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CERTAIN TYPES OF EDGE $m-$POLAR FUZZY GRAPHS

M. AKRAM, N. WASEEM AND W. A. DUDEK

Abstract. In this research paper, we present a novel frame work for handling m-polar information by combining the theory of $m-$polar fuzzy sets with graphs. We introduce certain types of edge regular $m-$polar fuzzy graphs and edge irregular $m-$polar fuzzy graphs. We describe some useful properties of edge regular, strongly edge irregular and strongly edge totally irregular $m-$polar fuzzy graphs. We discuss the relationship between degree of a vertex and degree of an edge in an $m-$polar fuzzy graph. We investigate edge irregularity on a path on $2n$ vertices and barbell graph $B_{n,n}$. We also present an application of $m-$polar fuzzy graph to decision making.

1. **Introduction**

A graph theory is a representation of the connected aspects of a system or network. It has become fashionable to mention that there are some applications of graph theory in different areas, including anthropology, linguistics, architecture, psychology, communication science, coding theory, computer programming, molecular structure, operational research, electrical networks, switching and combinatorics. In 1965, Zadeh [18] introduced the notion of fuzzy subset of a set. In 1994, Zhang [20] generalized the idea of a fuzzy set and gave the concept of bipolar fuzzy set on a given set X as a map which associates each element of X to a real number in the interval $[-1, 1]$. In 2014, Chen *et al.* [7] introduced the concept of m-polar fuzzy sets as an extension of bipolar fuzzy sets and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions, and we can obtain concisely one from the corresponding one in [7]. The idea behind this is that "multipolar information" (not just bipolar information which corresponds to two-valued logic) exists because data for a real World problem are sometimes from n agents ($n \geq 2$). For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n (n \approx 7 \times 10^9)$ because different persons have been monitored different times. There are many examples, including truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formula which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.
Kauffmann [8] gave the definition of a fuzzy graph in 1973 on the basis of Zadeh's

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fuzzy relations [19]. Rosenfeld [15] discussed the notion of fuzzy graph in 1975. Further remarks on fuzzy graphs were given by Bhattacharya [6]. Several concepts on fuzzy graphs were introduced by Mordeson and Nair [11]. Nagoorgani and Radha [13] initiated the idea of regular fuzzy graphs and totally regular fuzzy graphs. Radha and Kumaravel presented the idea of edge regular fuzzy graphs in [14]. Akram *et al.* [1-5] introduced many new concepts, including bipolar fuzzy graphs, regular bipolar fuzzy graphs, $m-$ polar fuzzy labeling graphs, certain metrics in $m-$polar fuzzy graphs, and certain types of irregular $m-$polar fuzzy graphs. In this research paper, we introduce certain types of edge regular $m-$polar fuzzy graphs and edge irregular $m-$polar fuzzy graphs. We describe some useful properties of edge regular, strongly edge irregular and strongly edge totally irregular $m-$polar fuzzy graphs. We discuss the relationship between degree of a vertex and degree of an edge in an $m-$polar fuzzy graph. We investigate edge irregularity on a path on $2n$ vertices and barbell graph $B_{n,n}$. We also present an application of $m-$polar fuzzy graph to decision making.

2. **Edge Regular $m-$polar Fuzzy Graphs**

An $m-$ *polar fuzzy set* [7] on V is a mapping $C : V \to [0,1]^m$. The set of all $m-$polar fuzzy sets on V is denoted by $m(V)$. It is easy to see that $\mathbf{0} = (0, 0, \ldots, 0)$ is the smallest element in $[0,1]^m$ and $\mathbf{1} = (1, 1, \ldots, 1)$ is the largest element in $[0,1]^m$. Let C be an $m-$polar fuzzy subset of a non-empty set V. An *$m-$polar fuzzy relation* [5] on C is an $m-$polar fuzzy subset D of $V \times V$ defined by the mapping $D : V \times V \to [0,1]^m$ such that $p_i \circ D(uv) \le \inf\{p_i \circ C(u), p_i \circ C(v)\},\ i = 1, 2, 3, \ldots, m$, for all $u, v \in V$, where $p_i \circ C(u)$ denotes the i-th degree of membership of the vertex u and $p_i \circ D(uv)$ denote the i-th degree of membership of the edge uv. An *$m-$polar fuzzy graph* [5, 7] is a pair $G = (C, D)$, where $C : V \to [0,1]^m$ is an $m-$polar fuzzy set on V and $D : V \times V \to [0,1]^m$ is an $m-$polar fuzzy relation on V such that $p_i \circ D(uv) \le \inf(p_i \circ C(u), p_i \circ C(v))$, for all $u, v \in V$, and $p_i \circ D(uv) = 0$, for all $uv \in V \times V - E,\ i = 1, 2, 3, \ldots, m$. D is a relation on C, C is called the $m-$polar fuzzy vertex set of G and D is called the $m-$polar fuzzy edge set of G, respectively. An $m-$polar fuzzy relation D is called symmetric if $p_i \circ D(uv) = p_i \circ D(vu)$, for all $u, v \in V$.
We now introduce certain types of edge regular $m-$polar fuzzy graphs.

Definition 2.1. Let $G = (C, D)$ be an $m-$polar fuzzy graph on a non-empty set V. If each edge in G has the same degree $(r_1, r_2, \ldots, r_m)$, G is called an *edge regular* $m-$polar fuzzy graph.

Remark 2.2. Let G be a $(r_1, r_2, \ldots, r_m)-$edge regular $m-$polar fuzzy graph. If $r_1 = r_2 = \ldots = r_m$, then G is called an *equally edge regular* $m-$polar fuzzy graph.

Example 2.3. Consider a graph $G^* = (V, E)$ such that $V = \{a, b, c, d, \}, E = \{ab, bc, cd, da\}$. Let C be a 5-polar fuzzy subset of V and let D be a 5-polar fuzzy subset of $E \subseteq V \times V$ defined by

C	a	b	c	d
$p_1 \circ C$	0.4	0.5	0.4	0.9
$p_2 \circ C$	0.6	0.5	0.7	0.5
$p_3 \circ C$	0.8	0.9	0.6	0.8
$p_4 \circ C$	0.2	0.4	0.4	0.5
$p_5 \circ C$	0.3	0.1	0.3	0.1

D	ab	bc	cd	da
$p_1 \circ D$	0.4	0.4	0.4	0.4
$p_2 \circ D$	0.5	0.5	0.5	0.5
$p_3 \circ D$	0.6	0.6	0.6	0.6
$p_4 \circ D$	0.2	0.2	0.2	0.2
$p_5 \circ D$	0.1	0.1	0.1	0.1

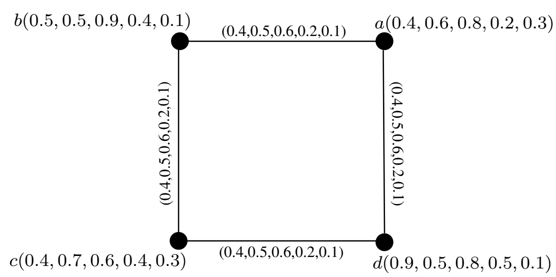

FIGURE 1. Edge Regular 5-polar Fuzzy Graph

By direct calculations, it is easy to see from Figure 1, that $d_G(a) = (0.8, 1.0, 1.2, 0.4, 0.2)$, $d_G(b) = (0.8, 1.0, 1.2, 0.4, 0.2)$, $d_G(c) = (0.8, 1.0, 1.2, 0.4, 0.2)$ and $d_G(d) = (0.8, 1.0, 1.2, 0.4, 0.2)$. By simple calculations, the degree of each edge is given below:

$$\begin{aligned}
d_G(ab) &= d_G(a) + d_G(b) - 2p_i \circ D(ab)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2) + (0.8, 1.0, 1.2, 0.4, 0.2) - 2(0.4, 0.5, 0.6, 0.2, 0.1)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2),\\
d_G(bc) &= d_G(b) + d_G(c) - 2p_i \circ D(bc)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2) + (0.8, 1.0, 1.2, 0.4, 0.2) - 2(0.4, 0.5, 0.6, 0.2, 0.1)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2),\\
d_G(cd) &= d_G(c) + d_G(d) - 2p_i \circ D(cd)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2) + (0.8, 1.0, 1.2, 0.4, 0.2) - 2(0.4, 0.5, 0.6, 0.2, 0.1)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2),\\
d_G(da) &= d_G(d) + d_G(a) - 2p_i \circ D(da)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2) + (0.8, 1.0, 1.2, 0.4, 0.2) - 2(0.4, 0.5, 0.6, 0.2, 0.1)\\
&= (0.8, 1.0, 1.2, 0.4, 0.2).
\end{aligned}$$

Clearly, degree of each edge in G is the same, therefore, G is an edge regular $5-$polar fuzzy graph.

Definition 2.4. Let G be an $m-$polar fuzzy graph on V. If each edge in G has the same total degree $(t_1, t_2, \ldots, t_m)$, then G is called a *totally edge regular* $m-$polar fuzzy graph.

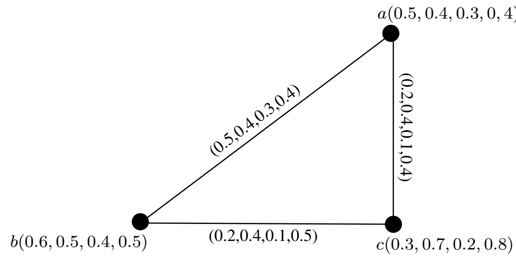

FIGURE 2. Totally Edge Regular 4-polar Fuzzy Graph

Example 2.5. Consider a $4-$polar fuzzy graph G on $V = \{a, b, c\}$, as shown in Figure 2. By direct calculations, it is easy to see from Figure 2, that

$$d_G(a) = (0.7, 0.8, 0.4, 0.8), \quad d_G(b) = (0.7, 0.8, 0.4, 0.9)$$

and $d_G(c) = (0.4, 0.8, 0.2, 0.9)$.

(i) By simple calculations, the degree of each edge is given below:

$$\begin{aligned} d_G(ab) &= d_G(a) + d_G(b) - 2p_i \circ D(ab) \\ &= (0.7, 0.8, 0.4, 0.8) + (0.7, 0.8, 0.4, 0.9) - 2(0.5, 0.4, 0.3, 0.4) \\ &= (0.4, 0.8, 0.2, 0.9), \end{aligned}$$

$$\begin{aligned} d_G(bc) &= d_G(b) + d_G(c) - 2p_i \circ D(bc) \\ &= (0.7, 0.8, 0.4, 0.9) + (0.4, 0.8, 0.2, 0.9) - 2(0.2, 0.4, 0.1, 0.5) \\ &= (0.7, 0.8, 0.4, 0.8), \end{aligned}$$

$$\begin{aligned} d_G(ca) &= d_G(c) + d_G(a) - 2p_i \circ D(ca) \\ &= (0.4, 0.8, 0.2, 0.9) + (0.7, 0.8, 0.4, 0.8) - 2(0.2, 0.4, 0.1, 0.4) \\ &= (0.7, 0.8, 0.4, 0.9). \end{aligned}$$

Clearly, $d_G(ab) \neq d_G(bc) \neq d_G(ca)$. So G is not an edge regular $4-$polar fuzzy graph.

(ii) The total degree of each edge is computed as:

$$td_G(ab) = d_G(ab) + p_i \circ D(ab) = (0.9, 1.2, 0.5, 1.3),$$

$$td_G(bc) = d_G(bc) + p_i \circ D(bc) = (0.9, 1.2, 0.5, 1.3),$$

$$td_G(ca) = d_G(ca) + p_i \circ D(ca) = (0.9, 1.2, 0.5, 1.3).$$

Clearly, total degree of each edge in G is the same. So, G is a totally edge regular $4-$polar fuzzy graph.

Remark 2.6. Note that

(i) $\delta_E(G) = \Delta_E(G) = r = (r_1, r_2, ..., r_m)$ if and only if G is a $r = (r_1, r_2, ..., r_m)$-edge regular $m-$polar fuzzy graph.

(ii) $\delta_{tE}(G) = \Delta_{tE}(G) = t = (t_1, t_2, ..., t_m)$ if and only if G is a $t = (t_1, t_2, ..., t_m)$-totally edge regular $m-$polar fuzzy graph.

Remark 2.7. A complete $m-$polar fuzzy graph G may not be edge regular $m-$polar fuzzy graph as seen in the following example.

Example 2.8. Consider a $4-$polar fuzzy graph G on $V = \{a, b, c, d\}$, as shown in Figure 3.

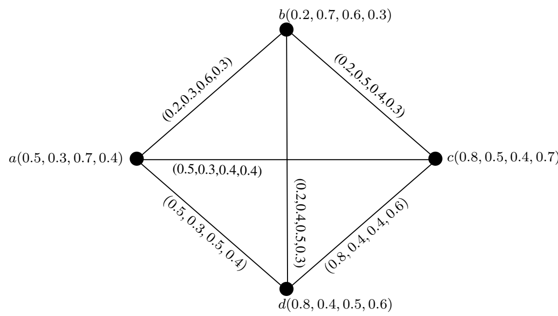

FIGURE 3. Complete 4-polar Fuzzy Graph

By direct calculations, it is easy to see from Figure 3, that $d_G(a) = (1.2, 0.9, 1.5, 1.1)$, $d_G(b) = (0.6, 1.2, 1.5, 0.9), d_G(c) = (1.5, 1.2, 1.2, 1.3)$ and $d_G(d) = (1.5, 1.1, 1.4, 1.3)$. The degrees of the edges are given below:

$$\begin{aligned}
d_G(ab) &= d_G(a) + d_G(b) - 2p_i \circ D(ab)\\
&= (1.2, 0.9, 1.5, 1.1) + (0.6, 1.2, 1.5, 0.9) - 2(0.2, 0.3, 0.6, 0.3)\\
&= (1.4, 1.5, 1.8, 1.4),\\
d_G(bc) &= d_G(b) + d_G(c) - 2p_i \circ D(bc)\\
&= (0.6, 1.2, 1.5, 0.9) + (1.5, 1.2, 1.2, 1.3) - 2(0.2, 0.5, 0.4, 0.3)\\
&= (1.7, 1.4, 1.9, 1.6),\\
d_G(cd) &= d_G(c) + d_G(d) - 2p_i \circ D(cd)\\
&= (1.5, 1.2, 1.2, 1.3) + (1.5, 1.1, 1.4, 1.3) - 2(0.8, 0.4, 0.4, 0.6)\\
&= (1.4, 1.5, 1.8, 1.4),\\
d_G(da) &= d_G(d) + d_G(a) - 2p_i \circ D(da)\\
&= (1.5, 1.1, 1.4, 1.3) + (1.2, 0.9, 1.5, 1.1) - 2(0.5, 0.3, 0.5, 0.4)\\
&= (1.7, 1.4, 1.9, 1.6),\\
d_G(ac) &= d_G(a) + d_G(c) - 2p_i \circ D(ac)\\
&= (1.2, 0.9, 1.5, 1.1) + (1.5, 1.2, 1.2, 1.3) - 2(0.5, 0.3, 0.4, 0.4)\\
&= (1.7, 1.5, 1.9, 1.6),\\
d_G(bd) &= d_G(b) + d_G(d) - 2p_i \circ D(bd)\\
&= (0.6, 1.2, 1.5, 0.9) + (1.5, 1.1, 1.4, 1.3) - 2(0.2, 0.4, 0.5, 0.3)\\
&= (1.7, 1.5, 1.9, 1.6).
\end{aligned}$$

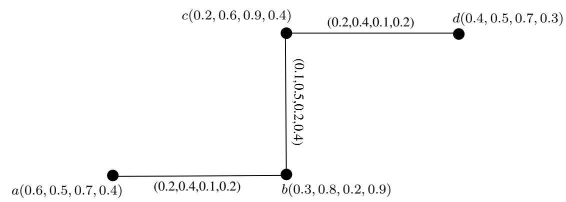

FIGURE 4. 4-polar Fuzzy Graph

Clearly, $d_G(ab) \neq d_G(bc) \neq d_G(ac)$. So, G is not an edge regular 4−polar fuzzy graph. Hence G is a complete 4−polar fuzzy graph but not an edge regular 4−polar fuzzy graph.

Example 2.9. Consider a 4−polar fuzzy graph G on non-empty set $V = \{a, b, c, d\}$ as shown in Figure 4. By direct calculations, it is easy to see from Figure 4, that $d_G(a) = (0.2, 0.4, 0.1, 0.2)$,
$d_G(b) = (0.3, 0.9, 0.3, 0.6), d_G(c) = (0.3, 0.9, 0.3, 0.6)$ and $d_G(d) = (0.2, 0.4, 0.1, 0.2)$. The degree of each edge is $d_G(ab) = (0.1, 0.5, 0.2, 0.4), d_G(bc) = (0.4, 0.8, 0.2, 0.4)$ and $d_G(cd) = (0.1, 0.5, 0.2, 0.4)$. Since $d_G(ab) \neq d_G(bc), G$ is not an edge regular 4−polar fuzzy graph.

Then $td_G(ab) = (0.3, 0.9, 0.3, 0.6)$, $td_G(bc) = (0.5, 1.3, 0.4, 0.8)$ and $td_G(cd) = (0.3, 0.9, 0.3, 0.6)$. Since $td_G(ab) \neq td_G(bc)$, G is not totally edge regular 4−polar fuzzy graph.

We state the following propositions with their proofs.

Proposition 2.10. *Let G be an $m-$polar fuzzy graph. Then*

$$\sum_{pq\in E} d_G(pq) = \sum_{pq\in E} d_{G^*}(pq)(p_i \circ D(pq)),$$

where $d_{G^*}(pq) = d_{G^*}(p) + d_{G^*}(q) - 2$, for all $p, q \in V$.

Proposition 2.11. *Let G be an $m-$polar fuzzy graph. Then*

$$\sum_{pq\in E} td_G(pq) = \sum_{pq\in E} d_{G^*}(pq)(p_i \circ D(pq)) + S(G).$$

Proposition 2.12. *Let G be a $(r_1, r_2, \ldots, r_m)-$edge regular $m-$polar fuzzy graph of a r-edge regular graph G^*. Then the size of G is $(\frac{nr_1}{r}, \frac{nr_2}{r}, \ldots, \frac{nr_m}{r})$, where $|E| = n$.*

Theorem 2.13. *Let G be a $(t_1, t_2, \ldots, t_m)-$totally edge regular $m-$polar fuzzy graph of a r-edge regular graph G^*. Then the size of G is $(\frac{nt_1}{r+1}, \frac{nt_2}{r+1}, \ldots, \frac{nt_m}{r+1})$, where $|E| = n$.*

Proof. Let G be a $(t_1, t_2, \ldots, t_m)-$totally edge regular $m-$polar fuzzy graph and G^* a r-edge regular crisp graph. That is, $td_G(pq) = (t_1, t_2, \ldots, t_m)$ and $d_{G^*}(pq) = r$, for all $pq \in E$. Since,

$$\begin{aligned} \sum_{pq\in E} td_G(pq) &= \sum_{pq\in E} d_{G^*}(pq)p_i \circ D(pq) + S(G) \\ \Rightarrow \sum_{pq\in E}(t_1, t_2, \ldots, t_m) &= r \sum_{pq\in E} p_i \circ D(pq) + S(G) \\ \Rightarrow n(t_1, t_2, \ldots, t_m) &= rS(G) + S(G) \\ \Rightarrow (nt_1, nt_2, \ldots, nt_m) &= (r+1)S(G). \end{aligned}$$

Hence $S(G) = (\frac{nt_1}{r+1}, \frac{nt_2}{r+1}, \ldots, \frac{nt_m}{r+1})$. This ends the proof. □

Theorem 2.14. *Let G be a $(r_1, r_2, \ldots, r_m)-$edge regular and $(t_1, t_2, \ldots, t_m)$-totally edge regular $m-$polar fuzzy graph of a graph $G^* = (V, E)$. Then the size of G is $n(t_1 - r_1, t_2 - r_2, \ldots, t_m - r_m)$, where $|E| = n$.*

Proof. Suppose that G is a $(r_1, r_2, \ldots, r_m)-$edge regular and $(t_1, t_2, \ldots, t_m)$-totally edge regular $m-$polar fuzzy graph, i.e., $d_G(pq) = (r_1, r_2, \ldots, r_m)$ and $td_G(pq) = (t_1, t_2, \ldots, t_m)$. Since $|E| = n$, so $\sum_{pq\in E} d_G(pq) = n(r_1, r_2, \ldots, r_m)$ and $\sum_{pq\in E} td_G(pq) = n(t_1, t_2, \ldots, t_m)$. Now,

$$\begin{aligned} td_G(pq) &= d_G(pq) + p_i \circ D(pq) \\ \sum_{pq\in E} td_G(pq) &= \sum_{pq\in E} d_G(pq) + \sum_{pq\in E} p_i \circ D(pq) \\ n(t_1, t_2, \ldots, t_m) &= n(r_1, r_2, \ldots, r_m) + S(G). \end{aligned}$$

This implies that,

$$S(G) = n(t_1 - r_1, t_2 - r_2, \ldots, t_m - r_m),$$

which ends the proof. □

Theorem 2.15. *Let G be an $m-$polar fuzzy graph of a graph G^*, which is a cycle on n vertices. Then*

$$\sum_{v_j\in V} d_G(v_j) = \sum_{v_jv_k\in E} d_G(v_jv_k).$$

Proof. Let G be an $m-$polar fuzzy graph of a graph G^*. Let G^* be a cycle $v_1, v_2, \ldots, v_n, v_1$ on n vertices. Then

$$\begin{aligned} \sum_{v_jv_k\in E} d_G(v_jv_k) &= d_G(v_1v_2) + d_G(v_2v_3) + \ldots + d_G(v_nv_1) \\ &= d_G(v_1) + d_G(v_2) - p_i \circ D(v_1v_2) + d_G(v_2) + d_G(v_3) - p_i \circ D(v_2v_3) \\ &\quad + \ldots + d_G(v_n) + d_G(v_1) - p_i \circ D(v_nv_1) \\ &= 2d_G(v_1) + 2d_G(v_2) + \ldots + 2d_G(v_n) - (p_i \circ D(v_1v_2) + p_i \circ D(v_2v_3) \\ &\quad + \ldots + p_i \circ D(v_nv_1)) \\ &= 2\sum_{v_j\in V} d_G(v_j) - 2\sum_{v_jv_k\in E} p_i \circ D(v_jv_k) \\ &= \sum_{v_j\in V} d_G(v_j) + \sum_{v_j\in V} d_G(v_j) - 2\sum_{v_jv_k\in E} p_i \circ D(v_jv_k) \\ &= \sum_{v_j\in V} d_G(v_j) + 2\sum_{v_jv_k\in E} p_i \circ D(v_jv_k) - 2\sum_{v_jv_k\in E} p_i \circ D(v_jv_k) \\ &= \sum_{v_j\in V} d_G(v_j). \end{aligned}$$

This completes the proof. □

Theorem 2.16. *Let $G = (C, D)$ be an $m-$polar fuzzy graph. Then D is a constant function if and only if the following statements are equivalent:*

(i) *G is an edge regular $m-$polar fuzzy graph,*
(ii) *G is a totally edge regular $m-$polar fuzzy graph.*

Proof. Assume that D is a constant function, i.e., $p_i \circ D(pq) = (k_1, k_2, \ldots, k_m)$, for all $pq \in E$.

$(i) \Rightarrow (ii)$: Suppose that G is a $(r_1, r_2, \ldots, r_m)-$edge regular $m-$polar fuzzy graph. So $d_G(pq) = (r_1, r_2, \ldots, r_m)$, for all $pq \in E$. Then obviously $td_G(pq) = d_G(pq) + p_i \circ D(pq)$, for all $pq \in E$, i.e., $td_G(pq) = (k_1 + r_1, k_2 + r_2, \ldots, k_m + r_m)$, for all $pq \in E$. Hence G is a $(k_1 + r_1, k_2 + r_2, \ldots, k_m + r_m)-$totally edge regular $m-$polar fuzzy graph.

$(ii) \Rightarrow (i)$: Assume that G is a $(t_1, t_2, \ldots, t_m)-$totally edge regular $m-$polar fuzzy graph. So, $td_G(pq) = (t_1, t_2, \ldots, t_m)$, for all $pq \in E$. Thus, $d_G(pq) + p_i \circ D(pq) = (t_1, t_2, \ldots, t_m)$. Consequently, $d_G(pq) = (t_1, t_2, \ldots, t_m) - p_i \circ D(pq)$. therefore, $d_G(pq) = (t_1 - k_1, t_2 - k_2, \ldots, t_m - k_m)$, for all $pq \in E$. Hence G is a $(t_1 - k_1, t_2 - k_2, \ldots, t_m - k_m)-$edge regular $m-$polar fuzzy graph.

Thus the statements (i) and (ii) are equivalent.

Conversely, assume that (i) and (ii) are equivalent. If D is not a constant function, then $p_i \circ D(pq) \neq p_i \circ D(vw)$ for at least one pair of edges $pq, vw \in E$. Suppose that G is a $(r_1, r_2, \ldots, r_m)-$edge regular $m-$polar fuzzy graph. Then it holds $d_G(pq) = d_G(vw) = (r_1, r_2, \ldots, r_m)$, i.e., $td_G(pq) = d_G(pq) + p_i \circ D(pq) = (r_1, r_2, \ldots, r_m) + p_i \circ D(pq)$ and $td_G(vw) = d_G(vw) + p_i \circ D(vw) = (r_1, r_2, \ldots, r_m) + p_i \circ D(vw)$. Since $p_i \circ D(pq) \neq p_i \circ D(vw)$, $td_G(pq) \neq td_G(vw)$. Hence G is not totally edge regular $m-$polar fuzzy graph, which is a contradiction to our supposition.

Now, assume that G is a totally edge regular $m-$polar fuzzy graph. Then $td_G(pq) = td_G(vw) = (t_1, t_2, \ldots, t_m) = d_G(pq) + p_i \circ D(pq) = d_G(vw) + p_i \circ D(vw)$. So, $d_G(pq) - d_G(vw) = p_i \circ D(pq) - p_i \circ D(vw)$. Since $p_i \circ D(pq) \neq p_i \circ D(vw)$, we have $d_G(pq) - d_G(vw) \neq 0$, i.e., $d_G(pq) \neq d_G(vw)$. Hence G is not edge regular $m-$polar fuzzy graph, which is a contradicts our supposition. Thus D is a constant function. □

Theorem 2.17. *Let $G = (C, D)$ be an $m-$polar fuzzy graph. Suppose that G is both edge regular and totally edge regular $m-$polar fuzzy graph. Then D is a constant function.*

Proof. Obvious. □

Remark 2.18. The converse of Theorem 2.17 may not be true in general, i.e., an $m-$polar fuzzy graph $G = (C, D)$, where D is a constant function on $V \times V$, may not be edge regular and totally edge regular $m-$polar fuzzy graph.

Example 2.19. Consider a $3-$polar fuzzy graph $G = (C, D)$ of a graph G^*, where $V = \{a, b, c, d\}$ and $E = \{ab, bc, ca, bd\}$, as shown in Figure 5.

Then $d_G(a) = (1.0, 0.8, 1.2), d_G(b) = (1.5, 1.2, 1.8), d_G(c) = (1.0, 0.8, 1.2)$ and $d_G(d) = (0.5, 0.4, 0.6)$. The degree of each edge is $d_G(ab) = (1.5, 1.2, 1.8)$, $d_G(bc) =$

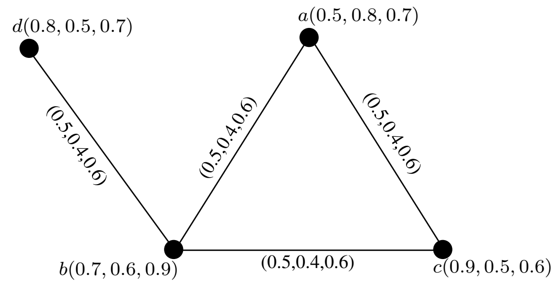

FIGURE 5. 3-polar Fuzzy Graph

$(1.5, 1.2, 1.8)$, $d_G(ac) = (1.0, 0.8, 1.2)$ and $d_G(bd) = (1.0, 0.8, 1.2)$. The total degree of each edge is $td_G(ab) = (2.0, 1.6, 2.4)$, $td_G(bc) = (2.0, 1.6, 2.4)$, $td_G(ac) = (1.5, 1.2, 1.8)$ and $td_G(bd) = (1.5, 1.2, 1.8)$.

It follows from the above calculations that G is neither edge regular nor totally edge regular 3-polar fuzzy graph.

Theorem 2.20. *Let $G = (C, D)$ be an $m-$polar fuzzy graph, where D is a constant function. If G is regular $m-$polar fuzzy graph. Then G is edge regular $m-$polar fuzzy graph.*

Proof. Let D be a constant function, i.e., $p_i \circ D(pq) = (k_1, k_2, \dots, k_m)$, for all $pq \in E$. Suppose that G is a regular $m-$polar fuzzy graph, i.e., $d_G(p) = (s_1, s_2, \dots, s_m)$, for all $p \in V$. Now,

$$\begin{aligned} d_G(pq) &= d_G(p) + d_G(q) - 2p_i \circ D(pq) \\ &= (s_1, s_2, \dots, s_m) + (s_1, s_2, \dots, s_m) - 2(k_1, k_2, \dots, k_m) \\ &= 2(s_1 - k_1, s_2 - k_, \dots, s_m - k_m), \quad \text{for all } pq \in E. \end{aligned}$$

Hence G is edge regular $m-$polar fuzzy graph. □

Theorem 2.21. *Let $G = (C, D)$ be an $m-$polar fuzzy graph, where D is a constant function. If G is regular $m-$polar fuzzy graph. Then G is totally edge regular $m-$polar fuzzy graph.*

Proof. Assume that D is a constant function, i.e., $p_i \circ D(pq) = (k_1, k_2, \dots, k_m)$, for all $pq \in E$. Suppose that G is a regular $m-$polar fuzzy graph, i.e., $d_G(p) = (s_1, s_2, \dots, s_m)$, for all $p \in V$. Then G is edge regular $m-$polar fuzzy graph, i.e., $d_G(pq) = (r_1, r_2, \dots, r_m)$. Now,

$$\begin{aligned} td_G(pq) &= d_G(pq) + p_i \circ D(pq) \\ &= (r_1, r_2, \dots, r_m) + (k_1, k_2, \dots, k_m) \\ &= (r_1 + k_1, r_2 + k_2, \dots, r_m + k_m). \end{aligned}$$

Hence G is a totally edge regular $m-$polar fuzzy graph. □

Theorem 2.22. *Let $G = (C, D)$ be an $m-$polar fuzzy graph of a graph $G^* = (V, E)$ which is s-regular. Then G is both regular and totally edge regular $m-$polar fuzzy graph if and only if D is a constant function.*

Proof. Let $G = (C, D)$ be an $m-$polar fuzzy graph of a graph $G^* = (V, E)$ which is s-regular. Suppose that G is both regular and totally edge regular $m-$polar fuzzy graph. Then $d_G(p) = (r_1, r_2, \ldots, r_m)$ and $td_G(pq) = (t_1, t_2, \ldots, t_m)$, for all $p \in V$ and $pq \in E$. Now, for all $pq \in E$, will be

$$\begin{aligned} td_G(pq) &= d_G(p) + d_G(q) - p_i \circ D(pq) \\ (t_1, t_2, \ldots, t_m) &= (r_1, r_2, \ldots, r_m) + (r_1, r_2, \ldots, r_m) - p_i \circ D(pq) \\ p_i \circ D(pq) &= (2r_1 - t_1, 2r_2 - t_2, \ldots, 2r_m - t_m). \end{aligned}$$

Thus, D is a constant function.

Conversely, if D is a constant function, then $p_i \circ D(pq) = (k_1, k_2, \ldots, k_m)$, for all $pq \in E$. Therefore, for all $p \in V$, we have

$$\begin{aligned} d_G(p) &= \sum_{pq \in E} p_i \circ D(pq) \\ &= \sum_{pq \in E} (r_1, r_2, \ldots, r_m) \\ &= (r_1, r_2, \ldots, r_m) d_{G^*}(p) \\ &= (r_1, r_2, \ldots, r_m) s \\ &= (sr_1, sr_2, \ldots, sr_m). \end{aligned}$$

So, G is a regular $m-$polar fuzzy graph.

$$\begin{aligned} td_G(pq) &= \sum_{pl \in El \neq q} p_i \circ D(pl) + \sum_{lq \in El \neq p} p_i \circ D(lq) + p_i \circ D(pq) \\ &= \sum_{pl \in El \neq q} (k_1, k_2, \ldots, k_m) + \sum_{lq \in El \neq p} (k_1, k_2, \ldots, k_m) + (k_1, k_2, \ldots, k_m) \\ &= (k_1, k_2, \ldots, k_m)(d_{G^*}(p) - 1) + (k_1, k_2, \ldots, k_m)(d_{G^*}(q) - 1) \\ &+ (k_1, k_2, \ldots, k_m) \\ &= (k_1, k_2, \ldots, k_m)(s - 1) + (k_1, k_2, \ldots, k_m)(s - 1) + (k_1, k_2, \ldots, k_m) \\ &= (2k_1, 2k_2, \ldots, 2k_m)(s - 1) + (k_1, k_2, \ldots, k_m), \qquad \text{for all } pq \in E. \end{aligned}$$

Thus, G is totally edge regular $m-$polar fuzzy graph. □

Theorem 2.23. *Let $G = (C, D)$ be a regular $m-$polar fuzzy graph. Then D is a constant function if and only if G is edge regular $m-$polar fuzzy graph.*

Proof. Suppose that G is $(s_1, s_2, \ldots, s_m)-$regular $m-$polar fuzzy graph i.e., $d_G(p) = (s_1, s_2, \ldots, s_m)$, for all $p \in V$. Let D be a constant function, i.e., $p_i \circ D(pq) = (k_1, k_2, \ldots, k_m)$, for all $pq \in E$. Now, for all $pq \in E$

$$\begin{aligned} d_G(pq) &= d_G(p) + d_G(q) - 2(p_i \circ D(pq)) \\ &= (s_1, s_2, \ldots, s_m) + (s_1, s_2, \ldots, s_m) - 2(k_1, k_2, \ldots, k_m) \\ &= 2(s_1, s_2, \ldots, s_m) - 2(k_1, k_2, \ldots, k_m). \end{aligned}$$

Therefore G is an edge regular $m-$polar fuzzy graph.

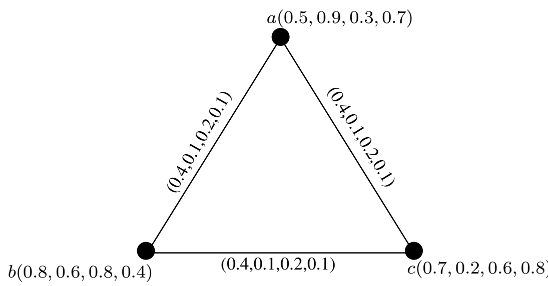

FIGURE 6. Full Edge Regular 4-polar Fuzzy Graph

Conversely, suppose that G is an edge regular $m-$polar fuzzy graph, i.e., $d_G(pq) = (r_1, r_2, \ldots, r_m)$, for all $pq \in E$. Now,

$$\begin{aligned} d_G(pq) &= d_G(p) + d_G(q) - 2(p_i \circ D(pq)) \\ (r_1, r_2, \ldots, r_m) &= (s_1, s_2, \ldots, s_m) + (s_1, s_2, \ldots, s_m) - 2(p_i \circ D(pq)) \\ p_i \circ D(pq) &= \frac{(r_1, r_2, \ldots, r_m) - (2s_1, 2s_2, \ldots, 2s_m)}{2}, \quad \text{for all } pq \in E. \end{aligned}$$

Hence D is a constant function. □

Definition 2.24. An $m-$polar fuzzy graph G of an edge regular crisp graph G^* is called a *partially edge regular* $m-$polar fuzzy graph.

Definition 2.25. An $m-$polar fuzzy graph G, which is both edge regular and partially edge regular is called a *full edge regular* $m-$polar fuzzy graph.

Example 2.26. Consider a $4-$polar fuzzy graph $G = (C, D)$ on $V = \{a, b, c\}$, as shown in Figure 6.

By direct calculations, it follows that G is full edge regular $4-$polar fuzzy graph.

Theorem 2.27. *Let G be an $m-$polar fuzzy graph of a graph G^*, where D is a constant function. If G is full regular $m-$polar fuzzy graph, then G is full edge regular $m-$polar fuzzy graph.*

Proof. Let D be a constant function, i.e., $p_i \circ D(pq) = (k_1, k_2, \ldots, k_m)$, for all $pq \in E$, where $k_1, k_2, \ldots, k_m$ are constants. Suppose that G is full regular $m-$polar fuzzy graph. That is G is both regular and partially regular. Then $d_G(p) = (s_1, s_2, \ldots, s_m)$ and $d_{G^*}(p) = s$, for all $p \in V$, where s and $s_1, s_2, \ldots, s_m$ are constants. Now, $d_{G^*}(pq) = d_{G^*}(p) + d_{G^*}(q) - 2 = 2s - 2$, which is a constant. Thus G^* is edge regular graph. Also, for all $pq \in E$, we have

$$\begin{aligned} d_G(pq) &= d_G(p) + d_G(q) - 2(p_i \circ D(pq)) \\ &= (s_1, s_2, \ldots, s_m) + (s_1, s_2, \ldots, s_m) - 2(k_1, k_2, \ldots, k_m) \\ &= 2(s_1 - k_1, s_2 - k_2, \ldots, s_m - k_m), \quad \text{which is a constant.} \end{aligned}$$

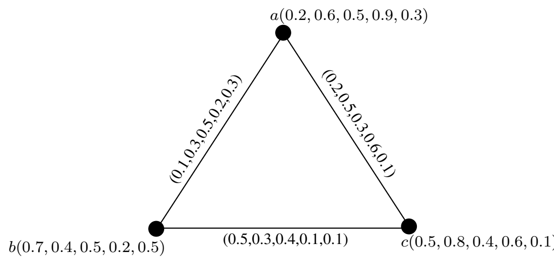

Figure 7. Strongly Edge Irregular 5-polar Fuzzy Graph

Thus, G is an edge regular $m-$polar fuzzy graph. Hence G is a full edge regular $m-$polar fuzzy graph. □

3. Edge Irregularity in $m-$polar Fuzzy Graphs

Definition 3.1. Let G be an $m-$polar fuzzy graph. If each edge in G has distinct degree, G is called a *strongly edge irregular* $m-$polar fuzzy graph. That is, in a *strongly edge irregular* $m-$polar fuzzy graph no two edges have the same degree.

Example 3.2. Consider a $5-$polar fuzzy graph G on $V = \{a, b, c\}$ as shown in Figure 7. In this case $d_G(a) = (0.3, 0.8, 0.8, 0.8, 0.4)$, $d_G(b) = (0.6, 0.6, 0.9, 0.3, 0.4)$, and $d_G(c) = (0.7, 0.8, 0.7, 0.7, 0.2)$.

The degrees of the edges are calculated as:

$$\begin{aligned} d_G(ab) &= d_G(a) + d_G(b) - 2p_i \circ D(ab) \\ &= (0.3, 0.8, 0.8, 0.8, 0.4) + (0.6, 0.6, 0.9, 0.3, 0.4) - 2(0.1, 0.3, 0.5, 0.2, 0.3) \\ &= (0.7, 0.8, 0.7, 0.7, 0.2), \end{aligned}$$

$$\begin{aligned} d_G(bc) &= d_G(b) + d_G(c) - 2p_i \circ D(bc) \\ &= (0.6, 0.6, 0.9, 0.3, 0.4) + (0.7, 0.8, 0.7, 0.7, 0.2) - 2(0.5, 0.3, 0.4, 0.1, 0.1) \\ &= (0.3, 0.8, 0.8, 0.8, 0.4), \end{aligned}$$

$$\begin{aligned} d_G(ca) &= d_G(c) + d_G(a) - 2p_i \circ D(ca) \\ &= (0.7, 0.8, 0.7, 0.7, 0.2) + (0.3, 0.8, 0.8, 0.8, 0.4) - 2(0.2, 0.5, 0.3, 0.6, 0.1) \\ &= (0.6, 0.6, 0.9, 0.3, 0.4). \end{aligned}$$

All edges have distinct degrees. Hence G is strongly edge irregular $5-$polar fuzzy graph.

Definition 3.3. Let G be an $m-$polar fuzzy graph. If each edge in G has distinct total degree, then G is called a *strongly edge totally irregular* $m-$polar fuzzy graph. That is, in a *strongly edge totally irregular* $m-$polar fuzzy graph no two edges have the same total degree.

Example 3.4. Consider a $5-$polar fuzzy graph G on $V = \{a, b, c, d\}$ as shown in Figure 8. We have $d_G(a) = (0.4, 0.2, 1.0, 0.8, 0.2)$, $d_G(b) = (0.5, 0.2, 0.9, 0.5, 0.4)$,

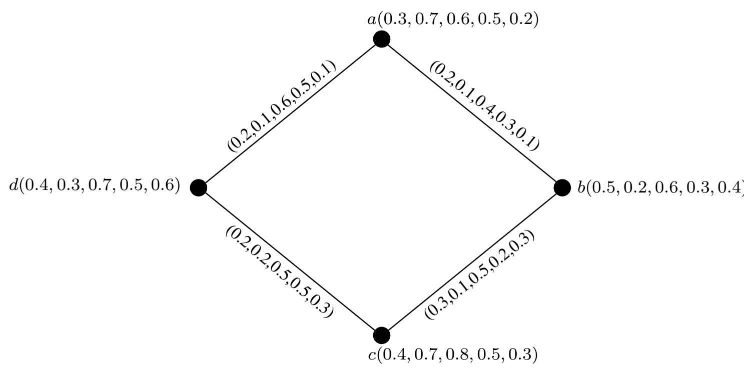

FIGURE 8. Strongly Edge Totally Irregular 5-polar Fuzzy Graph

$d_G(c) = (0.5, 0.3, 1.0, 0.7, 0.6)$ and $d_G(d) = (0.4, 0.3, 1.1, 1.0, 0.4)$.

(i) The degree of each edge is computed as:

$$\begin{aligned} d_G(ab) &= d_G(a) + d_G(b) - 2p_i \circ D(ab) \\ &= (0.4, 0.2, 1.0, 0.8, 0.2) + (0.5, 0.2, 0.9, 0.5, 0.4) - 2(0.2, 0.1, 0.4, 0.3, 0.1) \\ &= (0.5, 0.2, 1.1, 0.7, 0.4), \end{aligned}$$

$$\begin{aligned} d_G(bc) &= d_G(b) + d_G(c) - 2p_i \circ D(bc) \\ &= (0.5, 0.2, 0.9, 0.5, 0.4) + (0.5, 0.3, 1.0, 0.7, 0.6) - 2(0.3, 0.1, 0.5, 0.2, 0.3) \\ &= (0.4, 0.3, 0.9, 0.8, 0.4), \end{aligned}$$

$$\begin{aligned} d_G(cd) &= d_G(c) + d_G(d) - 2p_i \circ D(cd) \\ &= (0.5, 0.3, 1.0, 0.7, 0.6) + (0.4, 0.3, 1.1, 1.0, 0.4) - 2(0.2, 0.2, 0.5, 0.5, 0.3) \\ &= (0.5, 0.2, 1.1, 0.7, 0.4), \end{aligned}$$

$$\begin{aligned} d_G(da) &= d_G(d) + d_G(a) - 2p_i \circ D(da) \\ &= (0.4, 0.3, 1.1, 1.0, 0.4) + (0.4, 0.2, 1.0, 0.8, 0.2) - 2(0.2, 0.1, 0.6, 0.5, 0.1) \\ &= (0.4, 0.3, 0.9, 0.8, 0.4). \end{aligned}$$

(ii) The total degree of each edge is computed as:

$$\begin{aligned} td_G(ab) &= d_G(ab) + p_i \circ D(ab) \\ &= (0.5, 0.2, 1.1, 0.7, 0.4) + (0.2, 0.1, 0.4, 0.3, 0.1) = (0.7, 0.3, 1.5, 1.0, 0.5), \\ td_G(bc) &= d_G(bc) + p_i \circ D(bc) \\ &= (0.4, 0.3, 0.9, 0.8, 0.4) + (0.3, 0.1, 0.5, 0.2, 0.3) = (0.7, 0.4, 1.4, 1.0, 0.7), \\ td_G(cd) &= d_G(cd) + p_i \circ D(cd) \\ &= (0.5, 0.2, 1.1, 0.7, 0.4) + (0.2, 0.2, 0.5, 0.5, 0.3) = (0.7, 0.4, 1.6, 1.2, 0.7), \\ td_G(da) &= d_G(da) + p_i \circ D(da) \\ &= (0.4, 0.3, 0.9, 0.8, 0.4) + (0.2, 0.1, 0.6, 0.5, 0.1) = (0.6, 0.4, 1.5, 1.3, 0.5). \end{aligned}$$

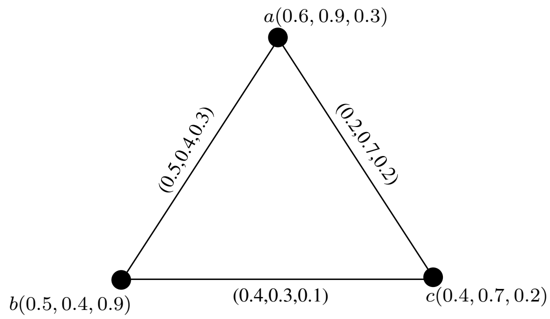

FIGURE 9. 3-polar Fuzzy Graph

Since all the edges have distinct total degrees, G is strongly edge totally irregular $5-$polar fuzzy graph.

Remark 3.5. A strongly edge irregular $m-$polar fuzzy graph G may not be strongly edge totally irregular $m-$polar fuzzy graph.

Example 3.6. Consider a $3-$polar fuzzy graph of a graph $G^* = (V, E)$, where $V = \{a, b, c\}$ and $E = \{ab, bc, ca\}$, as shown in Figure 9. By direct calculations, it is easy to see from Figure 9, that $d_G(a) = (0.7, 1.1, 0.5)$,
$d_G(b) = (0.9, 0.7, 0.4)$, $d_G(c) = (0.6, 1.0, 0.3)$.

(i) The degree of each edge is computed as:

$$d_G(ab) = d_G(a) + d_G(b) - 2p_i \circ D(ab) = (0.7, 1.1, 0.5) + (0.9, 0.7, 0.4) - 2(0.5, 0.4, 0.3) = (0.6, 1.0, 0.3),$$

$$d_G(bc) = d_G(b) + d_G(c) - 2p_i \circ D(bc) = (0.9, 0.7, 0.4) + (0.6, 1.0, 0.3) - 2(0.4, 0.3, 0.1) = (0.7, 1.1, 0.5),$$

$$d_G(ca) = d_G(c) + d_G(a) - 2p_i \circ D(ca) = (0.6, 1.0, 0.3) + (0.7, 1.1, 0.5) - 2(0.2, 0.7, 0.2) = (0.9, 0.7, 0.4).$$

Since all the edges have distinct degrees, G is strongly edge irregular $3-$polar fuzzy graph.

(ii) The total degree of each edge is computed as:

$$td_G(ab) = d_G(ab) + p_i \circ D(ab) = (1.1, 1.4, 0.6),$$
$$td_G(bc) = d_G(bc) + p_i \circ D(bc) = (1.1, 1.4, 0.6),$$
$$td_G(ca) = d_G(ca) + p_i \circ D(ca) = (1.1, 1.4, 0.6).$$

Since all the edges have same total degrees, G is not totally edge irregular 3-polar fuzzy graph, a strongly edge irregular $m-$polar fuzzy graph may not be strongly edge totally irregular $m-$polar fuzzy graph.

Remark 3.7. A strongly edge totally irregular $m-$polar fuzzy graph G may not be strongly edge irregular $m-$polar fuzzy graph.

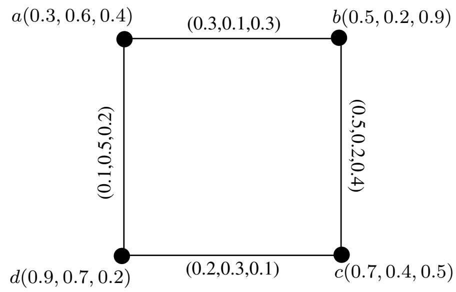

FIGURE 10. 3-polar Fuzzy Graph

Example 3.8. Consider a $3-$polar fuzzy graph G on $V = \{a, b, c, d\}$ as shown in Figure 10. From Figure 10, it follows that $d_G(a) = (0.4, 0.6, 0.5)$, $d_G(b) = (0.8, 0.3, 0.7)$,
$d_G(c) = (0.7, 0.5, 0.5)$ and $d_G(d) = (0.3, 0.8, 0.3)$.

The degree of each edge is $d_G(ab) = (0.6, 0.7, 0.6)$, $d_G(bc) = (0.5, 0.4, 0.4)$, $d_G(cd) = (0.6, 0.7, 0.6)$ and $d_G(da) = (0.5, 0.4, 0.4)$. Since $d_G(ab) = d_G(cd)$ and $d_G(bc) = d_G(da)$, G is not strongly edge irregular 3-polar fuzzy graph.

The total degree of each edge is $td_G(ab) = (0.9, 0.8, 0.9)$, $td_G(bc) = (1.0, 0.6, 0.8)$, $td_G(cd) = (0.8, 1.0, 0.7)$ and $td_G(da) = (0.6, 0.9, 0.6)$. Since all the edges have distinct total degrees, therefore G is a strongly edge totally irregular $3-$polar fuzzy graph but G is not strongly edge irregular $3-$polar fuzzy graph.

Theorem 3.9. *If $G = (C, D)$ is a strongly edge irregular connected $m-$polar fuzzy graph, where D is a constant function. Then G is a strongly edge totally irregular $m-$polar fuzzy graph.*

Proof. Let $G = (C, D)$ be a connected $m-$polar fuzzy graph. Suppose that D is a constant function, i.e., $p_i \circ D(vw) = b = (k_1, k_2, k_3, \dots, k_m)$, for all $pq \in E$, where $k_1, k_2, k_3, \dots, k_m$ are constants. Let st and vw be any pair of edges in E. Assume that G is strongly edge irregular $m-$polar fuzzy graph. Then $d_G(st) \neq d_G(vw)$, where st and vw are any pair of edges in E. This implies that $d_G(st) + b \neq d_G(vw) + b$. So, $d_G(st) + p_i \circ D(st) \neq d_G(vw) + p_i \circ D(vw)$, i.e., $td_G(st) \neq td_G(vw)$, where st and vw are any pair of edges in E. Thus every pair of edges in G have distinct total degrees. Hence G is a strongly edge totally irregular $m-$polar fuzzy graph. □

Theorem 3.10. *If $G = (C, D)$ is a strongly edge totally irregular connected $m-$polar fuzzy graph, where D is a constant function. Then G is a strongly edge irregular $m-$polar fuzzy graph.*

Proof. Let $G = (C, D)$ be a connected $m-$polar fuzzy graph. Suppose that D is a constant function, i.e., $p_i \circ D(vw) = b = (k_1, k_2, k_3, \dots, k_m)$, for all $pq \in E$, where $k_1, k_2, k_3, \dots, k_m$ are constants. Let st and vw be any pair of edges in E. Assume that G is strongly edge totally irregular $m-$polar fuzzy graph. Then $td_G(st) \neq td_G(vw)$, where st and vw are any pair of edges in E. This implies that

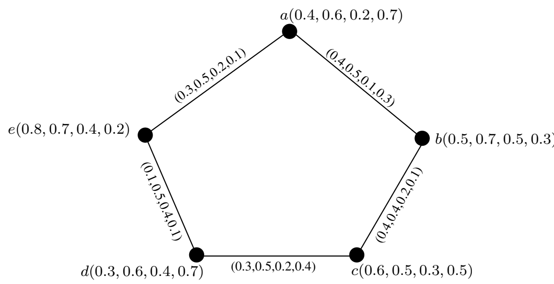

FIGURE 11. 4-polar Fuzzy Graph

$d_G(st) + p_i \circ D(st) \neq d_G(vw) + p_i \circ D(vw)$. Hence $d_G(st) + b \neq d_G(vw) + b$ and consequently, $d_G(st) \neq d_G(vw)$, where st and vw are any pair of edges in E. Thus every pair of edges in G have distinct degrees. Hence G is strongly edge irregular $m-$polar fuzzy graph. □

Remark 3.11. If $G = (C, D)$ is both strongly edge irregular $m-$polar fuzzy graph and strongly edge totally irregular $m-$polar fuzzy graph, then D may not be a constant function.

Example 3.12. Consider a $4-$polar fuzzy graph G on $V = \{a, b, c, d, e\}$ as shown in Figure 11. From Figure 11, we have $d_G(a) = (0.7, 1.0, 0.3, 0.4)$, $d_G(b) = (0.8, 0.9, 0.3, 0.4)$, $d_G(c) = (0.7, 0.9, 0.4, 0.5)$, $d_G(d) = (0.4, 1.0, 0.6, 0.5)$ and $d_G(e) = (0.4, 1.0, 0.6, 0.2)$.

(i) The degree of each edge is calculated as:

$$d_G(ab) = (0.7, 0.9, 0.4, 0.2),\ d_G(bc) = (0.7, 1.0, 0.3, 0.7),$$
$$d_G(cd) = (0.5, 0.9, 0.6, 0.2),\ d_G(de) = (0.6, 1.0, 0.4, 0.5),$$
$$d_G(ea) = (0.5, 1.0, 0.5, 0.4).$$

(ii) The total degree of each edge is calculated as:

$$td_G(ab) = (1.1, 1.4, 0.5, 0.5),\ td_G(bc) = (1.1, 1.4, 0.5, 0.8),$$
$$td_G(cd) = (0.8, 1.4, 0.8, 0.6),\ td_G(de) = (0.7, 1.5, 0.8, 0.6),$$
$$td_G(ea) = (0.8, 1.5, 0.7, 0.5).$$

Since all the edges have distinct degrees and total degrees, G is both strongly edge irregular and strongly edge totally irregular $4-$polar fuzzy graph but D is not a constant function.

Theorem 3.13. *Let G be a strongly edge irregular $m-$polar fuzzy graph. Then G is a neighborly edge irregular $m-$polar fuzzy graph.*

Proof. Suppose that G is a strongly edge irregular $m-$polar fuzzy graph. Then each edge in G has distinct degree. Therefore every pair of adjacent edges in G also have distinct degrees. Hence G is a neighborly edge irregular $m-$polar fuzzy graph. □

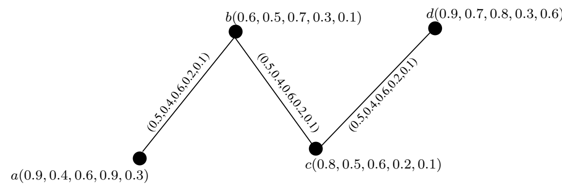

FIGURE 12. Neighborly Edge Irregular 5-polar Fuzzy Graph

Theorem 3.14. *Let G be a strongly edge totally irregular $m-$polar fuzzy graph. Then G is a neighborly edge totally irregular $m-$polar fuzzy graph.*

Proof. Suppose that G is a strongly edge totally irregular $m-$polar fuzzy graph. Then each edge in G has distinct total degree. Therefore, every pair of adjacent edges in G also have distinct total degrees. Hence $G = (C, D)$ is a neighborly edge totally irregular $m-$polar fuzzy graph. □

Remark 3.15. A neighborly edge irregular $m-$polar fuzzy graph G may not be strongly edge (totally) irregular $m-$polar fuzzy graph.

Example 3.16. Consider a $5-$polar fuzzy graph G on $V = \{a, b, c, d\}$ as shown in Figure 12. From Figure 12, we obtain

$d_G(a) = (0.5, 0.4, 0.6, 0.2, 0.1)$, $d_G(b) = (1.0, 0.8, 1.2, 0.4, 0.2)$,
$d_G(c) = (1.0, 0.8, 1.2, 0.4, 0.2)$, $d_G(d) = (0.5, 0.4, 0.6, 0.2, 0.1)$.

Then $d_G(ab) = (0.5, 0.4, 0.6, 0.2, 0.1)$, $d_G(bc) = (1.0, 0.8, 1.2, 0.4, 0.2)$ and $d_G(cd) = (0.5, 0.4, 0.6, 0.2, 0.1)$.

Note that $d_G(ab) \neq d_G(bc)$ and $d_G(bc) \neq d_G(cd)$, G is neighborly edge irregular $5-$polar fuzzy graph. Also, note that $d_G(ab) = d_G(cd)$. So, G is not strongly edge irregular $5-$polar fuzzy graph.

The total degrees of each edge is $td_G(ab) = (1.0, 0.8, 1.2, 0.4, 0.2)$, $td_G(bc) = (1.5, 1.2, 1.8, 0.6, 0.3)$ and $td_G(cd) = (1.0, 0.8, 1.2, 0.4, 0.2)$. We noted that $td_G(ab) \neq td_G(bc)$ and $td_G(bc) \neq td_G(cd)$, G is neighborly edge totally irregular $5-$polar fuzzy graph. Also, note that $td_G(ab) = td_G(cd)$. So, G is not strongly edge totally irregular $5-$polar fuzzy graph.

Theorem 3.17. *If $G = (C, D)$ is a strongly edge irregular connected $m-$polar fuzzy graph, where D is a constant function. Then G is an irregular $m-$polar fuzzy graph.*

Proof. Let $G = (C, D)$ be a connected $m-$polar fuzzy graph. Suppose that D is a constant function. That is, $p_i \circ D(pq) = b = (k_1, k_2, k_3, \ldots, k_m)$, for all $pq \in E$, where $k_1, k_2, k_3, \ldots, k_m$ are constants. Since G is strongly edge irregular $m-$polar fuzzy graph, so each edge in G has distinct degrees. Let vw and wx be any two adjacent edges in G such that $d_G(vw) \neq d_G(wx)$. So, $d_G(v)+d_G(w)-2(p_i \circ D(vw)) \neq d_G(w)+d_G(x)-2(p_i \circ D(wx))$. Hence, $d_G(v)+d_G(w)-2b \neq d_G(w)+d_G(x)-2b$, i.e., $d_G(v) \neq d_G(x)$. This means that there exists a vertex w in G which is adjacent to the vertices with distinct degrees. Hence G is an irregular $m-$polar fuzzy graph. □

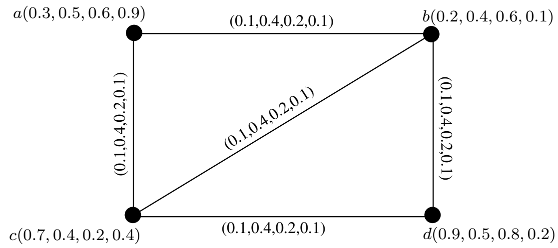

FIGURE 13. 4-polar Fuzzy Graph

Theorem 3.18. *If $G = (C, D)$ is a strongly edge totally irregular connected $m-$polar fuzzy graph, where D is a constant function. Then G is an irregular $m-$polar fuzzy graph.*

Proof. Let $G = (C, D)$ be a connected $m-$polar fuzzy graph. Suppose that D is a constant function. That is, $p_i \circ D(pq) = b = (k_1, k_2, k_3, \ldots, k_m)$, for all $pq \in E$, where $k_1, k_2, k_3, \ldots, k_m$ are constants. Since G is strongly edge totally irregular $m-$polar fuzzy graph, so each edge in G has distinct total degrees. Let vw and wx be any two adjacent edges in G such that $td_G(vw) \neq td_G(wx)$. Then obviously $d_G(vw) + p_i \circ D(vw) \neq d_G(wx) + p_i \circ D(wx)$. Hence $d_G(v) + d_G(w) - b \neq d_G(w) + d_G(x) - b$, i.e., $d_G(v) \neq d_G(x)$. Thus there exists a vertex w in G which is adjacent to the vertices with distinct degrees. Hence G is an irregular $m-$polar fuzzy graph. □

Remark 3.19. An irregular $m-$polar fuzzy graph $G = (C, D)$, where D is a constant function, may not be a strongly edge (totally) irregular $m-$polar fuzzy graph.

Example 3.20. Consider a 4$-$polar fuzzy graph of a graph $G^* = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{ab, bc, cd, bd, ac\}$, as shown in Figure 13. From Figure 13, we have $d_G(a) = (0.2, 0.8, 0.4, 0.2)$, $d_G(b) = (0.3, 1.2, 0.6, 0.3)$, $d_G(c) = (0.3, 1.2, 0.6, 0.3)$ and $d_G(d) = (0.2, 0.8, 0.4, 0.2)$. Hence G is an irregular 4$-$polar fuzzy graph.

(i) The degree of each edge is calculated as:

$$d_G(ab) = (0.3, 1.2, 0.6, 0.3), \quad d_G(bc) = (0.4, 1.6, 0.8, 0.4),$$
$$d_G(cd) = (0.3, 1.2, 0.6, 0.3), \quad d_G(bd) = (0.3, 1.2, 0.6, 0.3),$$
$$d_G(ac) = (0.3, 1.2, 0.6, 0.3).$$

Since $d_G(ab) = d_G(cd) = d_G(bd) = d_G(ac)$, G is not strongly edge irregular 4$-$polar fuzzy graph.

(ii) The total degree of edge is calculated as:

$$td_G(ab) = (0.4, 1.6, 0.8, 0.4), \quad td_G(cd) = (0.4, 1.6, 0.8, 0.4),$$
$$td_G(bd) = (0.4, 1.6, 0.8, 0.4), \quad td_G(ac) = (0.4, 1.6, 0.8, 0.4),$$
$$td_G(bc) = (0.5, 2.0, 1.0, 0.5).$$

Since $td_G(ab) = td_G(cd) = td_G(bd) = td_G(ac)$, G is not strongly edge totally irregular 4$-$polar fuzzy graph.

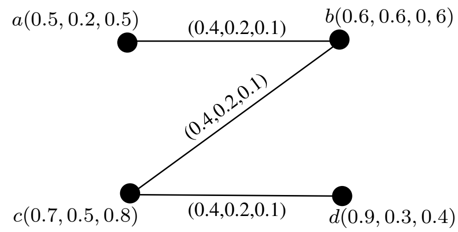

FIGURE 14. Highly Irregular 3-polar Fuzzy Graph

Theorem 3.21. *Let $G = (C, D)$ be a connected $m-$polar fuzzy graph, where D is a constant function. If G is strongly edge irregular $m-$polar fuzzy graph, then G is highly irregular $m-$polar fuzzy graph.*

Proof. Suppose that $G = (C, D)$ is a connected $m-$polar fuzzy graph, where D is a constant function. That is, $p_i \circ D(pq) = b = (k_1, k_2, k_3, \ldots, k_m)$. Assume that G is strongly edge irregular $m-$polar fuzzy graph. Then every pair of edges in G have distinct degrees. Also every pair of adjacent edges in G have distinct degrees. Let u be any arbitrary vertex in G which is adjacent to the vertices u_1 and u_2. Then the edges uu_1 and uu_2 have distinct degrees, i.e., $d_G(uu_1) \neq d_G(uu_2)$. This implies that $d_G(u) + d_G(u_1) - 2(p_i \circ D(uu_1)) \neq d_G(u) + d_G(u_2) - 2(p_i \circ D(uu_2))$. So, $d_G(u) + d_G(u_1) - b \neq d_G(u) + d_G(u_2) - b$, i.e., $d_G(u_1) \neq d_G(u_2)$. Thus there exists a vertex u in G which is adjacent to the vertices with distinct degrees. Since u is any arbitrary vertex in G, so all the vertices in G are adjacent to the vertices with distinct degrees. Hence G is a highly irregular $m-$polar fuzzy graph. □

Theorem 3.22. *Let $G = (C, D)$ be a connected $m-$polar fuzzy graph, where D is a constant function. If G is strongly edge totally irregular $m-$polar fuzzy graph, then G is highly irregular $m-$polar fuzzy graph.*

Proof. Proof is similar to the Theorem 3.21. □

Remark 3.23. A highly irregular $m-$polar fuzzy graph $G = (C, D)$, where D is a constant function, is not necessarily a strongly edge (totally) irregular $m-$polar fuzzy graph.

Example 3.24. Consider a 3$-$polar fuzzy graph of a graph $G^* = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{ab, bc, cd\}$, as shown in Figure 14. From Figure 14, we have $d_G(a) = (0.4, 0.2, 0.1)$, $d_G(b) = (0.8, 0.4, 0.2)$, $d_G(c) = (0.8, 0.4, 0.2)$ and $d_G(d) = (0.4, 0.2, 0.1)$. Since every vertex is adjacent to vertices with distinct degrees, G is highly irregular 3$-$polar fuzzy graph.

The degrees of each edge is calculated as: $d_G(ab) = (0.4, 0.2, 0.1)$, $d_G(bc) = (0.8, 0.4, 0.2)$, $d_G(cd) = (0.4, 0.2, 0.1)$. Since $d_G(ab) = d_G(cd)$. Hence G is not strongly edge irregular 3$-$polar fuzzy graph.

The total degree of each edge is calculated as: $td_G(ab) = (0.8, 0.4, 0.2)$, $td_G(bc) = (1.2, 0.6, 0.3)$, $td_G(cd) = (0.8, 0.4, 0.2)$. Since $td_G(ab) = td_G(cd)$. Hence G is not strongly edge totally irregular 3$-$polar fuzzy graph.

Theorem 3.25. *Let $G = (C, D)$ be an $m-$polar fuzzy graph of a graph $G^* = (V, E)$, the n-barbell graph $B_{n,n}$. If all the edges have distinct membership values, then G is strongly edge irregular fuzzy graph but G is not strongly edge totally irregular $m-$polar fuzzy graph.*

Proof. Let $G = (C, D)$ be an $m-$polar fuzzy graph of a graph $G^* = (V, E)$, the n-barbell graph $B_{n,n}$. Since G^* is a barbell graph, so there is a bridge wx connecting a pair of vertices w and x, and then connecting n new vertices to each of w and x. Let c be the membership value of the edge wx, where $c = (c_1, c_2, c_3, \ldots, c_m)$. Assume that $w_1, w_2, w_3, \ldots, w_n$ are the vertices adjacent to the vertex w and $x_1, x_2, x_3, \ldots, x_m$ are the vertices adjacent to the vertex x. Let $b_1, b_2, b_3, \ldots, b_n$ be the membership values of the edges $e_1, e_2, e_3, \ldots, e_n$ incident with the vertex w such that $b_1 < b_2 < b_3 < \ldots < b_n$, where each $b_j = (k_{1,j}, k_{2,j}, k_{3,j}, \ldots, k_{m,j})$, $j = 1, 2, 3, \ldots, n$. Let $d_1, d_2, d_3, \ldots, d_n$ be the membership values of the edges $f_1, f_2, f_3, \ldots, f_n$ incident with the vertex x such that $d_1 < d_2 < d_3 < \ldots < d_n$, where each $d_j = (l_{1,j}, l_{2,j}, l_{3,j}, \ldots, l_{m,j})$, $j = 1, 2, 3, \ldots, n$. Suppose that $b_1 < b_2 < b_3 < \ldots < b_n < d_1 < d_2 < d_3 < \ldots < d_n < c$. The degree of the edges are:

$$d_G(wx) = (b_1 + b_2 + b_3 + \ldots + b_n) + c + (d_1, d_2, d_3, \ldots, d_n) + c - 2c$$

$$\begin{aligned}
d_G(wx) &= (b_1 + b_2 + b_3 + \ldots + b_n) + (d_1, d_2, d_3, \ldots, d_n) \\
&= ((k_{1,1}, k_{2,1}, k_{3,1}, \ldots, k_{m,1}) + (k_{1,2}, k_{2,2}, k_{3,2}, \ldots, k_{m,2}) \\
&\quad + (k_{1,3}, k_{2,3}, k_{3,3}, \ldots, k_{m,3}) + \ldots + (k_{1,n}, k_{2,n}, k_{3,n}, \ldots, k_{m,n})) \\
&\quad + ((l_{1,1}, l_{2,1}, l_{3,1}, \ldots, l_{m,1}) + (l_{1,2}, l_{2,2}, l_{3,2}, \ldots, l_{m,2}) \\
&\quad + (l_{1,3}, l_{2,3}, l_{3,3}, \ldots, l_{m,3}) + \ldots + (l_{1,n}, l_{2,n}, l_{3,n}, \ldots, l_{m,n})) \\
&= ((k_{1,1} + k_{1,2} + k_{1,3} + \ldots + k_{1,n}), \ldots, (k_{m,1} + k_{m,2} + k_{m,3} + \ldots + k_{m,n})) \\
&\quad + ((l_{1,1} + l_{1,2} + l_{1,3} + \ldots + l_{1,n}), \ldots, (l_{m,1} + l_{m,2} + l_{m,3} + \ldots + l_{m,n})),
\end{aligned}$$

$$\begin{aligned}
d_G(e_j) &= (b_1 + b_2 + b_3 + \ldots + b_n + c) + b_j - 2b_j, \qquad j = 1, 2, 3, \ldots, n \\
&= (b_1 + b_2 + b_3 + \ldots + b_n + c) - b_j \\
&= ((k_{1,1}, k_{2,1}, k_{3,1}, \ldots, k_{m,1}) + (k_{1,2}, k_{2,2}, k_{3,2}, \ldots, k_{m,2}) \\
&\quad + \ldots + (k_{1,n}, k_{2,n}, k_{3,n}, \ldots, k_{m,n}) + (c_1, c_2, c_3, \ldots, c_m)) \\
&\quad - (k_{1,j}, k_{2,j}, k_{3,j}, \ldots, k_{m,j}) \\
&= ((k_{1,1} + k_{1,2} + \ldots + k_{1,n} + c_1), (k_{2,1} + k_{2,2} + \ldots + k_{2,n} + c_2), \\
&\quad \ldots, (k_{m,1} + k_{m,2} + \ldots + k_{m,n} + c_m)) - (k_{1,j}, k_{2,j}, k_{3,j}, \ldots, k_{m,j}),
\end{aligned}$$

$$\begin{aligned}
d_G(f_j) &= (d_1 + d_2 + d_3 + \ldots + d_n + c) + d_j - 2d_j, \; j = 1, 2, 3, \ldots, n. \\
&= (d_1 + d_2 + d_3 + \ldots + d_n + c) - d_j \\
&= ((l_{1,1}, l_{2,1}, l_{3,1}, \ldots, l_{m,1}) + (l_{1,2}, l_{2,2}, l_{3,2}, \ldots, l_{m,2}) \\
&\quad + \ldots + (l_{1,n}, l_{2,n}, l_{3,n}, \ldots, l_{m,n}) + (c_1, c_2, c_3, \ldots, c_m)) \\
&\quad - (l_{1,j}, l_{2,j}, l_{3,j}, \ldots, l_{m,j}) \\
&= ((l_{1,1} + l_{1,2} + \ldots + l_{1,n} + c_1), (l_{2,1} + l_{2,2} + \ldots + l_{2,n} + c_2), \\
&\quad \ldots, (l_{m,1} + l_{m,2} + \ldots + l_{m,n} + c_m)) - (l_{1,j}, l_{2,j}, l_{3,j}, \ldots, l_{m,j}).
\end{aligned}$$

Note that all the edges in G have distinct degrees. Hence G is strongly edge irregular $m-$polar fuzzy graph. The total degree of the edges are:

$$\begin{aligned} td_G(wx) &= (b_1 + b_2 + b_3 + \ldots + b_n) + (d_1 + d_2 + d_3 + \ldots + d_n) + c \\ &= ((k_{1,1}, k_{2,1}, k_{3,1}, \ldots, k_{m,1}) + (k_{1,2}, k_{2,2}, k_{3,2}, \ldots, k_{m,2}) \\ &\quad + \ldots + (k_{1,n}, k_{2,n}, k_{3,n}, \ldots, k_{m,n})) + ((l_{1,1}, l_{2,1}, l_{3,1}, \ldots, l_{m,1}) \\ &\quad + (l_{1,2}, l_{2,2}, l_{3,2}, \ldots, l_{m,2}) + \ldots + (l_{1,n}, l_{2,n}, l_{3,n}, \ldots, l_{m,n})) \\ &\quad + (c_1, c_2, c_3, \ldots, c_m) \\ &= ((k_{1,1} + k_{1,2} + \ldots + k_{1,n}) + \ldots + (k_{m,1} + k_{m,2} + \ldots + k_{m,n})) \\ &\quad + ((l_{1,1} + l_{1,2} + \ldots + l_{1,n}) + \ldots + (l_{m,1} + l_{m,2} + \ldots + l_{m,n})) \\ &\quad + (c_1, c_2, c_3, \ldots, c_m), \end{aligned}$$

$$\begin{aligned} td_G(e_j) &= (b_1 + b_2 + b_3 + \ldots + b_n) + c + b_j - b_j, \quad j = 1, 2, 3, \ldots, n. \\ &= (b_1 + b_2 + b_3 + \ldots + b_n) + c \\ &= ((k_{1,1}, k_{2,1}, k_{3,1}, \ldots, k_{m,1}) + (k_{1,2}, k_{2,2}, k_{3,2}, \ldots, k_{m,2}) \\ &\quad + \ldots + (k_{1,n}, k_{2,n}, k_{3,n}, \ldots, k_{m,n})) + (c_1, c_2, c_3, \ldots, c_m) \\ &= ((k_{1,1} + k_{1,2} + \ldots + k_{1,n} + c_1), (k_{2,1} + k_{2,2} + \ldots + k_{2,n} + c_2), \\ &\quad \ldots, (k_{m,1} + k_{m,2} + \ldots + k_{m,n} + c_m)), \end{aligned}$$

$$\begin{aligned} td_G(f_j) &= (d_1 + d_2 + d_3 + \ldots + d_n) + c + d_j - d_j, \quad j = 1, 2, 3, \ldots, n \\ &= (d_1 + d_2 + d_3 + \ldots + d_n) + c \\ &= ((l_{1,1}, l_{2,1}, l_{3,1}, \ldots, l_{m,1}) + (l_{1,2}, l_{2,2}, l_{3,2}, \ldots, l_{m,2}) \\ &\quad + \ldots + (l_{1,n}, l_{2,n}, l_{3,n}, \ldots, l_{m,n})) + (c_1, c_2, c_3, \ldots, c_m) \\ &= ((l_{1,1} + l_{1,2} + \ldots + l_{1,n} + c_1), (l_{2,1} + l_{2,2} + \ldots + l_{2,n} + c_2), \\ &\quad \ldots, (l_{m,1} + l_{m,2} + \ldots + l_{m,n} + c_m)). \end{aligned}$$

We note that all the edges e_j have the same total degree. All the edges f_j also have the same total degree. Hence G is not strongly edge totally irregular $m-$polar fuzzy graph. □

Using the similar method as used in above Theorem, we can prove the following Theorems.

Theorem 3.26. *Let $G = (C, D)$ be an $m-$polar fuzzy graph of a graph $G^* = (V, E)$, a star $K_{1,s}$. If all the edges have distinct membership values, then G is both strongly edge irregular and strongly edge totally regular $m-$polar fuzzy graph.*

Theorem 3.27. *Let $G = (C, D)$ be an $m-$polar fuzzy graph of a graph $G^* = (V, E)$, a path on $2n$ vertices. Let $e_1, e_2, e_3, \ldots, e_{2n-1}$ respectively, be the edges having membership values $b_1, b_2, b_3, \ldots, b_{2n-1}$ such that $b_1 < b_2 < b_3 < \ldots < b_{2n-1}$, where $b_j = (k_{1,j}, k_{2,j}, k_{3,j}, \ldots, k_{m,j})$, $1 \leq j \leq 2n-1$. Then G is both strongly edge irregular and strongly edge totally irregular $m-$polar fuzzy graph.*

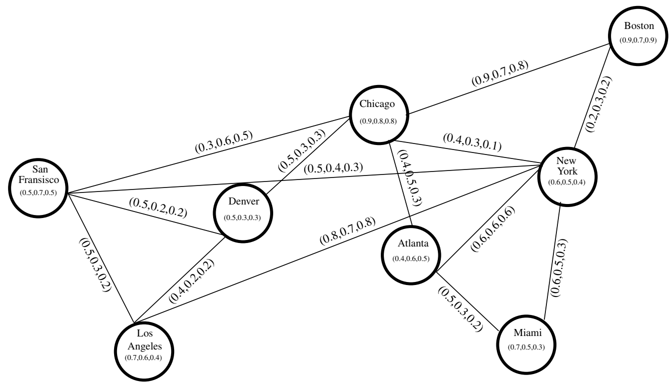

FIGURE 15. 3- polar Fuzzy Graph of an Airline Network

4. **Application**

Fuzzy graphs have become a very important part of mathematics with a number of applications in physics, biology, social networks and transportation networks. They take a vital role in flight networks. The airline companies aim to facilitate their passengers with high quality of service. Air traffic controllers have to make sure that company planes must arrive and depart at right time. This task is possible by planning efficient routes for the planes. Suppose we want to travel between different cities through an airline network. Consider the example of an airline network in which nodes represent the cities and edges represent the flights. Figure 15 shows an airline network between cities which is represented as a $3-$polar fuzzy graph $G = (C, R)$, where C is a $3-$polar fuzzy set of cities at which the distance, expenses and travel time is calculated, and R is a $3-$ polar fuzzy set of flights between the two cities. The membership degree of the edges can be calculated by using the relation

$$p_i \circ R(uv) \leq \inf\{p_i \circ C(u), p_i \circ C(v)\}.$$

We want to find the cheapest route between the cities such that distance, expenses and travel time is minimum. From the figure, it is easy to see that there exist nine routes between Boston and Chicago.

- $R_1 : Boston \to Chicago.$
- $R_2 : Boston \to NewYork \to Chicago.$
- $R_3 : Boston \to NewYork \to Miami \to Atlanta \to Chicago.$
- $R_4 : Boston \to NewYork \to LosAngeles \to Denver \to Chicago.$
- $R_5 : Boston \to NewYork \to LosAngeles \to SanFransisco \to Chicago.$
- $R_6 : Boston \to NewYork \to LosAngeles \to SanFransisco \to Denver \to Chicago.$
- $R_7 : Boston \to NewYork \to SanFransisco \to Denver \to Chicago.$
- $R_8 : Boston \to NewYork \to SanFransisco \to Chicago.$
- $R_9 : Boston \to NewYork \to Atlanta \to Chicago.$

Calculating the lengths of all the routes we obtain, $l(R_1) = (0.9, 0.7, 0.8)$, $l(R_2) = (0.6, 0.6, 0.3)$, $l(R_3) = (1.7, 1.6, 1.0)$, $l(R_4) = (1.9, 1.5, 1.5)$, $l(R_5) = (1.8, 1.9, 1.7)$, $l(R_6) = (2.5, 1.8, 1.7)$, $l(R_7) = (1.7, 1.2, 1.0)$, $l(R_8) = (1.0, 1.3, 1.0)$ and $l(R_9) = (1.2, 1.4, 1.1)$. It looks like traveling through Boston to Chicago is cheapest but actually after calculations we observed that the cheapest route is Boston to New York to Chicago. Similarly, the cheapest routes between other cities can be obtained. We present our method as an algorithm that is used in our application.

Algorithm 4.1.
1. *Input:* $C=$ *m-polar fuzzy set of vertices,* $R=$ *m-polar fuzzy set of edges,*
2. *Compute the m-polar fuzzy graph* $G = (C, R)$*,*
3. *Compute all possible routes* R_j *between cities,*
4. *Compute the length of all routes* R_j*, using formula*

$$l_i(P) = \sum_{j=1}^{n-1} \frac{1}{p_i \circ D(v_j v_{j+1})}, \; i = 1, 2, 3, \ldots, m.$$

5. *Find the route with minimum length.*

5. Conclusion and Future Work

Fuzzy graph theory has variety of applications in many disciplines, including control theory, expert system, and management sciences. An $m-$polar fuzzy model is a generalization of the fuzzy model which gives more precision, flexibility, and compatibility with a system when compared with the fuzzy model. We have applied the concept of $m-$polar fuzzy sets to graphs. We also discussed an application of an $m-$polar fuzzy graph to decision making. We are extending our research work to $m-$polar fuzzy soft graphs, $m-$ polar fuzzy hypergraphs, and $m-$polar fuzzy neutrosophic graphs.

Conflict of Interest: The authors declare that there is no conflict of interest regarding the publication of this research paper.

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References

[1] M. Akram, *Bipolar fuzzy graphs*, Information Sciences, **181** (2011), 5548-5564.
[2] M. Akram and A. Adeel, *m−polar fuzzy labeling graphs with application*, Math. Computer Sci., **10** (2016), 387-402.
[3] M. Akram and W. A. Dudek, *Regular bipolar fuzzy graphs*, Neural Computing & Applications, **21** (2012), 197-205.
[4] M. Akram and H. R. Younas, *Certain types of irregular m−polar fuzzy graphs*, J. Appl. Math. Computing, **53**(1) (2017), 365-382.
[5] M. Akram and N. Waseem, *Certain metrics in m−polar fuzzy graphs*, New Math. Natural Computation, **12** (2016), 135-155.
[6] P. Bhattacharya, *Some remarks on fuzzy graphs*, Pattern Recognition Letter, **6** (1987), 297-302.
[7] J. Chen, S. G. Li, S. Ma and X. Wang, *m−polar fuzzy sets*, The Scientific World Journal, Article ID 416530, **2014** (2014), 8 pages.

[8] A. Kauffman, *Introduction to la Theorie des Sous-emsembles Flous*, Masson et Cie, **1** (1973).
[9] S. Mathew and M. S. Sunitha, *Types of arcs in a fuzzy graph*, Information Sciences, **179** (2009), 1760-1768.
[10] N. R. S. Maheswari and C. Sekar, *On strongly edge irregular fuzzy graphs*, Kragujevac J. Math., **40** (2016), 125-135.
[11] J. N. Mordeson and P. S. Nair, *Fuzzy graphs and fuzzy hypergraphs*, Physica Verlag, Heidelberg, 2nd Edition, 2001.
[12] S. P. Nandhini and E. Nandhini, *Strongly irregular fuzzy graphs*, Internat. J. Math. Archive, **5** (2014), 110-114.
[13] A. Nagoorgani and K. Radha, *Regular property of fuzzy graphs*, Bull. Pure Appl. Sci., **27E** (2008), 411-419.
[14] K. Radha and N. Kumaravel, *On edge regular fuzzy graphs*, Internat. J. Math. Archive, **5**(9) (2014), 100-112.
[15] A. Rosenfeld, *Fuzzy graphs*, Fuzzy Sets and Their Applications, Academic Press, New York, (1975), 77-95.
[16] P. K. Singh and Ch. A. Kumar, *Bipolar fuzzy graph representation of concept lattice*, Information Sciences, **288** (2014), 437-448.
[17] H. L. Yang, S. G. Li, W. H. Yang and Y. Lu, *Notes on bipolar fuzzy graphs*, Information Sciences, **242** (2013), 113-121.
[18] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.
[19] L. A. Zadeh, *Similarity relations and fuzzy orderings*, Information Sciences, **3** (1971), 177-200.
[20] W. R. Zhang, *Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis*, Proc. of IEEE conf., (1994), 305-309.

Muhammad Akram, Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan
E-mail address: makrammath@yahoo.com

Neha Waseem, Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan
E-mail address: neha_waseem@yahoo.com

Wieslaw A. Dudek*, Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology, Wyb. Wyspianskiego 27, 50-370, Wroclaw, Poland
E-mail address: wieslaw.dudek@pwr.edu.pl

*Corresponding author