SOME PROBABILISTIC INEQUALITIES FOR FUZZY RANDOM VARIABLES

H. AHMADZADE, M. AMINI, S. M. TAHERI AND A. BOZORGNIA

Abstract. In this paper, the concepts of positive dependence and linearly positive quadrant dependence are introduced for fuzzy random variables. Also, an inequality is obtained for partial sums of linearly positive quadrant dependent fuzzy random variables. Moreover, a weak law of large numbers is established for linearly positive quadrant dependent fuzzy random variables. We extend some well known inequalities to independent fuzzy random variables. Furthermore, a weak law of large numbers for independent fuzzy random variables is stated and proved.

1. **Introduction**

The concept of fuzzy set is introduced by Zadeh [39] as an extension of the indicator function. As a mixture of randomness and fuzziness, the different versions of fuzzy random variable are introduced by Kwakernaak [22] and Puri and Ralescu [30] for modeling the indeterministic phenomena.

Over the last years, fuzzy random variable has been extensively applied in areas of statistics, stochastic processes and probability theory, see e.g. [13], [31] and [38]. For the purposes of this study, we review some works on this topic. Miyakoshi and Shimbo [26] obtained a strong law of large numbers for independent fuzzy random variables, by using a certain distance on the space of fuzzy numbers. Klement et al. [17] established a strong law of large numbers for fuzzy random variables, based on embedding theorem as well as certain probability techniques in the Banach spaces. Taylor et al. [34] proved a weak law of large numbers for fuzzy random variables in separable Banach spaces. Nguyen et al. [29] derived some limit theorems for independent fuzzy random variables based Choquet capacities in Banach spaces. Kratschmer [20] proved a strong law of large numbers and central limit theorem in Banach space by invoking L_p norm. Joo [14] obtained a strong law of large numbers for tight fuzzy random variables. Joo et al. [16] proved a strong law of large numbers for stationary fuzzy random variables respect to uniform metric d_∞. Joo et al. [15] established Chung's type strong law of large numbers for fuzzy random variables based on isomorphic isometric embedding theorem. Fu and Zhang [11] obtained some strong limit theorems for fuzzy random variables with slowly varying weight. In all of the pervious methods, the convergence theorems are established

Received: January 2016; Revised: January 2017; Accepted: August 2017

Key words and phrases: Fuzzy random variable, Linearly Positive Quadrant Dependence, Independence, Law of Large Numbers.

based on the mathematical technique without any statistical interesting. Thus, we review some topics based on statistical motivation. By using a certain metric d_*, Korner [19] and Feng [9] introduced the concept of variance and covariance for fuzzy random variables. It should be mentioned that, although the concept of variance has been found very convenient in studying limit theorems, but, as the authors know, it has not been developed the limit theorems for fuzzy random variables based on the concept of variance, except the works by Feng [9] and Korner [19]. By invoking the concept of variance, Feng [9] and Korner [19] proved strong and weak convergence theorems for independent fuzzy random variables. Since the concept of variance and covariance play important roles in convergence of partial and weighted sums of random variables, we prefer the metric d_* for establishing weak and strong convergence theorems. By invoking statistical concepts, Ahmadzade et al. [3] proved several convergence theorems for independent fuzzy random variables. Also, as an application of maximal inequalities, Ahmadzade et al. [4] established some limit theorems for fuzzy random variables. For investigating convergence theorems for dependent fuzzy random variables, Ahmadzade et al. [6] introduced the concept of negative dependence for fuzzy random variables and established some limit theorems for such fuzzy random variables. Furthermore, for illustration the role of suitable metric (distance) in convergence theorems, Ahmadzade et al. [5] obtained several limit theorems for fuzzy martingales.

It is mentioned that inequalities plays important roles in probability theory and statistics. As a direct generalization, Agahi et al. [1, 2] established some inequalities with respect to fuzzy measure and studied several applied instances in statistics. As a direct application of classical methods in probability theory, we prove some limit theorems for fuzzy random variables. In this paper, we extend some limit theorems for linearly positive quadrant dependent to the case of fuzzy random variables, thus, we review some topics in this field. There are many authors who have devoted their studies to linearly positive quadrant dependent random variables. For instance, the concept of linearly positive quadrant dependent sequence was introduced and investigated by Newman [28]. Birkel [7] stated and proved central limit theorem for linearly positive quadrant dependent random variables. Shao and Yu [33] obtained weak convergence for weighted empirical process of linearly positive quadrant dependent random variables. Louhichi [25] established Rosenthal's inequality for linearly positive quadrant dependent random variables. The purpose of this paper, is to introduce and investigate an approach to generalize the various limit theorems in classical probability theory to fuzzy random variables.

The structure of this paper is as follows. In Section 2, we recall some preliminaries of fuzzy arithmetic and fuzzy random variables. Section 3 provides concept of linearly positive quadrant dependence for fuzzy random variables and establishes a weak law of large numbers for such fuzzy random variables. In Section 4, we extend some well known inequalities for independent fuzzy random variables and prove a weak law of large numbers for such fuzzy random variables. Finally, in Section 5, we state and prove some limit theorems for independent and linearly positive quadrant dependent fuzzy random variables.

2. **Preliminaries**

In this section, we consider some elementary concepts of fuzzy set, fuzzy arithmetic and fuzzy random variables, based on [10, 27] . Suppose that $\mathbb{R}$ is the real line. Define $E = \{\tilde{u} : \mathbb{R} \to [0,1]\}$, where $\tilde{u}$ satisfies the following arguments:
(i) $\tilde{u}$ is normal; (ii) $\tilde{u}$ is convex fuzzy set; (iii) $\tilde{u}$ is upper semi-continuous. Any $\tilde{u} \in E$ is called a fuzzy number. For a $\tilde{u} \in E, [\tilde{u}]^r = \{x \in \mathbb{R} | \tilde{u}(x) \geq r, 0 < r \leq 1\}$ is called the r-level set of $\tilde{u}$. In fact, a fuzzy number $\tilde{u}$ is characterized by the end points of the interval $[\tilde{u}]^r = [\tilde{u}^-(r), \tilde{u}^+(r)]$. Let $\tilde{u}, \tilde{v} \in E$, and set

$$d_p(\tilde{u}, \tilde{v}) = (\int_0^1 h^p([\tilde{u}]^r, [\tilde{v}]^r)dr)^{\frac{1}{p}},\ 1 \leq p < \infty,$$

$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0<r\leq 1} h([\tilde{u}]^r, [\tilde{v}]^r)$, where h is Hausdorff metric i.e.
$h([\tilde{u}]^r, [\tilde{v}]^r) = \max\{|\tilde{u}^-(r) - \tilde{v}^-(r)|, |\tilde{u}^+(r) - \tilde{v}^+(r)|\}$. Norm $||\tilde{u}||_p$ of a fuzzy number $\tilde{u} \in E$ is defined by $||\tilde{u}||_p = d_p(\tilde{u}, \tilde{0})$, where $\tilde{0}$ is the fuzzy number in E whose membership function equals 1 at 0 and zero otherwise. The norm $||.||_\infty$of $\tilde{u}$ is defined by $||\tilde{u}||_\infty = d_\infty(\tilde{u}, \tilde{0})$.

In order to obtain our main results, we require to state propositions about fuzzy arithmetic.

Proposition 2.1. [18] *Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy numbers, the following properties hold:*
i) $[\tilde{u} \oplus \tilde{v}]^r = [\tilde{u}^-(r) + \tilde{v}^-(r), \tilde{u}^+(r) + \tilde{v}^+(r)]$,
ii) If $\lambda > 0$ *then* $[\lambda \odot \tilde{u}]^r = [\lambda\tilde{u}^-(r), \lambda\tilde{u}^+(r)]$,
iii) If $\lambda < 0$ *then* $[\lambda \odot \tilde{u}]^r = [\lambda\tilde{u}^+(r), \lambda\tilde{u}^-(r)]$,
iv) $[\tilde{u} \ominus \tilde{v}]^r = [\tilde{u}^-(r) - \tilde{v}^+(r), \tilde{u}^+(r) - \tilde{v}^-(r)]$.

The operation $\langle ., . \rangle : E \times E \to [-\infty, \infty]$ is defined by

$$\langle \tilde{u}, \tilde{v} \rangle = \int_0^1 (\tilde{u}^-(r)\tilde{v}^-(r) + \tilde{u}^+(r)\tilde{v}^+(r))dr.$$

If the indeterminacy of the form $\infty - \infty$ arises in the Lebesgue integral, then we say that $\langle \tilde{u}, \tilde{v} \rangle$ does not exist.

Proposition 2.2. [8] *The operation $\langle ., . \rangle$ has the following properties:*

$$\begin{aligned}
&(i) \quad \langle \tilde{u}, \tilde{u} \rangle \geq 0 \text{ and } \langle \tilde{u}, \tilde{u} \rangle = 0 \Leftrightarrow \tilde{u} = \tilde{0},\\
&(ii) \quad \langle \tilde{u}, \tilde{v} \rangle = \langle \tilde{v}, \tilde{u} \rangle,\\
&(iii) \quad \langle \tilde{u} + \tilde{v}, \tilde{w} \rangle = \langle \tilde{u}, \tilde{w} \rangle + \langle \tilde{v}, \tilde{w} \rangle,\\
&(iv) \quad \langle \lambda\tilde{u}, \tilde{v} \rangle = \lambda\langle \tilde{u}, \tilde{v} \rangle,\\
&(v) \quad |\langle \tilde{u}, \tilde{v} \rangle| < \sqrt{\langle \tilde{u}, \tilde{u} \rangle \langle \tilde{v}, \tilde{v} \rangle}.
\end{aligned}$$

For all $\tilde{u}, \tilde{v} \in E$, if $\langle \tilde{u}, \tilde{u} \rangle < \infty$ and $\langle \tilde{v}, \tilde{v} \rangle < \infty$, then the property (v) implies that $\langle \tilde{u}, \tilde{v} \rangle < \infty$. So, we can define

$$d_*(\tilde{u}, \tilde{v}) = \sqrt{\langle \tilde{u}, \tilde{u} \rangle - 2\langle \tilde{u}, \tilde{v} \rangle + \langle \tilde{v}, \tilde{v} \rangle}.$$

In fact, d_* is a metric in $\{\tilde{u} \in E | \langle \tilde{u}, \tilde{u} \rangle < \infty\}$.
Moreover, the norm $||\tilde{u}||_*$ of fuzzy number $\tilde{u} \in E$ is defined by $||\tilde{u}||_* = d_*(\tilde{u}, \tilde{0})$.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\mathcal{B}$ be the Borel σ-algebra. A fuzzy random variable (briefly: f.r.v.) is defined as a Borel measurable function $\tilde{X} : (\Omega, \mathcal{A}) \to (E, d_\infty)$.

Definition 2.3. [30] A fuzzy random variable is a function $\tilde{X} : (\Omega, \mathcal{A}) \to E$ such that

$$\{(\omega, x) : x \in [\tilde{X}]^r(\omega)\} \in \mathcal{A} \times \mathcal{B},$$

for every $r \in [0, 1]$, where $[\tilde{X}]^r(\omega)$ is defined by

$$[\tilde{X}]^r(\omega) = \{x \in \mathbb{R} : \tilde{X}(\omega)(x) \geq r\}.$$

Proposition 2.4. [30] *If $\tilde{X}$ is a f.r.v. defined on the probability space $(\Omega, \mathcal{A}, P)$, then $[\tilde{X}]^r = [\tilde{X}^-(r), \tilde{X}^+(r)], r \in (0, 1]$, is a random closed interval set, and $\tilde{X}^-(r)$ and $\tilde{X}^+(r)$ are real valued random variables, where*
$\tilde{X}^-(r) = \inf\{X_0 : X_0 \in [\tilde{X}]^r\}$ and $\tilde{X}^+(r) = \sup\{X_0 : X_0 \in [\tilde{X}]^r\}$.

A f.r.v. $\tilde{X}$ is called integrably bounded if $E||\tilde{X}||_\infty < \infty$. The expectation $E\tilde{X}$ is defined as the unique fuzzy number which satisfies the property $[E\tilde{X}]^r = E[\tilde{X}]^r$, $0 < r \leq 1$ [21, 24].

Definition 2.5. [36] Two f.r.v's $\tilde{X}$ and $\tilde{Y}$ are called independent if two σ-fields $\sigma(\tilde{X}) = \sigma(\{\tilde{X}^-(r), \tilde{X}^+(r) | r \in [0, 1]\})$ and $\sigma(\tilde{Y}) = \sigma(\{\tilde{Y}^-(r), \tilde{Y}^+(r) | r \in [0, 1]\})$ are independent.

Definition 2.6. A finite collection of f.r.v.'s $\{\tilde{X}_k,\ 1 \leq k \leq n\}$ is said to be independent if σ-fields $\sigma(\{\tilde{X}_k^-(r), \tilde{X}_k^+(r) | r \in [0, 1]\} | 1 \leq k \leq n)$ are independent. An infinite sequence $\{\tilde{X}_n, n \geq 1\}$ is called independent if every finite sub collection of it is independent.

Definition 2.7. [10] Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s in L_2 ($L_2 = \{\tilde{X} | \tilde{X}$ is f.r.v. and $E||\tilde{X}||_2^2 < \infty\}$), then

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2} \int_0^1 (Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r)))dr.$$

Specially, the variance of $\tilde{X}$ is defined by $Var(\tilde{X}) = Cov(\tilde{X}, \tilde{X})$.

Theorem 2.8. [10] *Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s in L_2 and $\tilde{u}, \tilde{v} \in E$ and $\lambda, k \in R$. Then*
i) $Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}(E\langle \tilde{X}, \tilde{Y} \rangle - \langle E\tilde{X}, E\tilde{Y} \rangle)$
ii) $Var(\tilde{X}) = \frac{1}{2} E d_*^2(\tilde{X}, E\tilde{X})$
iii) $Cov(\lambda\tilde{X} \oplus \tilde{u}, k\tilde{Y} \oplus \tilde{v}) = \lambda k Cov(\tilde{X}, \tilde{Y})$
iv) $Var(\lambda\tilde{X} \oplus u) = \lambda^2 Var(\tilde{X})$
v) $Var(\tilde{X} \oplus \tilde{Y}) = Var(\tilde{X}) + Var(\tilde{Y}) + 2Cov(\tilde{X}, \tilde{Y})$
vi) If $\tilde{X}$ and $\tilde{Y}$ are independent, then $Cov(\tilde{X}, \tilde{Y}) = 0$.

Definition 2.9. [32] The $D_{p,q}$ distance, indexed by parameters $1 \leq p \leq \infty$, $0 \leq q \leq 1$, between two fuzzy numbers u and v is a nonnegative function on $E \times E$ given as follows

$$D_{p,q}(\tilde{u}, \tilde{v}) = [(1-q)\int_0^1 |\tilde{u}^-(r) - \tilde{v}^-(r)|^p dr + q\int_0^1 |\tilde{u}^+(r) - \tilde{v}^+(r)|^p dr]^{\frac{1}{p}}.$$

Remark 2.10. If $p = 2, q = \frac{1}{2}$, then the metric $D_{p,q}$ is equal to the metric d_*, for more details, see [9].

In order to establish strong and weak convergence, we need the following definitions.

Definition 2.11. [37] Let $\tilde{X}$ and $\tilde{X}_n$ be f.r.v.'s are defined on the same probability space $(\Omega, \mathcal{A}, P)$. i) We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ in probability with respect to the metric d_* if, for all $\epsilon > 0$, $lim_{n\to\infty} P(d_*(\tilde{X}_n, \tilde{X}) > \epsilon) = 0$.
ii) We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ almost surely (briefly: a.s.) with respect to the metric d_* if $P(\omega : lim_{n\to\infty} d_*(\tilde{X}_n(\omega), \tilde{X}(\omega)) = 0) = 1$.

Throughout this paper it is assumed that all of f.r.v.'s are defined on the probability space $(\Omega, \mathcal{A}, P)$.

3. **Positive Dependence and Linearly Quadratic Dependence for f.r.v.s: Concepts and Some Results**

We introduce the concept of linearly positive quadrant dependence (LPQD, for short) for f.r.v.'s. Also, an inequality for such f.r.v.'s is established. In order to introduce the LPQD f.r.v.'s, we need to introduce the concept of positive dependence (PD, for short) for f.r.v.'s.

Concerning a generalization of law of large numbers for fuzzy random variables, Viertl [35] introduced the concept of independence for f.r.v.'s based on random sets as follows.

Definition 3.1. [35] Two f.r.v.'s $\tilde{X}$ and $\tilde{Y}$ are said independent if for any Borel set B_1 and B_2 and all $r \in (0,1]$

$$P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) = P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_2),$$

where, $P([\tilde{X}]^r \subset B) = P(\omega : [\tilde{X}]^r(\omega) \subset B)$.

A question may be arise: What is the opposite of independence, for f.r.v.'s? One possible answer is that $\tilde{X}$ and $\tilde{Y}$ should be as dependent as possible. Thus, we introduce the concept of positively dependent for f.r.v.'s.

Definition 3.2. Two f.r.v.'s $\tilde{X}$ and $\tilde{Y}$ are said positively dependent if for any Borel sets B_1 and B_2 and all $r \in (0,1]$,

$$P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) \geq P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_2),$$

where, $P([\tilde{X}]^r \subset B) = P(\omega : [\tilde{X}]^r(\omega) \subset B)$.

Remark 3.3. If $\tilde{X}$ and $\tilde{Y}$ reduce to real valued random variables and $B_1 = (-\infty, x_1]$ or (x_2, ∞) and $B_2 = (-\infty, y_1]$ or $(y_2, +\infty)$, definition 3.2 conclude the concept of positive dependence in the case of real valued random variables.

The following example explains in the above definition.

Example 3.4. If $\tilde{X}$ and $\tilde{Y}$ have following probability functions, then $\tilde{X}$ and $\tilde{Y}$ are PD f.r.v.'s,

$$P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{v})=\frac{2}{9}, P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{u})=0,$$
$$P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{w})=\frac{1}{9}, P(\tilde{X}=\tilde{u},\tilde{Y}=\tilde{v})=\frac{1}{9},$$
$$P(\tilde{X}=\tilde{u},\tilde{Y}=\tilde{u})=\frac{1}{9}, P(\tilde{X}=\tilde{u},\tilde{Y}=\tilde{w})=\frac{1}{9},$$
$$P(\tilde{X}=\tilde{w},\tilde{Y}=\tilde{v})=0, P(\tilde{X}=\tilde{w},\tilde{Y}=\tilde{u})=\frac{1}{9}$$
$$P(\tilde{X}=\tilde{w},\tilde{Y}=\tilde{w})=\frac{1}{9}.$$

where $\tilde{u}$ and $\tilde{v}$ are fuzzy numbers with the following membership function respectively.

$$\mu_{\tilde{u}}(x)=\begin{cases} x, & 0\le x<1,\\ 1, & 1\le x\le 2,\\ 3-x, & 2<x<3,\\ 0, & otherwise,\end{cases}$$

and

$$\mu_{\tilde{v}}(x)=\begin{cases} 2x-2, & 1\le x<\frac{3}{2},\\ 4-2x, & \frac{3}{2}\le x\le 2,\\ 0, & otherwise.\end{cases}$$

And,

$$\mu_{\tilde{w}}(x)=\begin{cases} 2x, & 0\le x<\frac{1}{2},\\ 1, & \frac{1}{2}\le x\le \frac{5}{2},\\ 6-2x, & \frac{5}{2}<x<3,\\ 0, & otherwise.\end{cases}$$

Since for $B_1 = [x,y],(x,y],[x,y),(x,y),(x,+\infty),[x,+\infty),(-\infty,y),(-\infty,y]$ where $1\le x\le \frac{3}{2}$ and $\frac{3}{2}\le y\le 2$ also for $B_2=[z,w],(z,w],[z,w),(z,w),(z,+\infty),[z,+\infty),$ $(-\infty,w),(-\infty,w]$ where $1\le z\le\frac{3}{2}$ and $\frac{3}{2}\le w\le 2$, we obtain

$$\begin{aligned} P([\tilde{X}]^r\subset B_1,[\tilde{Y}]^r\subset B_2) &= P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{v})=\frac{2}{9}\\ &> P([\tilde{X}]^r\subset B_1)P([\tilde{Y}]^r\subset B_2)\\ &= P(\tilde{X}=\tilde{v})P(\tilde{Y}=\tilde{v})=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}.\end{aligned}$$

For $B_3 = [x,y],(x,y],[x,y),(x,y),(x,+\infty),[x,+\infty),(-\infty,y),(-\infty,y]$ where $\frac{1}{2}\le x\le 1$ and $2\le y\le\frac{5}{2}$ also for $B_4=[z,w],(z,w],[z,w),(z,w),(z,+\infty),[z,+\infty),$ $(-\infty,w),(-\infty,w]$ where $\frac{1}{2}\le z\le 1$ and $2\le w\le\frac{5}{2}$, we obtain

$$\begin{aligned} P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_4) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) \\ &= \frac{2}{9} + 0 = \frac{2}{9} \\ &= P([\tilde{X}]^r \subset B_1) P([\tilde{Y}]^r \subset B_4) \\ &= P(\tilde{X} = \tilde{v})\{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v})\} \\ &= \frac{1}{3} \times \frac{2}{3}. \end{aligned}$$

Also, we can obtain the similar methods for other Borel sets.

Proposition 3.5. *Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\tilde{X}$ and $\tilde{Y}$ be PD f.r.v.'s. Then, $\tilde{X}^-(r)$ and $\tilde{Y}^-(r)$ as well as $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$ are PD.*

Proof. For all $x, y \in \mathbb{R}$ and $r \in (0, 1]$, we have

$$\begin{aligned} P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y) &= P([\tilde{X}]^r \subset (x, \infty), [\tilde{Y}]^r \subset (y, \infty)) \\ &\geq P([\tilde{X}]^r \subset (x, \infty)) P([\tilde{Y}]^r \subset (y, \infty)) \\ &= P(\tilde{X}^-(r) > x) P(\tilde{Y}^-(r) > y). \end{aligned}$$

A similar proof can be stated for $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$. □

Remark 3.6. The converse of previous proposition is not true in general. Namely, let $\tilde{X}$ and $\tilde{Y}$ have the following probability functions

$$P(\tilde{X} = \tilde{u}, \tilde{Y} = -\tilde{u}) = 0, P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) = \frac{1}{9}, P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) = \frac{2}{9},$$

$$P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{0}) = \frac{1}{9}, P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{0}) = \frac{1}{9}, P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{0}) = 0,$$

$$P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = \frac{2}{9}, P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{u}) = \frac{1}{9}, P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{u}) = \frac{1}{9},$$

where,

$$\mu_{\tilde{u}}(x) = \begin{cases} x, & 0 < x \leq 1, \\ 2 - x, & 1 < x \leq 2, \\ 0, & otherwise. \end{cases}$$

It is easy to see that $\tilde{X}$ and $\tilde{Y}$ are not PD f.r.v.'s, since

$$\begin{aligned} P([\tilde{X}]^r \subset [-\frac{3}{2}, -\frac{1}{2}], [\tilde{Y}]^r \subset [\frac{1}{2}, \frac{3}{2}]) = 0 < P([\tilde{X}]^r \subset [-\frac{3}{2}, -\frac{1}{2}]) P([\tilde{Y}]^r \subset [\frac{1}{2}, \frac{3}{2}]) \\ = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}. \end{aligned}$$

But, $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$ are PD random variables also $\tilde{X}^-(r)$ and $\tilde{Y}^-(r)$ are PD random variables. Since, for $x \in (-\infty, 0)$ and $y \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \leq x, \tilde{Y}^-(r) \leq y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) = \frac{2}{9} \\ &> P(\tilde{X}^-(r) \leq x) P(\tilde{Y}^-(r) \leq y) \\ &= P(\tilde{X} = -\tilde{u}) P(\tilde{Y} = -\tilde{u}) \\ &= \frac{3}{9} \times \frac{3}{9} = \frac{1}{9}. \end{aligned}$$

Also, for $x \in [0,1)$ and $y \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) \\ &\quad + P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) \\ &= \frac{2}{9} + \frac{1}{9} \\ &> P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{0})\} \times P(\tilde{Y} = -\tilde{u}) \\ &= (\frac{3}{9} + \frac{3}{9}) \times \frac{3}{9} = \frac{2}{9}. \end{aligned}$$

For $x \in [0,1)$ and $y \in [0,1)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y} \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{0}) \\ &\quad + P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{0}) \\ &= \frac{2}{9} + \frac{1}{9} + \frac{1}{9} = \frac{4}{9} \\ &> P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{0})\} \times \{P(\tilde{Y} = -\tilde{u}) + P(\tilde{Y} = \tilde{0})\} \\ &= \frac{2}{3} \times \frac{5}{9}. \end{aligned}$$

For $x \in [0,1)$ and $y \in [1, \infty)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{0}) \\ &\quad + P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{u}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = -\tilde{u}) \\ &\quad + P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{0}) + P(\tilde{X} = \tilde{0}, \tilde{Y} = \tilde{u}) = \frac{2}{3} \\ &= P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{0})\} \\ &\quad \times \{P(\tilde{Y} = -\tilde{u}) + P(\tilde{Y} = \tilde{0}) + P(\tilde{Y} = \tilde{u})\} = \frac{2}{3}. \end{aligned}$$

For $x \in [1, \infty)$ and $y, \in [1, \infty)$

$$P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) = 1 = P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) = 1 \times 1.$$

Definition 3.7. A sequence $\{\tilde{X}_n, n \ge 1\}$ of f.r.v.'s is said to be LPQD if for any disjoint subsets $A, B \in \mathcal{Z}^+$ and positive $r_j^\prime s$, $\oplus_{k \in A} \{r_k \tilde{X}_k\}$ and $\oplus_{j \in B} \{r_j \tilde{X}_j\}$ are PD f.r.v.'s.

Proposition 3.8. *Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\{\tilde{X}_n, n \ge 1\}$ be a sequence of LPQD f.r.v.'s. Then $\sum_{j \in B} \{r_j \tilde{X}_j^-(r)\}$ and $\sum_{k \in A} \{r_k \tilde{X}_k^-(r)\}$ as well as $\sum_{j \in B} \{r_j \tilde{X}_j^+(r)\}$ and $\sum_{k \in A} \{r_k \tilde{X}_k^+(r)\}$ are PD real valued random variables and consequently the sequences $\{\tilde{X}_n^-(r), n \ge 1\}$ and $\{\tilde{X}_n^+(r), n \ge 1\}$ are sequences of LPQD real valued random variables, for all $r \in (0,1]$.*

Proof. The proof can be done similar to that of Proposition 3.5. □

Lemma 3.9. *Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s, then $d_1(\tilde{X},\tilde{Y}) = (ED^p_{p,q}(\tilde{X},\tilde{Y}))^{\frac{1}{p}}$ is a metric on E (the space of fuzzy numbers) where $p \geq 1$ and $0 < q < 1$ and consequently $d_1(\tilde{X},\tilde{Y}) = (Ed^2_*(\tilde{X},\tilde{Y}))^{\frac{1}{2}}$ is a metric on E.*

Proof. By using triangle property of metric $D_{p,q}$ and Minkowski's inequality, proof is straightforward. □

The following theorem provides a moment inequality for f.r.v.'s as an extension of Theorem 11.5 in [23].

Theorem 3.10. *Let $\{\tilde{X}_n, n \geq 1\}$ be a sequence of LPQD f.r.v.'s, then*

$$\sup_{k\geq 0} Var(\tilde{S}_k(n)) \leq 4n(sup_{n\geq 1} Var(\tilde{X}_n) + \sum_{i=1}^{[logn]} max_{\{(\frac{n}{2^i})^{\frac{1}{3}} \leq j \leq \frac{n}{2^{i-1}}\}} \mu(j)),$$

where $\tilde{S}_k(n) = \oplus_{j=k+1}^{k+n} \tilde{X}_j$ and $\tilde{\mu}(n) = sup_{k\geq 1} \sum_{j:|j-k|\geq n} Cov(\tilde{X}_j, \tilde{X}_k)$, where $[x]$ is integer part of x.

Proof. Set $\tau_m = sup_{k\geq 1} d_1(\tilde{S}_k(m), E\tilde{S}_k(m))$ and $m_1 = m + [m^{\frac{1}{3}}]$. Using operations $\oplus$ and $\ominus$, we obtain

$$\tilde{S}_k(2m) = \tilde{S}_k(m) \oplus \tilde{S}_{k+m}([m^{\frac{1}{3}}]) \oplus \tilde{S}_{k+m_1}(m) \ominus \tilde{S}_{k+2m}([m^{\frac{1}{3}}]).$$

Invoking Lemma 3.9, we obtain

$$d_1(\tilde{S}_k(2m), E\tilde{S}_k(2m)) \leq d_1(\tilde{S}_k(m) \oplus \tilde{S}_{k+m_1}(m), E\tilde{S}_k(m) \oplus E\tilde{S}_{k+m_1}(m)) + 2[m^{\frac{1}{3}}]\tau_1.$$

Also,

$$\begin{aligned} Var(\tilde{S}_k(m) \oplus \tilde{S}_{k+m_1}(m)) &= Var(\tilde{S}_k(m)) + Var(\tilde{S}_{k+m_1}(m)) \\ &\quad + 2Cov(\tilde{S}_k(m), \tilde{S}_{k+m_1}(m)) \\ &\leq 2\tau_m^2 + 2Cov(\tilde{S}_k(m), \tilde{S}_{k+m_1}(m)) \\ &\leq 2\tau_m^2 + 2 \sum_{j=[m^{\frac{1}{3}}+1]}^{m_1} \mu(j). \end{aligned}$$

Now, using the similar proof as that of Theorem 11.5 of [23], the proof is complete. □

4. Some Probabilistic Inequalities for Independent f.r.v.s

In this section, some well-known probabilistic inequalities are extended to the case of independent f.r.v.'s. The first one is an extension of Skorohod's inequality ([12], p. 145) which is also known as Ottaviani's inequality.

Theorem 4.1. *Suppose that $\{\tilde{X}_n, n \geq 1\}$ is a sequence of independent f.r.v.'s, and let x and y be positive real numbers. If*

$$\beta = \max_{k\leq n} P(d_*(\tilde{S}_n, \tilde{S}_k) > y) < 1,$$

then

$$P(max_{1\leq k\leq n}||\tilde{S}_k||_* > x+y) \leq \frac{1}{1-\beta}P(||\tilde{S}_n||_* > x).$$

Proof. Subadditivity property of the metric d_* and independence imply that

$$\begin{aligned} P(||\tilde{S}_n||_* > x) &= \sum_{k=1}^{n} P(||\tilde{S}_n||_* > x, A_k) + \sum_{k=1}^{n} P(||\tilde{S}_n||_* > x, A_k^c) \\ &\geq \sum_{k=1}^{n} P(||\tilde{S}_n||_* > x, A_k) \geq \sum_{k=1}^{n} P(d_*(\tilde{S}_n, \tilde{S}_k) \leq y, A_k) \\ &= \sum_{k=1}^{n} P(d_*(\tilde{S}_k, \tilde{S}_n) \leq y) P(A_k) \\ &\geq (1-\beta) P(\cup_{k=1}^{n} A_k) \\ &= (1-\beta) P(\max_{1\leq k\leq n} ||\tilde{S}_k||_* > x+y), \end{aligned}$$

where,

$$A_k = \{\max_{1\leq j\leq k-1} ||\tilde{S}_j||_* \leq x+y, ||\tilde{S}_k||_* > x+y\}.$$

□

To establish the next results, we need the following lemma.

Lemma 4.2. *If $\tilde{X}$ and $\tilde{Y}$ are two f.r.v.'s then $D_{p,q}^p(E\tilde{X}, E\tilde{Y}) \leq ED_{p,q}^p(\tilde{X}, \tilde{Y})$ and consequently $||E\tilde{X}||_{p,q}^p \leq E||\tilde{X}||_{p,q}^p$, for all $p \geq 1$ and $0 \leq q \leq 1$.*

Proof. By using Jensen's inequality and Fubini's theorem, we obtain

$$\begin{aligned} D_{p,q}^p(E\tilde{X}, E\tilde{Y}) &= \int_0^1 q|E\tilde{X}^-(r) - E\tilde{Y}^-(r)|^p + (1-q)|E\tilde{X}^+(r) - E\tilde{Y}^+(r)|^p dr \\ &\leq \int_0^1 qE|\tilde{X}^-(r) - Y^-(r)|^p + (1-q)E|\tilde{X}^+(r) - \tilde{Y}^+(r)|^p dr \\ &= E\int_0^1 q|\tilde{X}^-(r) - Y^-(r)|^p + (1-q)|\tilde{X}^+(r) - \tilde{Y}^+(r)|^p dr \\ &= ED_{p,q}^p(\tilde{X}, \tilde{Y}). \end{aligned}$$

□

Theorem 4.3. *(A generalization of the second Kolmogorov's inequality)*
Let $\{\tilde{X}_n, n \geq 1\}$ be a sequence of independent fuzzy random variables such that for some constant A, $\sup_n ||\tilde{X}_n||_* \leq A$ then

$$P(\max_{1\leq k\leq n} d_*(\tilde{S}_k, E\tilde{S}_k) > x) \geq 1 - 2\frac{(x+2A)^2}{\sum_{k=1}^{n} Var(\tilde{X}_k)}.$$

Proof. For all $k = 1, 2, ..., n$, set $A_k = \{\max_{1\leq j\leq k-1} d_*(\tilde{S}_j, E\tilde{S}_j) \leq x, d_*(\tilde{S}_k, E\tilde{S}_k) > x\}$ and $B_k = \{\max_{j\leq k} d_*(\tilde{S}_j, E\tilde{S}_j) \leq x\}$. It is clear that for all k, $A_k \cap B_k = \phi$ and $\cup_{j=1}^{k} A_j = B_k^c$.

$$\begin{aligned} EI_{B_{k-1}} d_*^2(\tilde{S}_k, E\tilde{S}_k) &= EI_{B_{k-1}} d_*^2(\tilde{S}_{k-1} \oplus \tilde{X}_k, E\tilde{S}_{k-1} \oplus E\tilde{X}_k) \\ &= EI_{B_{k-1}} d_*^2(\tilde{S}_{k-1}, E\tilde{S}_{k-1}) \\ &\quad + EI_{B_{k-1}} d_*^2(\tilde{X}_k, E\tilde{X}_k) \\ &\quad + 2EI_{B_{k-1}} \langle \tilde{S}_{k-1}, \tilde{X}_k \rangle \\ &\quad - 2EI_{B_{k-1}} \langle E\tilde{S}_{k-1}, E\tilde{X}_k \rangle \\ &\quad - 2EI_{B_{k-1}} \langle \tilde{X}_k, E\tilde{S}_{k-1} \rangle. \end{aligned}$$

Since, $\tilde{S}_{k-1} I_{B_{k-1}}$ and $\tilde{X}_k$ are independent, hence;

$$\begin{aligned} EI_{B_{k-1}} \langle \tilde{S}_{k-1}, \tilde{X}_k \rangle &= \langle E\tilde{S}_{k-1} I_{B_{k-1}}, E\tilde{X}_k \rangle \\ &= EI_{B_{k-1}} \langle \tilde{S}_{k-1}, E\tilde{X}_k \rangle, \end{aligned}$$

and $EI_{B_{k-1}} \langle \tilde{X}_k, E\tilde{S}_{k-1} \rangle = EI_{B_{k-1}} \langle E\tilde{X}_k, E\tilde{S}_{k-1} \rangle$. Hence,

$$\begin{aligned} EI_{B_{k-1}} d_*^2(\tilde{S}_k, E\tilde{S}_k) &= EI_{B_{k-1}} d_*^2(\tilde{S}_{k-1}, E\tilde{S}_{k-1}) \\ &\quad + 2Var(\tilde{X}_k) P(B_{k-1}). \end{aligned} \tag{1}$$

But,

$$\begin{aligned} Ed_*^2(\tilde{S}_k, E\tilde{S}_k) I_{B_{k-1}} &= Ed_*^2(\tilde{S}_k, E\tilde{S}_k) I_{B_k} \\ &\quad + Ed_*^2(\tilde{S}_k, E\tilde{S}_k) I_{A_k}. \end{aligned}$$

Also, subadditivity property of the metric d_* and Lemma 4.2 imply that

$$\begin{aligned} d_*(\tilde{S}_{k-1} \oplus \tilde{X}_k, E\tilde{S}_{k-1} \oplus E\tilde{X}_k) I_{A_k} &\leq \{ d_*(\tilde{X}_k, E\tilde{X}_k) + ||E\tilde{X}_k||_* \\ &\quad + d_*(\tilde{S}_{k-1}, E\tilde{S}_{k-1}) \\ &\quad + ||E\tilde{X}_k||_* \} I_{A_k} \\ &\leq \{ ||\tilde{X}_k||_* + 3E^{\frac{1}{2}} ||\tilde{X}_k||_*^2 \\ &\quad + d_*(\tilde{S}_{k-1}, E\tilde{S}_{k-1}) \} I_{A_k} \\ &\leq (4A + 2x) I_{A_k}. \end{aligned}$$

Then,

$$\begin{aligned} Ed_*^2(\tilde{S}_k, E\tilde{S}_k) I_{B_{k-1}} &\leq Ed_*^2(\tilde{S}_k, E\tilde{S}_k) I_{B_k} \\ &\quad + (4A + 2x)^2 P(A_k). \end{aligned}$$

Combining (1) and above inequality, upon noticing that $B_n \subset B_k$ for all k, shows that

$$\begin{aligned} 2P(B_n) Var(\tilde{X}_k) \leq & \; Ed_*^2(\tilde{S}_k, E\tilde{S}_k) I_{B_k} - Ed_*^2(\tilde{S}_{k-1}, E\tilde{S}_{k-1}) I_{B_{k-1}} \\ & + (2x + 4A)^2 P(A_k), \end{aligned}$$

so that, after summing and telescoping

$$\begin{aligned} 2P(B_n) \sum_{k=1}^{n} Var(\tilde{X}_k) &\leq Ed_*^2(\tilde{S}_n, E\tilde{S}_n) I_{B_n} + (2x + 4A)^2 P(B_n^c) \\ &\leq (2x)^2 P(B_n) + (2x + 4A)^2 P(B_n^c) \\ &\leq (2x + 4A)^2. \end{aligned}$$

Therefore,

$$P(\max_{1\le k\le n} d_*(\tilde{S}_k, E\tilde{S}_k) > x) \ge 1 - 2\frac{(x+2A)^2}{\sum_{k=1}^{n} Var(\tilde{X}_k)}.$$

□

The above Theorem provides a necessary condition for condition of sums of fuzzy random variables.

Theorem 4.4. *Suppose that* $\{\tilde{X}_n, n \ge 1\}$ *is a sequence of independent f.r.v.'s, then*

$$ED^p_{p,q}(\tilde{S}_n, E\tilde{S}_n) \le B_p \sum_{k=1}^{n} ED^p_{p,q}(\tilde{X}_k, E\tilde{X}_k),\ \forall\ 1 \le p \le 2.$$

Proof. By invoking Fubini's Theorem, Marcinkiewicz-Zygmund's inequality and C_r inequality, we obtain

$$\begin{aligned}
ED^p_{p,q}(\tilde{S}_n, E\tilde{S}_n) &= E\int_0^1 q|\tilde{S}^-_n(r) - E\tilde{S}^-_n(r)|^p + (1-q)|\tilde{S}^+_n(r) - E\tilde{S}^+_n(r)|^p dr \\
&= \int_0^1 qE|\tilde{S}^-_n(r) - E\tilde{S}^-_n(r)|^p + (1-q)E|\tilde{S}^+_n(r) - E\tilde{S}^+_n(r)|^p dr \\
&\le B_p \int_0^1 qE(\sum_{k=1}^{n} |\tilde{X}^-_k(r) - E\tilde{X}^-_k(r)|^2)^{\frac{p}{2}} dr \\
&\quad + B_p \int_0^1 (1-q)E(\sum_{k=1}^{n} |\tilde{X}^+_k(r) - E\tilde{X}^+_k(r)|^2)^{\frac{p}{2}} dr \\
&\le B_p \sum_{k=1}^{n} \int_0^1 qE|\tilde{X}^-_k(r) - E\tilde{X}^-_k(r)|^p dr \\
&\quad + B_p \sum_{k=1}^{n} \int_0^1 (1-q)E|\tilde{X}^+_k(r) - E\tilde{X}^+_k(r)|^p dr \\
&= B_p \sum_{k=1}^{n} ED^p_{p,q}(\tilde{X}_k, E\tilde{X}_k).
\end{aligned}$$

□

The above theorem is generalization of Theorem 8.1 in [12].

5. Some Limit Theorems

In this section, invoking theorems of previous sections, we establish weak and strong convergence theorems for independent and LPQD f.r.v.'s. The following theorem provides a weak law of large numbers for LPQD f.r.v.'s as an extension of Theorem 11.6 in [23].

Theorem 5.1. *Let* $\{\tilde{X}_n, n \ge 1\}$ *be a sequence of LPQD f.r.v.'s. If*

$$\sum_{n=1}^{\infty} \frac{Var(\tilde{X}_n)}{n} < \infty,$$

and

$$\sum_{i=1}^{[logn]} max_{\{(\frac{n}{2^i})^{\frac{1}{3}} \leq j \leq \frac{n}{2^{i-1}}\}} \tilde{\mu}(j) \quad = \quad o(n) \text{ as } n \to \infty,$$

then $n^{-1}d_*(\tilde{S}_n, E\tilde{S}_n) \to 0$ *in probability i.e.* $\{\tilde{X}_n, n \geq 1\}$ *obeys the weak law of large numbers with respect to the metric* d_**, where* $\tilde{\mu}(j) = sup_{k\geq 1} \sum_{i:|i-k|>j} Cov(\tilde{X}_i, \tilde{X}_k)$*. In particular case, if* $Var(\tilde{X}_n) = O(n^{-\alpha})$ *and* $\max_{\{(\frac{n}{2^i})^{\frac{1}{3}} < j < \frac{n}{2^{i-1}}\}} \tilde{\mu}(j) = O(1)$ *as* $n \to \infty$ *for all* $i \geq 1$*, and* $\alpha > 0$ *then* $\{\tilde{X}_n, n \geq 1\}$ *obeys the weak law of large numbers.*

Proof. By using Markov's inequality and Theorem 3.10, we have

$$\begin{aligned} P(n^{-1}d_*(\tilde{S}_n, E\tilde{S}_n) > \epsilon) &\leq \frac{Var(\tilde{S}_n)}{n^2\epsilon^2} \\ &\leq sup_{k\geq 0} \frac{Var(\tilde{S}_k(n))}{n^2\epsilon^2} \\ &\leq \frac{4}{n}\{sup_{n\geq 1} Var(\tilde{X}_n) + \sum_{i=1}^{[logn]} max_{\{(\frac{n}{2^i})^{\frac{1}{3}} \leq j \leq \frac{n}{2^{i-1}}\}} \mu(j)\} \\ &\leq \frac{4}{n}\{\sum_{n=1}^{\infty} Var(\tilde{X}_n) + \sum_{i=1}^{n} max_{\{(\frac{n}{2^i})^{\frac{1}{3}} \leq j \leq \frac{n}{2^{i-1}}\}} \mu(j)\}. \end{aligned}$$

Thus, Kronecker's lemma implies that $n^{-1}d_*(\tilde{S}_n, E\tilde{S}_n) \to 0$ in probability. □

The following theorem is a generalization of Levy's theorem on the convergence of series(Theorem 5.5.7 in [12]).

Theorem 5.2. *Let* $\{\tilde{X}_n; n \geq 1\}$ *be a sequence of independent f.r.v.'s, then* $\tilde{S}_n$ *converges in probability with respect to the metric* d_* *iff* $\tilde{S}_n$ *converges almost surely with respect to the metric* d_**.*

Proof. We know that almost sure convergence with respect to any metric implies convergence in probability. It suffices to show that the opposite implication. The metric space (E, d_*) is complete, thus Cauchy convergent in probability is equivalent to convergent in probability. Now, suppose that $\{\tilde{S}_n, n \geq 1\}$ is Cauchy convergent in probability i.e., for $0 < \epsilon < \frac{1}{2}$ there exists n_0, such that, for all n, m, with $n_0 < n < m, P(d_*(\tilde{S}_n, \tilde{S}_m) > \epsilon) < \epsilon$, in particular,

$$\beta = \max_{n\leq k\leq m} P(d_*(\tilde{S}_m, \tilde{S}_k) > \epsilon) \leq \epsilon, for\ n_0 < n < m,$$

by invoking Theorem 4.1, we obtain

$$P(\max_{n\leq k\leq m} d_*(\tilde{S}_k, \tilde{S}_n) > 2\epsilon) \leq \frac{1}{1-\beta} P(d_*(\tilde{S}_m, \tilde{S}_n) > \epsilon) \leq \frac{\epsilon}{1-\epsilon} < 2\epsilon.$$

The above inequalities show that $sup_{m,n\geq n_0} d_*(\tilde{S}_n, \tilde{S}_m) \to 0$ almost surely i.e. $\{\tilde{S}_n, n \geq 1\}$ is almost surely Cauchy convergent. Since the metric space (E, d_*) is complete, the proof is complete. □

The following theorem is an extension of Theorem 5.5.3 in [12].

Theorem 5.3. *Let $\{\tilde{X}_n, n \geq 1\}$ be a sequence of independent fuzzy random variables, such that for some constant A, $sup_{n\geq 1}||\tilde{X}_n||_* \leq A$, then $d_*(\tilde{S}_n, E\tilde{S}_n) \to 0$ almost surely iff $\sum_{n=1}^{\infty} Var\tilde{X}_n < \infty$.*

Proof. If $\sum_{n=1}^{\infty} Var(\tilde{X}_n) < \infty$ then the Kolmogorov's inequality [9] implies that $d_*(\tilde{S}_n, E\tilde{S}_n) \to 0$ almost surely. If the sum of the variances diverges, then

$$P(\max_{n\leq k\leq m} d_*(\tilde{S}_k, E\tilde{S}_k) > \epsilon) \geq 1 - \frac{2(\epsilon + 2A)^2}{\sum_{k=n+1}^{m} Var(\tilde{X}_k)}$$

Letting $m \to \infty$, then

$$P(sup_{k\geq n} d_*(\tilde{S}_k, E\tilde{S}_k) > \epsilon) \geq 1,$$

so that almost sure convergence fails. □

In the following theorem, we obtain a weak law of large numbers for independent f.r.v.'s as an extension of Theorem 6.3.2 in [12].

Theorem 5.4. *If $\{\tilde{X}_n, n \geq 1\}$ be a sequence of independent f.r.v.'s such that $\sum_{k=1}^{n} ED_{p,q}^p(\tilde{X}_k, E\tilde{X}_k) = o(n^p)$, then $n^{-1}D_{p,q}(\tilde{S}_n, E\tilde{S}_n) \to 0$ in probability i.e. $\{\tilde{X}_n, n \geq 1\}$ obeys the weak law of large numbers with respect to the metric $D_{p,q}$ for $1 \leq p \leq 2$.*

Proof. Using Markov's inequality and Theorem 4.4, the proof is straightforward. □

Example 5.5. Let $\tilde{X}_k, k \geq 1$ be a sequence of independent fuzzy random variables with the following membership functions

$$\mu_{\tilde{X}_k(\omega)}(x) = \begin{cases} \dfrac{x - X_k(\omega)}{X_k(\omega)}, & X_k(\omega) < x \leq 2X_k(\omega), \\[2ex] \dfrac{3X_k(\omega) - x}{X_k(\omega)}, & 2X_k(\omega) < x < 3X_k(\omega), \\[2ex] 0, & otherwise, \end{cases}$$

where $\{X_k, k \geq 1\}$ is a sequence of independent random variables such that $Var(X_k) = k^\beta \sigma^2$. Considering $p = 2$ and $q = \frac{1}{2}$, we have $ED_{p,q}^p(\tilde{X}_k, E\tilde{X}_k) = Var(\tilde{X}_k)$. To obtain a weak law of large number for $\{\tilde{X}_n, n \geq 1\}$ with respect to the metric $D_{p,q}$, it remains to show that $\sum_{n=1}^{\infty} Var(\tilde{X}_n) = o(n^2)$. It is easy to see that $Var(\tilde{X}_k) = \frac{19}{6}k^\beta\sigma^2$, and, $\sum_{n=1}^{\infty} \frac{Var(\tilde{X}_n)}{n^2} = \frac{19}{6}\sum_{n=1}^{\infty} \frac{\sigma^2}{n^{2-\beta}} < \infty$, where $\beta < 1$. Therefore invoking Theorem 5.4 and Kronecker's lemma $\{\tilde{X}_n, n \geq 1\}$ obeys the weak law of large numbers.

6. **Conclusions**

In this article, we extended some well known inequality to f.r.v.'s. Also, we established weak law of large numbers for independent and LPQD f.r.v.'s. It is mentioned that, if fuzzy random variables reduce to ordinary (real valued) random variables, all results hold. The study of weak and strong convergence of LPQD f.r.v.'s is a potential work for future research. Moreover, investigation of the almost surely convergence theorems and strong law of large numbers for associated f.r.v.'s may be some topics for research.

Acknowledgements. Authors would like to express their sincere thanks to the anonymous reviewers and editor whose suggestions made this work better. Also, the research has been supported by a grant from Ferdowsi university of Mashhad (No. 2/42845).

References

[1] H. Agahi and E. Eslami, *A general inequality of Chebyshev type for semi(co)normed fuzzy integrals*, Soft Computing, **15** (2011), 771–780.

[2] H. Agahi, A. F. Franulic and S. M. Vaezpour, *Fatou's lemma for Sugeno integral*, Applied Mathematics and Computation, **217(13)** (2011), 6092–6096.

[3] H. Ahmadzade, M. Amini, S. M. Taheri and A. Bozorgnia, *Some limit theorems for independent fuzzy random variables*, Thai Journal of Mathematics, **12** (2014), 537–548.

[4] H. Ahmadzade, M. Amini, S. M. Taheri and A. Bozorgnia, *Some moment inequalities for fuzzy martingales and their applications,* Journal Uncertainty Analysis and Applications, **2** (2014), 1–14.

[5] H. Ahmadzade, M. Amini, S. M. Taheri and A. Bozorgnia, *Maximal inequalities and some convergence theorems for fuzzy random variables,* Kybernetika, **52(2)** (2016), 307–328.

[6] H. Ahmadzade, M. Amini, S. M. Taheri and A. Bozorgnia, *Negative Dependence for Fuzzy Random Variables: Basic Definitions and Some Limit Theorems,* Filomat, **30(9)** (2016), 2535–2549.

[7] T. Birkel, *A functional central limit theorem for positively dependent random variables,* Journal of Multivariate Analysis, **44** (1993), 314–320.

[8] P. Diamond and P. Kloeden, *Metric space of fuzzy sets*, Fuzzy Sets and Systems, **35** (1990), 241–249.

[9] Y. Feng, *An approach to generalize laws of large numbers for fuzzy random variables*, Fuzzy Sets and Systems, **128** (2002), 237–245.

[10] Y. Feng, L. Hu and H. Shu, *The variance and covariance of fuzzy random variables and their applications*, Fuzzy Sets and Systems, **120** (2001), 487–497.

[11] K. Fu and L. X. Zhang, *Strong limit theorems for random sets and fuzzy random sets with slowly varying weights*, Information Sciences, **178** (2008), 2648–2660.

[12] A. Gut, *Probability: A Graduate Course*, Springer, New York, 2005.

[13] L. Hongxing, *Probability representations of fuzzy systems*, Science China Information Sciences, **49** (2006), 339–363.

[14] S. Y. Joo, *Strong law of large numbers for tight fuzzy random variables*, Journal of the Korean Statistical Society, **31** (2002), 129–140.

[15] S. Y. Joo, Y. K. Kim and J. S. Kwon, *On Chung's type law of large numbers for fuzzy random variables*, Statistics and Probability Letteres, **74** (2005), 67–75.

[16] S. Y. Joo, S. S. Lee and Y. H. Yoo, *A strong law of large numbers for stationary fuzzy random variables*, Journal of the Korean Statistical Society, **30** (2001), 153-161.

[17] E. P. Klement, M. L. Puri and D. A. Ralescu, *Limit theorems for fuzzy random variables*, Proceedings of the Royal Society of London A, **407** (1986), 171-182.

[18] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic-Theory and Applications*, Prentice-Hall, Upper Saddle River, NJ, 1995.

[19] R. Korner, *On the variance of fuzzy random variables,* Fuzzy Sets and Systems **92** (1997), 83–93..

[20] V. Kratschmer, *Probability theory in fuzzy sample spaces*, Metrika, **60** (2004), 167–180.

[21] R. Kruse and K. D. Meyer, *Statistics with vague data*, Reidel Publishing Company, Dordrecht, 1987.

[22] H. Kwakernaak, *Fuzzy random variables I.* Information Sciences, **15** (1978), 1–29.
[23] Z. Lin and Z. Bai, *Probability inequalities,* Springer, New York, 2010.
[24] Y. K. Liu and B. Liu, *Fuzzy Random Variables: A Scalar Expected Value Operator*, Fuzzy Optimization and Decision Making, **2** (2003), 143–160.
[25] S. Louhichi, *Rosenthal's inequality for LPQD sequences,* Statistics and Probability Letters, **42** (1992), 139-144.
[26] M. Miyakoshi and M. Shimbo, *A strong law of large numbers for fuzzy random variables.* Fuzzy Sets and Systems **12** (1984), 133-142.
[27] R. E. Moore, R. B. Kearfott and M. J. Cloud, *Introduction to Interval Analysis*, Siam, Philadelphia, 2009.
[28] C. M. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, Statistics and Probability, **5** (1984), 127–140.
[29] H. T. Nguyen, T. Wang and B. Wu, *On probabilistic methods in fuzzy theory*, International Journal of Intelligent Systems, **19** (2004), 99-109.
[30] M. L. Puri and D. A. Ralescu, *Fuzzy random variables.* Journal of Mathematical Analysis and Applications, **114** (1986), 402–422.
[31] M. Rong, N. K. Mahapatra and M. Maiti, *A multi-objective wholesaler-retailers inventory-distribution model with controllable lead-time based on probabilistic fuzzy set and triangular fuzzy number*, Applied Mathematical Modelling, **32** (2008), 2670–2685.
[32] B. Sadeghpour Gildeh and S. Rahimpour, *A Fuzzy Bootstrap Test for the Mean with $D_{p,q}$-distance*, Fuzzy Information and Engineering, **4** (2011), 351–358.
[33] Q. M. Shao, H. Yu, *Weak convergence for weighted empirical processes of dependent sequences,* Annals of Probability, **24** (1996), 2052–2078.
[34] R. L. Taylor, L. Seymour and Y. Chen, *Weak laws of large numbers for fuzzy random sets*, Nonlinear Analysis, **47** (2001), 1245–125.
[35] R. Viertl. *Statistical Methods for Fuzzy Data*, John Wiley and Sons, Chichester, 2011.
[36] H. C. Wu, *The central limit theorems for fuzzy random variables*, Information Sciences. **120**, (1999), 239-256.
[37] H. C. Wu, *The laws of large numbers for fuzzy random variables*, Fuzzy Sets and Systems. **116** (2000), 245-262.
[38] H. C. Wu, *Fuzzy Bayesian system reliability assessment based on exponential distribution*, Applied Mathematical Modelling, **30** (2006), 509–530.
[39] L. A. Zadeh, *Fuzzy sets,* Information and Control, **8**, (1965), 338–353.

Hamed Ahmadzade, Department of Statistics, University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: ahmadzadeh.h.63@gmail.com

Mohammad Amini *, Department of Statistics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 91775, Iran
E-mail address: m-amini@um.ac.ir

Seyed Mahmoud Taheri, Faculty of Engineering Science, College of Engineering, University of Tehran, Tehran, Iran
E-mail address: sm _taheri@yahoo.com

Abolghasem Bozorgnia, Department of Statistics, Khayyam University, Mashhad, Iran
E-mail address: a.bozorgnia@khayyam.ac.ir