CREDIBILITY THEORY ORIENTED PREFERENCE INDEX FOR RANKING FUZZY NUMBERS

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Abstract. This paper suggests a novel approach for ranking the most applicable fuzzy numbers, i.e. LR-fuzzy numbers. Applying the α-optimistic values of a fuzzy number, a preference criterion is proposed for ranking fuzzy numbers using the Credibility index. The main properties of the proposed preference criterion are also studied. Moreover, the proposed method is applied for ranking fuzzy numbers using target-rank-based methods. Some numerical examples are used to illustrate the proposed ranking procedure. The proposed preference criterion is also examined in order to compare with some common methods and the feasibility and effectiveness of the proposed ranking method is cleared via some numerical comparisons.

1. Introduction

The problem of ordering fuzzy numbers plays very important roles in linguistic decision making and some other fuzzy application systems such as decision-making, data analysis, artificial intelligence, socioeconomic systems, statistical procedures, and etc. Because of the nature of measurements, many different strategies have been proposed for ranking fuzzy numbers. These include methods based on the coefficient of variation, distance measure, centroid point and original point, and weighted mean value, preference degree, and so on. Ranking fuzzy numbers were first proposed by Jain [22] for decision making in fuzzy environment. Since then, various procedures to rank fuzzy quantities are proposed by various researchers. Bortolan and Degani [4] considered the problem of ranking n fuzzy sets. They reviewed some methods suggested in the literature on a group of selected examples. Chen [8] presented ranking fuzzy numbers with maximizing set and minimizing set to decide the ordering value of each fuzzy number and used these values to determine the order of the n fuzzy numbers. He also gave a method for calculating the ordering value of each fuzzy number with triangular, trapezoidal, and a typical shaped membership functions. Dubois and Prade [16] developed the most known indices for ranking fuzzy numbers; the Pos and Nec criteria. Lee and Li [27] compared fuzzy numbers based on the probability measure of fuzzy events. Delgado et al. [13] proposed a procedure for ranking fuzzy numbers. De-Campos and Muñoz [12] presented a subjective approach for ranking fuzzy numbers. Kim and Park [23] introduced a method for comparing fuzzy numbers based on the combination of

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maximizing possibility and minimizing possibility using an criterion of optimism in $[0, 1]$ reflecting the decision maker's risk taking attitude. Yuan [47] proposed a criterion for evaluating fuzzy numbers. He also investigated the method based on some criteria for evaluating fuzzy ranking methods: fuzzy preference representation, rationality of fuzzy ordering, distinguishably, and robustness. Chanas and Zieliński [6] proved useful properties of fuzzy preference relations defined on the set of fuzzy numbers, which were introduced in [5]. Saade and Schwarzlander [35] presented ordering fuzzy sets over the real line from the point of view of ordering intervals. Liou and Wang [29] compared fuzzy numbers with integral value. They also used an criterion of optimism to reflect the decision maker's optimistic attitude in their proposed ranking method. Chang and Lee [7] formalized the concept of existence for the ranking of fuzzy sets. Since then several methods have been suggested by various researchers which includes ranking fuzzy numbers using area compensation, distance method, maximizing and minimizing set, decomposition principle, and signed distance. Wang and Kerre [40, 41] classified all the above ranking procedures into three classes: 1) ranking procedures based on fuzzy mean and spread, 2) ranking procedures based on fuzzy scoring, 3)ranking based on preference relations. Later on, the approaches for comparing fuzzy numbers have included ranking fuzzy numbers by preference ratio [32], left and right dominance [10], fuzzy distance measure [37], area between the centroid point and original point [11], preference weighting function expectations [31], sign distance [1], fuzzy simulation analysis method [36], an area method using radius of gyration [14], distance minimization [2], and fuzzy risk analysis based on the ranking of generalized trapezoidal fuzzy numbers [9]. Ekel et al. [18] presented some approaches for applying of fuzzy set based models and methods of multicriteria decision making for solving power engineering problems. The development in ranking fuzzy numbers can also be found in [19, 24, 28, 33, 34, 38, 39, 42, 43, 44, 45, 46].

The purpose of this paper is to provide a preference criterion to compare fuzzy numbers in setting of Credibility theory. Some main basic properties of the proposed preference criterion are put into the investigation including robustness, reciprocity. Then, we apply the proposed preference criterion to rank fuzzy numbers using a proposed unimodal fuzzy target as a basis for comparisons. Moreover, the proposed method is examined in order to compare with some common methods and the feasibility and effectiveness of the proposed methods are cleared via some theorems and numerical comparisons.

This paper is organized as follows: Section 2 reviews fuzzy numbers and some results in Credibility theory. Section 3 presents a method for constructing a preference criterion for comparing two fuzzy numbers based on the Credibility index. The basic properties of this criterion including robustness and reciprocity are being investigated in this section. The proposed preference criterion is then applied in order to compare fuzzy numbers using so-called fuzzy target-based evaluations. Several numerical examples used to clarify this study's discussion are presented. The proposed method is also compared with some other existing method. Finally, concluding remarks are made in another section.

2. **Fuzzy Numbers**

A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ (the real line) is defined by its membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0, 1]$. The set $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) \geq \alpha\}$ is called the α-level set (or α-cut) of the fuzzy set $\widetilde{A}$ at each level of $\alpha \in (0, 1]$. The set $\widetilde{A}[0]$ is also defined equal to the closure of the set $\{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) > 0\}$. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if it is normal, i.e. there exists a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ with $\mu_{\widetilde{A}}(x^*_{\widetilde{A}}) = 1$, and for every $\alpha \in (0, 1]$, the set $\widetilde{A}[\alpha]$ is a non-empty compact interval in $\mathbb{R}$. This interval will be denoted by $[\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. In what follows, a fuzzy number is called continuous if its membership function is continuous over $\mathbb{R}$. The set of all continuous fuzzy numbers will be denoted by $\mathcal{F}_c(\mathbb{R})$.

A LR-fuzzy number $\widetilde{A}$ is denoted simply by $(a^l, a, a^r)_{LR}$ with the following membership function:

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{a-x}{a-a^l}) & a^l < x \leq a, \\ R(\frac{x-a}{a^r-a}) & a < x < a^r, \\ 0 & x \in \mathbb{R} - [a^l, a^r], \end{cases}$$

where L and R are strictly decreasing functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. The LR-fuzzy numbers are very useful in practice since they can be characterized by means of three real numbers: the center, the left spread, and the right spread. The term LR is due to the left (L) and the right (R) shape of the membership function referred to the fuzzy number [26].

An special class of LR-fuzzy numbers is called triangular fuzzy numbers in which the shape functions L and R are given by $L(x) = R(x) = 1 - x$, for all $x \in [0, 1]$. The membership function of triangular fuzzy number, denoted by $\widetilde{A} = (a^l, a, a^r)_T$, is given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a^l}{a-a^l} & a^l \leq x \leq a, \\ \frac{a^r-x}{a^r-a} & a \leq x \leq a^r, \\ 0 & x \in \mathbb{R} - [a^l, a^r]. \end{cases}$$

Here, we recall some element of Credibility theory introduced by Liu [30]. We apply them to introduce a preference criterion to rank fuzzy numbers.

Definition 2.1. [17] Let $\widetilde{A} \in \mathcal{F}_c(\mathbb{R})$ and $b \in \mathbb{R}$. Then, the Credibility degree that "$\widetilde{A}$ is larger than or equal to b" is defined as follows:

$$Cr\{\widetilde{A} \succeq b\} = \frac{1}{2}(\sup_{x \geq b} \mu_{\widetilde{A}}(x) + 1 - \sup_{x < b} \mu_{\widetilde{A}}(x)). \tag{1}$$

In addition, the Credibility degree of $\widetilde{A} \prec b$ is similarly given by:

$$Cr\{\widetilde{A} \prec b\} = \frac{1}{2}(\sup_{x < b} \mu_{\widetilde{A}}(x) + 1 - \sup_{x \geq b} \mu_{\widetilde{A}}(x)). \tag{2}$$

Remark 2.2. It is worth to mention that for a given $\widetilde{A} \in \mathcal{F}_c(\mathbb{R})$, we have:

(1) $Cr\{\widetilde{A} \succeq b\} = 1$ if and only if $\widetilde{A}_0^L \geq b$.
(2) $Cr\{\widetilde{A} \succ b\} + Cr\{\widetilde{A} \preceq b\} = 1$.
(3) for $b_1 < b_2$, $Cr\{\widetilde{A} \succeq b_2\} \leq Cr\{\widetilde{A} \succeq b_1\}$.
(4) $Cr\{\widetilde{A} \succ b\} = Cr\{\widetilde{A} \succeq b\}$.

Example 2.3. let $\widetilde{A} = (a^l, a, a^r)_{LR}$ and b be a real number. Then, from equation (1), it is easy to verify that:

$$Cr\{\widetilde{A} \succeq b\} = \begin{cases} 1 & if\ x < a^l, \\ \frac{1}{2}(2 - L(\frac{k-(a-b)}{a-a^l})) & if\ a^l \leq x < a, \\ \frac{1}{2}R(\frac{b-a}{a^r-a}) & if\ a \leq x < a^r, \\ 0 & if\ x \geq a^r. \end{cases} \tag{3}$$

Especially, if let $\widetilde{A} = (a, a^l, a^r)_T$ then:

$$Cr\{\widetilde{A} \succeq b\} = \begin{cases} 1 & if\ x < a^l, \\ \frac{2a-a^l-b}{2(a-a^l)} & if\ a^l \leq x < a, \\ \frac{a^r-b}{2(a^r-a)} & if\ a \leq x < a^r, \\ 0 & if\ x \geq a^r. \end{cases} \tag{4}$$

Remark 2.4. For a given $\widetilde{A} \in \mathcal{F}_c(\mathbb{R})$, the α-optimistic values of a fuzzy number $\widetilde{A}$ is defined for each $\alpha \in [0, 1]$ by:

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^U_{2\alpha} & \alpha \in [0, 0.5], \\ \widetilde{A}^L_{2(1-\alpha)} & \alpha \in (0.5, 1]. \end{cases} \tag{5}$$

Then, it is easy to verify that the α-cuts of a fuzzy number $\widetilde{A} \in \mathcal{F}_c(\mathbb{R})$ is equivalent to $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2}, \widetilde{A}_{\alpha/2}]$, $\alpha \in [0, 1]$ (for more see [20]).

Example 2.5. For a given LR-fuzzy number $\widetilde{A} = (a^l, a, a^r)_{LR}$, it is easily to check that:

$$\widetilde{A}_\alpha = \begin{cases} a + (a^r - a)R^{-1}(2\alpha) & 0.0 \leq \alpha \leq 0.5, \\ a - (a - a^l)L^{-1}(2(1-\alpha)) & 0.5 \leq \alpha \leq 1.0. \end{cases}$$

Specially, if $\widetilde{A} = (a^l, a, a^r)_T$ is a triangular fuzzy number, then:

$$\widetilde{A}_\alpha = \begin{cases} a^r - 2\alpha(a^r - a) & 0.0 \leq \alpha \leq 0.5, \\ 2a - a^l - 2\alpha(a - a^l) & 0.5 \leq \alpha \leq 1.0. \end{cases}$$

For instance, let $\widetilde{A} = (-2, 0, 1)_T$. Therefore:

$$\widetilde{A}_\alpha = \begin{cases} 1 - 2\alpha & \text{for} \quad 0.0 \leq \alpha \leq 0.5, \\ 2 - 4\alpha & \text{for} \quad 0.5 \leq \alpha \leq 1.0. \end{cases}$$

Here, a well-known distance measure between fuzzy numbers is applied to fuzzy numbers. We apply this measure to examine the robustness property of the proposed preference criterion in Section 3.2.

Remark 2.6. [15] The Hausdorff distance on the set of fuzzy numbers, denoted in this paper by d_H, is defined as follows:

$$\forall \widetilde{A}, \widetilde{B} \in \mathcal{F}_c(\mathbb{R}), \quad d_H(\widetilde{A}, \widetilde{B}) = \sup_{\alpha \in [0,1]} H_1(\widetilde{A}[\alpha], \widetilde{B}[\alpha]).$$

According to the above consideration, $d_H(\widetilde{A}, \widetilde{B})$ equals to:

$$\sup_{\alpha \in [0,1]} \max\{|\widetilde{A}^L_\alpha - \widetilde{B}^L_\alpha|, |\widetilde{A}^U_\alpha - \widetilde{B}^U_\alpha|\} = \sup_{\alpha \in [0,1]} |\widetilde{A}_\alpha - \widetilde{B}_\alpha|,$$

which is the maximum difference between $\widetilde{A}$ and $\widetilde{B}$ introduced by Yuan [47]. It is easy to verify that, $d_H : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, +\infty)$ meets the following properties:

1) $d(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$, i.e. $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x)$ for all $x \in \mathbb{R}$,
2) $d(\widetilde{A}, \widetilde{B}) = d(\widetilde{B}, \widetilde{A})$,
3) $d(\widetilde{A}, \widetilde{C}) \leq d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C})$.

The following property of this distance will be applied in Section 3.2.

Lemma 2.7. *Let $\widetilde{A} \in \mathcal{F}_c(\mathbb{R})$ and suppose $\{\widetilde{A}_n\}$ is a sequence in $\mathcal{F}_c(\mathbb{R})$ with $\widetilde{A}_n \xrightarrow{d_H} \widetilde{A}$. Then $\mu_{\widetilde{A}_n}(x) \to \mu_{\widetilde{A}}(x)$, uniformly for all $x \in \mathbb{R}$.*

Proof. We show that $\sup_{x\in\mathbb{R}} |\mu_{\widetilde{A}_n}(x) - \mu_{\widetilde{A}}(x)| \to 0$. Suppose, on the contrary, that there exist $\epsilon_0 > 0$ and a subsequence of $\{\widetilde{A}_n\}$, denoted by $\{\widetilde{A}_n\}$ itself, for which:

$$\forall n \in \mathbb{N}, \quad \sup_{x\in\mathbb{R}} |\mu_{\widetilde{A}_n}(x) - \mu_{\widetilde{A}}(x)| > \epsilon_0.$$

Hence for each $n \in \mathbb{N}$, there exists $x_n \in \mathbb{R}$ such that:

$$|\mu_{\widetilde{A}_n}(x_n) - \mu_{\widetilde{A}}(x_n)| > \epsilon_0. \tag{6}$$

Clearly the sequence $\{x_n\}$ is situated in a compact subset of $\mathbb{R}$, and therefore it has a convergent subsequence which will be denoted again by $\{x_n\}$ itself. Let $x_0 := \lim x_n$. Using the continuity of the function $\mu_{\widetilde{A}}$, we have $|\mu_{\widetilde{A}}(x_n) - \mu_{\widetilde{A}}(x_0)| \to 0$. Hence, using equation (6), we have:

$$|\mu_{\widetilde{A}_n}(x_n) - \mu_{\widetilde{A}}(x_0)| > \epsilon_0$$

for $n \in \mathbb{N}$ large enough. Passing to a subsequence, if needed, and without loss of generality, we may assume that:

$$\mu_{\widetilde{A}_n}(x_n) < \mu_{\widetilde{A}}(x_0) - \epsilon_0, \tag{7}$$

for all $n \in \mathbb{N}$. Let $\alpha \in (0, 1)$ be chosen with $\mu_{\widetilde{A}}(x_0) - \epsilon_0 \leq \alpha < \mu_{\widetilde{A}}(x_0)$. Then the continuity of $\widetilde{A}$ implies that $x_0 \in (\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha)$. Therefore, we may choose $r > 0$ with $(x_0 - r, x_0 + r) \subset [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$. On the other hand, using equation (7) and the choice of α, we have $x_n \notin [(\widetilde{A}_n)^L_\alpha, (\widetilde{A}_n)^U_\alpha]$, for all $n \in \mathbb{N}$. Since $\widetilde{A}_n \xrightarrow{d_H} \widetilde{A}$, $(\widetilde{A}_n)^L_\alpha$ and $(\widetilde{A}_n)^U_\alpha$ converge $(\widetilde{A}_n)^L_\alpha$ and $(\widetilde{A}_n)^U_\alpha$, for large values of n, we get $x_n \notin (x_0 - r, x_0 + r)$. This contradicts the assumption that $x_n \to x_0$ as $n \to \infty$. Thus the proof is completed. □

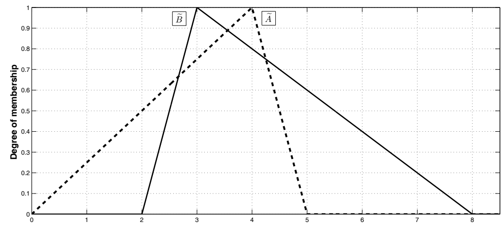

FIGURE 1. Fuzzy Numbers $\widetilde{A}$ and $\widetilde{B}$ in Example 3.3

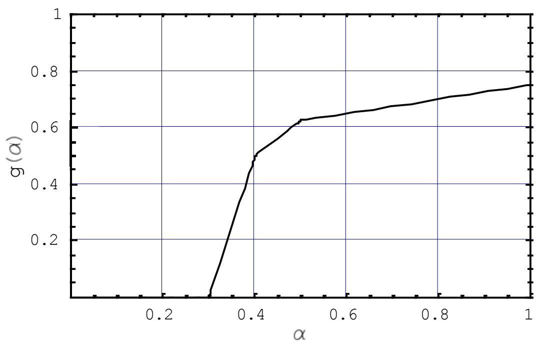

FIGURE 2. The Plot of $g(\alpha)$ in Example 3.3

3. A Preference Criterion for Comparing Fuzzy Numbers

In this section, we propose a novel preference criterion for ranking fuzzy numbers using the elements of the Credibility theory. Some of its useful properties are also investigated in this section.

Definition 3.1. For two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, based on the Credibility index introduced in equation (1), the preference criterion $D : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0,1]$ is defined as follows:

$$D\{\widetilde{A} \succ \widetilde{B}\} = \int_0^1 Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\} d\alpha, \tag{8}$$

where $\widetilde{B}_\alpha$ is defined in equation (5). The quantity $D\{\widetilde{A} \succ \widetilde{B}\}$ is called the preference degree that $\widetilde{A}$ dominates $\widetilde{B}$. The preference degree of $\widetilde{A} \preceq \widetilde{B}$ is also defined similarly.

Remark 3.2. Due to the monotonicity of function $g(\alpha) = Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\}$ on $[0,1]$, it is concluded that the definition of $D\{\widetilde{A} \succ \widetilde{B}\}$ is well-defined. Moreover, $D\{\widetilde{A} \succ \widetilde{B}\}$ may be viewed as the expected Credibility that $\widetilde{A}$ dominates $\widetilde{B}$.

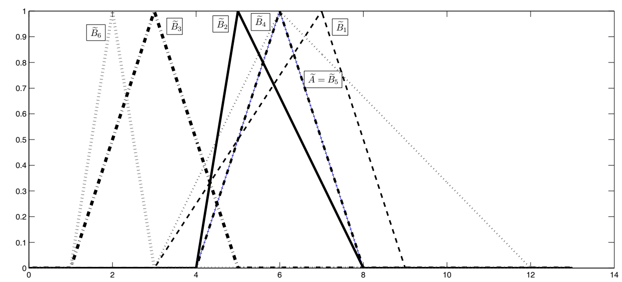

FIGURE 3. Fuzzy Numbers $\widetilde{A}$ and $\widetilde{B}_i$, $i = 1, 2, \ldots, 6$ in Example 3.3

Example 3.3. Suppose $\widetilde{A} = (0, 4, 5)_T$ and $\widetilde{B} = (2, 3, 8)_T$ are two triangular fuzzy numbers as shown in Figure (1). The plot of $g(\alpha) = Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\}$, $\alpha \in [0, 1]$ is drown in Figure (2). As it is observed, the square $[0, 1] \times [0, 1]$ is divided in two parts: 1) the area in which $\widetilde{A} \succ \widetilde{B}$ (the area of under g), and 2) the area in which $\widetilde{A} \preceq \widetilde{B}$ (the area of above g). Numerically, from equation (8), it is concluded that $D\{\widetilde{A} \succ \widetilde{B}\} = \int_0^1 g(\alpha) d\alpha = 0.425$. Moreover, for given triangular fuzzy numbers $\widetilde{B}_i$, $i = 1, 2, 3, 4, 5, 6$ as shown in Figure (3), the preference degrees to which $\widetilde{A} = (4, 6, 8)_T$ is larger than $\widetilde{B}_i$ are shown in Table 1.

i	$\widetilde{B}_i$	$D\{\widetilde{A} \succ \widetilde{B}_i\}$
1	$(3, 7, 9)_T$	0.390
2	$(4, 5, 8)_T$	0.625
3	$(1, 3, 5)_T$	0.968
4	$(3, 6, 12)_T$	0.458
5	$(4, 6, 8)_T$	0.500
6	$(1, 2, 3)_T$	1.000

TABLE 1. Preference Degrees Between $\widetilde{A} = (4, 6, 8)_T$ and $\widetilde{B}_i$s in Example 3.3

Theorem 3.4. *Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers in $\mathcal{F}_c(\mathbb{R})$. Then:*

1) $D\{\widetilde{A} \succ \widetilde{B}\} = 1 - D\{\widetilde{A} \preceq \widetilde{B}\}$ *(Reciprocity),*
2) $D\{\widetilde{A} \succ \widetilde{B}\} = 1$ *if and only if* $\widetilde{B}_0^U \leq \widetilde{A}_0^L$.
3) $D\{\widetilde{A} \succ \widetilde{A}\} = 0.5$ *(Reflexivity).*

No.	range of $d = D(\widetilde{A} \succ \widetilde{B})$	interpretation
1	$d \in [0.0, 0.10)$	$\widetilde{B}$ is absolutely dominates $\widetilde{A}$
2	$d \in [0.1, 0.25)$	$\widetilde{B}$ strongly dominates $\widetilde{A}$
3	$d \in [0.25, 0.35)$	$\widetilde{B}$ moderately dominates $\widetilde{A}$
4	$d \in [0.35, 0.45)$	$\widetilde{B}$ weakly dominates $\widetilde{A}$
5	$d \in [0.45, 0.55)$	$\widetilde{A}$ and $\widetilde{B}$ are equal
6	$d \in [0.55, 0.65)$	$\widetilde{A}$ weakly dominates $\widetilde{B}$
7	$d \in [0.65, 0.75)$	$\widetilde{A}$ is moderately greater than $\widetilde{B}$
8	$d \in [0.75, 0.90)$	$\widetilde{A}$ strongly dominates $\widetilde{B}$
9	$d \in [0.90, 1]$	$\widetilde{A}$ absolutely dominates $\widetilde{B}$

TABLE 2. A Linguistic Interpretation of $d = D\{\widetilde{A} \succ \widetilde{B}\}$

Proof. First note that $\int_0^1 Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\}d\alpha = \int_0^1 Cr\{\widetilde{B} \prec \widetilde{A}_\alpha\}d\alpha$. Therefore, we get:

$$D\{\widetilde{A} \succ \widetilde{B}\} = \int_0^1 \frac{Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\} + Cr\{\widetilde{B} \prec \widetilde{A}_\alpha\}}{2} d\alpha.$$

From Remark 2.2, therefore:

$$\begin{aligned} &1 - D\{\widetilde{A} \preceq \widetilde{B}\} = \\ &\int_0^1 \frac{(1-Cr\{\widetilde{A} \prec \widetilde{B}_\alpha\}) + (1-Cr\{\widetilde{B} \succeq \widetilde{A}_\alpha\})}{2} d\alpha = \\ &\int_0^1 \frac{Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\} + Cr\{\widetilde{B} \prec \widetilde{A}_\alpha\}}{2} d\alpha = \\ &D\{\widetilde{A} \succ \widetilde{B}\}, \end{aligned}$$

which concludes item 1). To prove item 2), note that $D\{\widetilde{A} \succ \widetilde{B}\} = 1$ if and only if $Cr\{\widetilde{A} \succeq \widetilde{B}_\alpha\} = 1$ for all $\alpha \in [0, 1]$. From Remark 2.2, it is equivalent to $\widetilde{B}_\alpha \leq \widetilde{A}_0^L$ for any $\alpha \in [0, 1]$ which is concludes item 2). The item 3) is immediately followed from 1) and this fact that $D\{\widetilde{A} \succ \widetilde{A}\} = D\{\widetilde{A} \succeq \widetilde{A}\}$ for every $\widetilde{A} \in \mathcal{F}_c(\mathbb{R})$. □

Inspire by Yuan [47], it is pointed out that the linguistic interpretation of D can be derived in nine cases. Therefore, an illustration can be shown in Table 2.

Remark 3.5. Assume that the membership functions of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ reduce to the non-fuzzy values a and b that is $\mu_{\widetilde{A}}(x) = I(x = a)$ and $\mu_{\widetilde{B}}(y) = I(y = b)$ where I is the indicator function denoted by:

$$I(\rho) = \begin{cases} 1 & if\ \rho\ is\ true, \\ 0 & if\ \rho\ is\ false. \end{cases}$$

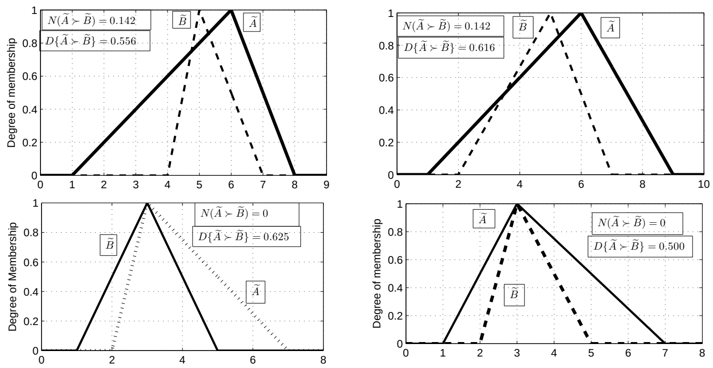

FIGURE 4. Comparison Between D and Nec in Some Cases in Remark 3.6

Then it is observed that $Cr\{\widetilde{A} \succ \widetilde{B}_\alpha\} = \frac{I(a>b)+1-I(a\leq b)}{2}$, for any $\alpha \in [0,1]$. Thus:

$$D(\widetilde{A} \succ \widetilde{B}) = \begin{cases} 1 & \text{if } a > b, \\ 0 & \text{if } a \leq b. \end{cases}$$

Therefore, the proposed ranking method reduces to the ordinary ordering of real numbers whenever the fuzzy numbers reduce to the non-fuzzy ones.

Remark 3.6. As the authors know, one of the most interesting indices to interpret the degree of dominance between fuzzy numbers is the Nec index introduced by Dubois and Prade [16]. The degree of necessity to which a fuzzy number "$\widetilde{A}$ dominates a non-fuzzy number b" is fulfilled by $Nec\ (\widetilde{A} \succ b) = 1 - \sup_{x \leq b} \mu_{\widetilde{A}}(x)$. In addition, the degree of possibility to which "$\widetilde{A}$ dominates b" is defined to be $Pos\ (\widetilde{A} \succ \widetilde{B}) = \sup_{x>b} \mu_{\widetilde{A}}(x)$. The quantities $Nec\ (\widetilde{A} \preceq b)$ and $Pos\ (\widetilde{A} \preceq b)$, which can be interpreted as the necessity and possibility that $\widetilde{A}$ does not dominate b, are given as: $Nec\ (\widetilde{A} \preceq b) = 1 - Pos\ (\widetilde{A} \succ b)$ and $Pos\ (\widetilde{A} \preceq b) = 1 - Nec\ (\widetilde{A} \succ b)$. Therefore, the Credibility degree of $\{\widetilde{A} \succ b\}$ is just the mean of $Pos\ (\widetilde{A} \succ b)$ and $Nec\ (\widetilde{A} \succ b)$. However, the Nec index doesn't have the reciprocity property while D has. Moreover, as it is shown in Figure (4), one can observe that the necessity index can not distinguish the order between two fuzzy numbers in some situations.

3.1. **Ranking Procedure.** Ranking fuzzy numbers plays an important rule in many filed of decision making based on imprecise information. In this regards, many methods for ranking fuzzy numbers have been proposed during the last decades and their acceptability are discussed based on some criteria such as counterintuitive, inconsistency, nondiscriminating and so on (for more see [21]). One of the interesting proposed methods is the comparison of fuzzy numbers based on a fuzzy target. This procedure firstly proposed by Lee-Kwang and Lee [25]. The ranking procedure relies on a satisfaction function defined as $S(\widetilde{A} \succ \widetilde{T}) = \frac{\int \int_y \mu_{\widetilde{A}}(x) \odot \mu_{\widetilde{T}}(y) dx dy}{\int \int \mu_{\widetilde{A}}(x) \odot \mu_{\widetilde{T}}(y) dx dy}$, where $\odot$ is a t-norm and $\widetilde{T}$ is a base to interpret "the possibility that is greater than".

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$
$D\{\widetilde{A}_i \succ \widetilde{T}\}$	0.578	0.502	0.235	0.593

Table 3. The Evaluation Values Between $\widetilde{A}_1 - \widetilde{A}_4$ in Example 3.7 Based on the Unimodal Fuzzy Target $\widetilde{T} = (1, 5.25, 12)_T$

A fuzzy target $\widetilde{T}$ is a fuzzy set satisfying the following conditions: 1) $\widetilde{T} \in \mathcal{F}_c(\mathbb{R})$, 2) $\int_{\mathbb{R}} \mu_{\widetilde{T}}(x)dx > 0$, and 3) $supp(\widetilde{A}_i) \subseteq supp(\widetilde{T})$ for any $i = 1, 2, \ldots, n$. In this paper, we apply the method introduced by Lee-Kwang and Lee for ranking fuzzy numbers. Assume that $\mathbb{S} = \{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ is a finite set of fuzzy numbers that we want to rank them. The satisfaction function method is based on evaluations of the satisfaction function of every fuzzy number $\widetilde{A}_i$, $i = 1, 2, \ldots, n$, with the fuzzy target. Based on the specified target $\widetilde{T}$, the fuzzy numbers $\widetilde{A}_i$ are ranked to their evaluation values $E(\widetilde{A}_i \succ \widetilde{T})$, $i = 1, 2, \ldots, n$. For more discussions on choosing a proper fuzzy target see [21]. Suppose that $supp(\widetilde{T}) = [t_{min}, t_{max}]$ where $t_{min} = \inf_i supp(\widetilde{A}_i)$ and $t_{max} = \sup_i supp(\widetilde{A}_i)$. Now, let us consider three interesting prototypical triangular fuzzy targets so-called pessimist $\widetilde{O} = (t_{min}, t_{max}, t_{max})_T$, optimist $\widetilde{P} = (t_{min}, t_{min}, t_{max})_T$ and unimodal fuzzy target $\widetilde{T} = (t_{min}, t_0, t_{max})_T$ represented as "about t_0" where $t_{min} < t_0 < t_{max}$. This fuzzy target represents a modal value as the most likely target and assesses the possibilistic uncertain target as distributed around it [21]. In the sequel, a unimodal fuzzy target is suggested as an unimodal fuzzy target denoted by $\widetilde{T} = (t_{min}, t_0, t_{max})_T$ where $t_0 = \frac{1}{n}\sum_{i=1}^{n} \widetilde{A}_i[1]$. Therefore, this type of unimodal fuzzy target can be regarded as a weight-based fuzzy target. However, the fuzzy target $\widetilde{T}$ is subjective and so the results of the present work will not be lost by altering this choice to the ones which fit the demands of the decision makers.

Example 3.7. Consider the following four triangular fuzzy numbers:

$$\begin{aligned} \widetilde{A}_1 &= (3, 7, 9)_T, & \widetilde{A}_2 &= (4, 5, 8)_T, \\ \widetilde{A}_3 &= (1, 3, 5)_T, & \widetilde{A}_4 &= (3, 6, 12)_T. \end{aligned}$$

The evaluation values $D\{\widetilde{A}_i \succ \widetilde{T}\}$ based on the proposed unimodal fuzzy target $\widetilde{T} = (1, 5.25, 12)_T$ are listed in Table 3. Therefore, the fuzzy triangular numbers $\widetilde{A}_i$ are ranked as follows: $\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$.

3.2. **Robustness.** One of the important properties of a preference criterion is the robustness. Naturally, a preference criterion should be able to tolerate minor errors in the estimation of membership functions that is the degree of the preference between two fuzzy numbers should not change if the change of the membership function is sufficiently small (for more, see [47]). For instance, assume that $\widetilde{A}$ dominates $\widetilde{B}$ with respect to a specific fuzzy target $\widetilde{T}$ that is $D\{\widetilde{A} \succ \widetilde{T}\} \geq D\{\widetilde{B} \succ \widetilde{T}\}$. Notably, the robustness property says that if $\widetilde{A}^{'} \simeq \widetilde{A}$ (and $\widetilde{B}^{'} \simeq \widetilde{B}$) then it is expected that $D\{\widetilde{A}' \succ \widetilde{T}\} \geq D\{\widetilde{B} \succ \widetilde{T}\}$ ($D\{\widetilde{A}' \succ \widetilde{T}\} \geq D\{\widetilde{B}' \succ \widetilde{T}\}$) which is a reasonable property for a given preference criterion. To investigate the robustness mathematically, we use here the Hausdorff distance as the measurement of a membership estimation error. Then, a concept of robustness for a given preference criterion is developed as follows.

Definition 3.8. Let $d : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, +\infty)$ be a distance on $\mathcal{F}_c(\mathbb{R})$. A preference criterion $S : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, 1]$ is called robust, if for any given pair of fuzzy numbers $(\widetilde{A}, \widetilde{B})$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all pairs $(\widetilde{A'}, \widetilde{B'}) \in \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$ with $d(\widetilde{A}, \widetilde{A'}) < \delta$ and $d(\widetilde{B}, \widetilde{B'}) < \delta$, we have $|S(\widetilde{A}, \widetilde{B}) - S(\widetilde{A'}, \widetilde{B'})| < \epsilon$.

Remark 3.9. It is clear that the above definition is equivalent to the following assertion. A preference criterion $S : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, 1]$ is robust with respect to a distance $d : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, +\infty)$, if for any given pair of fuzzy numbers $(\widetilde{A}, \widetilde{B})$ and a sequence $\{(\widetilde{A}_n, \widetilde{B}_n)\}$ in $\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$ with $\widetilde{A}_n \xrightarrow{d} \widetilde{A}$ and $\widetilde{B}_n \xrightarrow{d} \widetilde{B}$, we have $S(\widetilde{A}_n, \widetilde{B}_n) \to S(\widetilde{A}, \widetilde{B})$.

Now, by the following theorem, we examine the robustness property of the proposed preference criterion D.

Theorem 3.10. *The preference criterion* $D : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, 1]$ *is robust.*

Proof. Let $\widetilde{A}$ and $\widetilde{B}$ be two continuous fuzzy numbers and suppose $\{(\widetilde{A}_n, \widetilde{B}_n)\}$ is a sequence in $\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$ with $\widetilde{A}_n \xrightarrow{d_H} \widetilde{A}$ and $\widetilde{B}_n \xrightarrow{d_H} \widetilde{B}$. First note that:

$$\sup_{x > (\widetilde{B}_n)_\alpha} \mu_{\widetilde{A}_n}(x) = \sup_{x > (\widetilde{B}_n)_\alpha} \min\{\mu_{\widetilde{A}_n}(x), I(x = (\widetilde{B}_n)_\alpha)\}$$

for all $\alpha \in [0, 1]$. For simplicity, let $h, h_n : \mathbb{R} \to [0, 1]$ be defined, for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, as follows:

$$h_n^\alpha(x) = \min\{\mu_{\widetilde{A}_n}(x), I(x = (\widetilde{B}_n)_\alpha)\}$$

and

$$h^\alpha(x) = \min\{\mu_{\widetilde{A}}(x), I(x = (\widetilde{B}_n)_\alpha)\}.$$

Using Lemma 2.7, the sequence $\{h_n^\alpha(x)\}$ converges uniformly on $\mathbb{R}$ to $h^\alpha(x)$ for any $\alpha \in [0, 1]$ and any $x \in \mathbb{R}$. Hence, for any $\alpha \in [0, 1]$, we easily see that

$$\sup_{x \in K_n^\alpha} h_n^\alpha(x) \to \sup_{x \in K^\alpha} h^\alpha(x),$$

where $K_n^\alpha = \{x \in \mathbb{R} : x > (\widetilde{B}_n)_\alpha\}$ and $K^\alpha = \{x \in \mathbb{R} : x > (\widetilde{B})_\alpha\}$. Therefore, according to the definition of the Credibility index, it follows that $Cr\{\widetilde{A_n} \succ (\widetilde{B}_n)_\alpha\} \to Cr\{\widetilde{A} \succ (\widetilde{B})_\alpha\}$ for all $\alpha \in [0, 1]$. Thus from equation (8) and using Dominated Convergence Theorem [3], we have $D\{\widetilde{A_n} \succ \widetilde{B_n}\} \to D\{\widetilde{A} \succ \widetilde{B}\}$ which completes the proof. □

Remark 3.11. Here, the proposed criterion D is investigated in comparison with some common methods. In particular, for the purpose of comparative study, we select two common target-rank-based methods introduced by Huynh et al. [21]

and Lee-Kwang and Lee [25]. Huynh et al. defined a evaluation value with respect to a fuzzy target $\widetilde{T}$. In this procedure the preference criterion is denoted by $\mathcal{P}(\widetilde{A} \succ \widetilde{T}) = \int_0^1 P(\widetilde{A}[\alpha] \geq \widetilde{T}[\alpha])d\alpha$ where $\widetilde{A}[\alpha]$ and $\widetilde{T}[\alpha]$ are assumed to be independent random variables distributed uniformly on $[\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$ and $[\widetilde{T}^L_\alpha, \widetilde{T}^U_\alpha]$, respectively. $\mathcal{P}(\widetilde{A} \succ \widetilde{T})$ is viewed as expected probability of $\widetilde{A}$ dominating $\widetilde{T}$. Lee-Kwang and Lee [25] proposed a satisfaction criterion to compare two fuzzy numbers with respect to a fuzzy target $\widetilde{T}$ defined by $S(\widetilde{A} \succ \widetilde{T}) = \int \int_y f(x, y)dxdy$ where $f(x, y) = \frac{\mu_{\widetilde{A}}(x)\mu_{\widetilde{T}}(y)}{\int \mu_{\widetilde{A}}(x)dx \int \mu_{\widetilde{A}}(y)dy}$. This criterion is often interpreted as the probability that $\widetilde{A}$ dominates $\widetilde{T}$. It is worth noting that, all conditions of Theorem 3.4 are also satisfied for both preference criteria $\mathcal{P}$ and S. However, we show that the proposed preference criterion D meets the robustness property, too. In Figure (5), the degrees to which $\widetilde{A}$ dominates $\widetilde{B}$ are compared based on the proposed preference criterion, Lee-Kwang and Lee and Huynh et al.'s ones in some cases. In addition, to compare the proposed ranking procedure with respect to the mentioned methods, the fuzzy triangular fuzzy numbers $\widetilde{A}_1$, $\widetilde{A}_2$, $\widetilde{A}_3$ and $\widetilde{A}_4$ given in Example 3.7 are employed here. The ranking orders based on the common fuzzy targets $\widetilde{O}$, $\widetilde{P}$ and the introduced unimodal fuzzy target $\widetilde{T}$ are then given in Table 4. As it is seen, all methods obtain the same results for such a case to rank $\widetilde{A}_1$, $\widetilde{A}_2$, $\widetilde{A}_3$, and $\widetilde{A}_4$ for all fuzzy targets optimistic, pessimistic and unimodal.

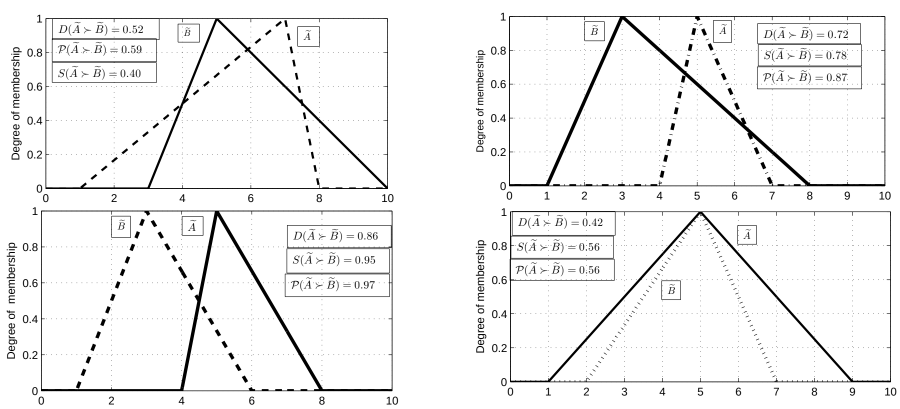

FIGURE 5. Numerical Comparison Between the Preference Criteria D, S and $\mathcal{P}$ in Some Specific Cases in Remark 3.11

4. **Conclusion**

This paper proposes a new preference criterion for ranking fuzzy numbers inspired by the Credibility index and α-optimistic values of a fuzzy number. As an illustration, the case of ranking rule for typical LR-fuzzy numbers is examined. Some basic properties of the proposed criterion are investigated including: 1) reciprocity, and 2) robustness. The proposed preference criterion is then applied to rank fuzzy numbers in terms of a proposed unimodal fuzzy target as well as the

Method	Target	Evaluation values				Ranking order
		$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	
Proposed method	Optimistic	0.250	0.204	0.09	0.261	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
	Pessimistic	0.75	0.705	0.590	0.761	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
	$\widetilde{T}$	0.578	0.502	0.235	0.593	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
Lee-Kwang and Lee's method	Optimistic	0.248	0.186	0.039	0.326	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
	Pessimistic	0.720	0.662	0.325	0.764	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
	$\widetilde{T}$	0.552	0.453	0.010	0.621	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
Huynh et al.'s method	Optimistic	0.146	0.114	0.022	0.203	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
	Pessimistic	0.827	0.785	0.490	0.864	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$
	$\widetilde{T}$	0.688	0.386	0.060	0.712	$\widetilde{A}_4 \succ \widetilde{A}_1 \succ \widetilde{A}_2 \succ \widetilde{A}_3$

Table 4. Comparison Results Between the Proposed Method, Lee-Kwang and Lee and Huynh et al.'s Ones in Remark 3.11

most commonly used fuzzy targets. The proposed method is also compared with some common methods and its effeteness is shown by some graphical and numerical results. Through specific theoretical and numerical results, it is shown that the proposed preference criterion performs well in ranking fuzzy numbers. Therefore, the suggested method provide us with a useful and valuable way to handle fuzzy numbers in many practical applications of decision-making. Although, in this paper, we focus on unimodal fuzzy target in ranking procedure, the propose method can be applied for other decision maker's attitudes.

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