F-TRANSFORM FOR NUMERICAL SOLUTION
OF TWO-POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. We propose a fuzzy-based approach aiming at finding numerical solutions to some classical problems. We use the technique of F-transform to solve a second-order ordinary differential equation with boundary conditions. We reduce the problem to a system of linear equations and make experiments that demonstrate applicability of the proposed method. We estimate the order of accuracy of the proposed method. We show that the F-transform-based approach does not only extend the set of its applications, but has a certain advantage in the solution of ill-posed problems.

1. Introduction

In most cases, differential equations cannot be solved analytically so that numerical solutions are expected. In this contribution, we continue study of numerical methods for ordinary differential equations (ODEs) that are based on the F-transform [9]. We discuss a new F-transform based numerical method for solving a boundary value problem (BVP) for a second-order ODE with Dirichlet boundary conditions.

Let us remark that the BVP is more complicated than the initial value problem (IVP): the analysis of existence and uniqueness of a solution and solution methods are more sophisticated. Moreover, BVP requires solution of a corresponding linear or non-linear system of equations which may cause additional technical efforts [3].

The proposed approach stems from [10] where the Euler-like method for the IVP was discussed. The same approach has been successfully used in [5] for a second order IVP. For the same problem, a number of numerical schemes based on the higher degree F-transform have been introduced in [7]. All of them outperform the second order Runge-Kutta method.

The analysis of the F-transform-based approach to BVP has been initiated in [8] where one particular case of this problem was analyzed and efficiently solved. In this contribution, we continue this analysis and apply the F-transform method to another group of second order BVPs. Our primary (still lasting) goal is to demonstrate the applicability of the F-transform method to all kinds of second order BVPs and to find cases where this method works better than its classical counterpart.

In the current paper, we pay attention to the theoretical estimation of an order of accuracy of the proposed method. We show that although it has the same order as its numerical prototype (the method of finite differences), it has a certain advantage in the ability to produce stable solutions for some ill-posed problems, see section 3.3. We give numerous examples and compare the obtained results with exact solutions and the method of finite differences.

The structure of the paper is as follows: preliminaries with basic facts about the F-transforms and BVPs are in section 2. A new F-transform based method for a numerical solution of the two-point problem for a linear second-order differential equation is given in section 3, and its advantages are discussed in section 3.3. Numerical examples are given and commented in section 4, and conclusions are in section 5.

2. Preliminaries: BVP and F-transform

In this section, we recall some basic facts about particular BVPs that are discussed in the paper, and repeat main definitions and properties about the F-transform as they were considered in [9].

2.1. **BVP for a Second Order Differential Equation.** Second order differential equations and systems constitute a majority of BVPs. They arise in ballistics, theory of elasticity, etc. We will be working with the following two-point problem for the second-order linear equation:

$$y'' = p(x)y' + q(x)y + f(x), \quad x \in (a, b), \tag{1}$$

$$y(a) = y_a, \quad y(b) = y_b, \tag{2}$$

where y is an unknown real function on (a, b), p, q, f are real functions and y_a, y_b are boundary values of y. In mathematical physics, the equation (1) is considered in connection with the integral equation [1, 4]

$$\int_a^b (q(x)y(x) + f(x))dx = k(x)y'|_a^b,$$

where functions k, k', f, f' can have continuity breaks.

We say that the BVP (1)-(2) is *homogeneous*, if $f \equiv 0$ and $y_a = y_b = 0$.

Let us remark that the case of BVP is much more complicated than that of the initial value problem (IVP). There are many examples where BVP and IVP share the same differential equation, both are homogeneous, but have different solution sets, see, e.g., [1]. In particular, we can justify this claim by considering the following equation:

$$y'' + y = 0,$$

and the two related to it problems:

(1) the IVP with initial values $y(-\pi/4) = 0$, $y'(-\pi/4) = 0$;
(2) the BVP with boundary conditions $y(-\pi/4) = 0$, $y(3\pi/4) = 0$.

The first problem (IVP) has a unique trivial solution, while the second problem (BVP) has the infinite set of solutions whose parametric representation is given by $y(x) = C \sin x + C \cos x$.

In the present contribution, we will be dealing with the following particular case of (1)-(2):

$$y'' + d_1 y' + d_0 y = f(x), \quad x \in (a, b),$$

$$y(x_a) = y_a, \quad y(x_b) = y_b, \tag{3}$$

where y is an unknown real function on (a, b), f is a real function on $[a, b]$ (usually called a *source function*), d_1, d_0, y_a, y_b are reals and y_a, y_b are boundary values of y.

Let us describe necessary and sufficient conditions that guarantee the existence of a unique solution to this problem. The uniqueness of a solution assures correctness of numerical methods developed for searching approximate solutions. The below given theorem is an adapted version of a more general sentence from [1].

Theorem 2.1. *Let ϕ_1, ϕ_2 be a fundamental pair of solutions to the homogeneous ODE $y'' + d_1 y' + d_0 y = 0$ in (3). Then the following are equivalent.*

(1) *The nonhomogeneous boundary value problem (3) has a unique solution for any given constants y_a and y_b, and a given continuous function f on the interval $[a, b]$.*

(2) *The associated homogeneous boundary value problem has only trivial solution.*

(3) *The determinant*

$$\begin{vmatrix} \phi_1(a) & \phi_2(a) \\ \phi_1(b) & \phi_2(b) \end{vmatrix} \neq 0. \tag{4}$$

Recall that a fundamental pair of solutions to the homogeneous ODE $y'' + d_1 y' + d_0 y = 0$ always exists.

2.2. **F-transform.** In this section, we recall the main facts (see [9] for more details) about F-transform — the technique, which will be used for the numerical solution of BVP (1)–(2).

Definition 2.2. Let $n > 2$, $a = x_0 = x_1 < \ldots < x_n = x_{n+1} = b$ be fixed nodes within $[a, b] \subseteq \mathbb{R}$. Fuzzy sets $A_1, \ldots, A_n : [a, b] \to [0, 1]$, identified with their membership functions defined on $[a, b]$, establish a fuzzy partition of $[a, b]$, if they fulfill the following conditions for $k = 1, \ldots, n$:

(1) $A_k(x_k) = 1$;

(2) $A_k(x) = 0$ if $x \in [a, b] \setminus (x_{k-1}, x_{k+1})$;

(3) $A_k(x)$ is continuous on $[x_{k-1}, x_{k+1}]$;

(4) $A_k(x)$ for $k = 2, \ldots, n$ strictly increases on $[x_{k-1}, x_k]$ and for $k = 1, \ldots, n-1$ strictly decreases on $[x_k, x_{k+1}]$;

(5) for all $x \in [a, b]$ the Ruspini condition holds

$$\sum_{k=1}^{n} A_k(x) = 1.$$

We say that n is a *size of fuzzy partition* $\{A_1, \ldots, A_n\}$ with the elements to which we refer as to *basic functions*.

If the nodes $x_1, \ldots, x_n$ are $h-equidistant$, i.e. $x_k = x_{k-1} + h$, $k = 2, \ldots, n$, where $h = (b-a)/(n-1)$, and for all $k = 2, \ldots, n-1$, the following two additional properties

(6) $A_k(x_k - x) = A_k(x_k + x)$ for all $x \in [0, h]$,

(7) $A_k(x) = A_{k-1}(x - h)$ and $A_{k+1}(x) = A_k(x - h)$ for all $x \in [x_k, x_{k+1}]$,

are met, then the fuzzy partition $A_1, \ldots, A_n$ is h-uniform.

An h-uniform fuzzy partition of $[a, b]$ can be also defined using a *generating function*

$$A_0 : [-1, 1] \to [0, 1],$$

which is assumed to be even, continuous and positive everywhere except for on boundaries, where it vanishes. Then, basic functions $A_2, \ldots, A_{n-1}$ of an h-uniform fuzzy partition are re-scaled and shifted copies of A_0 in the sense that for all $k = 2, \ldots, n-1$;

$$A_k(x) = \begin{cases} A_0(\frac{x-x_k}{h}), & x \in [x_k - h, x_k + h], \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. Let $A_1, \ldots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$ and f be any continuous function defined on $[a, b]$. We say that the n-tuple of real numbers (*components*) $\mathbf{F}_n[f] = (F_1, \ldots, F_n)$, where

$$F_k = \frac{\int_a^b f(x) A_k(x) dx}{\int_a^b A_k(x) dx}, \quad k = 1, \ldots, n,$$

is the *(direct) F-transform* of f with respect to $A_1, \ldots, A_n$.

Definition 2.4. Let $\mathbf{F}_n[f] = (F_1, \ldots, F_n)$ be the direct F-transform of f with respect to $A_1, \ldots, A_n$. Then the function

$$\hat{f}_n(x) = \sum_{k=1}^{n} F_k A_k(x), \qquad x \in [a, b],$$

is called the *inverse F-transform* of f.

It has been shown in [9] that the inverse F-transform of f can approximate the original function f with an arbitrary precision, when $h \to 0$. Moreover, the components of the direct F-transform are close to the function values at the corresponding nodes. The estimation of closeness is given in the Proposition below.

Proposition 2.5. *Let function f be twice continuously differentiable in (a, b), and $A_1, \ldots, A_n$, $n \geq 3$, be basic functions which form a uniform fuzzy partition of $[a, b]$. Then for each $k = 1, \ldots, n$,*

$$F_k = f(x_k) + O(h^2).$$

3. Numerical Solution Based on F-transform

In this section, we propose a new F-transform based method for numerical solution of the two-point BVP for the second-order linear equation (3) with constant coefficients and Dirichlet boundary conditions. We also estimate the order of accuracy of the proposed method.

3.1. **Numerical Methods.** There is a number of methods used for solving boundary value problems: shooting, finite differences, collocation, etc., see [2, 3, 6]. For the second-order linear equation (3), the easiest method with a sufficient order of accuracy is based on finite differences [6].

The F-transform provides with a (finite) vector representation of a function, continuous or discrete. This representation is in the form of a vector of components. It is a result of the linear map $\mathbf{F}_n : \mathcal{F} \to \mathbb{R}^n$ where n is a size of the corresponding fuzzy partition, $\mathcal{F}$ a certain set of functions and $\mathbb{R}^n$ the set of n-dimensional vectors. The main idea of the F-transform method for solving BVP consists in applying the map $\mathbf{F}_n$ to both sides of differential equation (1) and transform the problem from the space of functions to the space of finite n-dimensional vectors.

Below, we modify the general scheme of the method of finite differences to the case where exact values of involved functions and their derivatives are replaced by the F-transform components.

Let us repeat the formulation of the problem (3), which we are going to solve numerically. We consider

$$y'' + d_1 y' + d_0 y = f(x), \quad x \in (a, b),$$
$$y(x_a) = y_a, \quad y(x_b) = y_b,$$

where y is an unknown real function on (a, b), f is a real function on (a, b), d_1, d_0, y_a, y_b are reals and y_a, y_b are boundary values of y.

Let us fix the problem (3) and assume that it has a unique solution, i.e. the assumptions of Theorem 2.1 are fulfilled and the condition (4) is satisfied. Let us choose $n > 2$ and create an h-uniform fuzzy partition $A_1, \ldots, A_n$ of $[a, b]$ where $h = \frac{b-a}{n-1}$. We denote the F-transforms of the functions on left and right sides of (3) by $\mathbf{F}_n[y] = (Y_1, \ldots, Y_n)^T$, $\mathbf{F}_n[y'] = (Y_1', \ldots, Y_n')^T$, $\mathbf{F}_n[y''] = (Y_1'', \ldots, Y_n'')^T$ and $\mathbf{F}_n[f] = (F_1, \ldots, F_n)^T$, respectively. Because the map $\mathbf{F}_n$ obeys the property of linearity, we come to the following discrete version of (3):

$$Y_k'' + d_1 Y_k' + d_0 Y_k = F_k, \; k = 2, \ldots, n - 1,$$

$$Y_1 = y_a, \; Y_n = y_b, \tag{5}$$

where using schemes of the method of finite differences, we replace Y_k' and Y_k'' by

$$Y_k' = \frac{Y_{k+1} - Y_{k-1}}{2h}, \; k = 2, \ldots, n - 1,$$
$$Y_k'' = \frac{Y_{k+1} - 2Y_k + Y_{k-1}}{h^2}, \; k = 2, \ldots, n - 1.$$

We rewrite (5) to the following system of linear equations:

$$(D_2 + d_1 D_1 + d_0 D_0)\mathbf{F}_n[y] = \tilde{\mathbf{F}}_n[f], \tag{6}$$

where vectors $\mathbf{F}_n[y]$ and $\tilde{\mathbf{F}}_n[f]$ are respectively, the unknown vector of the F-transform components of y, and the vector that differs from $\mathbf{F}_n[f]$ in the first and last coordinates so that $\tilde{\mathbf{F}}_n[f] = (\frac{y_a}{h}, F_2, \ldots, F_{n-1}, \frac{y_b}{h})^T$. Matrix D_0 differs

from the unit $n \times n$ matrix by the first and last rows, which are zero, D_1, D_2 are the following $n \times n$ matrices:

$$D_1 = \frac{1}{2h} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & ... & 0 \\ -1 & 0 & 1 & 0 & 0 & ... & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & ... & -1 & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \end{bmatrix},$$

$$D_2 = \frac{1}{h^2} \begin{bmatrix} h & 0 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & ... & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & ... & 1 & -2 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 & h \end{bmatrix}.$$

Our final (equivalent) transformation of (6) leads to the system

$$D\mathbf{F}_n[y] = \tilde{\mathbf{F}}_n[f], \tag{7}$$

with the following tridiagonal $n \times n$ matrix:

$$D = D_2 + d_1 D_1 + d_0 D_0 = \begin{bmatrix} 1/h & 0 & 0 & 0 & 0 & ... & 0 \\ a & b & c & 0 & 0 & ... & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & ... & a & b & c \\ 0 & 0 & 0 & 0 & ... & 0 & 1/h \end{bmatrix}, \tag{8}$$

where

$$a = \frac{1}{h^2} - \frac{d_1}{2h}, \quad b = d_0 - \frac{2}{h^2}, \quad c = \frac{1}{h^2} + \frac{d_1}{2h}.$$

To assure solvability of the system (7), we need to verify that matrix D (8) is non-singular (see **Appendix** at the end of this paper). Then we apply the *Thomas algorithm* (simplified version of the *Gaussian elimination*, see e.g., [12]), that results in the following recurrent scheme:

$$Y_k + \gamma_k \cdot Y_{k+1} = \rho_k, \quad k = 2, \ldots, n-2,$$

where

$$\gamma_k = \frac{c}{b - a \cdot \gamma_{k-1}}, \quad \gamma_1 = 0,$$

$$\rho_k = \frac{F_k - a \cdot \rho_{k-1}}{b - a \cdot \gamma_{k-1}}, \quad \rho_1 = Y_1 = y_a,$$

$$Y_n = y_b.$$

Remark 3.1. The implementation of Thomas algorithm is two stage: at first, components ρ_k, γ_k, $k = 2, \ldots, n-2$, are computed and then, the backward computation of $Y_{n-1}, \ldots, Y_2$, is executed.

The solution vector $\mathbf{F}_n[y] = (Y_1, \ldots, Y_n)$ approximates the (unique) solution y of the initial BVP (3) in the sense that $y(x_k) \approx Y_k$, $k = 1, \ldots, n$, where nodes $x_1, \ldots, x_n$ are determined by fuzzy partition $\{A_1, \ldots, A_n\}$. The quality of approximation is estimated in the next section.

Applying the inverse F-transform to the extended vector $\mathbf{F}_n[y] = (Y_1, \ldots, Y_n)$, we obtain an approximate solution $\hat{y}$ of (3) in the form of a function:

$$\hat{y}(x) = \sum_{k=1}^{n} Y_k A_k(x), \quad x \in [a, b].$$

3.2. The Order of Accuracy of the F-transform-based Numerical Method.
In this section, we show that the F-transform method and the method of finite differences have the same order of accuracy.

Theorem 3.2. *Let the BVP be given by the second-order linear equation (3) where function f is twice continuously differentiable in (a, b). We choose $n > 2$ and create an h-uniform fuzzy partition $A_1, \ldots, A_n$ of $[a, b]$ with nodes $a = x_0 = x_1 < \ldots < x_n = x_{n+1} = b$ where $h = \frac{b-a}{n-1}$. Let $(y_1, \ldots, y_n)$ and $(Y_1, \ldots, Y_n)$ be (approximate) numerical solutions to (3), obtained by the method of finite differences and the F-transform respectively, and such that $y_1 = Y_1 = y_a$, $y_n = Y_n = y_b$. Then the following estimation is valid:*

$$\max_{1 \leq k \leq n} |y_k - Y_k| \leq Ch^2, \tag{9}$$

where $C > 0$ is some constant.

Proof. Let the assumptions be fulfilled. Then from the description of the method of finite differences [3, 6] and the F-transform-based method we infer that vectors $\mathbf{y} = (y_1, \ldots, y_n)^T$ and $\mathbf{F}_n[y] = (Y_1, \ldots, Y_n)^T$ are solutions of the respective systems of linear equations

$$D\mathbf{y} = \mathbf{f},$$
$$D\mathbf{F}_n[y] = \tilde{\mathbf{F}}_n[f],$$

where $D = D_2 + d_1 D_1 + d_0 D_0$ is the $n \times n$ non-singular matrix given by (8), $\tilde{\mathbf{F}}_n[f]$ is the right-hand side of (7) and $\mathbf{f} = (f_1, \ldots, f_n)$ where $f_k = f(x_k)$, $k = 1, \ldots, n$. It is easy to infer that

$$\|\mathbf{y} - \mathbf{F}_n[y]\|_\infty \leq \|D^{-1}\|_\infty \|\mathbf{f} - \tilde{\mathbf{F}}_n[f]\|_\infty,$$

where $\|\cdot\|_\infty$ is the max norm of a vector and $\|\cdot\|_\infty$ is the induced matrix norm. By Proposition 2.5, $\|\mathbf{f} - \tilde{\mathbf{F}}_n[f]\|_\infty \leq C_1 h^2$ where $C_1 > 0$ is some constant. Finally, let us take $C = C_1 \|D^{-1}\|_\infty$ and obtain the desired inequality (9). $\square$

The following corollary easily follows from the following sentence [6]: if the source function of (3) is sufficiently differentiable, then the order of accuracy of the method of finite differences for (3) is $O(h^2)$.

Corollary 3.3. *Let the assumptions of Theorem 3.2 be fulfilled. Then*

$$\max_{1 \leq k \leq n} |y(x_k) - Y_k| = O(h^2),$$

where y is the exact solution of (3).

3.3. **Advantages of the F-transform Based Method.** From previous sections, we see that both methods: the proposed F-transform-based one and the method of finite differences have similar computation algorithm and the order of accuracy. In the current section, we clarify differences and highlight advantages of the proposed method. We will be focused on the following two characteristics.

(1) Formally, the F-transform-based method works with the F-transform components Y_k and F_k instead of exact values y_k and f_k of the unknown and given functions respectively, where $k = 1, \ldots, n$. It may be advantageous in the situation where we are interested in a rough estimation of a solution. The reason is that for small values of n, it is difficult to select representative values $f_1, \ldots, f_n$ at the corresponding nodes $x_1, \ldots, x_n$, which characterize the behavior of this function on the whole domain $[a, b]$. On the other hand, the F-transform components $F_1, \ldots, F_n$ provide with aggregated information about the behavior of f in the neighborhood of the same nodes. Therefore, the computation that is based on the F-transform components is more reliable.

(2) Assume that the values of the source function f in (3) are affected by noise. This can happen, if we work with real data. The method of finite differences propagates the noisy values and thus, the obtained numerical solution is noisy as well. On the other hand, the F-transform components of an original and noisy functions are very much similar to each other, because the F-transform has a strong filtering effect, see [11]. Therefore, the F-transform-based numerical solutions of (3) for noisy and non-noisy source functions are almost identical. The illustration is given in the next section and visualized in Figure 1 and Figure 2.

4. Tests and Comparisons

In this section, we present several examples and tests with the purpose to demonstrate the applicability and universality of the proposed F-transform-based method. Moreover, we demonstrate the ability of this method to cope with BVP problems affected by noise.

For the first test, we selected 10 second-order differential equations with various source functions and Dirichlet boundary conditions, see Table 1. For all these equations, we know their analytically expressed exact solutions and use them as benchmarks for our numerical approximations.

In Table 2, the obtained numerical solutions are compared with the similar ones obtained by the method of finite differences. For the comparison we used the same number $n = 20$ of nodes. In all cases, the F-transform-based method was applied for the h-uniform fuzzy partition with triangular basic functions where $h = \frac{b-a}{n-1}$. The comparison is based on average accuracies, i.e. on arithmetic means of absolute errors between exact and numerical solutions computed over all nodes.

Let us make two comments, taking into account that the quality of approximation is measured by average accuracy.

	Equation	Boundary conditions
Equation 1	$y'' = 12x^2$	y(0)=0, y(1)=0
Equation 2	$y'' = 3\cos x + 6\sin x$	$y(0) = 0, y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
Equation 3	$y'' + y' = -\sin x + \cos 2x$	y(0)=0, y(1)=1
Equation 4	$y'' - y' = x^2 - 2x + 3$	y(0)=0, y(1)=0
Equation 5	$y'' - 2y' = 6e^{3x}$	y(0)=0, y(1)=0
Equation 6	$y'' + 2y' = 8\cos 4x$	$y(0) = 0, y\left(\frac{\pi}{2}\right) = 0$
Equation 7	$y'' + 4y = 3\cos x + 6\sin x$	$y(0) = 0, y(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
Equation 8	$y'' + (\frac{\pi}{2})^2 y = -x$	y(0)=0, y(1)=0
Equation 9	$y'' + 2y' + y = e^{-x}$	y(0)=2, y(2)=0
Equation 10	$y'' - 7y' + 12y = 4e^{2x}$	$y(0) = 4, y(1) = 2e^3$

TABLE 1. Examples of Various BVPs

	Average errors	
	F-transform	Finite difference
Equation 1	3,95833e-06	0,000395833
Equation 2	3,13166e-07	3,1319e-05
Equation 3	2,25323e-05	4,45193e-05
Equation 4	1,9567e-05	1,25627e-05
Equation 5	0,004295825	0,000276053
Equation 6	0,001360801	0,004413392
Equation 7	0,000159328	0,000200571
Equation 8	3,60019e-05	3,60019e-05
Equation 9	0,001883458	0,001776987
Equation 10	0,035334804	0,034420399

TABLE 2. Comparison of Numerical Solutions in Terms of Average Errors

(1) In the majority of examples (Equations 3–4, 6–10), the difference between both methods is almost negligible. The order of accuracy is better with the F-transform-based method in Examples 1, 2, and with the method of finite differences in Example 5. This confirms our result expressed in Theorem 3.2 about the same order of accuracy of both methods.

(2) The highest average errors of both methods are in Examples 5, 9, 10. We explain this fact referring to the sampling theorem with respect to ordinary samples and the F-transform components, see [11]. The source functions of all three above mentioned examples are given by exponential functions with relatively high rates of growth or decrease. Because exponential functions have full spectrum of Fourier frequencies, their reconstruction requires smaller sample step than that in the case of slowly changing functions. This means that in order to keep the same level of accuracy in numerical computations, exponential functions require more samples than slowly changing ones.
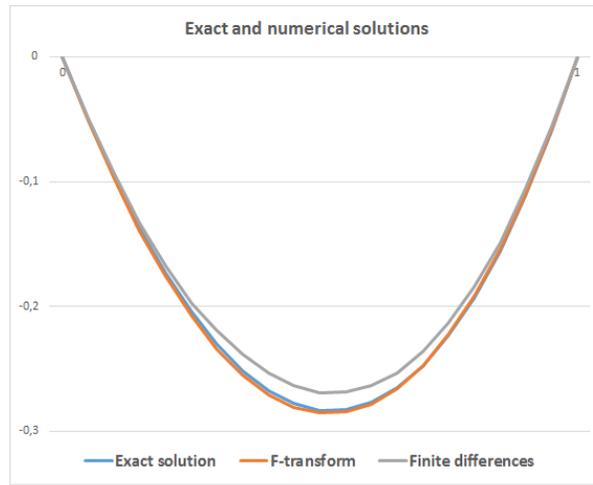
FIGURE 1. Exact Solution of Equation 4 and Two Numerical Solutions of Equation 4 with the Source Function Affected by a Gaussian Noise

In the second test, we are focused on the robustness of both numerical methods with respect to a noisy right-hand side. We chose equation Equation 4 and added a Gaussian noise to the source function $x^2 - 2x + 3$ in the right-hand side. We applied both methods with the described above settings to equation Equation 4 with the noisy source function and obtained two numerical solutions. Each of them was compared with the exact solution to the equation Equation 4 without added noise, see illustrations in Figure 1 and Figure 2. As it can be observed from these Figures, the F-transform-based solution is almost identical with the exact solution of Equation 4 that has not been affected by noise. On the other hand, the numerical solution by the method of finite differences visibly deviates from the exact and non-noisy one.

5. Conclusion

The problem of solving the two-point boundary value problem for a second-order ordinary linear differential equation with Dirichlet boundary conditions was investigated. A new numeric method based on the F-transform was proposed and its accuracy was estimated. The advantage of the proposed method over its numerical counterpart was analyzed.

We demonstrated robustness of the proposed method with respect to a noisy right-hand side of the corresponding differential equation. We analyzed various examples and performed a detailed comparison with exact and ordinary numerical solutions.
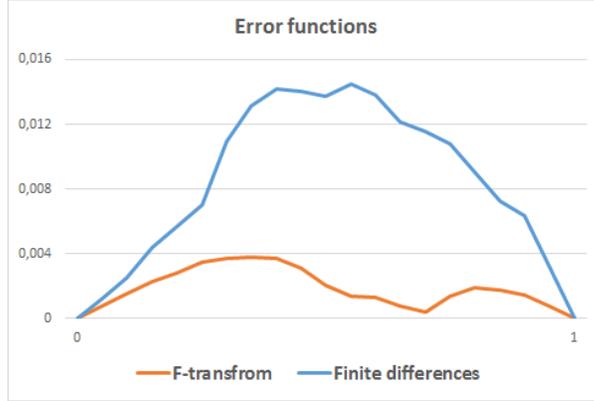
FIGURE 2. Error Functions of the Two Numerical Solutions (Based on the F-transform and Finite Differences) to Equation 4 with the Source Function Affected by a Gaussian Noise

Our current research is focused on solving a boundary value problem for a second-order ordinary differential equation with fuzzy boundary conditions (FBVP), see preliminary results in [13]. In the future work, we plan to give more details to this particular technique and compare it with existing approaches.

6. Appendix

Let us show that under certain conditions, the system of linear equations (7) with matrix D given by (8) is solvable. This is equivalent to show that D is a non-singular matrix.

By the representation in (8), we observe that D is a tridiagonal matrix. To show that it is non-singular we first transform it to a certain canonical form and then apply one of sufficient criteria learned from [3]). The mentioned transformation results in the below given $(n-2) \times (n-2)$ matrix

$$
D' =
\begin{bmatrix}
b & c & 0 & 0 & 0 & ... & 0 \\
a & b & c & 0 & 0 & ... & 0 \\
 & & & \ddots & & & \\
0 & 0 & 0 & ... & a & b & c \\
0 & 0 & 0 & 0 & ... & a & b
\end{bmatrix},
\tag{10}
$$

where

$$
a = \frac{1}{h^2} - \frac{d_1}{2h}, \quad b = d_0 - \frac{2}{h^2}, \quad c = \frac{1}{h^2} + \frac{d_1}{2h}.
\tag{11}
$$

Observe that $|D| \neq 0$, if and only if $|D'| \neq 0$. The latter condition can be assured by the following statement.

Proposition 6.1. *Let D' be a tridiagonal $k \times k$ matrix represented in* (10) *with elements given by* (11) *and $k \geq 2$. Then for a sufficiently small $h > 0$, $|D'| \neq 0$, if $d_0 \leq 0$.*

Proof. Let us reproduce the sufficient condition (denotation agrees with (10)) from [3] that guarantees the fact $|D'| \neq 0$.

    (1) $|b| > |c| > 0$,
    (2) $|b| \geq |a| + |c|$,
    (3) $|b| > |a| > 0$.

We will prove that under the given above assumptions all three inequalities are fulfilled.

At first, we observe that according to (11), a, b, c behave monotonously when h is sufficiently small and decreases. Therefore, there exists $h_0 < 1$ such that for all $0 < h < h_0$, $a > 0$, $c > 0$ and $b < 0$. For this particular h, inequalities $|b| > |a|$ and $|b| > |c|$ follow from $|b| \geq |a| + |c|$. Therefore, we aim at finding conditions that guarantee the latter inequality solely.

By (11) and the choice of h, we easily have

$$a + c = d_0 - b,$$

and

$$|a| + |c| = |d_0 - b|.$$

If $d_0 \leq 0$, then $|d_0 - b| = d_0 + |b| \leq |b|$, and thus, $|b| \geq |a| + |c|$. This completes the proof.   □

Let us emphasize that the above given Proposition provides with the sufficient condition only. The direct way to verify that the matrix D' in (10) is non-singular consists in computation of its determinant. For this purpose, we can propose the following recurrence equation:

$$D_k = bD_{k-1} - acD_{k-2}, \; k \geq 2,$$
$$D_0 = 1, D_1 = b.$$

Then, matrix D' is non-singular if $D_{n-2} \neq 0$.

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