ON CONVERGENCE THEOREMS FOR FUZZY HENSTOCK INTEGRALS

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Abstract. The main purpose of this paper is to establish different types of convergence theorems for fuzzy Henstock integrable functions, introduced by Wu and Gong [12]. In fact, we have proved fuzzy uniform convergence theorem, convergence theorem for fuzzy uniform Henstock integrable functions and fuzzy monotone convergence theorem. Finally, a necessary and sufficient condition under which the point-wise limit of a sequence of fuzzy Henstock integrable functions is fuzzy Henstock integrable has been established.

1. Introduction

The concept of Henstock integration (=gauge integration) for real-valued functions was introduced by Henstock [8] and Kurzweil [9] independently and is considered as one of the powerful integration theory in modern days. It not only generalizes the concepts of Riemann integration as well as Lebesgue integration but also is equivalent to the Denjoy integration and Perron integration of real valued functions. In addition, this integration theory satisfies most of the desired properties of integral.

Because of growing importance, generalization of such concept in fuzzy setting is almost inevitable; in fact, in 2001, the concept of Henstock integral was fuzzyfied by Wu and Gong [12]. Some recent works related to fuzzy Henstock integrals are found in literature in the form of published papers of Bongiorno and Piazza [2], Gong and Wang [6] and Musiał [11]. As convergence theory is one of the fundamental concepts in measure theory and has various applications in integration theory as well, we are tempted to establish some convergence theorems for the fuzzy Henstock integrable functions. In fact, we have proved fuzzy uniform convergence theorem, convergence theorem for fuzzy uniform Henstock integrable functions and fuzzy monotone convergence theorem. At the end, we have given a necessary and sufficient condition under which the point-wise limit of a sequence of fuzzy Henstock integrable functions is fuzzy Henstock integrable.

It is to be mentioned that, if $\{\psi_k\}$ is a sequence of fuzzy Henstock integrable functions on $[a, b]$ which pointwise converges to $\psi : [a, b] \to E^1$ in the metric space

Received: December 2015; Revised: December 2016; Accepted: April 2017

Key words and phrases: Fuzzy number, Fuzzy number function, Fuzzy Henstock integral, Fuzzy monotone sequence.

$(E^1, \hat{\rho})$, then

$$\lim(H)\int_a^b \psi_k = (H)\int_a^b \psi.$$

is not true in general (see Example 3.1).

2. Preliminaries

Throughout this paper, symbols $\mathbf{R}, \mathbf{R}_+$ and $\mathbf{N}$ stand for the real line (with usual topology), the set of all positive real numbers and the set of all positive integers respectively. For any two given sets A and B, B^A denotes the set of all mappings with domain A and codomain B.

Definition 2.1. A mapping $\alpha \in [0,1]^{\mathbf{R}}$ is called fuzzy number if
(i) α is normal, i.e. $\alpha(r) = 1$ for some $r \in \mathbf{R}$,
(ii) α is convex, i.e. $\alpha(\lambda r_1 + (1-\lambda)r_2) \geq \min\{\alpha(r_1), \alpha(r_2)\}$ for all $r_1, r_2 \in \mathbf{R}$ and $\lambda \in [0,1]$,
(iii) α is semi-continuous, i.e. for every $\lambda \in [0,1]$, the set $\{x \in \mathbf{R} : \alpha(x) \geq \lambda\}$ is closed,
(iv) $cl([\alpha]^0) = cl(\{x \in \mathbf{R} : \alpha(x) > 0\})$ is compact, where $cl(A)$ is the closure of $A \subset \mathbf{R}$.

The set of all fuzzy numbers is denoted by E^1.

Let $\alpha \in E^1$. Then Wu and Ming [13] showed that for each $\lambda \in (0,1]$, $[\alpha]^\lambda = \{x \in \mathbf{R} : \alpha(x) \geq \lambda\}$ is a closed and bounded interval and $[\alpha]^1 \neq \emptyset$. For each $\lambda \in (0,1]$, let $[\alpha]^\lambda = [\alpha_1^\lambda, \alpha_2^\lambda]$.

Goetschel and Voxman [3] established the following lemma.

Lemma 2.2. [3] *Let $\alpha_1, \alpha_2 \in \mathbf{R}^{[0,1]}$ be two mapping sending each $\lambda \in [0,1]$ to $\alpha_1(\lambda) = \alpha_1^\lambda$ and $\alpha_2(\lambda) = \alpha_2^\lambda$ respectively with the properties:*
(i) α_1 is a bounded increasing function,
(ii) α_2 is a bounded decreasing function,
(iii) $\alpha_1(1) \leq \alpha_2(1)$ and
(iv) α_1 and α_2 are both left continuous on $(0,1]$ and right continuous at 0.
Then there exists a unique fuzzy number $\alpha \in E^1$ such that $[\alpha]^\lambda = [\alpha_1^\lambda, \alpha_2^\lambda]$ for each $\lambda \in [0,1]$.

Let $\Omega = \{a = [\underline{a}, \bar{a}] : \underline{a}, \bar{a} \in \mathbf{R}, \underline{a} \leq \bar{a}\}$ be the family of all bounded closed intervals. Let $a, b \in \Omega$. We define

(i) $a = b$ if and only if $\underline{a} = \underline{b}, \bar{a} = \bar{b}$,

(ii) $a \leq b$ if and only if $\underline{a} \leq \underline{b}, \bar{a} \leq \bar{b}$,

(iii) $a + b = [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$,

(iv) $a.b = \{st : s \in a, t \in b\}$,

(v) $\underline{a.b} = \min\{\underline{a}.\underline{b}, \underline{a}.\bar{b}, \bar{a}.\underline{b}, \bar{a}.\bar{b}\}$,

(vi) $\overline{a.b} = \max\{\underline{a}.\underline{b}, \underline{a}.\bar{b}, \bar{a}.\underline{b}, \bar{a}.\bar{b}\}$.

Here we observe that "$\leq$" is a partial order in Ω and the mapping $\rho : \Omega \times \Omega \to \mathbf{R}$ defined by $\rho(a,b) = \max\{| \underline{a} - \underline{b} |, | \bar{a} - \bar{b} |\}$ for all $a, b \in \Omega$ is a metric (called Hausdorff metric) on Ω.

Now it is easy to verify that the mapping $\hat{\rho} : E^1 \times E^1 \to \mathbf{R}$ defined by $\hat{\rho}(\alpha,\beta) = \sup\{\rho([\alpha]^\lambda, [\beta]^\lambda) : \lambda \in [0,1]\}$ for all $\alpha, \beta \in E^1$ is a metric on E^1.

The results of the following theorem have been used frequently in this paper.

Theorem 2.3. [7, 10, 13] *(i) $(E^1, \hat{\rho})$ is a complete metric space.*
(ii) $\hat{\rho}(\alpha+\gamma, \beta+\gamma) = \hat{\rho}(\alpha,\beta)$ for all $\alpha, \beta, \gamma \in E^1$.
(iii) $\hat{\rho}(\lambda\alpha, \lambda\beta) = | \lambda | \hat{\rho}(\alpha,\beta)$ for all $\alpha, \beta \in E^1$ and $\lambda \in \mathbf{R}$.
(iv) $\hat{\rho}(\alpha+\gamma, \beta+\eta) \leq \hat{\rho}(\alpha,\beta) + \hat{\rho}(\gamma,\eta)$ for all $\alpha, \beta, \gamma, \eta \in E^1$.
(v) $\hat{\rho}(\alpha+\beta, \theta) \leq \hat{\rho}(\alpha,\theta) + \hat{\rho}(\beta,\theta)$ for all $\alpha, \beta \in E^1$ and θ is the characteristic function of zero.
(vi) $\hat{\rho}(\alpha+\beta, \gamma) \leq \hat{\rho}(\alpha,\gamma) + \hat{\rho}(\beta,\theta)$ for all $\alpha, \beta, \gamma \in E^1$.
(vii) If $\alpha, \beta, \gamma \in E^1$, $\hat{\rho}(\alpha,\beta) \leq \hat{\rho}(\alpha,\gamma)$ and $\hat{\rho}(\beta,\gamma) \leq \hat{\rho}(\alpha,\gamma)$.

Definition 2.4. [8] A *tagged partition* of $[a,b]$ consist of a partition $\Sigma = \{x_0, x_1, x_2, ..., x_{i-1}, x_i, ..., x_n\}$, where $a = x_0 < x_1 < x_2 < ... < x_{i-1} < x_i, ... < x_n = b$ of $[a,b]$ and $\xi = \{\xi_i : i = 1, 2, ..., n\}$, where $\xi_i \in [x_{i-1}, x_i], i = 1, 2, ..., n$ and it is denoted by (Σ, ξ). Also let $\sigma_i = \xi_i - \xi_{i-1}, i = 1, 2, ..., n$.

Let $\delta \in \mathbf{R}_+^{[a,b]}$, $\Sigma = \{x_0, x_1, x_2, ..., x_{i-1}, x_i, ..., x_n\} \subset [a,b]$ and $\xi = \{\xi_i : i = 1, 2, ..., n\}$, where $\xi_i \in [x_{i-1}, x_i]$ for each $i = 1, 2, ..., n$.
(i) The pair (Σ, ξ) is called a *δ-fine division* of $[a,b]$ if $a = x_0 < x_1 < x_2 < ... < x_{i-1} < x_i, ... < x_n = b$ and $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.
(ii) The pair (Σ, ξ) is called a *δ-fine subdivision* of $[a,b]$ if $a \leq x_0 \leq x_1 \leq x_2 \leq ... \leq x_{i-1} \leq x_i, ... < x_n \leq b$ and $[x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

Now we recall the definition of Henstock integral [8] for a function $\psi \in \mathbf{R}^{[a,b]}$.

Definition 2.5. [8] A mapping $\psi \in \mathbf{R}^{[a,b]}$ is called *Henstock integrable* on $[a,b]$ with real value l if for each $\varepsilon > 0$, there exists a $\delta \in \mathbf{R}_+^{[a,b]}$ such that $| \sum_{i=1}^{n} \psi(\xi_i)\sigma_i - l | < \varepsilon$ for every δ-fine division (Σ, ξ) of $[a,b]$.

Any function $\psi \in {E^1}^{[a,b]}$ is called a *fuzzy function* defined on $[a,b]$. Wu and Gong introduced the notion of fuzzy Henstock integral of fuzzy function defined on a closed interval $[a,b]$.

Definition 2.6. A fuzzy function ψ defined on $[a,b]$ is called *fuzzy Henstock integrable* [12] on $[a,b]$ with value $\alpha \in E^1$ if for each $\varepsilon > 0$, there exists a $\delta \in \mathbf{R}_+^{[a,b]}$ such that $\hat{\rho}(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \alpha) < \varepsilon$ for every δ-fine division (Σ, ξ) of $[a,b]$.

In symbol, we write

$$(H)\int_a^b \psi = \alpha$$

or

$$(H)\int_{[a,b]} \psi = \alpha$$

if it exists. Here α is called Henstock integral value of ψ on $[a,b]$. The set of all fuzzy Henstock integrable fuzzy functions defined on $[a,b]$ is denoted by $FH[a,b]$.

Wu and Gong [12] have achieved the following basic results of fuzzy Henstock integrable function.

Theorem 2.7. [12] *Let* $\psi, \psi_1, \psi_2 \in {E^1}^{[a,b]}$. *Then*
(i) If $(H)\int_a^b \psi$ *exists, then its value is unique.*
(ii) $\psi \in FH[a,b]$ *if and only if for each* $\varepsilon > 0$, *there exists a* $\delta \in \mathbf{R}_+^{[a,b]}$ *such that all* δ*-fine divisions* (Σ, ξ) *and* (Σ', ξ') *of* $[a,b]$ *satisfy*

$$\hat{\rho}(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \sum_{i=1}^{m} \psi(\xi_i')\sigma_i') < \varepsilon.$$

(iii) If $\psi_1, \psi_2 \in FH[a,b]$, *then*

$$(H)\int_a^b (\psi_1 + \psi_2) = (H)\int_a^b \psi_1 + (H)\int_a^b \psi_2.$$

(iv) If $\psi \in FH[a,b]$, *then*

$$(H)\int_a^b (\lambda\psi) = \lambda(H)\int_a^b \psi$$

for any $\lambda \in \mathbf{R}$.
(v) If $\psi \in FH[a,b]$ *and* $[c,d] \subset [a,b]$, *then* $\psi \in FH[c,d]$.
(vi) If $c \in [a,b]$, $\psi \in FH[a,c]$ *and* $\psi \in FH[c,b]$, *then* $\psi \in FH[a,b]$ *with*

$$(H)\int_a^b \psi = (H)\int_a^c \psi + (H)\int_c^d \psi.$$

(vii) If $\psi = \theta$ *almost everywhere on [a,b], then*

$$(H)\int_a^b \psi = \theta.$$

(viii) If $\psi = \phi$ *almost everywhere on [a,b], then*

$$(H)\int_a^b \psi = (H)\int_a^b \phi.$$

Let μ be a real constant, i.e. $\mu \in \mathbf{R}$. Then define $\mu : \mathbf{R} \to [0,1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \mu \\ 0 & \text{if } x \neq \mu \end{cases}$$

Clearly, $\mu \in E^1$. Thus a real number can be viewed as a fuzzy number in this way.

For $\alpha, \beta \in E^1$, we define the relation "$\alpha \le \beta$" if and only if $\alpha(x) \le \beta(x)$ for all $x \in \mathbf{R}$.

Zhang Guang-Quan [7] introduced the concepts of bounds of a set of fuzzy numbers.

Definition 2.8. [7] A fuzzy number $\alpha_0 \in E^1$ is called the least upper bound (or suprimum) of $A \subset E^1$ if
(i) $\alpha \leq \alpha_0$ for all $\alpha \in A$ (i.e. α is an upper bound of A) and
(ii) for any $\varepsilon > 0$, there exists at least one $\beta \in A$ such that $\alpha_0 < \beta + \varepsilon$. We write $\alpha_0 = \sup A$.
Similarly, the greatest lower bound (or infimum) [7] of $A \subset E^1$ has been defined and is denoted by $\inf A$.
A sequence $\{\alpha_k\}, \alpha_k \in E^1$ is said to be monotonically increasing (resp. monotonically decreasing) [7] if $\alpha_k \leq \alpha_{k+1}$ (resp. $\alpha_{k+1} \leq \alpha_k$) for all $k \in \mathbf{N}$.

Zhang Guang-Quan [7] established the following simple but important theorem.

Theorem 2.9. [7] *Every monotonically increasing (resp. monotonically decreasing) sequence $\{\alpha_k\}, \alpha_k \in E^1$ with an upper bound (resp. a lower bound) converges to $\sup\{\alpha_k : k \in \mathbf{N}\}$ (resp. $\inf\{\alpha_k : k \in \mathbf{N}\}$) in the metric space $(E^1, \hat{\rho})$.*

3. Convergence Theorems

Let $\{\psi_k\}$ be a sequence of fuzzy Henstock integrable functions in ${E^1}^{[a,b]}$ that fuzzy converges to the fuzzy function $\psi \in {E^1}^{[a,b]}$ in the metric space $(E^1, \hat{\rho})$. It is quite natural to expect that $\psi \in FH[a,b]$ and

$$(H)\int_a^b \psi = \lim (H)\int_a^b \psi_k.$$

But the following example shows that this is not true in general.

Example 3.1. For each $k \in \mathbf{N}$, let $A_k = (0, \frac{1}{k})$ and define $\phi_k : [0,1] \to E^1$ by

$$\phi_k(t) = \begin{cases} \lambda & \text{if } t \in A_k \\ \theta & \text{if } t \notin A_k \end{cases}$$

where $\lambda \in E^1$ is defined by

$$\lambda(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1. \end{cases}$$

Now consider for each $k \in \mathbf{N}$, $\psi_k = k \cdot \phi_k$. Since each ϕ_k is a step function having only two discontinuities 0 and $\frac{1}{k}$, clearly $\phi_k \in FH[0,1]$ and so by Theorem 2.7, for each $k \in \mathbf{N}$, $\psi_k \in FH[0,1]$. Now by Theorem 2.7,

$$\begin{aligned}(H)\int_0^1 \psi_k &= (H)\int_0^{\frac{1}{k}} \psi_k + (H)\int_{\frac{1}{k}}^1 \psi_k \\ &= k \cdot (H)\int_0^{\frac{1}{k}} \phi_k + \theta = k \cdot (H)\int_0^{\frac{1}{k}} \lambda.\end{aligned}$$

Using Riemann type sum, it is easy to verify that

$$\int_0^{\frac{1}{k}} \lambda = \lambda \cdot \frac{1}{k}.$$

So

$$(H)\int_0^1 \psi_k = \lambda.$$

Now consider the fuzzy function $\psi : [0,1] \to E^1$ defined by

$$\psi(t) = \theta$$

for all $t \in [0,1]$. Then $\{\psi_k\}$ fuzzy converges to the fuzzy function $\psi \in E^{1[a,b]}$ in the metric space $(E^1, \hat{\rho})$ and by (vii) of Theorem 2.7,

$$(H)\int_0^1 \psi = \theta.$$

Thus

$$(H)\int_a^b \psi \neq \lim(H)\int_a^b \psi_k.$$

The main purpose of this paper is to establish some sufficient conditions such that the limit $\psi \in E^{1[a,b]}$ of a sequence $\{\psi_k\}$ of fuzzy Henstock integrable functions in $E^{1[a,b]}$ is fuzzy Henstock integrable on $[a,b]$ and

$$(H)\int_a^b \psi = \lim(H)\int_a^b \psi_k.$$

Definition 3.2. A sequence $\{\psi_k\}$ in $E^{1[a,b]}$ is said to be fuzzy uniformly converge to $\psi \in E^{1[a,b]}$ on $[a,b]$ if for each $\varepsilon > 0$, there exists a $k_0 \in \mathbf{N}$ such that $\hat{\rho}\left(\psi_k(x), \psi(x)\right) < \varepsilon$ for all $k \geq k_0$ and for all $x \in [a,b]$.

Theorem 3.3. *(Fuzzy uniform convergence theorem). Let $\{\psi_k\}$ be a sequence of fuzzy Henstock integrable functions in $E^{1[a,b]}$ that fuzzy uniformly converges to the fuzzy function $\psi \in E^{1[a,b]}$. Then*
(i) ψ is Henstock integrable on $[a,b]$ and
(ii)

$$(H)\int_a^b \psi = \lim(H)\int_a^b \psi_k.$$

Proof. First we shall show that $\{(H)\int_a^b \psi_k\}$ is a Cauchy sequence in $(E^1, \hat{\rho})$. Let $\varepsilon > 0$. Since for each $k \in \mathbf{N}$, there exists a $\delta_k \in \mathbf{R}_+^{[a,b]}$ such that

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, (H)\int_a^b \psi_k\right) < \frac{\varepsilon}{3}$$

for every δ_k-fine division (Σ, ξ) of $[a,b]$. Again since $\{\psi_k\}$ in $E^{1[a,b]}$ fuzzy uniformly converges to $\psi \in E^{1[a,b]}$, there exists a $k_0 \in \mathbf{N}$ such that

$$\hat{\rho}\left(\psi_k(\xi_i), \psi(\xi_i)\right) < \frac{\varepsilon}{3(b-a)}$$

for all $k \geq k_0$ and for all $i = 1, 2, ..., n$.

So, for all $k, l \geq k_0$, taking an arbitrary partition (Σ, ξ) simultaneously δ_k- and δ_l-fine, we have

$$\begin{aligned}
&\hat{\rho}\left((H)\int_a^b \psi_k, (H)\int_a^b \psi_l\right) \\
&\leq \hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, (H)\int_a^b \psi_k\right) + \hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_l(\xi_i)\sigma_i\right) + \hat{\rho}\left(\sum_{i=1}^{n} \psi_l(\xi_i)\sigma_i, (H)\int_a^b \psi_l\right) \\
&< \frac{\varepsilon}{3} + \sum_{i=1}^{n} \hat{\rho}(\psi_k(\xi_i), \psi_l(\xi_i))\sigma_i + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus $\{(H)\int_a^b \psi_k\}$ is a Cauchy sequence in $(E^1, \hat{\rho})$.

Since by Theorem 2.3, $(E^1, \hat{\rho})$ is complete, $\{(H)\int_a^b \psi_k\}$ converges in the metric space $(E^1, \hat{\rho})$. Suppose

$$\lim (H)\int_a^b \psi_k = \alpha.$$

Let $\varepsilon > 0$ be given. By the condition of the theorem there exists a $k_0 \in \mathbf{N}$ such that

$$\hat{\rho}\left(\psi_k(x), \psi(x)\right) < \frac{\varepsilon}{3(b-a)}$$

for all $k \geq k_0$ and for all $x \in [a, b]$. Now for any tagged partition (Σ, ξ) of $[a, b]$ and $k \geq k_0$, by Theorem 2.3, we get

$$\begin{aligned}
\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi(\xi_i)\sigma_i\right) &\leq \sum_{i=1}^{n} \sigma_i \hat{\rho}\left(\psi_k(\xi_i), \psi(\xi_i)\right) \\
&\leq (b-a)\sum_{i=1}^{n} \hat{\rho}\left(\psi_k(\xi_i), \psi(\xi_i)\right) < \frac{\varepsilon}{3}.
\end{aligned}$$

Now since $\lim (H)\int_a^b \psi_k = \alpha$, there exists a $p(\geq k_0) \in \mathbf{N}$ such that

$$\hat{\rho}\left((H)\int_a^b \psi_p, \alpha\right) < \frac{\varepsilon}{3}.$$

Since $(H)\int_a^b \psi_p$ exists, there exists a $\delta_p \in \mathbf{R}_+{}^{[a,b]}$ such that every δ_p-fine division (Σ, ξ) of $[a, b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_p(\xi_i)\sigma_i, (H)\int_a^b \psi_p\right) < \frac{\varepsilon}{3}.$$

Thus

$$\begin{aligned}
&\hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \alpha\right) \\
&\leq \hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_p(\xi_i)\sigma_i\right) + \hat{\rho}\left(\sum_{i=1}^{n} \psi_p(\xi_i)\sigma_i, (H)\int_a^b \psi_p\right) + \hat{\rho}\left((H)\int_a^b \psi_p, \alpha\right) < \varepsilon
\end{aligned}$$

for every δ_p-fine division (Σ, ξ) of $[a, b]$.

Hence $(H)\int_a^b \psi$ exists and

$$(H)\int_a^b \psi = \alpha.$$

□

Definition 3.4. A sequence $\{\psi_k\}$ of fuzzy Henstock integrable functions in $E^{1^{[a,b]}}$ is called fuzzy uniform Henstock integrable on $[a,b]$ if for each $\varepsilon > 0$, there exists a $\delta \in \mathbf{R}_+^{[a,b]}$ such that every δ-fine division (Σ, ξ) of $[a,b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, (H)\int_a^b \psi_k\right) < \varepsilon$$

for all $k \in \mathbf{N}$.

Z. Gong and Y. Shao [5] proved a convergence theorem (see Theorem 5.1 of [5]) for the strongly fuzzy Henstock integral [4] which showed that the controlled convergence [5] of a sequence of strongly fuzzy Henstock integrable functions implies the equi-integrability [5] of a subsequence of the sequence. In the next theorem, we shall prove a convergence theorem for fuzzy uniform Henstock integrable functions following the argument used in Theorem 5.1 of [5].

Theorem 3.5. *(Convergence theorem for fuzzy uniform Henstock integrable functions). Let $\{\psi_k\}$ be a fuzzy uniform Henstock integrable sequence of fuzzy Henstock integrable functions in $E^{1^{[a,b]}}$ and $\psi \in E^{1^{[a,b]}}$ be such that for each $x \in [a,b]$, $\{\psi_k(x)\}$ converges to $\psi(x)$ in the metric space $(E^{1^{[a,b]}}, \hat{\rho})$. Then*
(i) ψ is Henstock integrable on $[a,b]$ and
(ii)

$$(H)\int_a^b \psi = \lim (H)\int_a^b \psi_k.$$

Proof. Let $\varepsilon > 0$ be given. Since $\{\psi_k\}$ is a fuzzy uniform Henstock integrable sequence on $[a,b]$, there exists a δ-fine division (Σ, ξ) of $[a,b]$ that satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, (H)\int_a^b \psi_k\right) < \frac{\varepsilon}{3}$$

for all $k \in \mathbf{N}$.

Again by the condition of the theorem there exists a $k_0 \in \mathbf{N}$ such that

$$\hat{\rho}\left(\psi_k(\xi_i), \psi_l(\xi_i)\right) < \frac{\varepsilon}{3(b-a)}$$

for all $k, l > k_0$ and so

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_l(\xi_i)\sigma_i\right) \le \sum_{i=1}^{n} \sigma_i \hat{\rho}\left(\psi_k(\xi_i), \psi_l(\xi_i)\right)$$
$$\le (b-a)\sum_{i=1}^{n} \hat{\rho}\left(\psi_k(\xi_i), \psi_l(\xi_i)\right) < \frac{\varepsilon}{3}$$

for all $k, l > k_0$. Then

$$\hat{\rho}\left((H)\int_a^b \psi_k, (H)\int_a^b \psi_l\right)$$
$$\le \hat{\rho}\left((H)\int_a^b \psi_k, \sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i\right) + \hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_l(\xi_i)\sigma_i\right)$$
$$+ \hat{\rho}\left(\sum_{i=1}^{n} \psi_l(\xi_i)\sigma_i, (H)\int_a^b \psi_l\right) < \varepsilon$$

for all $k, l > k_0$.

So $\{(H)\int_a^b \psi_k\}$ is a Cauchy sequence in the complete metric space $(E^1,\hat{\rho})$. Therefore $\{(H)\int_a^b \psi_k\}$ converges in the metric space $(E^1,\hat{\rho})$. Suppose

$$\lim (H)\int_a^b \psi_k = \alpha.$$

We claim that

$$(H)\int_a^b \psi = \alpha.$$

To show this, let $\varepsilon > 0$ be given. Since $\{\psi_k\}$ is a fuzzy uniformly Henstock integrable sequence on $[a,b]$, there exists a $\delta \in \mathbf{R}_+^{[a,b]}$ such that every δ-fine division (Σ,ξ) of $[a,b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, (H)\int_a^b \psi_k\right) < \frac{\varepsilon}{3}$$

for all $k \in \mathbf{N}$.

Since $\lim (H)\int_a^b \psi_k = \alpha$, there exists a $k_1 \in \mathbf{N}$ such that

$$\hat{\rho}\left((H)\int_a^b \psi_k, \alpha\right) < \frac{\varepsilon}{3}$$

for all $k \geq k_1$. Again by the given condition of the theorem, there exists a $k_2(\geq k_1) \in \mathbf{N}$ such that

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi(\xi_i)\sigma_i\right) \leq (b-a)\sum_{i=1}^{n} \hat{\rho}\left(\psi_k(\xi_i), \psi(\xi_i)\right) < \frac{\varepsilon}{3}$$

Thus

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \alpha\right)$$
$$\leq \hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i\right) + \hat{\rho}\left(\sum_{i=1}^{n} \psi_k(\xi_i)\sigma_i, (H)\int_a^b \psi_k\right) + \hat{\rho}\left((H)\int_a^b \psi_k, \alpha\right) < \varepsilon$$

for every δ-fine division (Σ,ξ) of $[a,b]$. Hence $(H)\int_a^b \psi$ exists and

$$(H)\int_a^b \psi = \alpha.$$

□

Saks-Henstock lemma plays an important role in Henstock integration theory. Now we shall establish the fuzzy version of this lemma.

Lemma 3.6. *Let $\psi \in {E^1}^{[a,b]}$ be a fuzzy Henstock integrable function in ${E^1}^{[a,b]}$, let*

$$\phi(x) = (H)\int_a^x \psi$$

for all $x \in [a,b]$ and let $\varepsilon > 0$. Further suppose
(i) $\delta \in \mathbf{R}_+^{[a,b]}$ is a positive real-valued function such that

$$\hat{\rho}\left(\sum_{i=1}^{n}\psi(\xi_i)\sigma_i, \sum_{i=1}^{n}\phi(\xi_i)\sigma_i\right) < \varepsilon$$

for every δ-fine division (Σ, ξ) of $[a,b]$.

If $(\Sigma', \xi') = \{(\xi_i', [a_i', b_i']) : i = 1, 2, ..., n\}$ is a δ-fine subdivision of $[a,b]$, then

$$\hat{\rho}\left(\sum_{i=1}^{m}\psi(\xi_i')\sigma_i', \sum_{i=1}^{m}\phi(\xi_i')\sigma_i'\right) \leq \varepsilon.$$

Proof. Let $\varepsilon_0 > 0$ and $\{F_i : i = 1, 2, ..., r\}$ be the family of closed intervals in $[a,b]$ such that $\{F_i : i = 1, 2, ..., r\} \cup \Sigma'$ is a partition of $[a,b]$. Here $\psi \in E^{1^{[a,b]}}$ is fuzzy Henstock integrable on each of the intervals $F_1, F_2, ..., F_r$ and hence for each $k \in \{1, 2, ..., r\}$, there exists a $\delta_k \in \mathbf{R}_+^{[a,b]}$ such that every δ_k-fine division (Σ^k, ξ^k), $\Sigma^k = \{x_0^k, x_1^k, ..., x_{i-1}^k, x_i^k, ..., x_{n_k}^k\}, \xi^k = \{\xi_1^k, ..., \xi_i^k, ..., \xi_{n_k}^k\}$ of $[a,b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_k}\psi(\xi_i^k)\sigma_i^k, (H)\int_{F_t}\psi\right) < \frac{\varepsilon_0}{r}.$$

Without loss of generality, we can assume that $\delta_k(x) \leq \delta(x)$ for all $x \in F_k$, $k \in \{1, 2, ..., r\}$. If we take $\Sigma = \Sigma^1 \cup \Sigma^2 \cup ... \cup \Sigma^r \cup \Sigma'$ and $\xi = \xi^1 \cup \xi^2 \cup ... \cup \xi^r \cup \xi'$, then (Σ, ξ) is a δ-fine division of $[a,b]$.

Thus using condition (i) and Theorem 2.3, we get

$$\hat{\rho}\left(\sum_{k=1}^{r}\left(\sum_{i=1}^{n_k}\psi(\xi_i^k)\sigma_i^k\right) + \sum_{i=1}^{m}\psi(\xi_i')\sigma_i', \sum_{k=1}^{r}\left(\sum_{i=1}^{n_k}\phi(\xi_i^k)\sigma_i^k\right) + \sum_{i=1}^{m}\phi(\xi_i')\sigma_i'\right) < \varepsilon.$$

Now

$$\hat{\rho}\left(\sum_{i=1}^{m}\psi(\xi_i')\sigma_i', \sum_{i=1}^{m}\phi(\xi_i')\sigma_i'\right)$$
$$= \hat{\rho}((\sum_{k=1}^{r}(\sum_{i=1}^{n_k}\psi(\xi_i^k)\sigma_i^k) + \sum_{i=1}^{m}\psi(\xi_i')\sigma_i') - \sum_{k=1}^{r}(\sum_{i=1}^{n_k}\psi(\xi_i^k)\sigma_i^k),$$
$$(\sum_{k=1}^{m}(\sum_{i=1}^{n_k}\phi(\xi_i^k)\sigma_i^k) + \sum_{i=1}^{m}\phi(\xi_i')\sigma_i') - \sum_{k=1}^{m}(\sum_{i=1}^{n_k}\phi(\xi_i^k)\sigma_i^k))$$
$$\leq \hat{\rho}\left(\left(\sum_{k=1}^{r}\left(\sum_{i=1}^{n_k}\psi(\xi_i^k)\sigma_i^k\right) + \sum_{i=1}^{m}\psi(\xi_i')\sigma'\right), \left(\sum_{k=1}^{m}\left(\sum_{i=1}^{n_k}\phi(\xi_i^k)\sigma_i^k\right) + \sum_{i=1}^{m}\phi(\xi_i')\sigma_i'\right)\right) +$$
$$\sum_{k=1}^{r}\hat{\rho}(\sum_{i=1}^{n_k}\psi(\xi_i^k)\sigma_i^k, \sum_{i=1}^{n_k}\phi(\xi_i^k)\sigma_i^k)$$
$$< \varepsilon + r\frac{\varepsilon_0}{r} = \varepsilon + \varepsilon_0$$

and consequently,

$$\hat{\rho}(\sum_{i=1}^{m}\psi(\xi_i')\sigma_i', \sum_{i=1}^{m}\phi(\xi_i')\sigma_i') \leq \varepsilon.$$

□

A sequence $\{\psi_k\}, \psi_k \in E^{1^{[a,b]}}$ is called *fuzzy increasing (resp. fuzzy decreasing)* in $[a,b]$ if $\psi_k(x) \leq \psi_{k+1}(x)$ (resp. $\psi_{k+1}(x) \leq \psi_k(x)$) for all $x \in [a,b]$ and $k \in \mathbf{N}$. A sequence $\{\psi_k\}$ is called *fuzzy monotone* on $[a,b]$ if it is either fuzzy increasing or fuzzy decreasing in $[a,b]$.

Theorem 3.7. *(Fuzzy monotone convergence theorem). Let $\{\psi_k\}$ be a fuzzy monotone sequence of fuzzy Henstock integrable functions in $E^{1^{[a,b]}}$, $\{(H)\int_a^b \psi_k\}$ be fuzzy bounded and $\psi \in E^{1^{[a,b]}}$ be such that for each $x \in [a,b], \{\psi_k(x)\}$ converges to $\psi(x)$ in the metric space $(E^{1^{[a,b]}}, \hat{\rho})$. Then*
(i) ψ is Henstock integrable on $[a,b]$ and
(ii)

$$(H)\int_a^b \psi = \lim (H)\int_a^b \psi_k.$$

Proof. Let $\{\psi_k\}$ be a fuzzy increasing sequence of fuzzy Henstock integrable functions in $E^{1^{[a,b]}}$. Then $\{(H)\int_a^b \psi_k\}$ is fuzzy increasing and bounded. Then by Theorem 2.9, $\{(H)\int_a^b \psi_k\}$ must be fuzzy converges to $\alpha = \sup\{(H)\int_a^b \psi_k\}$.

Let $\varepsilon > 0$ be given. Then we can choose an $r \in \mathbf{N}$ such that $\frac{1}{2^{r-2}} < \frac{\varepsilon}{3}$ and

$$\hat{\rho}\left((H)\int_a^b \psi_r, \alpha\right) < \frac{\varepsilon}{3}.$$

Again since $\{\psi_k\}$ is a sequence of fuzzy Henstock integrable functions on $[a,b]$, for each $k \in \mathbf{N}$ there exists a $\delta_k \in \mathbf{R}_+^{[a,b]}$ such that every δ_k-fine division (Σ^k, ξ^k) of $[a,b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_k} \psi_k(\xi_i^k)\sigma_i^k, ((H)\int_a^b \psi_k\right) < \frac{1}{2^k}.$$

Again by the condition, for each $x \in [a,b]$, we can select a $k_x(\geq r) \in \mathbf{N}$ such that

$$\hat{\rho}\left(\psi_{k_x}(x), \psi(x)\right) < \frac{\varepsilon}{3(b-a)}.$$

Consider the function $\delta = \delta_{k_x}$ and let $(\Sigma, \xi) = \{(\Sigma_i, \xi_i) : i = 1, 2, ..., n\}$ be any δ-fine division of $[a,b]$. Here

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \alpha\right) \leq \hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_{k_{\xi_i}}(\xi_i)\sigma_i\right) + \hat{\rho}\left(\sum_{i=1}^{n} \psi_{k_{\xi_i}}(\xi_i)\sigma_i, \sum_{i=1}^{n}(H)\int_{\Sigma_i} \psi_{k_{\xi_i}}\right) + \hat{\rho}\left(\sum_{i=1}^{n}(H)\int_{\Sigma_i} \psi_{k_{\xi_i}}, \alpha\right).$$

Now we estimate the three values in the right-handed sum of the last inequality.
(a) Estimation of

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_{k_{\xi_i}}(\xi_i)\sigma_i\right):$$

By Theorem 2.3,

$$\hat{\rho}\left(\sum_{i=1}^{n} \psi(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi_{k_{\xi_i}}(\xi_i)\sigma_i\right) \leq \sum_{i=1}^{n} \hat{\rho}\left(\psi(\xi_i), \psi_{k_{\xi_i}}(\xi_i)\right)\sigma_i < \frac{\varepsilon}{3(b-a)}(b-a) = \frac{\varepsilon}{3}.$$

(b) Estimation of
$$\hat{\rho}\left(\sum_{i=1}^{n}\psi_{k_{\xi_i}}(\xi_i)\sigma_i, \sum_{i=1}^{n}(H)\int_{\Sigma_i}\psi_{k_{\xi_i}}\right):$$
Suppose $p = \max\{k_{\xi_i} : i = 1, 2, ..., n\}$. Then
$$\hat{\rho}\left(\sum_{i=1}^{n}\psi_{k_{\xi_i}}(\xi_i)\sigma_i, \sum_{i=1}^{n}(H)\int_{\Sigma_i}\psi_{k_{\xi_i}}\right) \leq \sum_{t=r}^{p}\left(\sum_{i\in\{1,2,...,n:k_{\xi_i}=t\}}\hat{\rho}\left(\psi_{k_{\xi_i}}(\xi_i), (H)\int_{\Sigma_i}\psi_{k_{\xi_i}}\right)\right).$$

Now applying Lemma 3.6,
$$\sum_{i\in\{1,2,...,n:k_{\xi_i}=t\}}\hat{\rho}\left(\psi_{k_{\xi_i}}(\xi_i), (H)\int_{\Sigma_i}\psi_{k_{\xi_i}}\right) \leq \frac{1}{2^{t-1}}$$
and hence
$$\hat{\rho}\left(\sum_{i=1}^{n}\psi_{k_{\xi_i}}(\xi_i)\sigma_i, \sum_{i=1}^{n}(H)\int_{\Sigma_i}\psi_{k_{\xi_i}}\right) < \sum_{t=r}^{p}\frac{1}{2^{t-1}} < \frac{1}{2^{r-2}} < \frac{\varepsilon}{3}.$$
(c) Estimation of
$$\hat{\rho}\left(\sum_{i=1}^{n}(H)\int_{\Sigma_i}\psi_{k_{\xi_i}}, \alpha\right):$$
Here $r \leq k_{\xi_i} \leq p$ implies $\psi_r(x) \leq \psi_{k_{\xi_i}}(x) \leq \psi_p(x)$ for all $x \in [a, b]$ and so
$$\int_{\Sigma_i}\psi_r \leq \int_{\Sigma_i}\psi_{k_{\xi_i}} \leq \int_{\Sigma_i}\psi_p.$$
Hence
$$\int_a^b\psi_r \leq \sum_{i=1}^{n}\int_{\Sigma_i}\psi_{k_{\xi_i}} \leq \int_a^b\psi_p \leq \alpha.$$
Therefore by Theorem 2.3,
$$\hat{\rho}\left(\sum_{i=1}^{n}(H)\int_{\Sigma_i}\psi_{k_{\xi_i}}, \alpha\right) \leq \hat{\rho}\left(\int_a^b\psi_r, \alpha\right) < \frac{\varepsilon}{3}.$$
Thus
$$\hat{\rho}\left(\sum_{i=1}^{n}\psi(\xi_i)\sigma_i, \alpha\right) < \varepsilon.$$
So ψ is Henstock integrable on $[a, b]$ and
$$(H)\int_a^b\psi = \alpha = \lim(H)\int_a^b\psi_k.$$
□

Bartle [1] found necessary and sufficient conditions for a Henstock integral convergence theorem of real functions. This paper of Bartle [1] inspires to establish the final theorem of this section. Actually, this theorem provides us a necessary and sufficient condition such that the point-wise limit $\psi \in E^{1^{[a,b]}}$ of a sequence $\{\psi_k\}$ of fuzzy Henstock integrable functions is to be fuzzy Henstock integrable on $[a, b]$ and the equality

$$(H)\int_a^b \psi = \lim (H)\int_a^b \psi_k,$$

holds.

Theorem 3.8. *Let $\{\psi_k\}$ be a sequence of fuzzy Henstock integrable functions in $E^{1^{[a,b]}}$ and $\psi \in E^{1^{[a,b]}}$ be such that for each $x \in [a, b], \{\psi_k(x)\}$ converges to $\psi(x)$ in the metric space $(E^{1^{[a,b]}}, \hat{\rho})$. Then the following conditions are equivalent:*
(i) ψ is fuzzy Henstock integrable on $[a, b]$ and

$$(H)\int_a^b \psi = \lim (H)\int_a^b \psi_k,$$

(ii) for each $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that for each $k \geq m$, there exists $\delta \in \mathbf{R}_+^{[a,b]}$ such that every δ-fine division (Σ^k, ξ^k) of $[a, b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_k} \psi_k(\xi_i^k)\sigma_i^k, \sum_{i=1}^{n_k} \psi(\xi_i^k)\sigma_i^k\right) < \varepsilon.$$

Proof. $(i) \Rightarrow (ii)$. Let $\varepsilon > 0$. Since

$$\lim (H)\int_a^b \psi_k = (H)\int_a^b \psi,$$

there exists an $m \in \mathbf{N}$ such that

$$\hat{\rho}\left((H)\int_a^b \psi_k, (H)\int_a^b \psi\right) < \frac{\varepsilon}{3}$$

for all $k \geq m$. Again since $\{\psi_k\}$ is a sequence of fuzzy Henstock integrable functions, for each $k(\geq m)$, we can find $\delta_k \in \mathbf{R}_+^{[a,b]}$ such that every δ_k-fine division (Σ^k, ξ^k) of $[a, b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_k} \psi_k(\xi_i^k)\sigma_i^k, (H)\int_a^b \psi_k\right) < \frac{\varepsilon}{3}.$$

Again since ψ is a fuzzy Henstock integrable on $[a, b]$, we can find $\delta_0 \in \mathbf{R}_+^{[a,b]}$ such that every δ_0-fine division (Σ^0, ξ^0) of $[a, b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_0} \psi(\xi_i^0)\sigma_i^0, (H)\int_a^b \psi\right) < \frac{\varepsilon}{3}.$$

We take $\delta \in \mathbf{R}_+^{[a,b]}$ defined by $\delta(x) = \min\{\delta_0(x), \delta_k(x)\}$. Then

$$\hat{\rho}\left(\sum_{i=1}^{n_k} \psi_k(\xi_i^k)\sigma_i^k, \sum_{i=1}^{n_k} \psi(\xi_i^k)\sigma_i^k\right) \leq \hat{\rho}\left(\sum_{i=1}^{n_k} \psi_k(\xi_i^k)\sigma_i^k, (H)\int_a^b \psi_k\right) + \hat{\rho}\left((H)\int_a^b \psi_k, (H)\int_a^b \psi\right) + \hat{\rho}\left((H)\int_a^b \psi, \sum_{i=1}^{n_k} \psi(\xi_i^k)\sigma_i^k\right) < \varepsilon$$

for every δ-fine division (Σ^k, ξ^k) of $[a,b]$ and $k \geq m$.

$(ii) \Rightarrow (i)$. Let $\varepsilon > 0$ and (ii) holds. Then we claim that $\{(H)\int_a^b \psi_k\}$ is a Cauchy sequence.

By (ii), there exists $m \in \mathbf{N}$ such that for each $k, l \geq m$, there exist $\delta_k, \delta_l \in \mathbf{R}_+^{[a,b]}$ such that

$$\hat{\rho}\left(\sum_{i=1}^{n_k} \psi_k(\xi_i^k)\sigma_i^k, \sum_{i=1}^{n_k} \psi(\xi_i^k)\sigma_i^k\right) < \frac{\varepsilon}{4}$$

for every δ_k-fine division (Σ^k, ξ^k) of $[a,b]$ and

$$\hat{\rho}\left(\sum_{i=1}^{n_l} \psi_l(\xi_i^l)\sigma_i^l, \sum_{i=1}^{n_l} \psi(\xi_i^l)\sigma_i^l\right) < \frac{\varepsilon}{4}$$

every δ_l-fine division (Σ^l, ξ^l) of $[a,b]$. Since ψ_k and ψ_l are fuzzy Henstock integrable functions on $[a,b]$, we can find $\varsigma_k, \varsigma_l \in \mathbf{R}_+^{[a,b]}$ such that

$$\hat{\rho}\left(\sum_{i=1}^{s_k} \psi_k(\tau_i^k)\varrho_i^k, (H)\int_a^b \psi_k\right) < \frac{\varepsilon}{4}$$

for every ς_k-fine division (Δ^k, τ^k) of $[a,b]$ and

$$\hat{\rho}\left(\sum_{i=1}^{s_l} \psi_k(\tau_i^l)\varrho_i^l, (H)\int_a^b \psi_l\right) < \frac{\varepsilon}{4}$$

for every ς_l-fine division (Δ^l, τ^l) of $[a,b]$.

Now define $\delta \in \mathbf{R}_+^{[a,b]}$ by $\delta(x) = \min\{\delta_k(x), \delta_l(x), \varsigma_k(x), \varsigma_l(x)\}$. Then for all $k, l \geq m$,

$$\hat{\rho}\left((H)\int_a^b \psi_k, (H)\int_a^b \psi_l\right) < \varepsilon$$

and so $\{(H)\int_a^b \psi_k\}$ is a Cauchy sequence. Completeness of metric space $(E^1, \hat{\rho})$ ensures that

$$\lim (H)\int_a^b \psi_k$$

exists in E^1. Suppose

$$\lim (H)\int_a^b \psi_k = \alpha.$$

Then we can choose a $p(\geq m) \in \mathbf{N}$ such that

$$\hat{\rho}\left((H)\int_a^b \psi_k, \alpha\right) < \frac{\varepsilon}{3}.$$

Also by (ii), there exists $\delta_1 \in \mathbf{R}_+^{[a,b]}$ such that every δ_1-fine division (Σ^1, ξ^1) of $[a,b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_1} \psi_p(\xi_i^1)\sigma_i^1, \sum_{i=1}^{n_1} \psi(\xi_i^1)\sigma_i^1\right) < \frac{\varepsilon}{3}.$$

Since ψ_p is a fuzzy Henstock integrable function, there exists an $\delta_2 \in \mathbf{R}^{+[a,b]}$ such that every δ_2-fine division (Σ^2, ξ^2) of $[a,b]$ satisfies

$$\hat{\rho}\left(\sum_{i=1}^{n_2} \psi_p(\xi_i^2)\sigma_i, (H)\int_a^b \psi_p\right) < \frac{\varepsilon}{3}.$$

We define $\delta \in \mathbf{R}_+^{[a,b]}$ by $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then

$$\hat{\rho}\left(\alpha, \sum_{i=1}^{n} \psi(\xi_i)\sigma_i\right) \leq \hat{\rho}\left(\alpha, (H)\int_a^b \psi_p\right) + \hat{\rho}\left((H)\int_a^b \psi_k, \sum_{i=1}^{n} \psi_p(\xi_i)\sigma_i\right) +$$
$$\hat{\rho}\left(\sum_{i=1}^{n} \psi_p(\xi_i)\sigma_i, \sum_{i=1}^{n} \psi(\xi_i)\sigma_i\right) < \varepsilon.$$

So ψ is Henstock integrable on $[a,b]$ and

$$(H)\int_a^b \psi = \alpha = \lim (H)\int_a^b \psi_k.$$

□

4. **Conclusions**

Let $\{\psi_k\}$ be a sequence of fuzzy Henstock integrable functions in $E^{1[a,b]}$ which pointwise converges to $\psi \in E^{1[a,b]}$ in the metric space $(E^1, \hat{\rho})$. In Example 3.1, we have shown that

$$\lim (H)\int_a^b \psi_k = (H)\int_a^b \psi.$$

is not true in general. As a result, finding various sufficient conditions as when the above equality will hold, are very much desired for fuzzy Henstock integrable functions. Being tempted, we have established, in this paper, three coveted convergence theorems for fuzzy Henstock integrable functions: “fuzzy uniform convergence theorem”, “convergence theorem for fuzzy uniform Henstock integrable functions” and “fuzzy monotone convergence theorem”; we have also achieved in finding a necessary and sufficient condition under which the point-wise limit of a sequence of fuzzy Henstock integrable functions is fuzzy Henstock integrable. In this paper, attempts have been made in establishing some basic convergence theorems, but more and more subsequent venture in this arena will emerge many non trivial results that will definitely enrich the Henstock integration theory in fuzzy setting.

Acknowledgements. The author is grateful to the learned reviewers for their constructive comments and valuable suggestions, which improved the paper to a great extent.

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