STABILITY OF THE JENSEN'S FUNCTIONAL EQUATION IN MULTI-FUZZY NORMED SPACES

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Abstract. In this paper, we define the notion of (dual) multi-fuzzy normed spaces and describe some properties of them. We then investigate Ulam-Hyers stability of Jensen's functional equation for mappings from linear spaces into multi-fuzzy normed spaces. We establish an asymptotic behavior of the Jensen equation in the framework of multi-fuzzy normed spaces.

1. **Introduction**

The notion of multi-normed spaces was initiated by H. G. Dales and M. E. Polyakov [6]. This concept is somewhat similar to the operator sequence space and has some connections with operator spaces and Banach lattices. An incentive for the study of multi-normed spaces and many examples are given in [6]. Some results of multi-normed spaces are stable under fuzzy normed spaces [3, 4]. In this paper, using some ideas from [6, 18] we first introduce the concept of (dual) multi-fuzzy normed spaces and then we investigate the Ulam-Hyers stability of Jensen's functional equation for mappings from linear spaces into multi-fuzzy normed spaces.

In 1984, A. K. Katsaras [11] defined the notion of fuzzy norm on a linear space to construct a fuzzy vector topological structure. Thereafter, some mathematicians introduced and discussed several notions of fuzzy norms from various points of view [7, 13, 24]. In particular, in 2003 T. Bag and S. K. Samanta [3], following Cheng and Mordeson [5] gave an idea of a fuzzy norm in such a way that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They established a decomposition theorem of a fuzzy norm into a family of "crisp norms" and also described some nice properties of the fuzzy norm in [4].

In 1940, a question that was given by S. M. Ulam [22] concerning the stability of group homomorphisms gave rise to the stability problem of functional equations. D. H. Hyers [9] was the first to come out with a partial affirmative answer to solve the question posed by Ulam on Banach spaces. Hyers's theorem was extended by T. Aoki [2] for additive mappings in 1950 and by Th. M. Rassias [20] for linear mappings by taking into consideration an unbounded Cauchy difference in 1978. Rassias's paper [20] has significantly influenced in the expansion of what we now call Ulam-Hyers-Rassias stability of functional equations. Recently, several fuzzy stability problems of various functional equations and in particular, Jensen

Received: March 2015; Revised: September 2016; Accepted: November 2016

Key words and phrases: Fuzzy normed space, Ulam-Hyers stability, Jensen's functional equation, Multi-normed space.

equation and its generalizations have been studied by some mathematicians; we refer the interested reader to [1, 8, 10, 14, 15, 17, 18, 21].

2. Preliminaries

Let $(E, \|.\|)$ be a complex normed space and let $k \in \mathbb{N}$. We denote by E^k, the linear space $E \oplus \ldots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, \ldots, k\}$ and denote by $\mathfrak{S}_k$ the group of permutations on k symbols. For $\sigma \in \mathfrak{S}_k$, $x = (x_1, \ldots, x_k) \in E^k$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$ define $A_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ and $M_\alpha(x) = (\alpha_1 x_1, \ldots, \alpha_k x_k)$. Let $n \in \mathbb{N}$, we set $x^{[n]} = (x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k) \in E^{nk}$, where $x^{[n]}$ consists of n copies of each block $(x_1, \ldots, x_k)$.
Take $k \in \mathbb{N}$ and let S be a subset of $\mathbb{N}_k$. For $(x_1, \ldots, x_k) \in E^k$, we set $Q_S(x_1, \ldots, x_k) = (y_1, \ldots, y_k)$, where $y_i = x_i$ $(i \notin S)$ and $y_i = 0$ $(i \in S)$. Thus Q_S is the projection onto the complement of S. Indeed, we observe that $Q_S^2 = Q_S^* = Q_S$, where $Q_S^*(x_1, \ldots, x_k) = (\overline{y_1}, \ldots, \overline{y_k})$ (we mean by $\overline{z}$, the complex conjugate of a complex number z).

Definition 2.1. [6] Let $(E, \|.\|)$ be a complex (real) normed space. A multi-norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence $\{\|.\|_k\}_{k\in\mathbb{N}}$ of norms on E^k $(k \in \mathbb{N})$ such that $\|x\|_1 = \|x\|$, for each $x \in E$ and the following axioms $(A1)$-$(A4)$ are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:
$(A1)$ for each $\sigma \in \mathfrak{S}_k$ and $x \in E^k$ we have

$$\|A_\sigma(x)\|_k = \|x\|_k;$$

$(A2)$ for each $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ $(\mathbb{R})$ and $x \in E^k$ we have

$$\|M_\alpha(x)\|_k \leq (\max_{1\leq i\leq k} |\alpha_i|)\|x\|_k;$$

$(A3)$ for each $x_1, \ldots, x_{k-1} \in E$ we have

$$\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1};$$

$(A4)$ for each $x_1, \ldots, x_{k-1} \in E$ we have

$$\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}.$$

In this case, we say that $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is a multi-normed space. Now if the axiom $(A4)$ is replaced by the following axiom, then $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is called a dual multi-normed space.
$(B4)$ for each $x_1, \ldots, x_{k-1} \in E$, $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, 2x_{k-1})\|_{k-1}$.

Remark 2.2. [6] Suppose that $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is a (dual) multi-normed space, and take $k \in \mathbb{N}$. The following property is an almost immediate consequence of the axioms $(A1), (A2)$ and $(A3)$.

$$\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\| \ (x_1, \ldots, x_k \in E). \tag{1}$$

Applying (1) one concludes that if $(E, \|.\|)$ is a Banach space, then $(E^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case, $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ is called a (dual) multi-Banach space.

Definition 2.3. [4] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is a fuzzy norm on X if for all $x, y \in X$ and $s, t \in \mathbb{R}$, the following conditions hold:
$(N1)$ $N(x, t) = 0$ for $t \leq 0$;
$(N2)$ $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
$(N3)$ $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
$(N4)$ $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
$(N5)$ $N(x, .)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
$(N6)$ for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed space.

Definition 2.4. Let (X, N) be a fuzzy-normed space. Then
(i) A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$ and we write $N\text{-}\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is called Cauchy if for each $0 < \varepsilon < 1$ and each $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_m - x_n, \delta) > 1 - \varepsilon$ for all $n, m \geq n_0$.
The fuzzy-normed space (X, N) is complete if every Cauchy sequence in X converges in X. In this case fuzzy normed space is called a fuzzy Banach space.

For more details on this issue see [8, 17, 23] and the bibliography quoted there.

3. Multi-fuzzy Normed Spaces

We begin this section with the notion of (dual) multi-fuzzy normed space.

Definition 3.1. Let (E, N) be a fuzzy normed space. A multi-fuzzy norm on $\{E^k, k \in \mathbb{N}\}$ is a sequence $\{N_k\}$ such that N_k is a fuzzy norm on E^k $(k \in \mathbb{N})$, $N_1(x, t) = N(x, t)$ for each $x \in E$ and $t \in \mathbb{R}$ and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:
$(MF1)$ for each $\sigma \in \mathfrak{S}_k$, $x \in E^k$ and $t \in \mathbb{R}$,

$$N_k(A_\sigma(x), t) = N_k(x, t);$$

$(MF2)$ for each $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $x \in E^k$ and $t \in \mathbb{R}$,

$$N_k(M_\alpha(x), t) \geq N_k(\max_{i\in\mathbb{N}_k}|\alpha_i|x, t);$$

$(MF3)$ for each $x_1, \ldots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}((x_1, \ldots, x_k, 0), t) = N_k((x_1, \ldots, x_k), t);$$

$(MF4)$ for each $x_1, \ldots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}((x_1, \ldots, x_k, x_k), t) = N_k((x_1, \ldots, x_k), t).$$

In such a case $\{(E^k, N_k), k \in \mathbb{N}\}$ is called a multi-fuzzy normed space. Moreover, if axiom $(MF4)$ is replaced by the following axiom, then $\{N_k\}$ is called a dual

multi-fuzzy norm and $\{(E^k, N_k), k \in \mathbb{N}\}$ is called a dual multi-fuzzy normed space.
$(DF4)$ for each $x_1, \dots, x_k \in E$ and $t \in \mathbb{R}$,

$$N_{k+1}((x_1, \dots, x_k, x_k), t) = N_k((x_1, \dots, 2x_k), t).$$

We present the definition just in the case where the index set is $\mathbb{N}$, however there is also an obvious definition of (dual) multi-fuzzy normed space of level n ($n \in \mathbb{N}$), that is the index set is $\mathbb{N}_n$.

The following examples guarantee the existence of notable source of examples of (dual) multi-fuzzy normed spaces.

Example 3.2. Let (E, N) be a fuzzy normed space. For each $k \in \mathbb{N}$, set

$$N_k((x_1, \dots, x_k), t) = \min\{N(x_i, t), i = 1, ..., k\}, \quad (x_1, \dots, x_k \in E \text{ and } t \in \mathbb{R}).$$

It is easily verified that, $\{(E^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy normed space.

Example 3.3. Let $\{(E^k, N_k^\alpha), k \in \mathbb{N}\}_\alpha$ be a family of (dual) multi-fuzzy normed spaces. For each $k \in \mathbb{N}$, $x_1, \dots, x_k \in E$ and $t \in \mathbb{R}$ define

$$N_k((x_1, \dots, x_k), t) = \inf_\alpha N_k^\alpha((x_1, \dots, x_k), t).$$

Then $\{(E^k, N_k), k \in \mathbb{N}\}$ is a (dual) multi-fuzzy normed space, too.

Example 3.4. Let $\{(E^k, \|.\|_k), k \in \mathbb{N}\}$ be a (dual) multi-normed space. For each $x \in E^k$, $t \in \mathbb{R}$ and $\alpha, \beta \geq 0$, define N_k^1, N_k^2 and N_k^3 by setting

$$N_k^1(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|_k}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

$$N_k^2(x,t) = \begin{cases} 0, & t \leq \|x\|_k, \\ 1, & t > \|x\|_k. \end{cases}$$

$$N_k^3(x,t) = \begin{cases} \frac{t^2 - \|x\|_k^2}{t^2 + \|x\|_k^2}, & t > \|x\|_k, \\ 0, & t \leq \|x\|_k. \end{cases}$$

It is rutin to check that N_k^1, N_k^2 and N_k^3 are (dual) multi-fuzzy normed spaces (see also [8, 19]).

According to Example 3.4 and also using some examples in [6], one can give examples that show axioms for multi- and dual multi-fuzzy normed spaces are independent of each other.

• *Independece of the axiom* $(MF1)$

(1) Assume that $(E, \|.\|)$ is a normed space.

(I) For each $k \in \mathbb{N}$, set $\|(x_1, \dots, x_k)\|_k = \max\{\|x_1\|, \frac{\|x_2\|}{2}, \dots, \frac{\|x_k\|}{k}\}$, where $x_1, \dots, x_k$ are in E. Let $x \in E$ with $\|x\| = 1$, then for some $t \in \mathbb{R}^+$, $N_2^i((2x, 3x), t) \neq N_2^i((3x, 2x), t)$ $(i = 1, 2, 3)$. Clearly $\{(E^k, N_k^i), k \in \mathbb{N}\}$ $(i = 1, 2, 3)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF2)$, $(MF3)$ and $(MF4)$, however it does not satisfy axiom $(MF1)$.

(II) Define $\|x\|_1 = \|x\|$ $(x \in E)$ and $\|(x_1, x_2)\|_2 = \max\{\|x_1\|, 2\|x_2\|\}$, where $x_1, x_2 \in E$. Let $x \in E$ with $\|x\| = 1$, then for some $t \in \mathbb{R}^+$, $N_2^i((2x, 3x), t) \neq N_2^i((3x, 2x), t)$ $(i = 1, 2, 3)$. It is relatively easy to see that $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$

$(i = 1, 2, 3)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF2)$, $(MF3)$ and $(DF4)$, however it does not satisfy axiom $(MF1)$.

• *Independece of the axiom* $(MF2)$

(2) Let $E = \mathbb{R}$ and $\|x\|_1 = |x|$ for each $x \in E$.

(III) Define $\|(x_1, \ldots, x_k)\|_k = \max\{\max_{i\in\mathbb{N}_k}|x_i|, \max_{i,j\in\mathbb{N}_k}|x_i - x_j|\}$, where $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E$ ([6, Example 2.4]). Then $\{(E^k, N_k^i), k \in \mathbb{N}\}$ $(i = 1, 2)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF1)$, $(MF3)$ and $(MF4)$ but $(MF2)$ fails.

(IV) Define $\|(x, y)\|_2 = \frac{1}{2}(|x + y| + |x| + |y|)$, where $x, y \in E$. It is immediately checked that $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$ $(i = 1, 2)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF1)$, $(MF3)$ and $(DF4)$ but $(MF2)$ fails.

• *Independece of the axiom* $(MF3)$

(3) (V) Suppose $E = \mathbb{R}$. Set $\|x\|_1 = |x|$ and $\|(x, y)\|_2 = \frac{1}{2}(|x| + |y|)$ $(x, y \in E)$ ([6, Example 2.5]). Clearly, $N_2^i((1, 0), t) \neq N_1^i(1, t)$ $(i = 1, 2, 3)$ for some $t \in \mathbb{R}^+$. Therefore $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$ $(i = 1, 2, 3)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF1)$, $(MF2)$ and $(MF4)$ but $(MF3)$ does not hold.

(VI) Assume that $E = \mathbb{R}^2$, $\|(x, y)\|_1 = |x - y| + |x| + |y|$ and $\|((x, y), (z, w))\|_2 = |x| + |y| + |z| + |w| + 2\max\{|x - y|, |z - w|\}$ $(x, y, z, w \in \mathbb{R})$. It is readily verified that $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$ $(i = 1, 2, 3)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF1)$, $(MF2)$ and $(DF4)$ but $(MF3)$ is not true.

• *Independece of the axiom* $(MF4)$ *and* $(DF4)$

(4) Let $(E, \|.\|)$ be a normed space and $p > 1$. For each $k \in \mathbb{N}$, define $\|(x_1, \ldots, x_k)\|_k = (\sum_{i=1}^{k} \|x_i\|^p)^{\frac{1}{p}}$ ([6, Example 2.6]). Then $\{(E^k, N_k^i), k \in \mathbb{N}\}$ $(i = 1, 2, 3)$ is a family of fuzzy normed spaces in which satisfies axioms $(MF1)$, $(MF2)$ and $(MF3)$ but it does not satisfy $(MF4)$ and $(DF4)$.

The following lemma is an immediate consequence of the definition of multi-fuzzy normed space.

Lemma 3.5. *Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a (dual) multi-fuzzy normed space, $k, n \in \mathbb{N}, x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+n} \in E$ and $\eta_1, \ldots, \eta_k$ be real numbers of absolute value 1, then we have*

(i) $N_k((\eta_1 x_1, \ldots, \eta_k x_k), t) = N_k((x_1, \ldots, x_k), t)$.

(ii) $N_k((x_1, \ldots, x_k), t) \geq N_{k+1}((x_1, \ldots, x_k, x_{k+1}), t)$.

(iii) $N_{k+n}((x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+n}), t) \geq$

$$\min\{N_k((x_1, \ldots, x_k), \alpha t), N_n((x_{k+1}, \ldots, x_{k+n}), \beta t)\},$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

(iv) $\min_{i\in\mathbb{N}_k} N(x_i, t) \geq N_k((x_1, \ldots, x_k), t) \geq \min_{i\in\mathbb{N}_k} N(kx_i, t)$.

As an important result of the preceding lemma we get the following corollary.

Corollary 3.6. *Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a (dual) multi-fuzzy normed space, and (E, N_1) be a fuzzy Banach space. Then for each $k \in \mathbb{N}$, (E^k, N_k) is a fuzzy Banach space, too.*

Proof. Fix an arbitrary $k \in \mathbb{N}$. Let $\{X_n\}$ be a Cauchy sequence in E^k, where $X_n = (x_{n,1}, \ldots, x_{n,k})$. Also let $0 < \varepsilon < 1$ and $\delta > 0$ be given, then there exists $n_0 \in \mathbb{N}$ such that $N_k(X_m - X_n, \delta) > 1 - \varepsilon$ for all $n, m \geq n_0$. As a result of Lemma 3.5, we observe that for $i = 1, \ldots, k$, $N_1(x_{m,i} - x_{n,i}, \delta) > 1 - \varepsilon$ for all $n, m \geq n_0$. Hence $\{x_{n,i}\}$, $i = 1, \ldots, k$, is a Cauchy sequence in fuzzy Banach space (E, N_1) and therefore it converges to some $x_i \in E$. Thus there exists n_i, $i = 1, \ldots, k$ for which $N_1(x_{n,i} - x_i, \delta) > 1 - \varepsilon$ for all $n \geq n_i$. Take $n' = \max\{n_1, \ldots, n_k\}$. Applying again Lemma 3.5, we obtain that

$$\begin{aligned} N_k((x_{n,1} - x_1, \ldots, x_{n,k} - x_k), \delta) &\geq \min_{i \in \mathbb{N}_k} N_1(x_{n,i} - x_i, \frac{\delta}{k}) \\ &\geq 1 - \varepsilon, \end{aligned}$$

for all $n \geq n'$. This proves $N\text{-}\lim_{n\to\infty} X_n = X$, where $X = (x_1, \ldots, x_k)$. Therefore (E^k, N_k) is a fuzzy Banach space. □

In the light of the previous corollary, the following definition is reasonable.

Definition 3.7. Suppose that $\{(E^k, N_k), k \in \mathbb{N}\}$ is a (dual) multi-fuzzy normed space, for which (E, N_1) is a fuzzy Banach space, then we say $\{(E^k, N_k), k \in \mathbb{N}\}$ is a (dual) multi-fuzzy Banach space.

Using the similar techniques applied in [6, Lemmata 2.16, 2.19] we get the following two propositions.

Proposition 3.8. *Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $k \in \mathbb{N}$ with $k \geq 2$, $x_1, \ldots, x_k \in E$, α, β be nonnegative real numbers with $\alpha + \beta = 1$ and $x = \alpha x_{k-1} + \beta x_k$. Then for each $t \in \mathbb{R}$, $N_k((x_1, \ldots, x_{k-2}, x, x), t) \geq N_k((x_1, \ldots, x_{k-2}, x_{k-1}, x_k), t)$.*

Proposition 3.9. *Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a dual multi-fuzzy normed space, $k \in \mathbb{N}$ with $k \geq 2$, $x_1, \ldots, x_k \in E$ and $t \in \mathbb{R}$. Then for each $t \in \mathbb{R}$, $N_{k-1}((x_1, \ldots, x_{k-1} + x_k), t) \geq N_k((x_1, \ldots, x_{k-1}, x_k), t)$.*

The following proposition is proved in framework of multi-normed spaces ([6, Proposition 2.7]). Slightly modification in the proof shows this proposition holds in the category of multi-fuzzy normed spaces.

Proposition 3.10. *Let (E, N) be a fuzzy normed space, $\{N_k\}$ be a sequence such that N_k is a fuzzy norm on E^k for each $k \in \mathbb{N}$ and $N_1(x, t) = N(x, t)$ for each $x \in E$ and $t \in \mathbb{R}$. Also assume that axioms $(MF1), (MF2)$ and $(MF4)$ are satisfied for each $k \in \mathbb{N}$. Then $\{(E^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy normed space.*

Proof. By Definition 3.1, it is enough to show that axiom $(MF3)$ holds. Let $k \in \mathbb{N}$ and $x = (x_1, \ldots, x_k)$ be an arbitrary element of E^k. Obviously if $x = 0$, then $N_{k+1}((x_1, \ldots, x_k, 0), t) = N_k((x_1, \ldots, x_k), t)$ by $(MF4)$ (or $(N2)$) and so in this case $(MF3)$ holds. Now assume that x is nonzero and $n \in \mathbb{N}$ is arbitrary. For $i = 1, ..., n+1$, let $B_i = \{(i-1)k + 1, \ldots, ik\}$ be the subset of $\mathbb{N}_{k(n+1)}$, and Q_{B_i} be the projection onto the complement of B_i. From $(N4)$ it follows that

$$N_{k(n+1)}(\sum_{i=1}^{n+1} Q_{B_i}(x^{[n+1]}), (n+1)t) \geq \min\{N_{k(n+1)}(Q_{B_i}(x^{[n+1]}), t), i = 1, \ldots, n+1\}. \quad (2)$$

Since $\sum_{i=1}^{n+1} Q_{B_i}(x^{[n+1]}) = nx^{[n+1]}$, so the left hand side of (2) is equal to $N_{k(n+1)}(nx^{[n+1]}, (n+1)t) = N_k((x_1, \ldots, x_k), \frac{n+1}{n}t)$ by $(N3)$, $(MF1)$ and $(MF4)$. On the other hand, the right hand side of (2) is equal to $N_{k+1}((x_1, \ldots, x_k, 0), t)$ by $(MF1)$ and $(MF4)$. Consequently $N_k((x_1, \ldots, x_k), \frac{n+1}{n}t) \geq N_{k+1}((x_1, \ldots, x_k, 0), t)$. Since n is arbitrary, then let $n \to \infty$, by $(N6)$ we get

$$N_k((x_1, \ldots, x_k), t) \geq N_{k+1}((x_1, \ldots, x_k, 0), t). \quad (3)$$

For the reverse direction applying $(MF2)$ and $(MF4)$, we deduce that

$$\begin{aligned} N_{k+1}((x_1, \ldots, x_k, 0), t) &= N_{k+1}(M_\alpha(x_1, \ldots, x_k, x_k), t) \\ &\geq N_{k+1}((x_1, \ldots, x_k, x_k), t) \\ &= N_k((x_1, \ldots, x_k), t), \end{aligned} \quad (4)$$

where $\alpha = (1, \ldots, 1, 0) \in \mathbb{C}^{k+1}$. Now from (3) and (4) we achieve our goal. □

In the following, we are going to show that the above result holds for dual multi-fuzzy norms, too.

Proposition 3.11. *Let (E, N) be a fuzzy normed space, $\{N_k\}$ be a sequence such that N_k is a fuzzy norm on E^k for each $k \in \mathbb{N}$ and $N_1(x, t) = N(x, t)$ for each $x \in E$ and $t \in \mathbb{R}$. Also assume that axioms $(MF1), (MF2)$ and $(DF4)$ are satisfied for each $k \in \mathbb{N}$. Then $\{(E^k, N_k), k \in \mathbb{N}\}$ is a dual multi-fuzzy normed space.*

Proof. Suppose that $x = (x_1, \ldots, x_k)$ is in E^k $(k \in \mathbb{N})$, n is an arbitrary element of $\mathbb{N}$ and $t \in \mathbb{R}$. For $i = 1, \ldots, 2^n$, let B_i be the subset $\{(i-1)k+1, \ldots, ik\}$ of $\mathbb{N}_{(2^n)k}$, and Q_{B_i} be the projection onto the complement of B_i. Then we obtain

$$\begin{aligned} N_k(x, t) &= N_k(2^n x, 2^n t) \\ &= N_{2^n k}((2^n - 1)x^{[2^n]}, 2^n(2^n - 1)t) \quad (DF4) \\ &= N_{2^n k}(\sum_{i=1}^{2^n} Q_{B_i}(x^{[2^n]}), 2^n(2^n - 1)t) \\ &\geq \min\{N_{2^n k}(Q_{B_i}(x^{[2^n]}), (2^n - 1)t), i = 1, \ldots, 2^n\} \\ &= N_{k+1}((2^n - 1)(x_1, \ldots, x_k, 0), (2^n - 1)t) \end{aligned}$$

and so

$$N_k(x, t) \geq N_{k+1}((x_1, \ldots, x_k, 0), t). \quad (5)$$

For the reverse direction assume that $x = (x_1, \ldots, x_k, 0)$. Without loss of generality we may assume that (using $(MF1)$) $x^{[2^n]} = (x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k, 0, \ldots, 0)$, where the number of repetitions of each item is 2^n. For $i = 1, \ldots, k$, let $C_i = \{(i-1)2^n + 1, \ldots, i2^n\}$ and Q_{C_i} be the projection onto the complement of C_i. Also for $i = 1, \ldots, k-2$, put
$X_1^i = (x_1, \ldots, x_1, \ldots, x_{i-1}, \ldots, x_{i-1}, 0, \ldots, 0, x_{i+1}, \ldots, x_{i+1}, \ldots, x_k, \ldots, x_k, 0, \ldots, 0)$,
$X_2^i = (x_1, \ldots, x_1, \ldots, x_{i-1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+1}, \ldots, x_k, \ldots, x_k, 0, \ldots, 0)$,
where the number of repetitions of each item x_t, $t = 1, \ldots, i-1, i+1, \ldots, k$ is 2^n and also zero has repeated 2×2^n times,
$X_3^i = (x_1, \ldots, x_1, \ldots, x_{i-1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+1}, \ldots, x_k, \ldots, x_k, 0, \ldots, 0)$,

$X_4^i = (x_1, \ldots, x_1, \ldots, x_{i-1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+1}, \ldots, x_k, \ldots, x_k, x_i, \ldots, x_i)$,
$X_5 = (x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k)$,
where the number of repetitions of each item in X_3^i,X_4^i and X_5 is 2^n.
Take $A = N_{2^n(k+1)}(Q_{C_{k-1}}(x^{[2^n]}) + Q_{C_k}(x^{[2^n]}), 2^n t)$. Then

$$\begin{aligned}
A = N_{2^n(k+1)}((&2x_1, \ldots, 2x_1, \ldots, 2x_{k-2}, \ldots, 2x_{k-2}, \\
&x_{k-1}, \ldots, x_{k-1}, x_k, \ldots, x_k, 0, \ldots, 0), 2^n t) \\
\geq\ & N_{2^n(k+1)}(2M_\alpha(x_1, \ldots, x_1, \ldots, x_{k-1}, \ldots, x_{k-1}, x_k, \ldots, x_k, x_k, \ldots, x_k), 2^n t) \\
\geq\ & N_{k+1}(2^{n+1}(x_1, \ldots, x_{k-1}, x_k, x_k), 2^n t) \\
\geq\ & N_k((x_1, \ldots, x_k), \frac{2^n}{2^{n+2}} t),
\end{aligned}$$

where $\alpha = (\underbrace{1, \ldots, 1}_{2^n k-tuples}, \underbrace{0, \ldots, 0}_{2^n-tuples})$. Also, we observe that

$$\begin{aligned}
N_{2^n(k+1)}(\sum_{i=1}^{k} Q_{C_i}(x^{[2^n]}), 2^n(k-1)t)& \\
\geq\ & \min\{A, N_{2^n(k+1)}(Q_{C_i}(x^{[2^n]}), 2^n t) : i = 1, \ldots, k-2\} \\
=\ & \min\{A, N_{2^n(k+1)}(X_1^i, 2^n t) : i = 1, \ldots, k-2\} \\
=\ & \min\{A, N_{2^n(k+1)}(X_2^i, 2^n t) : i = 1, \ldots, k-2\} \quad (MF1) \\
=\ & \min\{A, N_{2^n k}(X_3^i, 2^n t) : i = 1, \ldots, k-2\} \quad (DF4) \\
=\ & \min\{A, N_{2^n k}(M_\beta X_4^i, 2^n t) : i = 1, \ldots, k-2\} \\
\geq\ & \min\{A, N_{2^n k}(X_5, 2^n t) : i = 1, \ldots, k-2\} \quad (MF1), (MF2) \\
=\ & \min\{A, N_{2^n k}(2^n(x_1, \ldots, x_k), 2^n t)\} \\
\geq\ & N_k((x_1, \ldots, x_k), \frac{2^n}{2^{n+2}} t),
\end{aligned}$$

where $\beta = (\underbrace{1, \ldots, 1}_{2^n(k-1)-tuples}, \underbrace{0, \ldots, 0}_{2^n-tuples})$.

Furthermore, let $x'^{[2^n]} = (x_1, \ldots, x_k, 0, ..., x_1, \ldots, x_k, 0)$. It is easily verified that $\sum_{i=1}^{k} Q_{C_i}(x^{[2^n]}) = (k-1)x^{[2^n]}$. Take n sufficiently large in which $k \leq 2^n$, then we deduce

$$\begin{aligned}
N_{k+1}(x, \frac{2^n}{2^n - 1} t) &= N_{k+1}(2^n x, \frac{2^{2n}}{2^n - 1} t) \\
&\geq N_{2^n(k+1)}(x'^{[2^n]}, 2^n \frac{2^n - 1}{2^n - 1} t) \\
&= N_{2^n(k+1)}(x'^{[2^n]}, 2^n \frac{k-1}{k-1} t) \\
&= N_{2^n(k+1)}(\sum_{i=1}^{k} Q_{C_i}(x^{[2^n]}), 2^n(k-1)t) \\
&\geq N_k((x_1, \ldots, x_k), \frac{2^n}{2^{n+2}} t).
\end{aligned}$$

Thus we find that $N_{k+1}(x, \frac{2^n}{2^n-1} t) \geq N_k((x_1, \ldots, x_k), \frac{2^n}{2^{n+2}} t)$. Let $n \to \infty$, it obtains that

$$N_{k+1}((x_1, \ldots, x_k, 0), t) \geq N_k((x_1, \ldots, x_k), t). \tag{6}$$

Now the result follows by (5) and (6). □

4. Ulam-Hyers Stability of the Jensen's Functional Equation

This section is devoted to establishing Ulam-Hyers stability of the Jensen's functional equation for mappings from linear spaces into multi-fuzzy normed spaces. To this end, we extend some results of [18] in the framework of multi-fuzzy normed spaces.

Theorem 4.1. *Let E be a linear space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space, (F, N') be a fuzzy normed space and $f : E \to F$ be a mapping satisfying $f(0) = 0$. Suppose that α is a nonzero fixed vector in F such that*

$$N_k((f(\frac{x_1+y_1}{2}) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \ldots, f(\frac{x_k+y_k}{2}) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}), t) \geq N'(\alpha, t), \tag{7}$$

for all $x_1, ..., x_k, y_1, ..., y_k \in E$ and for all $t > 0$. Then there exists a unique additive mapping $T : E \to F$ such that

$$N_k((f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k)), 2t) \geq N'(\alpha, t), \tag{8}$$

where $x_1, ..., x_k \in E$ and $t > 0$.

Proof. Substituting $y_i = 0$ for $i = 1, \cdots, k$ and replacing $x_1, ..., x_k$ by $2^n x_1, ..., 2^n x_k$ in (7), we get

$$N_k((\frac{f(2^{n-1}x_1)}{2^{n-1}} - \frac{f(2^n x_1)}{2^n}, \ldots, \frac{f(2^{n-1}x_k)}{2^{n-1}} - \frac{f(2^n x_k)}{2^n}), \frac{t}{2^{n-1}}) \geq N'(\alpha, t). \tag{9}$$

Regarding (9), we conclude that

$$N_k((\frac{f(2^m x_1)}{2^m} - \frac{f(2^n x_1)}{2^n}, \ldots, \frac{f(2^m x_k)}{2^m} - \frac{f(2^n x_k)}{2^n}), t(\sum_{i=m}^{n-1} 2^{-i})) \geq N'(\alpha, t), \tag{10}$$

for nonnegative integer numbers m, n.
Fix nonzero $x \in E$, put $x_1 = \ldots = x_k = x$ in (10) we thus find that

$$N(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, t(\sum_{i=m}^{n-1} 2^{-i})) \geq N'(\alpha, t).$$

Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} N'(\alpha, t) = 1$, there is some $t_0 > 0$ such that $N'(\alpha, t_0) > 1-\varepsilon$. Since $\sum_{i=1}^{\infty} t_0 2^{-i} < \infty$, there is some $n_0 \in \mathbb{N}$ such that $\sum_{i=m}^{n-1} t_0 2^{-i} < \delta$ for all $m \geq n_0 + 1$. Therefore we derive

$$N(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \delta) \geq N(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, t_0(\sum_{i=m}^{n-1} 2^{-i})) \geq N'(\alpha, t_0) > 1 - \varepsilon.$$

It follows that $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy and so is convergent in the complete fuzzy normed space F. Set

$$T(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

In view of Lemma 3.5, we deduce

$$N\text{-}\lim_{n\to\infty}(\frac{f(2^nx_1)}{2^n},\ldots,\frac{f(2^nx_k)}{2^n}) = (T(x_1),\ldots,T(x_k)).$$

Moreover, if we put $m = 0$ in (10), we observe that

$$N_k((f(x_1)-\frac{f(2^nx_1)}{2^n},\ldots,f(x_k)-\frac{f(2^nx_k)}{2^n}),t) \geq N'(\alpha,t). \tag{11}$$

Therefore

$$N_k((f(x_1)-T(x_1),\ldots,f(x_k)-T(x_k)),2t) \geq$$
$$\min\{N_k((f(x_1)-\frac{f(2^nx_1)}{2^n},\ldots,f(x_k)-\frac{f(2^nx_k)}{2^n}),t),$$
$$N_k((\frac{f(2^nx_1)}{2^n}-T(x_1),\ldots,\frac{f(2^nx_k)}{2^n}-T(x_k)),t)\}.$$

The second term on the right hand side of the above inequality tends to 1 as $n\to\infty$ and the first term, by (11) is grater than or equal to $N'(\alpha,t)$. It gives us relation (8). Next, we will show that T is additive. Let $x,y\in E$. Put $x_1 = ... = x_k = 2^nx$, $y_1 = ... = y_k = 2^ny$ and replace t by 2^nt in (7) to obtain

$$N_k(\frac{f(\frac{2^n(x+y)}{2})}{2^n}-\frac{1}{2}\frac{f(2^nx)}{2^n}-\frac{1}{2}\frac{f(2^ny)}{2^n}),t) \geq N'(\alpha,2^nt). \tag{12}$$

On the other hand,

$$N(T(\tfrac{x+y}{2})-\tfrac{T(x)}{2}-\tfrac{T(y)}{2},4t) \geq \min\Big\{N(T(\tfrac{x+y}{2})-\tfrac{f(2^n(\frac{x+y}{2}))}{2^n},t),$$
$$N(\tfrac{T(x)}{2}-\tfrac{1}{2}\tfrac{f(2^nx)}{2^n},t), N(\tfrac{T(y)}{2}-\tfrac{1}{2}\tfrac{f(2^ny)}{2^n},t), N(\tfrac{f(2^n(\frac{x+y}{2}))}{2^n}-\tfrac{1}{2}\tfrac{f(2^nx)}{2^n}-\tfrac{1}{2}\tfrac{f(2^ny)}{2^n},t)\Big\},$$

for each $x,y\in E$ and $t>0$. The first three terms on the right hand side of the above inequality tend to 1 as $n\to\infty$ and the forth term, by (12) is greater than or equal to $N'(\alpha,2^nt)$, which tends to 1 as $n\to\infty$. Therefore

$$N(T(\frac{x+y}{2})-\frac{T(x)}{2}-\frac{T(y)}{2},4t) = 1,$$

for each $x,y\in E$ and $t>0$. It enforces that T satisfies the Jensen equation and by virtue of the fact that $T(0)=0$, T is additive. To end the proof, let T' be another additive mapping satisfying (8) and x be nonzero element of E, then

$$\begin{aligned} N(T'(x)-T(x),2t) &= N(\frac{T'(2^nx)}{2^n}-\frac{T(2^nx)}{2^n},2t) \\ &\geq \min\{N(T'(2^nx)-f(2^nx),2^nt), N(T(2^nx)-f(2^nx),2^nt)\} \\ &\geq N'(\alpha,2^{n-1}t). \end{aligned}$$

By taking the limit as n tends to infinity we get $T=T'$. This proves the uniqueness assertion. □

Example 4.2. Let E and F be fuzzy normed space $(\mathbb{R},N)$, where

$$N(x,t) = \begin{cases} 1, & t>|x|, \\ 0, & t\leq |x|. \end{cases}$$

Consider $N_k((x_1,\dots,x_k),t) = \min\{N(x_i,t) : i = 1,\dots,k\}$ $((x_1,\dots,x_k) \in F^k,\ t \in \mathbb{R})$. Then $\{(F^k, N_k), k \in \mathbb{N}\}$ is a multi-fuzzy Banach space. Also consider $f(x) = \mathrm{sgn}(x)$, $(x \in \mathbb{R})$, take $\alpha = 4$ and $N' = N$. Now, all conditions of Theorem 4.1 are satisfied and so there is a unique additive mapping T fulfills condition (8).

Next, we show that the condition $f(0) = 0$ in Theorem 4.1 is necessary.

Example 4.3. In Theorem 4.1, let E and F be fuzzy normed space $(\mathbb{R}, N)$, where $N(x,t) = \frac{t}{t+|x|}$ $(x \in E$ or $F,\ t > 0)$ and $N_k((x_1,\dots,x_k),t) = \min\{N(x_i,t) : i = 1,\dots,k\}$ $((x_1,\dots,x_k) \in F^k,\ t \in \mathbb{R})$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 10$. Also take $\alpha = 1$ and $N' = N$. Evidently, condition (7) holds. If on the contrary, there exists an additive mapping $T : \mathbb{R} \to \mathbb{R}$ satisfying (8), then take $x = n$ ($n \in \mathbb{N}$ large enough), we get $N(10 - nT(1), 2t) \geq N(1,t)$ for all $t > 0$. Hence

$$\frac{2t}{2t + |10 - nT(1)|} \geq \frac{t}{t+1}, \tag{13}$$

for all $t > 0$. In turn this proves

$$|\frac{10}{n} - T(1)| \leq \frac{2}{n}. \tag{14}$$

In the light of the relation (14) we conclude that $T(1) = 0$, which leads to a contradiction with the condition (13).

Remark 4.4. A significant fact about Ulam-Hyers stability in the setup multi-fuzzy normed spaces which is interesting in own right is that we can obtain some results in multi-normed spaces as particular cases in such spaces. For instance [18, Theorem 1] can be obtained from Theorem 4.1, when we consider Banach space $(F, \|.\|)$ with the fuzzy norm $N(x,t) = \frac{t}{t+\|x\|}$ $(x \in F,\ t > 0)$, (F^k, N_k) is defined as Example 3.2 and $N'(\alpha,t) = \frac{t}{t+\|\alpha\|}$.

The following example shows the usefulness of our results. Indeed, we give an example of a multi-fuzzy normed space such that its topology is not multi-normable.

Example 4.5. Let E be the space of complex-valued continuous functions on the real line. Then E is not normable [25]. Define

$$N(f,t) = \begin{cases} 0, & t \leq 0, \\ \sup\{\frac{n}{n+1},\ \|f\|_n \leq t\}, & t > 0, \end{cases}$$

where $\|.\|_n$ denotes the supremum norm on $[-n, n]$, $n \in \mathbb{N}$. By applying a similar argument as in the proof of [3, Theorem 2.2], one can show that (E, N) is a fuzzy normed space (see [16]). Now consider $\{(E^k, N_k), k \in \mathbb{N}\}$ as Example 3.2. Then it is a multi-fuzzy normed space which is not multi-normable.

In the following we give a result by utilizing the strategy applied in [18, Page 458]. Note that this theorem in comparison with the similar one in multi-normed spaces is applicable in a different area.

Theorem 4.6. *Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space and $(E, N'), (F, N')$ be fuzzy normed spaces. Suppose that $\{\beta_k\}$ is a sequence in E, α is a nonzero fixed vector in F and $f : E \to F$ is a mapping satisfying $f(0) = 0$ and*

$$N_k((f(\frac{x_1+y_1}{2}) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f(\frac{x_k+y_k}{2}) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2})), \frac{t}{5}) \geq N'(\alpha, t), \tag{15}$$

for all $k \in \mathbb{N}$, $x_1, \dots, x_k, y_1, \dots, y_k \in E$ and all $t > 0$ with

$$\min\{N_k((x_1, \dots, x_k), t), N_k((y_1, \dots, y_k), t)\} \leq N'(\beta_k, t).$$

Then there exists a unique additive mapping $T : E \to F$ such that

$$N_k((f(x_1) - T(x_1), \dots, f(x_k) - T(x_k)), 2t) \geq N'(\alpha, t), \tag{16}$$

for all $x_1, \dots, x_k \in E$ and all $t > 0$.

Proof. Fix $k \in \mathbb{N}$ and $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in E^k. Assume that $\min\{N_k(x, t), N_k(y, t)\} > N'(\beta_k, t)$. Obviously, if $x = y = 0$, then the condition (7) of Theorem 4.1 holds. Now if x and y are nonzero, then there exist sufficiently large natural numbers k_1 and k_2, in which $N_k(x, \frac{t}{k_1}) \leq N'(\beta_k, t)$ and $N_k(y, \frac{t}{k_2-1}) \leq N'(\beta_k, t)$. Put $z_1 = k_1 x$ and $z_2 = k_2 y$, then $N_k(z_1 + x, t) \leq N'(\beta_k, t)$ and $N_k(z_2 - y, t) \leq N'(\beta_k, t)$. Set

$$z := \begin{cases} z_1 + x, & \text{if} \quad N_k(x, t) \leq N_k(y, t), \\ z_2 - y, & \text{if} \quad N_k(x, t) > N_k(y, t), \end{cases}$$

for all $t > 0$. Clearly, $N_k(z, t) \leq N'(\beta_k, t)$. Now if $N_k(x, t) > N_k(y, t)$, then $\min\{N_k(x - z, t), N_k(y + z, t)\} \leq N_k(y + z, t) = N_k(z_2, t) \leq N'(\beta_k, t)$, and if $N_k(x, t) \leq N_k(y, t)$, then $\min\{N_k(x-z, t), N_k(y+z, t)\} \leq N_k(x-z, t) = N_k(z_1, t) \leq N'(\beta_k, t)$. Taking all these considerations into account we observe that, the following relations hold.

$$\begin{aligned} \min\{N_k(x, t), N_k(z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(y, t), N_k(2z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(y + z, t), N_k(z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(2z, t), N_k(x - z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(x - z, t), N_k(y + z, t)\} &\leq N'(\beta_k, t). \end{aligned} \tag{17}$$

If $x = 0$ and $y \neq 0$ (or $x \neq 0$ and $y = 0$), then an easy verification proves that (17) holds, too. From (15) and (17) we get

$$\begin{aligned} &N_k((f(\tfrac{x_1+y_1}{2}) - \tfrac{f(x_1)}{2} - \tfrac{f(y_1)}{2}, \dots, f(\tfrac{x_k+y_k}{2}) - \tfrac{f(x_k)}{2} - \tfrac{f(y_k)}{2}), t) \geq \\ &\min\Big\{N_k(f(\tfrac{x_1+y_1}{2}) - \tfrac{f(x_1-z_1)+f(y_1+z_1)}{2}, \dots, f(\tfrac{x_k+y_k}{2}) - \tfrac{f(x_k-z_k)+f(y_k+z_k)}{2}, \tfrac{t}{5}), \\ &N_k(f(\tfrac{x_1+z_1}{2}) - \tfrac{f(2z_1)+f(x_1-z_1)}{2}, \dots, f(\tfrac{x_k+z_k}{2}) - \tfrac{f(2z_k)+f(x_k-z_k)}{2}, \tfrac{t}{5}), \\ &N_k(f(\tfrac{y_1+2z_1}{2}) - \tfrac{f(y_1)+f(2z_1)}{2}, \dots, f(\tfrac{y_k+2z_k}{2}) - \tfrac{f(y_k)+f(2z_k)}{2}, \tfrac{t}{5}), \\ &N_k(f(\tfrac{y_1+2z_1}{2}) - \tfrac{f(y_1+z_1)+f(z_1)}{2}, \dots, f(\tfrac{y_k+2z_k}{2}) - \tfrac{f(y_k+z_k)+f(z_k)}{2}, \tfrac{t}{5}), \\ &N_k(f(\tfrac{x_1+z_1}{2}) - \tfrac{f(x_1)+f(z_1)}{2}, \dots, f(\tfrac{x_k+z_k}{2}) - \tfrac{f(x_k)+f(z_k)}{2}, \tfrac{t}{5})\Big\} \\ &\geq N'(\alpha, t). \end{aligned}$$

This inequality holds for all $k \in \mathbb{N}$, all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ and all $t > 0$ $\big($if $\min\{N_k(x,t), N_k(y,t)\} \leq N'(\beta_k, t)$ we get it immediately by (15)$\big)$. Now, the result is deduced from Theorem 4.1. □

Example 4.7. Consider $\{(E^k, \|.\|), k \in \mathbb{N}\}$ and $\{(F^k, \|.\|), k \in \mathbb{N}\}$ are multi-normed space and multi-Banach space, respectively. Equip E^k and F^k with the multi-fuzzy norm N_k^i ($i = 1$ or 2 or 3) as Example 3.2 and take $N' = N_1^i$. Suppose that α is in F and $\{\beta_k\}$, $k \in \mathbb{N}$ is a sequence in E. Now take $f : E \to F$ is a mapping satisfying all conditions of Theorem 4.6. It follows that if

$$\|\Big(f(\frac{x_1+y_1}{2}) - \frac{f(x_1)+f(y_1)}{2}, \ldots, f(\frac{x_k+y_k}{2}) - \frac{f(x_k)+f(y_k)}{2}\Big)\|_k \leq \frac{\|\alpha\|}{5},$$

for all $k \in \mathbb{N}$ and for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ with

$$\max\Big\{\|(x_1, \ldots, x_k)\|_k, \|(y_1, \ldots, y_k)\|_k\Big\} \geq \|\beta_k\|,$$

then there exists a unique additive mapping $T : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|\big(f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k)\big)\|_k \leq 2\|\alpha\|,$$

for all $x_1, \ldots, x_k \in E$.

Example 4.8. Let $\{(E^k, \|.\|), k \in \mathbb{N}\}$ and $\{(F^k, \|.\|), k \in \mathbb{N}\}$ be as Example 4.7. Also let $N'(x,t) = \frac{t^2-\|x\|^2}{t^2+\|x\|^2}$, $x \in E$ or F, when $t > \|x\|$. Suppose that $\alpha \in F$ and $\beta_k \in E$ ($k \in \mathbb{N}$) with $\|\alpha\| = \|\beta_k\| = 1$ and $f : E \to F$ is a mapping satisfying all conditions of Theorem 4.6. It follows that if

$$\|\Big(f(\frac{x_1+y_1}{2}) - \frac{f(x_1)+f(y_1)}{2}, \ldots, f(\frac{x_k+y_k}{2}) - \frac{f(x_k)+f(y_k)}{2}\Big)\|_k \leq \frac{2t}{5(t^2-1)},$$

for all $k \in \mathbb{N}$, all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ and all $t > 1$ with

$$\max\Big\{\|(x_1, \ldots, x_k)\|_k, \|(y_1, \ldots, y_k)\|_k\Big\} \geq \frac{2t}{t^2-1},$$

then there exists a unique additive mapping $T : E \to F$ such that

$$\sup_{k\in\mathbb{N}} \|\big(f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k)\big)\|_k \leq \frac{4t}{t^2-1},$$

for all $x_1, \ldots, x_k \in E$ and all $t > 1$.

In the above example, we observe that the stability of Jensen's functional equation depends on another parameter t such that for each fixed $t > 1$ it reduces to the same result as Example 4.7.

Example 4.9. Under the hypotheses of Example 4.3 except $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = f(\frac{1}{2}) = 0$ and $f(x) = 1$ elsewhere, $\alpha = 100$ and $\beta_k = 1$ ($k \in \mathbb{N}$), then all conditions of Theorem 4.6 hold. Therefore there exists a unique additive mapping $T : \mathbb{R} \to \mathbb{R}$ fulfilling condition (16).

Now we are ready to prove our main result.

Theorem 4.10. *Let $\{(E^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy normed space, $\{(F^k, N_k), k \in \mathbb{N}\}$ be a multi-fuzzy Banach space and $f : E \to F$ be a mapping such that $f(0) = 0$. Then f is additive if and only if for all $k \in \mathbb{N}$*

$$N_k((f(\frac{x_1+y_1}{2}) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f(\frac{x_k+y_k}{2}) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}), \frac{t}{5}) \to 1 \quad (18)$$

as

$$\min\{N_k((x_1, \dots, x_k), t), N_k((y_1, \dots, y_k), t)\} \to 0.$$

Proof. If f is additive, then evidently (18) holds. Conversely, let (E, N') and (F, N') be fuzzy-normed spaces. Fix nonzero vector α in F. Employing the condition (18) we can find for each $n \in \mathbb{N}$ a sequence $\{\beta_{n_k}\}$ in E such that

$$N_k((f(\frac{x_1+y_1}{2}) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f(\frac{x_k+y_k}{2}) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}), \frac{t}{5}) \geq N'(\frac{\alpha}{n}, t),$$

for all $k \in \mathbb{N}$ and all $x_1, \cdots, x_k, y_1, \cdots, y_k \in E$, with

$$\min\{N_k((x_1, \dots, x_k), t), N_k((y_1, \dots, y_k), t)\} \leq N'(\beta_{n_k}, t).$$

In view of Theorem 4.6, for every $n \in \mathbb{N}$ there exists a unique additive mapping T_n such that

$$N(T_n(x) - f(x), 2t) \geq N'(\frac{\alpha}{n}, t), \quad (19)$$

for all $x \in E$ and all $t > 0$. Since $N(T_1(x) - f(x), 2t) \geq N'(\alpha, t)$ and $N(T_n(x) - f(x), 2t) \geq N'(\frac{\alpha}{n}, t) \geq N'(\alpha, t)$ by the uniqueness of T_1 we conclude that $T_n = T_1$ for each $n \in \mathbb{N}$. Now tending with n to infinity in (19), we deduce that $f = T_1$ and hence f is additive. □

Acknowledgements. Author would like to express her sincere thanks to the anonymous reviewers for several useful comments.

References

[1] A. Alotaibi, M. Mursaleen, H. Dutta and S. A. Mohiuddine, *On the Ulam stability of Cauchy functional equation in IFN-spaces*, Appl. Math. Inf. Sci. **8(3)** (2014), 1135–1143.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.

[3] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11(3)** (2003), 687–705.

[4] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Syst. **151** (2005), 513–547.

[5] S. C. Cheng and J. N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., **86(5)** (1994), 429–436.

[6] H. G. Dales and M. E. Polyakov, *Multi-normed spaces and multi-Banach algebras*, University of Leeds, preprint, (2008), 1–156.

[7] C. Felbin, *Finite-dimensional fuzzy normed linear space*, Fuzzy Sets and Syst., **48(2)** (1992), 239–248.

[8] A. Ghaffari and A. Alinejad, *Stabilities of cubic mappings in fuzzy normed spaces*, Adv. Difference Equ., **2010** (2010), 1–15.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.

[10] S. M. Jung, *Hyers-Ulam-Rassias stability of Jensens equation and its application*, Proc. Amer. Math. Soc., **126** (1998), 3137–3143.
[11] A. K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Syst., **12(2)** (1984), 143–154.
[12] I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11(5)** (1975), 336-344.
[13] S. V. Krishna and K. K. M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Syst., **63(2)** (1994), 207–217.
[14] L. Li, J. Chung and D. Kim, *Stability of Jensen equations in the space of generalized functions*, J. Math. Anal. Appl., **299** (2004), 578–586.
[15] T. Li, A. Zada and S. Faisal, *Hyers-Ulam stability of n-th order linear differential equations*, J. Nonlinear Sci. Appl., **9(5)** (2016), 2070–2075.
[16] A. K. Mirmostafaee, *Perturbation of generalized derivations in fuzzy Menger normed algebras*, Fuzzy Sets and Syst., **195** (2012), 109-117.
[17] A. K. Mirmostafaee, M. Mirzavaziri and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Syst., **159(6)** (2008), 730–738.
[18] M. S. Moslehian and H. M. Sirvastava, *Jensen's functional equation in multi-normed spaces*, Taiwanese J. Math., **14(2)** (2010), 453–462.
[19] E. Movahednia, S. Eshtehar and Y. Son, *Stability of quadratic functional equations in fuzzy normed spaces*, Int. J. Math. Anal., **6(48)** (2012), 2405–2412.
[20] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.
[21] Y. Shen, *An integrating factor approach to the Hyers-Ulam stability of a class of exact differential equations of second order*, J. Nonlinear Sci. Appl., **9(5)** (2016), 2520–2526.
[22] S. M. Ulam, *A collection of mathematical problems*, Interscience Publ., New York, 1960.
[23] S. M. Vaezpour and F. Karimi, *t-Best approximation in fuzzy normed spaces*, Iranian Journal of Fuzzy Systems, **5(2)** (2008), 93–99.
[24] J. Z. Xiao and X. H. Zhu, *Fuzzy normed space of operators and its completeness*, Fuzzy Sets and Syst., **133(3)** (2003), 389–399.
[25] B. Yood, *On the non-existence of norms for some algebras of functions*, Stud. Math., **111(1)** (1994), 97-101.

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