\ 0 & \text{, otherwise,} \end{cases}$$

$$(f * g)(x) = \begin{cases} \sup_{x=a(b+c)-ab} \{\min\{f(a), g(c)\}\}, & \text{if } x = a(b + c) - ab, \\ 0 & \text{, otherwise,} \end{cases}$$

**Definition 2.8.** For any  $x \in N$  and  $t \in (0, 1]$ , define a *fuzzy point*  $x_t$  as

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

If  $x_t$  is a fuzzy point and  $\mu$  is any fuzzy subset of  $N$  and  $x_t \leq \mu$ , then we write  $x_t \in \mu$ . Note that  $x_t \in \mu$  if and only if  $x \in \mu_t$  where  $\mu_t$  is a level subset of  $\mu$ .

**Definition 2.9.** A fuzzy subset  $\mu$  of  $N$  is called a *fuzzy subnear-ring* of  $N$  if for all  $x, y \in N$ ;

- (1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ ,
- (2)  $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ .

**Definition 2.10.** Let  $\mu$  be a nonempty fuzzy subset of  $N$ .  $\mu$  is a *fuzzy ideal* of  $N$  if for all  $x, y, i$  in  $N$ ;

- (1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ ,
- (2)  $\mu(x) = \mu(y + x - y)$ ,
- (3)  $\mu(xy) \geq \mu(x)$ ,
- (4)  $\mu(x(y + i) - xy) \geq \mu(i)$ .

If  $\mu$  satisfies (1), (2) and (3), then it is called a *fuzzy right ideal* of  $N$ . If  $\mu$  satisfies (1), (2) and (4), then it is called a *fuzzy left ideal* of  $N$ . If  $\mu$  is both fuzzy right as well as fuzzy left ideal of  $N$ , then  $\mu$  is called a *fuzzy ideal* of  $N$ .

**Definition 2.11.** A fuzzy point  $x_t$  is said to *belong to* (resp. be *quasi-coincident with*) a fuzzy subset  $\mu$ , written as  $x_t \in \mu$  (resp.  $x_t q \mu$ ) if  $\mu(x) \geq t$  (resp.  $\mu(x) + t > 1$ ). If  $x_t \in \mu$  or  $x_t q \mu$ , then we write  $x_t \in \vee q \mu$ .

**Definition 2.12.** A fuzzy subset  $\mu$  of a ring  $R$  is said to be an  $(\in, \in \vee q)$ -fuzzy subring of  $R$  if for all  $x, y \in R$  and  $t, r \in (0, 1]$ ;

(1)  $x_t, y_r \in \mu$  implies  $(x + y)_{\min\{t, r\}} \in \vee q \mu$ ,

(2)  $x_t \in \mu$  implies  $(-x)_t \in \vee q \mu$ ,

(3)  $x_t, y_r \in \mu$  implies  $(xy)_{\min\{t, r\}} \in \vee q \mu$ .

**Definition 2.13.** [4] A fuzzy subset  $\mu$  of a ring  $R$  is said to be an  $(\in, \in \vee q)$ -fuzzy ideal of  $R$  if:

(1)  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy subring of  $R$ ,

(2)  $x_t \in \mu$  and  $y \in R$  implies  $(xy)_t, (yx)_t \in \vee q \mu$ , for all  $x, y \in R$  and  $t \in (0, 1]$ .

**Example 2.14.** [4] Consider the ring  $R = J/(4)$ . Let  $\lambda : R \rightarrow [0, 1]$  be defined by  $\lambda(0) = 0.6, \lambda(1) = \lambda(3) = 0.4, \lambda(2) = 0.7$ . Then  $\lambda$  is an  $(\in, \in \vee q)$ -fuzzy ideal of  $R$  but not a fuzzy ideal according to Definition 2.4.

**Definition 2.15.** [15] A fuzzy subset  $\mu$  is said to be an  $(\in, \in \vee q)$ -fuzzy subnear-ring of  $N$  if for all  $x, y \in N$  and  $t, r \in (0, 1]$ ;

(1)  $x_t, y_r \in \mu$  implies  $(x + y)_{\min\{t, r\}} \in \vee q \mu$ ,

(2)  $x_t \in \mu$  implies  $(-x)_t \in \vee q \mu$ ,

(3)  $x_t, y_r \in \mu$  implies  $(xy)_{\min\{t, r\}} \in \vee q \mu$ .

**Definition 2.16.** [15] A fuzzy subset  $\mu$  of  $N$  is said to be an  $(\in, \in \vee q)$ -fuzzy ideal of  $N$  if for all  $x, y, z \in N$  and for all  $r, t \in (0, 1]$ ;

(1)  $x_r, y_t \in \mu$  implies  $(x - y)_{\min(r, t)} \in \vee q \mu$ ,

(2)  $x_t \in \mu$  and  $y \in N$  implies  $(y + x - y)_t \in \vee q \mu$ ,

(3)  $x_t \in \mu$  and  $y \in N$  implies  $(xy)_t \in \vee q \mu$ ,

(4)  $z_t \in \mu$  and  $x, y \in N$  implies  $(x(y + z) - xy)_t \in \vee q \mu$ .

If  $\mu$  satisfies (1), (2) and (3), then it is called an  $(\in, \in \vee q)$ -fuzzy right ideal of  $N$ . If  $\mu$  satisfies (1), (2) and (4), then it is called an  $(\in, \in \vee q)$ -fuzzy left ideal of  $N$ .

**Example 2.17.** ([15]) Let  $N = \{0, a, b, c\}$  be Klein's four group. Define multiplication in  $N$  as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	a	a	a	c

Then  $(N, +, \bullet)$  is a near-ring ( see [ [16], P-408 ] scheme 19 ). Let  $\mu : N \rightarrow [0, 1]$  be a fuzzy subset of  $N$  such that  $\mu(0) = 0.7, \mu(a) = \mu(c) = 0.4, \mu(b) = 0.8$ . Then

$\mu$  is an  $(\in, \in \vee q)$ -fuzzy subnear-ring of  $N$ . But, since  $0.7 = \mu(0) = \mu(b - b) \not\geq \min\{\mu(b), \mu(b)\} = 0.8$ ,  $\mu$  is not a fuzzy subnear-ring of  $N$ . Further,  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy ideal of  $N$ . Since  $0.7 = \mu(0) = \mu(b0) \not\geq \mu(b) = 0.8$ ,  $\mu$  is not a fuzzy ideal of  $N$ .

**Lemma 2.18.** [1] *Let  $\mu$  be a fuzzy subset of  $N$ .  $\mu$  is a fuzzy left (right)  $N$ -subgroup of  $N$  if and only if the level subset  $\mu_t$ ,  $t \in \text{Im } \mu$ , is a left (right)  $N$ -subgroup of  $N$ .*

**Lemma 2.19.** [1] *Let  $I$  be a subset of  $N$ .  $I$  is an (left or right) ideal of  $N$  if and only if  $f_I$  is a fuzzy (left or right) ideal of  $N$ .*

**Lemma 2.20.** [1] *Let  $\mu$  be a fuzzy subset of  $N$ .  $\mu$  is a fuzzy (left or right) ideal of  $N$  if and only if the level subset  $\mu_t$ ,  $t \in \text{Im } \mu$ , is an ideal of  $N$ .*

### 3. $(\in, \in \vee q_k)$ -fuzzy Subnear-rings and $(\in, \in \vee q_k)$ -fuzzy Ideals in $N$

In this section, we introduce the notion of  $\vee q_k$ -fuzzy sets which are a generalization of fuzzy sets.

**Definition 3.1.** A fuzzy point  $x_t$  is said to *belong to* (resp., be  *$k$ -quasi-coincident with*) a fuzzy subset  $\mu$ , written as  $x_t \in \mu$  (resp.,  $x_t q_k \mu$ ) if  $\mu(x) \geq t$  (resp.,  $\mu(x) + t > 1 - k$ , where  $k \in [0, 1)$ ).

For any  $t \in (0, 1]$ ,  $x_t \in \mu$  or  $x_t q_k \mu$  will be denoted by  $x_t \vee q_k \mu$ .  $x_t \in \bar{\mu}$ ,  $x_t \in \overline{\vee q_k \mu}$  will respectively mean  $x_t \in \mu$  and  $x_t \in \vee q_k \mu$  do not hold.

If  $k = 0$ , then  $x_t \in \vee q_k \mu$  if and only if  $x_t \in \vee q \mu$ . Thus the two definitions Definition 2.11 and Definition 3.1 will coincide when  $k = 0$ . Throughout this paper,  $k \in [0, 1)$  is arbitrary, but fixed.

**Definition 3.2.** A fuzzy subset  $\mu$  is said to be an  $(\in, \in \vee q_k)$ -fuzzy subnear-ring of  $N$  if for all  $x, y \in N$  and  $t, r \in (0, 1]$ :

- (1)  $x_t, y_r \in \mu$  implies  $(x + y)_{\min\{t, r\}} \in \vee q_k \mu$ ,
- (2)  $x_t \in \mu$  implies  $(-x)_t \in \vee q_k \mu$ ,
- (3)  $x_t, y_r \in \mu$  implies  $(xy)_{\min\{t, r\}} \in \vee q_k \mu$ .

**Lemma 3.3.** *Let  $\mu$  be a fuzzy subset of  $N$  and  $t, r \in (0, 1]$ . Then:*

- (1) (a)  $x_t, y_r \in \mu$  implies  $(x + y)_{\min\{t, r\}} \in \vee q_k \mu$ , and  
(b)  $\mu(x + y) \geq \min(\mu(x), \mu(y), \frac{1-k}{2})$  for all  $x, y \in N$  are equivalent.
- (2) (c)  $x_t \in \mu$  implies  $(-x)_t \in \vee q_k \mu$ , and  
(d)  $\mu(-x) \geq \min(\mu(x), \frac{1-k}{2})$  for all  $x \in N$  are equivalent.
- (3) (e)  $x_t, y_r \in \mu$  implies  $(xy)_{\min\{t, r\}} \in \vee q_k \mu$ , and  
(f)  $\mu(xy) \geq \min(\mu(x), \mu(y), \frac{1-k}{2})$  for all  $x, y \in N$  are equivalent.

*Proof.* (1) (a)  $\Rightarrow$  (b). Let  $x, y \in N$  and  $\min(\mu(x), \mu(y)) < \frac{1-k}{2}$ . Assume that  $\mu(x + y) < \min(\mu(x), \mu(y))$ . Choose  $t$  such that  $\mu(x + y) < t < \min(\mu(x), \mu(y))$ . This implies  $x_t, y_t \in \mu$  but  $(x + y)_t \in \overline{\vee q_k \mu}$ , which contradicts (a). Next, let  $\min(\mu(x), \mu(y)) \geq \frac{1-k}{2}$ . Assume that  $\mu(x + y) < \frac{1-k}{2}$ . Then  $x_{\frac{1-k}{2}}, y_{\frac{1-k}{2}} \in \mu$  but  $(x + y)_{\frac{1-k}{2}} \in \overline{\vee q_k \mu}$ , which contradicts (a). Thus  $\mu(x + y) \geq \min(\mu(x), \mu(y), \frac{1-k}{2})$ .

(b)  $\Rightarrow$  (a). Let  $a_t, b_r \in \mu$ . Then  $\mu(a+b) \geq \min(\mu(a), \mu(b), \frac{1-k}{2}) \geq \min(t, r, \frac{1-k}{2})$ . Thus  $\mu(a+b) \geq \min(t, r)$  if  $t < \frac{1-k}{2}$  or  $r < \frac{1-k}{2}$  and  $\mu(a+b) \geq \frac{1-k}{2}$  if  $t \geq \frac{1-k}{2}$  and  $r \geq \frac{1-k}{2}$ . Hence  $(a+b)_{\min(t,r)} \in \forall q_k \mu$ .

(2) (c)  $\Rightarrow$  (d). Let  $x \in N$ .  $\mu(x) = t < \frac{1-k}{2}$ . Suppose  $\mu(-x) < \mu(x)$ . Let  $\mu(-x) = r$ . Choose  $p$  such that  $r < p < t$  and  $r+p < 1-k$ . Then  $x_p \in \mu$  but  $(-x)_p \in \sqrt{q_k \mu}$ , which is a contradiction to (c). So,  $\mu(-x) \geq \mu(x)$ . Next, let  $\mu(x) \geq \frac{1-k}{2}$ . If  $\mu(-x) < \frac{1-k}{2}$ , then  $x_{\frac{1-k}{2}} \in \mu$  but  $(-x)_{\frac{1-k}{2}} \in \sqrt{q_k \mu}$ , which contradicts (c). Hence  $\mu(-x) \geq \min(\mu(x), \frac{1-k}{2})$ .

(d)  $\Rightarrow$  (c). Let  $x_t \in \mu$ . Then  $\mu(x) \geq t$ . Now  $\mu(-x) \geq \min(\mu(x), \frac{1-k}{2}) \geq \min(t, \frac{1-k}{2})$ . That is  $\mu(-x) \geq t$  or  $\frac{1-k}{2}$  according as  $t \leq \frac{1-k}{2}$  or  $t > \frac{1-k}{2}$ . Hence  $(-x)_t \in \forall q_k \mu$ .

(3) follows easily from (2).  $\square$

**Theorem 3.4.** A fuzzy subset  $\mu$  of  $N$  is an  $(\in, \in \forall q_k)$ -fuzzy subnear-ring of  $N$  if and only if  $\mu(x-y), \mu(xy) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ , for all  $x, y \in N$ .

*Proof.* It follows from Lemma 3.3.  $\square$

**Corollary 3.5.** [[15], Lemma 3.2.] A fuzzy subset  $\mu$  of  $N$  is an  $(\in, \in \forall q)$ -fuzzy subnear-ring of  $N$  if and only if  $\mu(x-y), \mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$ , for all  $x, y \in N$ .

*Proof.* The result follows easily from Lemma 3.3 if we take  $k = 0$ .  $\square$

**Corollary 3.6.** [[4], Theorem 3.3.]  $\mu$  is an  $(\in, \in \forall q)$ -fuzzy subring if and only if  $\mu(x-y), \mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}$ , for all  $x, y \in R$ .

**Remark 3.7.** Every fuzzy subnear-ring and  $(\in, \in \forall q)$ -fuzzy subnear-ring of  $N$  is an  $(\in, \in \forall q_k)$ -fuzzy subnear-ring of  $N$ , but, as the following example shows, the converse is not necessarily true.

**Example 3.8.** Consider the near-ring  $(N, +, \bullet)$  as defined in Example 2.17. Define a fuzzy subset  $\mu : N \rightarrow [0, 1]$  by  $\mu(0) = 0.42$ ,  $\mu(a) = \mu(c) = 0.4$ ,  $\mu(b) = 0.44$ . Then  $\mu$  is an  $(\in, \in \forall q_{0.2})$ -fuzzy subnear-ring of  $N$ . But, since  $\mu(0) = \mu(b-b) \not\geq \min\{\mu(b), \mu(b)\}$  and  $\mu(0) = \mu(b-b) \not\geq \min\{\mu(b), \mu(b), 0.5\}$ ,  $\mu$  is neither a fuzzy subnear-ring of  $N$  nor an  $(\in, \in \forall q)$ -fuzzy subnear-ring of  $N$ .

Now we generalize the notions of fuzzy ideals of  $N$  defined by Zaid [1] and  $(\in, \in \forall q)$ -fuzzy ideals of  $N$  defined by Narayana and Manikantan [15].

**Definition 3.9.** A fuzzy subset  $\mu$  of  $N$  is said to be an  $(\in, \in \forall q_k)$ -fuzzy subgroup of  $N$  if  $x_r, y_t \in \mu$  implies  $(x-y)_{\min(r,t)} \in \forall q_k \mu$ , for all  $x, y \in N$  and for all  $r, t \in (0, 1]$ .

**Definition 3.10.** A fuzzy subset  $\mu$  of  $N$  is said to be an  $(\in, \in \forall q_k)$ -fuzzy ideal of  $N$  if for all  $x, y, z \in N$  and for all  $r, t \in (0, 1]$ :

- (1)  $x_r, y_t \in \mu$  implies  $(x-y)_{\min(r,t)} \in \forall q_k \mu$ ,
- (2)  $x_t \in \mu$  and  $y \in N$  implies  $(y+x-y)_t \in \forall q_k \mu$ ,
- (3)  $x_t \in \mu$  and  $y \in N$  implies  $(xy)_t \in \forall q_k \mu$ ,
- (4)  $z_t \in \mu$  and  $x, y \in N$  implies  $(x(y+z)-xy)_t \in \forall q_k \mu$ .

A fuzzy subset  $\mu$  with conditions (1), (2) and (3) is called an  $(\in, \in \vee q_k)$ -fuzzy right ideal of  $N$ . If  $\mu$  satisfies (1), (2) and (4), then it is called an  $(\in, \in \vee q_k)$ -fuzzy left ideal of  $N$ .

**Lemma 3.11.** *Let  $\mu$  be a fuzzy subset of  $N$ . Then:*

- (1) (a)  $x_r, y_t \in \mu$  implies  $(x - y)_{\min(r,t)} \in \vee q_k \mu$  and  
(b)  $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$  for all  $x, y \in N$  are equivalent.
- (2) (c)  $x_t \in \mu$  and  $y \in N$  implies  $(y + x - y)_t \in \vee q_k \mu$  and  
(d)  $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\}$  for all  $x, y \in N$  are equivalent.
- (3) (e)  $x_t \in \mu$  and  $y \in N$  implies  $(xy)_t \in \vee q_k \mu$  and  
(f)  $\mu(xy) \geq \min\{\mu(x), \frac{1-k}{2}\}$  for all  $x, y \in N$  are equivalent.
- (4) (g)  $z_t \in \mu$  and  $x, y \in N$  implies  $(x(y + z) - xy)_t \in \vee q_k \mu$  and  
(h)  $\mu(x(y + z) - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$  for all  $x, y, z \in N$  are equivalent.

*Proof.* (a)  $\Leftrightarrow$  (b). It follows from Lemma 3.3.

(c)  $\Rightarrow$  (d). Let  $x, y \in N$  and  $\mu(x) < \frac{1-k}{2}$ . Assume that  $\mu(y + x - y) < \mu(x)$ . Choose  $t$  such that  $\mu(y + x - y) < t < \mu(x)$ . Then  $x_t \in \mu$  and  $(y + x - y)_t \in \overline{\vee q_k \mu}$ , which contradicts (c). Thus  $\mu(y + x - y) \geq \mu(x)$ . Next let  $\mu(x) \geq \frac{1-k}{2}$ . Assume that  $\mu(y + x - y) < \frac{1-k}{2}$ . Then  $x_{\frac{1-k}{2}} \in \mu$  and  $y + x - y \in \overline{\vee q_k \mu}$ , which contradicts (c). Hence (d) holds.

(d)  $\Rightarrow$  (c). Let  $x_t \in \mu$  and  $y \in N$ . Then  $\mu(x) \geq t$  and, by (d),  $\mu(y + x - y) \geq \min\{\mu(x), \frac{1-k}{2}\}$ . This implies  $\mu(y + x - y) \geq \min\{t, \frac{1-k}{2}\}$ . Then  $\mu(y + x - y) \geq t$  if  $t \leq \frac{1-k}{2}$  and  $\mu(y + x - y) \geq \frac{1-k}{2}$  if  $\frac{1-k}{2} < t$ . Thus  $y + x - y \in \vee q_k \mu$ . Hence (c) holds.

(e)  $\Rightarrow$  (f). Let  $x, y \in N$ . Let  $\mu(x) < \frac{1-k}{2}$ . Assume that  $\mu(xy) < \mu(x)$ . Choose  $t$  such that  $\mu(xy) < t < \mu(x)$ . Then  $x_t \in \mu$  and  $(xy)_t \in \overline{\vee q_k \mu}$  which contradicts (e). So  $\mu(xy) \geq \mu(x)$ . Next let  $\mu(x) \geq \frac{1-k}{2}$ . Assume that  $\mu(xy) < \frac{1-k}{2}$ . Then  $x_{\frac{1-k}{2}} \in \mu$  and  $(xy)_{\frac{1-k}{2}} \in \overline{\vee q_k \mu}$ , which contradicts (e). Hence (f) holds.

(f)  $\Rightarrow$  (e). Let  $x_t \in \mu$  and  $y \in N$ . By (f),  $\mu(xy) \geq \min\{\mu(x), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$ . Then  $\mu(xy) \geq t$  if  $t \leq \frac{1-k}{2}$  and  $\mu(xy) \geq \frac{1-k}{2}$  if  $t > \frac{1-k}{2}$ . Hence  $(xy)_t \in \vee q_k \mu$ . Thus (e) holds.

(g)  $\Rightarrow$  (h). Let  $x, y, z \in N$  and  $\mu(z) < \frac{1-k}{2}$ . Assume that  $\mu(x(y + z) - xy) < \mu(z)$ . Then there exists  $t$  such that  $\mu(x(y + z) - xy) < t < \mu(z)$ . This implies that  $z_t \in \mu$  and  $x(y + z) - xy \in \overline{\vee q_k \mu}$ , which contradicts (g). Assume that  $\mu(z) \geq \frac{1-k}{2}$ . Then  $z_{\frac{1-k}{2}} \in \mu$ . Suppose  $\mu(x(y + z) - xy) < \frac{1-k}{2}$ . Then  $(x(y + z) - xy)_{\frac{1-k}{2}} \in \overline{\vee q_k \mu}$ , which contradicts (h). Thus  $\mu(x(y + z) - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$  for all  $x, y, z \in N$ . Therefore (h) holds.

(h)  $\Rightarrow$  (g). Let  $z_t \in \mu$  and  $x, y \in N$ . Then, by (h),  $\mu(x(y + z) - xy) \geq \min\{\mu(z), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$ . Thus  $\mu(x(y + z) - xy) \geq t$  if  $t \leq \frac{1-k}{2}$  or  $\mu(x(y + z) - xy) \geq \frac{1-k}{2}$  if  $t > \frac{1-k}{2}$ . Therefore  $(x(y + z) - xy)_t \in \vee q_k \mu$ . Hence (g) holds.  $\square$

**Corollary 3.12.** [[15], Theorem 3.8.] *A fuzzy subset  $\mu$  of  $N$  is an  $(\in, \in \vee q)$ -fuzzy ideal of  $N$  if and only if for all  $x, y, z \in N$ :*

- (1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y), 0.5\}$ ,
- (2)  $\mu(y + x - y) \geq \min\{\mu(x), 0.5\}$ ,

- (3)  $\mu(xy) \geq \min\{\mu(x), 0.5\}$ ,  
(4)  $\mu(x(y+z) - xy) \geq \min\{\mu(z), 0.5\}$ .

*Proof.* The result follows from Lemma 3.11, if we take  $k = 0$ .  $\square$

**Corollary 3.13.** [[4] BHDF, Theorem 3.5.]  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy ideal of ring  $R$  if and only if:

- (1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y), 0.5\}$ ,  
(2)  $\mu(xy), \mu(yx) \geq \min\{\mu(x), 0.5\}$ , for all  $x, y \in R$ .

**Remark 3.14.** A fuzzy ideal and an  $(\in, \in \vee q)$ -fuzzy ideal of  $N$  are  $(\in, \in \vee q_k)$ -fuzzy ideals of  $N$ . However, as the following example shows, the converse is not necessarily true.

**Example 3.15.** Consider the near-ring  $(N, +, \bullet)$  as defined in Example 2.17. Define a fuzzy subset  $\mu : N \rightarrow [0, 1]$  by  $\mu(0) = 0.42$ ,  $\mu(a) = \mu(c) = 0.4$ ,  $\mu(b) = 0.44$ . Then  $\mu$  is an  $(\in, \in \vee q_{0.2})$ -fuzzy ideal of  $N$ . But, since  $\mu(b0) = \mu(0) \not\geq \mu(b)$  and  $\mu(b0) = \mu(0) \not\geq \min\{\mu(b), 0.5\}$ ,  $\mu$  is neither a fuzzy ideal nor an  $(\in, \in \vee q)$ -fuzzy ideal of  $N$ .

**Theorem 3.16.** Let  $\{\mu_i\}_{i=1}^n$  be a family of  $(\in, \in \vee q_k)$ -fuzzy subnear-rings (ideals) of  $N$ . Then  $\mu = \cap \mu_i$ , is an  $(\in, \in \vee q_k)$ -fuzzy subnear-ring (ideal) of  $N$ .

*Proof.* Let  $x, y \in \mu$ .

$$\begin{aligned} \mu(x + y) &= \bigcap_{i=1}^n \mu_i(x + y) \\ &\geq \min_{1 \leq i \leq n} \{ \min\{\mu_i(x), \mu_i(y), \frac{1-k}{2}\} \} \\ &\geq \min\{ \min_{1 \leq i \leq n} \{\mu_i(x)\}, \min_{1 \leq i \leq n} \{\mu_i(y)\}, \frac{1-k}{2} \} \\ &= \min\{ (\bigcap_{i=1}^n \mu_i)(x), (\bigcap_{i=1}^n \mu_i)(y), \frac{1-k}{2} \} \\ &= \min\{ \mu(x), \mu(y), \frac{1-k}{2} \}. \end{aligned}$$

$$\begin{aligned} \mu(-x) &= \bigcap_{i=1}^n \mu_i(-x) \\ &\geq \min_{1 \leq i \leq n} \{ \min\{\mu_i(x), \frac{1-k}{2}\} \} \\ &\geq \min\{ \min_{1 \leq i \leq n} \{\mu_i(x)\}, \frac{1-k}{2} \} \\ &= \min\{ (\bigcap_{i=1}^n \mu_i)(x), \frac{1-k}{2} \} \\ &= \min\{ \mu(x), \frac{1-k}{2} \}. \end{aligned}$$

$$\begin{aligned} \mu(xy) &= \bigcap_{i=1}^n \mu_i(xy) \\ &\geq \min_{1 \leq i \leq n} \{ \min\{\mu_i(x), \mu_i(y), \frac{1-k}{2}\} \} \\ &\geq \min\{ \min_{1 \leq i \leq n} \{\mu_i(x)\}, \min_{1 \leq i \leq n} \{\mu_i(y)\}, \frac{1-k}{2} \} \\ &= \min\{ (\bigcap_{i=1}^n \mu_i)(x), (\bigcap_{i=1}^n \mu_i)(y), \frac{1-k}{2} \} \\ &= \min\{ \mu(x), \mu(y), \frac{1-k}{2} \}. \end{aligned}$$

Thus  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy subnear-ring of  $N$ .  $\square$

**Theorem 3.17.** A non-empty subset  $I$  of  $N$  is an ideal (subnear-ring) of  $N$  if and only if  $f_I$  is an  $(\in, \in \vee q_k)$ -fuzzy ideal (subnear-ring) of  $N$ .

*Proof.* Let  $I$  be an ideal of  $N$ . Then  $f_I$  is an  $(\in, \in \vee q_k)$ -fuzzy ideal of  $N$ .

Conversely, let  $f_I$  be an  $(\in, \in \vee q_k)$ -fuzzy ideal of  $N$ . For any  $x, y \in I$ , we have  $f_I(x - y) \geq \min\{f_I(x), f_I(y), \frac{1-k}{2}\} = \min\{1, 1, \frac{1-k}{2}\}$ . Since  $k \in [0, 1)$ ,  $x - y \in I$ . Let  $x \in I$  and  $y \in N$ . Then  $f_I(y + x - y) \geq \min\{f_I(x), \frac{1-k}{2}\} = \frac{1-k}{2} \neq 0$ . This implies that  $y + x - y \in I$ . Now let  $a \in N$  and  $x \in I$ . Then  $f_I(xa) \geq \min\{f_I(x), \frac{1-k}{2}\} = \frac{1-k}{2}$ . This implies that  $xa \in I$ . Let  $x, y \in N$  and  $z \in I$ ,  $f_I(x(y + z) - xy) \geq \min\{f_I(z), \frac{1-k}{2}\} = \frac{1-k}{2}$ . This implies that  $x(y + z) - xy \in I$ . Thus  $I$  is an ideal in  $N$ .  $\square$

**Corollary 3.18.** [[15], Theorem 3.12.] *A non-empty subset  $I$  of  $N$  is a subnear-ring (ideal) of  $N$  if and only if  $f_I$  is an  $(\in, \in \vee q)$ -fuzzy subnear-ring (ideal) of  $N$ .*

*Proof.* The result follows from Theorem 3.17, if we take  $k = 0$ .  $\square$

**Corollary 3.19.** [[4], Theorem 3.10.] *A non-empty subset  $S$  of  $R$  is a subring (ideal) of  $R$  if and only if  $f_S$  is an  $(\in, \in \vee q)$ -fuzzy subring (ideal) of  $R$ .*

**Theorem 3.20.** *A fuzzy subset  $\mu$  of  $N$  is an  $(\in, \in \vee q_k)$ -fuzzy ideal (subnear-ring) of  $N$  if and only if the level subset  $\mu_t$  is an ideal (subnear-ring) of  $N$ , for all  $0 < t \leq \frac{1-k}{2}$  and  $k \in [0, 1)$ .*

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy ideal of  $N$ . Let  $0 < t \leq \frac{1-k}{2}$  and  $x, y, z \in \mu_t$ . Then  $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = t$  and hence  $x - y \in \mu_t$ . Now  $\mu(xy) \geq \min\{\mu(x), \frac{1-k}{2}\} = t$ . Thus  $xy \in \mu_t$ .  $\mu(x + y - x) \geq \min\{\mu(y), \frac{1-k}{2}\} = t$  implies  $x + y - x \in \mu_t$ . Hence  $\mu(a(b + z) - ab) \geq \min\{\mu(z), \frac{1-k}{2}\} = t$  for every  $a, b, z \in N$ . This implies that  $a(b + z) - ab \in \mu_t$ . So  $\mu_t$  is an ideal of  $N$ .

Conversely, let  $\mu_t$  be an ideal of  $N$  for all  $0 < t \leq \frac{1-k}{2}$ . Let  $x, y \in N$ . Suppose  $\mu(x - y) < \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ . Choose  $t$  such that  $\mu(x - y) < t < \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ . This implies  $x, y \in \mu_t$ . Then  $x - y \in \mu_t$ , since  $\mu_t$  is an ideal of  $N$ . This implies  $\mu(x - y) \geq t$ , a contradiction. Thus  $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{1-k}{2}\}$ . Suppose  $\mu(x + y - x) < \min\{\mu(x), \frac{1-k}{2}\}$ . Choose  $t$  such that  $\mu(x + y - x) < t < \min\{\mu(x), \frac{1-k}{2}\}$ . Then  $y \in \mu_t$ . Since  $\mu_t$  is an ideal of  $N$ ,  $x + y - x \in \mu_t$  and  $\mu(x + y - x) \geq t$ , a contradiction. Thus  $\mu(x + y - x) \geq \min\{\mu(x), \frac{1-k}{2}\}$ . Similarly, it can be shown that  $\mu(x(y + z) - xy) \geq \min\{\mu(z), \frac{1-k}{2}\}$  for all  $x, y \in N$ . Thus  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy ideal of  $N$ .  $\square$

**Corollary 3.21.** [[15], Theorem 3.13.] *A fuzzy subset  $\mu$  of  $N$  is an  $(\in, \in \vee q)$ -fuzzy subnear-ring (ideal) of  $N$  if and only if the level subset  $\mu_t$  is a subnear-ring (ideal) of  $N$  for all  $0 < t \leq 0.5$ .*

*Proof.* The proof follows from Theorem 3.20, if we take  $k = 0$ .  $\square$

**Corollary 3.22.** [[4], Theorem 3.12.] *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q)$ -fuzzy subring (ideal) of ring  $R$  if and only if the level subset  $\mu_t$  is a subring (ideal) of  $R$  for all  $0 < t \leq 0.5$ .*

**Remark 3.23.** Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy subnear-ring (ideal) of  $N$ . Then the level subset  $\mu_t$  is not necessarily a subnear-ring (ideal) in  $N$ . In Example 3.8, if we take  $t = 0.43$ , then  $\mu_t = \{b\}$  which is not a subnear-ring in  $N$ , because  $t \notin (0, 0.4]$ .

#### 4. Fuzzy Quasi-ideal and Fuzzy Bi-ideal

In this section, we introduce the notion of fuzzy bi-ideals of  $N$ . We characterize fuzzy quasi-ideals and fuzzy bi-ideals of  $N$ .

**Definition 4.1.** A subgroup  $Q$  of  $(N, +)$  is said to be a *quasi-ideal* of  $N$  if  $QN \cap NQ \cap N * Q \subseteq Q$ .

A subgroup  $B$  of  $(N, +)$  is said to be a *bi-ideal* of  $N$  if  $BNB \cap (BN) * B \subseteq B$ .

**Definition 4.2.** A fuzzy subgroup  $\mu$  of  $N$  is called a *fuzzy quasi-ideal* of  $N$  if  $(\mu \mathbf{N}) \cap (\mathbf{N} \mu) \cap (\mathbf{N} * \mu) \leq \mu$ .

A fuzzy subgroup  $\mu$  of  $N$  is called a *fuzzy bi-ideal* of  $N$  if  $(\mu \mathbf{N} \mu) \cap (\mu \mathbf{N} * \mu) \leq \mu$ .

Note that

$$\begin{aligned} (\mu \mathbf{N} * \mu)(z) &= \sup_{z=x(y+c)-xy} \{\min\{(\mu \mathbf{N})(x), \mu(c)\}\} \\ &= \sup_{z=x(y+c)-xy} \{\min\{\sup_{x=x_1 x_2} \{\mu(x_1), \mu(c)\}\}\} \\ &= \sup_{z=x_1 x_2 (y+c)-x_1 x_2 y} \{\min\{\mu(x_1), \mu(c)\}\} \\ &= 0 \text{ otherwise.} \end{aligned}$$

It is clear that if  $N$  is a zero-symmetric near-ring, then  $\mu \mathbf{N} \mu \leq \mu$  for every bi-ideal  $\mu$ .

**Lemma 4.3.** Let  $\mu$  be a fuzzy subset of  $N$ . If  $\mu$  is a fuzzy left ideal (right ideal,  $N$ -subgroup, subnear-ring) of  $N$ , then  $\mu$  is a fuzzy quasi-ideal of  $N$ .

*Proof.* Let  $x \in N$  and  $x = ab = n_1(n_2 + c) - n_1 n_2$ , where  $a, b, n_1, n_2$  and  $c$  are in  $N$ . Consider

$$\begin{aligned} (\mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu)(x) &= \min\{(\mu \mathbf{N})(x), (\mathbf{N} \mu)(x), (\mathbf{N} * \mu)(x)\} \\ &= \min\{\sup_{x=ab} \{\mu(a)\}, \sup_{x=ab} \{\mu(b)\}, \sup_{x=n_1(n_2+c)-n_1 n_2} \{\mu(c)\}\} \\ &\quad (\text{since } \mu \text{ is the fuzzy left ideal, } \mu(n_1(n_2 + c) - n_1 n_2) \geq \mu(c).) \\ &\leq \min\{1, 1, \sup_{x=n_1(n_2+c)-n_1 n_2} \{\mu(n_1(n_2 + c) - n_1 n_2)\}\}; \\ &\leq \mu(x). \end{aligned}$$

We remark that if  $x$  is not expressed as  $x = ab = n_1(n_2 + c) - n_1 n_2$ , then  $(\mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu)(x) = 0 \leq \mu(x)$ . Thus  $\mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu \leq \mu$ . Hence  $\mu$  is a fuzzy quasi-ideal of  $N$ .  $\square$

**Lemma 4.4.** For any non-empty subsets  $A$  and  $B$  of  $N$ ;

- (1)  $f_A f_B = f_{AB}$ ,
- (2)  $f_A \cap f_B = f_{A \cap B}$ ,
- (3)  $f_A * f_B = f_{A * B}$ .

*Proof.* The proof is straightforward. □

**Lemma 4.5.** *Let  $Q$  be a subgroup of  $N$ .*

- (1)  *$Q$  is a quasi-ideal of  $N$  if and only if  $f_Q$  is a fuzzy quasi-ideal of  $N$ .*
- (2)  *$Q$  is a bi-ideal of  $N$  if and only if  $f_Q$  is a fuzzy bi-ideal of  $N$ .*

*Proof.* (1) can be easily proved.

(2) Assume that  $Q$  is a bi-ideal of  $N$ . Then  $f_Q$  is a fuzzy subgroup of  $N$ . Also,  $f_Q f_N f_Q \cap f_Q f_N * f_Q \leq f_{QNQ \cap QN * Q} \leq f_Q$ . This means that  $f_Q$  is a fuzzy bi-ideal of  $N$ .

Conversely, let  $y$  be any element of  $QNQ \cap QN * Q$ . Then we have

$$f_Q(y) \geq (f_Q f_N f_Q \cap f_Q f_N * f_Q)(y) = f_{QNQ \cap QN * Q}(y) = 1$$

Thus  $y \in Q$  and this implies  $QNQ \cap QN * Q \subseteq Q$ , which, in turn, implies that  $Q$  is a bi-ideal of  $N$ . This completes the proof. □

**Lemma 4.6.** *Any fuzzy quasi-ideal of  $N$  is a fuzzy bi-ideal of  $N$ .*

*Proof.* Let  $\mu$  be any fuzzy quasi-ideal of  $N$ . Then we have

$$\begin{aligned} \mu \mathbf{N} \mu &\leq \mu(\mathbf{N} \mathbf{N}) \leq \mu \mathbf{N} \\ \mu \mathbf{N} \mu &\leq (\mathbf{N} \mathbf{N}) \mu \leq \mathbf{N} \mu \\ \mu \mathbf{N} * \mu &\leq (\mathbf{N} \mathbf{N}) * \mu \leq \mathbf{N} * \mu \end{aligned}$$

Hence

$$\mu \mathbf{N} \mu \cap \mu \mathbf{N} * \mu \leq \mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu \leq \mu$$

Thus  $\mu$  is a fuzzy bi-ideal of  $N$ . □

However, as shown by the following example, the converse of the Lemma 4.6 need not be true in general.

**Example 4.7.** Let  $N = \{0, a, b, c\}$  be the Klein's four group. Define multiplication in  $N$  as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

•	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Then  $(N, +, \bullet)$  is a near-ring ( see [ [16], P.408 ] scheme 15 ).

Define  $\mu : N \rightarrow [0, 1]$  by  $\mu(x) := \begin{cases} 1 & \text{if } x = 0, a. \\ 0 & \text{if otherwise.} \end{cases}$

For any  $t \in [0, 1]$ ,  $\mu_t = \{0, a\}$  or  $\{0, a, b, c\}$ . Since  $\{0, a\}$  and  $\{0, a, b, c\}$  are the bi-ideals in  $N$ ,  $\mu_t$  is a bi-ideal in  $N$  for all  $t$ . Hence  $\mu$  is a fuzzy bi-ideal of  $N$ .

$$\begin{aligned} \text{Now } (\mu \mathbf{N})(b) &= \sup_{b=xy} \{\min\{\mu(x), \mathbf{N}(y)\}\} \\ &= \sup \{\min\{\mu(a), \mathbf{N}(a)\}, \min\{\mu(c), \mathbf{N}(c)\}, \min\{\mu(a), \mathbf{N}(c)\}, \min\{\mu(c), \mathbf{N}(a)\}\}, \\ &\qquad \text{as } b = a.a = c.c = a.c = c.a. \end{aligned}$$

$$= \sup \{ \min\{1, 1\}, \min\{0, 1\}, \min\{1, 1\}, \min\{0, 1\} \} = \sup \{1, 0, 1, 0\} = 1.$$

Similarly, we have  $(\mathbf{N}\mu)(b) = 1$ . Thus

$$(\mu\mathbf{N} \cap \mathbf{N}\mu)(b) = \min\{(\mu\mathbf{N})(b), (\mathbf{N}\mu)(b)\} = \min\{1, 1\} = 1.$$

But  $\mu(b) = 0$ . Hence  $(\mu\mathbf{N} \cap \mathbf{N}\mu)(b) = 1 \not\leq \mu(b) = 0$ . Therefore  $\mu$  is not a fuzzy quasi-ideal of  $N$ .

**Lemma 4.8.** *Let  $\mu$  be a fuzzy subset of  $N$ . If  $\mu$  is a fuzzy left ideal (right ideal,  $N$ -subgroup, subnear-ring) of  $N$ , then  $\mu$  is a fuzzy bi-ideal of  $N$ .*

*Proof.* Since  $\mu$  is a left ideal of  $N$ ,  $\mu$  is a fuzzy quasi-ideal of  $N$  by Lemma 4.3. Hence  $\mu$  is a fuzzy bi-ideal of  $N$ , by Lemma 4.6.  $\square$

**Theorem 4.9.** *Let  $\mu$  be a fuzzy subset of  $N$ .  $\mu$  is a fuzzy quasi-ideal of  $N$  if and only if  $\mu_t$  is a quasi-ideal of  $N$ , for all  $t \in \text{Im}(\mu)$ .*

*Proof.* Let  $\mu$  be a fuzzy quasi-ideal of  $N$ . Let  $t \in \text{Im}(\mu)$ . Suppose  $x, y \in N$  such that  $x, y \in \mu_t$ . Then  $\mu(x) \geq t$  and  $\mu(y) \geq t$ . This implies that  $\min\{\mu(x), \mu(y)\} \geq t$ . Since  $\mu$  is a fuzzy quasi-ideal,  $\mu(x-y) \geq t$  and hence  $x-y \in \mu_t$ . Let  $x \in N$ . Suppose  $x \in \mu_t N \cap N\mu_t \cap N * \mu_t$ . Then there exist  $a, b, c \in \mu_t$  and  $n_1, n_2, n_3, n_4 \in N$  such that  $x = an_1 = n_2b = n_3(n_4 + c) - n_3n_4$ . Thus  $\mu(a) \geq t$ ,  $\mu(b) \geq t$  and  $\mu(c) \geq t$ . Then

$$\begin{aligned} (\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) &= \min\{(\mu\mathbf{N})(x), (\mathbf{N}\mu)(x), (\mathbf{N} * \mu)(x)\} \\ &= \min\left\{ \sup_{x=an_1} \{\mu(a)\}, \sup_{x=n_2b} \{\mu(b)\}, \sup_{x=n_3(n_4+c)-n_3n_4} \{\mu(c)\} \right\} \\ &\geq t. \end{aligned}$$

Since  $\mu$  is a fuzzy quasi-ideal of  $N$ ,  $\mu(x) \geq t$ . Thus  $x \in \mu_t$  and hence  $\mu_t$  is a quasi-ideal of  $N$ .

Conversely, let us assume that  $\mu_t, t \in \text{Im}(\mu)$ , is a quasi-ideal of  $N$ . Let  $x \in N$ . Consider

$$\begin{aligned} (\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) &= \min\{(\mu\mathbf{N})(x), (\mathbf{N}\mu)(x), (\mathbf{N} * \mu)(x)\} \\ &= \min\left\{ \sup_{x=ab} \{\min\{\mu(a), \mathbf{N}(b)\}\}, \sup_{x=ab} \{\min\{\mathbf{N}(a), \mu(b)\}\}, \right. \\ &\quad \left. \sup_{x=n_1(n_2+c)-n_1n_2} \{\min\{\mathbf{N}(n_1), \mu(c)\}\} \right\} \\ &= \min\left\{ \sup_{x=ab} \{\mu(a)\}, \sup_{x=ab} \{\mu(b)\}, \sup_{x=n_1(n_2+c)-n_1n_2} \{\mu(c)\} \right\}. \end{aligned}$$

Let  $\sup_{x=ab} \{\mu(a)\} = t_1$ ,  $\sup_{x=ab} \{\mu(b)\} = t_2$  and  $\sup_{x=n_1(n_2+c)-n_1n_2} \{\mu(c)\} = t_3$  for some  $a, b, n_1, n_2$  and  $c$  in  $N$ . Assume that  $\min\{t_1, t_2, t_3\} = t$ . Then  $a, b, c \in \mu_t$ . Since  $\mu_t$  is a quasi-ideal of  $N$ , then  $x = ab \in N\mu_t$ ,  $x = ab \in \mu_t N$  and  $x = n_1(n_2 + c) - n_1n_2 \in N * \mu_t$ . This implies  $x \in \mu_t N \cap N\mu_t \cap N * \mu_t \subseteq \mu_t$ . Thus  $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) \leq t \leq \mu(x)$ . Hence  $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) \leq \mu(x)$ , for all  $x \in N$ . This shows that  $\mu$  is a fuzzy quasi-ideal of  $N$ .  $\square$

**Theorem 4.10.** *Let  $\mu$  be a fuzzy subset of  $N$ .  $\mu$  is a fuzzy bi-ideal of  $N$  if and only if  $\mu_t$  is a bi-ideal of  $N$ , for all  $t \in \text{Im}(\mu)$ .*

*Proof.* Let  $\mu$  be a fuzzy bi-ideal of  $N$ . Let  $t \in Im(\mu)$ . Suppose  $x, y \in N$  such that  $x, y \in \mu_t$ . Then  $\mu(x) \geq t, \mu(y) \geq t$  and  $\min\{\mu(x), \mu(y)\} \geq t$ . Since  $\mu$  is a fuzzy bi-ideal,  $\mu(x - y) \geq t$  and thus  $x - y \in \mu_t$ .

Let  $z \in N$ . Suppose  $z \in \mu_t N \mu_t \cap \mu_t N * \mu_t$ . Then there exist  $x, y, a_1, a_2, b \in \mu_t$  and  $n_1, n_2, n_3 \in N$  such that  $z = xn_1y = a_1n_2(a_2n_3 + b) - a_1n_2a_2n_3$ . Then  $(\mu N \mu \cap \mu N * \mu)(z) = \min\{(\mu N \mu)(z), (\mu N * \mu)(z)\}$ .

Now  $(\mu N \mu)(z) = \sup_{z=xn_1y} \{\min\{\mu(x), \mu(y)\}\} \geq t$ .

$$(\mu N * \mu)(z) = \sup_{z=a_1n_2(a_2n_3+b)-a_1n_2a_2n_3} \{\min\{\mu(a_1), \mu(b)\}\} \geq t.$$

Therefore  $\min\{(\mu N \mu)(z), (\mu N * \mu)(z)\} \geq t$  and thus  $(\mu N \mu \cap \mu N * \mu)(z) \geq t$ . Since  $\mu$  is a fuzzy bi-ideal of  $N$ ,  $\mu(z) \geq t$  implies  $z \in \mu_t$ . Hence  $\mu_t$  is a bi-ideal in  $N$ .

Conversely, let us assume that  $\mu_t$  is a bi-ideal of  $N$ ,  $t \in Im(\mu)$ . Let  $p \in N$ . Consider

$$\begin{aligned} (\mu N \mu \cap \mu N * \mu)(p) &= \min\{(\mu N \mu)(p), (\mu N * \mu)(p)\} \\ &= \min\left\{ \sup_{p=xny} \{\min\{\mu(x), \mu(y)\}\}, \right. \\ &\quad \left. \sup_{p=a_1n_1(b+c)-a_1n_1b} \{\min\{\mu(a_1), \mu(c)\}\} \right\} \\ &= \sup_{p=xny=a_1n_1(b+c)-a_1n_1b} \{\min\{\mu(x), \mu(y), \mu(a_1), \mu(c)\}\}. \end{aligned}$$

Let  $\mu(x) = t_1 < \mu(y) = t_2 < \mu(a_1) = t_3 < \mu(c) = t_4$ . Then  $\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3} \supseteq \mu_{t_4}$ . Thus  $x, y, a_1, c \in \mu_{t_1}$  and  $p = xny \in \mu_{t_1} N \mu_{t_1}$  and  $p = a_1n_1(b + c) - a_1n_1b \in \mu_{t_1} N * \mu_{t_1}$ . Thus  $p \in \mu_{t_1} N \mu_{t_1} \cap \mu_{t_1} N * \mu_{t_1} \subseteq \mu_{t_1}$ . This implies that  $\mu(p) \geq t_1$  and hence  $\mu N \mu \cap \mu N * \mu \leq \mu$ . Therefore  $\mu$  is a fuzzy bi-ideal of  $N$ .  $\square$

**Proposition 4.11.** *Let  $\lambda$  and  $\mu$  be any two fuzzy quasi-ideals of  $N$ . Then  $\lambda \cap \mu$  is also a fuzzy quasi-ideal of  $N$ .*

*Proof.* The proof is straightforward.  $\square$

**Proposition 4.12.** *Let  $\lambda$  and  $\mu$  be any two fuzzy bi-ideals of  $N$ . Then  $\lambda \cap \mu$  is also a fuzzy bi-ideal of  $N$ .*

*Proof.* The proof is straightforward.  $\square$

**Corollary 4.13.** *If  $\mu$  is a fuzzy bi-ideal of  $N$  and  $\lambda$  is a fuzzy subnear-ring of  $N$ , then  $\mu \cap \lambda$  is a fuzzy bi-ideal of  $N$ .*

*Proof.* The result follows from Lemma 4.8 and Proposition 4.12.  $\square$

### 5. $(\in, \in \vee q_k)$ -fuzzy Quasi-ideal and $(\in, \in \vee q_k)$ -fuzzy Bi-ideal

In this section, we introduce the notions of  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals and  $(\in, \in \vee q_k)$ -fuzzy bi-ideals of  $N$  which, respectively, are generalizations of fuzzy quasi-ideals and bi-ideals of  $N$ .

**Definition 5.1.** An  $(\in, \in \vee q)$ -fuzzy subgroup  $\mu$  of  $N$  is called an  $(\in, \in \vee q)$ -fuzzy quasi-ideal of  $N$  if for all  $x \in N$ ,  $\mu(x) \geq \min\{((\mu N) \cap (N \mu) \cap (N * \mu))(x), 0.5\}$ .

An  $(\in, \in \vee q)$ -fuzzy subgroup  $\mu$  of  $N$  is called an  $(\in, \in \vee q)$ -fuzzy bi-ideal of  $N$  if for all  $x \in N$ ,  $\mu(x) \geq \min\{((\mu \mathbf{N} \mu) \cap (\mu \mathbf{N} * \mu))(x), 0.5\}$ .

**Definition 5.2.** An  $(\in, \in \vee q_k)$ -fuzzy subgroup  $\mu$  of  $N$  is called an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$  if for all  $x \in N$ ,  $\mu(x) \geq \min\{((\mu \mathbf{N}) \cap (\mathbf{N} \mu) \cap (\mathbf{N} * \mu))(x), \frac{1-k}{2}\}$ , where  $k \in [0, 1)$ .

An  $(\in, \in \vee q_k)$ -fuzzy subgroup  $\mu$  of  $N$  is called an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$  if for all  $x \in N$ ,  $\mu(x) \geq \min\{((\mu \mathbf{N} \mu) \cap (\mu \mathbf{N} * \mu))(x), \frac{1-k}{2}\}$ , where  $k \in [0, 1)$ .

**Remark 5.3.** Every fuzzy quasi-ideal and  $(\in, \in \vee q)$ -fuzzy quasi-ideal of  $N$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ . Also every fuzzy bi-ideal and  $(\in, \in \vee q)$ -fuzzy bi-ideal of  $N$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ . However, as the following example shows, the converse is not necessarily true.

**Example 5.4.** Let  $N = \{0, 1, 2, 3\}$  be the group under addition modulo 4. Define multiplication as follows:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

•	0	1	2	3
0	0	0	0	0
1	0	3	0	1
2	0	2	0	2
3	0	1	0	3

Then  $(N, +, \bullet)$  is a near-ring ( see [ 16], P.407 ] scheme 7 ). Let  $\mu : N \rightarrow [0, 1]$  be a fuzzy subset of  $N$  such that  $\mu(0) = 0.4$ ,  $\mu(1) = \mu(3) = 0.3$ ,  $\mu(2) = 0.42$ . Then  $\mu$  is an  $(\in, \in \vee q_{0.2})$ -fuzzy quasi-ideal of  $N$ . Since  $\mu(0) = 0.4 \not\geq \min\{(\mu \mathbf{N})(0), (\mathbf{N} \mu)(0), (\mathbf{N} * \mu)(0)\} = 0.42$  and  $\mu(0) = 0.4 \not\geq \min\{(\mu \mathbf{N})(0), (\mathbf{N} \mu)(0), (\mathbf{N} * \mu)(0), 0.5\} = 0.42$ ,  $\mu$  is neither a fuzzy quasi-ideal of  $N$  nor an  $(\in, \in \vee q)$ -fuzzy quasi-ideal of  $N$ . Also  $\mu$  is an  $(\in, \in \vee q_{0.2})$ -fuzzy bi-ideal of  $N$ . Since  $\mu(0) \not\geq \min\{(\mu \mathbf{N} \mu)(0), (\mu \mathbf{N} * \mu)(0)\}$  and  $\mu(0) \not\geq \min\{(\mu \mathbf{N} \mu)(0), (\mu \mathbf{N} * \mu)(0), 0.5\}$ ,  $\mu$  is neither a fuzzy bi-ideal of  $N$  nor an  $(\in, \in \vee q)$ -fuzzy bi-ideal of  $N$ .

**Lemma 5.5.** Let  $Q$  be any nonempty subset of  $N$ . Then

- (1)  $Q$  is a quasi-ideal of  $N$  if and only if  $f_Q$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .
- (2)  $Q$  is a bi-ideal of  $N$  if and only if  $f_Q$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .

*Proof.* (1) Let  $Q$  be a quasi-ideal of  $N$ . By Lemma 4.5,  $f_Q$  is a fuzzy quasi-ideal of  $N$  and by Remark 5.3,  $f_Q$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .

Conversely, let  $f_Q$  be an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ . Let  $x$  be any element of  $Q \mathbf{N} \cap \mathbf{N} \mathbf{Q} \cap \mathbf{N} * \mathbf{Q}$ . Then we have

$$\begin{aligned} f_Q(x) &\geq \min\{(f_Q f_N \cap f_N f_Q \cap f_N * f_Q)(x), \frac{1-k}{2}\} \\ &= \min\{f_{(Q \mathbf{N} \cap \mathbf{N} \mathbf{Q} \cap \mathbf{N} * \mathbf{Q})}(x), \frac{1-k}{2}\} \\ &= \min\{1, \frac{1-k}{2}\} \\ &= \frac{1-k}{2}. \end{aligned}$$

This implies that  $x \in Q$  and so  $Q \mathbf{N} \cap \mathbf{N} \mathbf{Q} \cap \mathbf{N} * \mathbf{Q} \subseteq Q$ . This means that  $Q$  is a quasi-ideal of  $N$ .

(2) Let  $Q$  be a bi-ideal of  $N$ . By Lemma 4.5,  $f_Q$  is a fuzzy bi-ideal of  $N$  and by Remark 5.3,  $f_Q$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .

Conversely, let  $f_Q$  be an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ . Let  $y$  be an element of  $Q N Q \cap Q N * Q$ . Then we have

$$\begin{aligned} f_Q(y) &\geq \min\{(f_Q f_N f_Q \cap f_Q f_N * f_Q)(y), \frac{1-k}{2}\} \\ &= \min\{f_{(Q N Q \cap Q N * Q)}(y), \frac{1-k}{2}\} \\ &= \min\{1, \frac{1-k}{2}\} \\ &= \frac{1-k}{2}. \end{aligned}$$

This implies that  $y \in Q$  and so  $Q N Q \cap Q N * Q \subseteq Q$ . This means that  $Q$  is a bi-ideal of  $N$ .  $\square$

**Corollary 5.6.** [[15], Theorem 4.5.] *A nonempty subset  $Q$  of  $N$  is a quasi-ideal of  $N$  if and only if  $f_Q$  is an  $(\in, \in \vee q)$ -fuzzy quasi-ideal of  $N$ .*

**Lemma 5.7.** *Any  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .*

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ . Then we have

$$\begin{aligned} \mu \mathbf{N} \mu &\leq \mu(\mathbf{N} \mathbf{N}) \leq \mu \mathbf{N} \\ \mu \mathbf{N} \mu &\leq (\mathbf{N} \mathbf{N}) \mu \leq \mathbf{N} \mu \\ \mu \mathbf{N} * \mu &\leq (\mathbf{N} \mathbf{N}) * \mu \leq \mathbf{N} * \mu. \end{aligned}$$

Hence  $\mu \mathbf{N} \mu \cap \mu \mathbf{N} * \mu \leq \mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu \leq \mu$ .

Let  $x \in N$ . Now

$$\min\{(\mu \mathbf{N} \mu \cap \mu \mathbf{N} * \mu)(x), \frac{1-k}{2}\} \leq \min\{(\mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu)(x), \frac{1-k}{2}\} \leq \mu(x).$$

It follows that  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .  $\square$

However, as the following example shows, the converse of the Lemma 5.7 is not necessarily true.

**Example 5.8.** Consider the near-ring  $(N, +, \bullet)$  as defined in Example 4.7.

Define a fuzzy subset  $\mu : N \rightarrow [0, 1]$  by:

$$\mu(x) := \begin{cases} 0.3 & \text{if } x = 0, a \\ 0.2 & \text{if otherwise.} \end{cases}$$

Let  $k = 0.2$ . For all  $t \in (0, \frac{1-k}{2}]$ ,  $\mu_t$  is the bi-ideal of  $N$ . Hence  $\mu$  is an  $(\in, \in \vee q_{0.2})$ -fuzzy bi-ideal of  $N$ . For  $t = 0.24$ ,  $\mu_t = \{0, a\}$  and  $N \mu_t \cap \mu_t N \cap N * \mu_t = \{0, b\} \not\subseteq \{0, a\}$ . Thus  $\mu_t$  is not a quasi-ideal of  $N$ . Hence  $\mu$  is not an  $(\in, \in \vee q_{0.2})$ -fuzzy quasi-ideal of  $N$ .

**Theorem 5.9.** *Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy subset of  $N$ . If  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy left ideal (right ideal,  $N$ -subgroup, subnear-ring) of  $N$ , then  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .*

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy left ideal of  $N$ . Let  $x \in N$ . Suppose  $x = ab = n_1(n_2 + c) - n_1 n_2$ , where  $a, b, n_1, n_2$  and  $c$  are in  $N$ . We have

$$\begin{aligned} (\mu \mathbf{N} \cap \mathbf{N} \mu \cap \mathbf{N} * \mu)(x) &= \min\{(\mu \mathbf{N})(x), (\mathbf{N} \mu)(x), (\mathbf{N} * \mu)(x)\} \\ &= \min\{\sup_{x=ab} \mu(a), \sup_{x=ab} \mu(b), \sup_{x=n_1(n_2+c)-n_1 n_2} \mu(c)\}. \end{aligned}$$

Now

$$\begin{aligned} \min\{(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x), \frac{1-k}{2}\} &= \min\{\min\{\sup_{x=ab} \mu(a), \sup_{x=ab} \mu(b), \\ &\quad \sup_{x=n_1(n_2+c)-n_1n_2} \mu(c)\}, \frac{1-k}{2}\} \\ &\leq \min\{\min\{1, 1, \sup_{x=n_1(n_2+c)-n_1n_2} \mu(c)\}, \frac{1-k}{2}\} \\ &= \min\{\sup_{x=n_1(n_2+c)-n_1n_2} \mu(c), \frac{1-k}{2}\}, \end{aligned}$$

[since  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy left ideal,  $\mu(n_1(n_2+c) - n_1n_2) \geq \min\{\mu(c), \frac{1-k}{2}\}$ ]  
 $\leq \mu(n_1(n_2+c) - n_1n_2) = \mu(x)$ .

We remark that if  $x$  is not expressed as  $x = ab = n_1(n_2+c) - n_1n_2$ , then  $(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) = 0 \leq \mu(x)$  and  $\min\{(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x), \frac{1-k}{2}\} = 0 \leq \mu(x)$ . Thus  $\mu(x) \geq \min\{(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x), \frac{1-k}{2}\}$ , for all  $x \in N$ . Hence  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .  $\square$

**Corollary 5.10.** [[15], Theorem 4.9, Theorem 4.10, Theorem 4.9] *Every  $(\in, \in \vee q)$ -fuzzy left ideal (right ideal, two-sided ideal) of  $N$  is an  $(\in, \in \vee q)$ -fuzzy quasi-ideal of  $N$ .*

**Theorem 5.11.** *Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy subset of  $N$ . If  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy left ideal (right ideal,  $N$ -subgroup, subnear-ring) of  $N$ , then  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .*

*Proof.* By Theorem 5.9, Every  $(\in, \in \vee q_k)$ -fuzzy left ideal of  $N$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$  and by Lemma 5.7, it is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .  $\square$

**Theorem 5.12.** *Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy subset of  $N$ .  $\mu$  is a  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$  if and only if  $\mu_t$  is a quasi-ideal of  $N$ , for all  $t \in (0, \frac{1-k}{2}]$ ,  $k \in [0, 1)$ .*

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ . Let  $x, y \in N$ . Suppose  $x, y \in \mu_t$ ,  $t \in (0, \frac{1-k}{2}]$ ,  $k \in [0, 1)$ . Then  $\mu(x) \geq t$  and  $\mu(y) \geq t$ . This implies that  $\min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq t$ . Since  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal,  $\mu(x-y) \geq t$  and hence  $x-y \in \mu_t$ . Next, let  $x \in \mu_t N \cap N \mu_t \cap N * \mu_t$ . Then there exist  $a, b, c \in \mu_t$  and  $n_1, n_2, n_3, n_4 \in N$  such that  $x = an_1 = n_2b = n_3(n_4+c) - n_3n_4$ . Thus  $\mu(a) \geq t$ ,  $\mu(b) \geq t$  and  $\mu(c) \geq t$ . Then

$$\begin{aligned} (\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x) &= \min\{(\mu\mathbf{N})(x), (\mathbf{N}\mu)(x), (\mathbf{N} * \mu)(x)\} \\ &= \min\{\sup_{x=an_1} \mu(a), \sup_{x=n_2b} \mu(b), \sup_{x=n_3(n_4+c)-n_3n_4} \mu(c)\} \\ &\geq t. \end{aligned}$$

Now

$$\min\{(\mu\mathbf{N} \cap \mathbf{N}\mu \cap \mathbf{N} * \mu)(x), \frac{1-k}{2}\} = \min \left[ \min \left\{ \sup_{x=an_1} \mu(a), \sup_{x=n_2b} \mu(b), \sup_{x=n_3(n_4+c)-n_3n_4} \mu(c) \right\}, \frac{1-k}{2} \right]$$

$$\begin{aligned} &\geq \min\{t, \frac{1-k}{2}\} \\ &= t. \end{aligned}$$

Since  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ ,  $\mu(x) \geq t$ . Thus  $x \in \mu_t$  and hence  $\mu_t$  is a quasi-ideal of  $N$ .

Conversely, let us assume that  $\mu_t$ ,  $t \in (0, \frac{1-k}{2}]$ ,  $k \in [0, 1)$ , is a quasi-ideal of  $N$ . By Theorem 4.9,  $\mu$  is a fuzzy quasi-ideal of  $N$ . By Remark 5.3,  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .  $\square$

**Remark 5.13.** Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ . Then the level subset  $\mu_t$  is not necessarily a quasi-ideal in  $N$ . In Example 5.4,  $\mu$  is an  $(\in, \in \vee q_{0.2})$ -fuzzy quasi-ideal of  $N$ . If we take  $t = 0.42$ , then  $\mu_t = \{2\}$  and  $\mu_t N \cap N \mu_t \cap N * \mu_t = \{0\} \not\subseteq \{2\} = \mu_t$ . Hence  $\mu_t$  is not a quasi-ideal in  $N$ , because  $t \notin (0, 0.4]$ .

**Theorem 5.14.** Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy subset of  $N$ .  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$  if and only if  $\mu_t$  is a bi-ideal in  $N$ , for all  $t \in (0, \frac{1-k}{2}]$ ,  $k \in [0, 1)$ .

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ . Let  $t \in (0, \frac{1-k}{2}]$ ,  $k \in [0, 1)$ . Suppose  $x, y \in N$  such that  $x, y \in \mu_t$ . Then  $\mu(x) \geq t$ ;  $\mu(y) \geq t$  and  $\min\{\mu(x), \mu(y), \frac{1-k}{2}\} \geq t$ . Since  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal,  $\mu(x-y) \geq t$  and hence  $x-y \in \mu_t$ . Let  $z \in N$ . Suppose  $x \in \mu_t N \mu_t \cap \mu_t N * \mu_t$ . Then there exist  $x, y, a_1, a_2, b \in \mu_t$  and  $n_1, n_2, n_3 \in N$  such that  $z = xn_1y = a_1n_2(a_2n_3 + b) - a_1n_2a_2n_3$ . Thus  $\mu(x) \geq t$ ,  $\mu(y) \geq t$ ,  $\mu(a_1) \geq t$ ,  $\mu(a_2) \geq t$  and  $\mu(b) \geq t$ . Now

$$\begin{aligned} (\mu \mathbf{N} \mu \cap \mu \mathbf{N} * \mu)(z) &= \min\{(\mu \mathbf{N} \mu)(z), (\mu \mathbf{N} * \mu)(z)\} \\ &= \min \left\{ \sup_{z=xn_1y} \{\min\{\mu(x), \mu(y)\}\}, \right. \\ &\quad \left. \sup_{z=a_1n_2(a_2n_3+b)-a_1n_2a_2n_3} \{\min\{\mu(a_1), \mu(b)\}\} \right\} \\ &\geq t. \end{aligned}$$

We have

$$\begin{aligned} \min\{(\mu \mathbf{N} \mu \cap \mu \mathbf{N} * \mu)(x), \frac{1-k}{2}\} &= \min \left[ \min \left\{ \sup_{z=xn_1y} \{\min\{\mu(x), \mu(y)\}\}, \right. \right. \\ &\quad \left. \left. \sup_{z=a_1n_2(a_2n_3+b)-a_1n_2a_2n_3} \{\min\{\mu(a_1), \mu(b)\}\} \right\}, \frac{1-k}{2} \right] \\ &\geq \min\{t, \frac{1-k}{2}\} \\ &= t. \end{aligned}$$

Since  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ ,  $\mu(z) \geq t$ . Thus  $z \in \mu_t$  and hence  $\mu_t$  is a bi-ideal of  $N$ .

Conversely, let us assume that  $\mu_t$ ,  $t \in (0, \frac{1-k}{2}]$ ,  $k \in [0, 1)$ , is a bi-ideal of  $N$ . By Theorem 4.10,  $\mu$  is a fuzzy bi-ideal of  $N$ . By Remark 5.3,  $\mu$  is an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .  $\square$

**Remark 5.15.** Let  $\mu$  be an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ . Then the level subset  $\mu_t$  is not necessarily a bi-ideal in  $N$ . In Example 5.4,  $\mu$  is an  $(\in, \in \vee q_{0.2})$ -fuzzy bi-ideal of  $N$ . If we take  $t = 0.42$ , then  $\mu_t = \{2\}$  and  $\mu_t N \mu_t \cap \mu_t N * \mu_t = \{0\} \not\subseteq \{2\} = \mu_t$ . Hence  $\mu_t$  is not a bi-ideal in  $N$ , because  $t \notin (0, 0.4]$ .

**Theorem 5.16.** *Let  $\lambda$  and  $\mu$  be any two  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals of  $N$ . Then  $\lambda \cap \mu$  is also an  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .*

*Proof.* Let  $\lambda$  and  $\mu$  be any two  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals of  $N$ . Then, by a proof similar to that of Theorem 3.16, we can show that  $(\lambda \cap \mu)$  is an  $(\in, \in \vee q_k)$ -fuzzy subgroup of  $N$ . Let  $x \in N$ . Choose  $a, b, a_1, b_1, c \in N$  such that  $x = ab = a_1(b_1 + c) - a_1b_1$ . Since  $\lambda$  and  $\mu$  are the  $(\in, \in \vee q_k)$ -fuzzy quasi-ideals of  $N$ , we have

$$(1) \quad \min \left\{ \min \left[ \sup_{x=ab} \{\mu(a)\}, \sup_{x=ab} \{\mu(b)\}, \sup_{x=a_1(b_1+c)-a_1b_1} \{\mu(c)\} \right], \frac{1-k}{2} \right\} \leq \mu(x)$$

and

$$(2) \quad \min \left\{ \min \left[ \sup_{x=ab} \{\lambda(a)\}, \sup_{x=ab} \{\lambda(b)\}, \sup_{x=a_1(b_1+c)-a_1b_1} \{\lambda(c)\} \right], \frac{1-k}{2} \right\} \leq \lambda(x).$$

Now

$$\begin{aligned} & \min \left[ \left( (\lambda \cap \mu) \mathbf{N} \cap \mathbf{N} (\lambda \cap \mu) \cap \mathbf{N} * (\lambda \cap \mu) \right)(x), \frac{1-k}{2} \right] \\ &= \min \left[ \min \left\{ \sup_{x=ab} \{(\lambda \cap \mu)(a)\}, \sup_{x=ab} \{(\lambda \cap \mu)(b)\}, \sup_{x=a_1(b_1+c)-a_1b_1} \{(\lambda \cap \mu)(c)\} \right\}, \frac{1-k}{2} \right] \\ &= \min \left[ \min \left\{ \sup_{x=ab} \{\min\{\lambda(a), \mu(a)\}\}, \sup_{x=ab} \{\min\{\lambda(b), \mu(b)\}\}, \right. \right. \end{aligned}$$

$$\left. \sup_{x=a_1(b_1+c)-a_1b_1} \{\min\{\lambda(c), \mu(c)\}\} \right], \frac{1-k}{2} \right]$$

$$\leq \min \left[ \min \left\{ \min \left[ \sup_{x=ab} \{\lambda(a)\}, \sup_{x=ab} \{\lambda(b)\}, \sup_{x=a_1(b_1+c)-a_1b_1} \{\lambda(c)\} \right], \frac{1-k}{2} \right\}, \right.$$

$$\left. \min \left\{ \min \left[ \sup_{x=ab} \{\mu(a)\}, \sup_{x=ab} \{\mu(b)\}, \sup_{x=a_1(b_1+c)-a_1b_1} \{\mu(c)\} \right], \frac{1-k}{2} \right\} \right]$$

$$\leq \min\{\lambda(x), \mu(x)\}, \text{ from (1) and (2).}$$

$$= (\lambda \cap \mu)(x).$$

Thus  $\lambda \cap \mu$  is a  $(\in, \in \vee q_k)$ -fuzzy quasi-ideal of  $N$ .  $\square$

**Theorem 5.17.** *Let  $\lambda$  and  $\mu$  be any two  $(\in, \in \vee q_k)$ -fuzzy bi-ideals of  $N$ . Then  $\lambda \cap \mu$  is also an  $(\in, \in \vee q_k)$ -fuzzy bi-ideal of  $N$ .*

*Proof.* The proof is similar to that of Theorem 5.16.  $\square$

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