ON STRATIFIED LATTICE-VALUED CONVERGENCE SPACES

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Abstract. In this paper we provide a common framework for different stratified LM-convergence spaces introduced recently. To this end, we slightly alter the definition of a stratified LMN-convergence tower space. We briefly discuss the categorical properties and show that the category of these spaces is a Cartesian closed and extensional topological category. We also study the relationship of our category to the categories of stratified L-topological spaces and of enriched LM-fuzzy topological spaces.

1. Introduction

Recently, two interesting definitions of stratified lattice-valued convergence spaces appeared [10, 11]. Both have nice categorical properties (i.e. are well-fibred and topological and Cartesian closed) and contain the category of enriched LM-fuzzy topological spaces as a reflective subcategory. The lattice background in these papers are completely distributive lattices L and M and L is equipped with an order-reversing involution. Another paper that dealt with stratified LM-filters and their convergence is [8]. Here, the lattice situation is more general (L, M are assumed to be frames) and the stratification condition seems quite different, using a so-called stratification mapping $s : L \longrightarrow M$. It is the purpose of this paper to establish a common framework for the three approaches. This is achieved by firstly showing that the approaches in [10, 11] actually use only special LM-filters for the stratification condition and then showing that these can be captured in the framework of s-stratified LM-filters used in [7, 8] with a suitable stratification mapping. Secondly, it is shown that using this framework, the lattice-valued convergence spaces of [10, 11] can be described by "level structures" and these level structures resemble the approach in [8]. Finally, the lattice-valued convergence tower construction used in [8] is slightly generalized by allowing for each "level" a different stratification mapping and as an "index set" a poset. This generalization then contains all three approaches of [8, 10, 11], has nice categorical properties and it is possible to view the category of enriched LM-fuzzy topological spaces as a subcategory.

The paper is organized as follows. In the second section we collect the lattice background and notations that we use and collect the basic facts about stratified LM-filters. The third section then introduces the different stratified lattice-valued

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convergence spaces that are considered in this paper. In particular, the stratification conditions in [10, 11] are rewritten in terms of s-stratified LM-filters. The fourth section then characterizes the stratified lattice-valued convergence spaces of [10, 11] by their level structures and the fifth section then briefly defines a common framework for all three approaches and states the categorical properties of the new category. The sixth section generalizes an approach of [11] and discusses the relations of our category with categories of lattice-valued topological spaces. Finally, some conclusions are drawn.

2. Preliminaries

We consider in the paper frames $L = (L, \wedge, \vee)$, i.e. complete lattices that satisfy the distributive law $\bigvee_{i \in J}(\alpha \wedge \beta_i) = \alpha \wedge \bigvee_{i \in J} \beta_i$ for all $\alpha, \beta_i \in L$, $(i \in J)$. An element $\alpha \in L$ is called *coprime* if $\alpha \leq \beta \vee \gamma$ implies that $\alpha \leq \beta$ or $\alpha \leq \gamma$. Dually, we define a prime element. The set of all non-zero coprime elements in L is denoted by $J(L)$. In a complete lattice, we can define the *well-below relation* $\alpha \lhd \beta$ if for all $D \subseteq L$ with $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. A complete lattice is called completely distributive, if for each $\alpha \in L$, we have $\alpha = \bigvee\{\beta \in L \ : \ \beta \lhd \alpha\}$. This can be sharpened using coprimes. If L is completely distributive, then for $\alpha \in L$ we have $\alpha = \bigvee\{\beta \in J(L) \ : \ \beta \lhd \alpha\}$, see e.g. the proof of Proposition 1-7.6 in [9]. For more results on lattices see e.g. [4].

For a frame L with top element $\top^L$ and bottom element $\bot^L$ and a set X, we denote the set of all L-sets on X, $a, b, c, ... : X \longrightarrow L$, by L^X. We define, for $\alpha \in L$ and $A \subseteq X$, the L-set α_A by $\alpha_A(x) = \alpha$ if $x \in A$ and $\alpha_A(x) = \bot^L$ if $x \notin A$. In particular, we denote a constant L-set with value $\alpha \in L$ by α_X and we write $\top^L_X$, resp. $\bot^L_X$, for the constant L-sets with value $\top^L$, resp. $\bot^L$. A *fuzzy point* is defined as $\alpha_{\{x\}}$ for $\alpha \in J(L)$ and we write α_x for short. It is not difficult to show that $J(L^X) = \{\alpha_x \ : \ \alpha \in J(L), x \in X\}$. For $\alpha_x \in J(L^X)$ and $a \in L^X$ we say, in case $(L,')$ is equipped with an order-reversing involution, i.e. if L is a *complete DeMorgan algebra*, that α_x *quasi-coincides with* a if $\alpha \not\leq (a(x))'$ and we write $\alpha_x \, \hat{q} \, a$ in this case. The lattice operations are extended pointwise from L to L^X, i.e. we define $(a \wedge b)(x) = a(x) \wedge b(x)$, $(a \vee b)(x) = a(x) \vee b(x)$, $(\bigwedge_{i \in J} a_i)(x) = \bigwedge_{i \in J}(a_i(x))$, $(\bigvee_{i \in J} a_i)(x) = \bigvee_{i \in J}(a_i(x))$. For a mapping $f : X \longrightarrow Y$ and $a \in L^X$ and $b \in L^Y$ the image of a under f, $f(a) \in L^Y$, is defined by $f(a)(y) = \bigvee_{f(x)=y} a(x)$ and the preimage of b under f, $f^{\leftarrow}(b) \in L^X$, is defined by $f^{\leftarrow}(b) = b \circ f$.

For notions from category theory, we refer to the textbook [1].

Let L, M be frames. A mapping $s : L \longrightarrow M$ with the properties (S1) $s(\bot^L) = \bot^M$; (S2) $s(\top^L) = \top^M$ and (S3) $s(\alpha \wedge \beta) = s(\alpha) \wedge s(\beta)$ for all $\alpha, \beta \in L$ is called a *stratification mapping*, [7, 8]. In the sequel, we consider a fixed stratification mapping $s : L \longrightarrow M$.

Definition 2.1 ([5] for $L = M$, [13] and [7])**.** A mapping $\mathcal{F} : L^X \longrightarrow M$ is an *LM-filter on X* if

(F1) $\mathcal{F}(\bot^L_X) = \bot^M$ and $\mathcal{F}(\top^L_X) = \top^M$;
(F2) $\mathcal{F}(a) \leq \mathcal{F}(b)$ whenever $a \leq b$;
(F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$ for all $a, b \in L^X$.

Moreover, if $\mathcal{F}$ satisfies

(Fs) $s(\alpha) \leq \mathcal{F}(\alpha_X)$ for all $\alpha \in L$,

then it is called an *s-stratified LM-filter on X.*

Note that the *stratification condition* (Fs) is equivalent to $s(\alpha) \wedge \mathcal{F}(a) \leq \mathcal{F}(\alpha_X \wedge a)$ for all $\alpha \in L$, $a \in L^X$. We denote the set of all LM-filters on X by $\mathcal{F}_{LM}(X)$ and the set of all s-stratified LM-filters on X by $\mathcal{F}^s_{LM}(X)$.

An example of an LM-filter is given in [10]. We define for $\alpha_x \in J(L^X)$, $\hat{q}(\alpha_x) \in \mathcal{F}_{LM}(X)$ by

$$\hat{q}(\alpha_x)(a) = \begin{cases} \top^M & \text{if } \alpha_x \,\hat{q}\, a \\ \bot^M & \text{if } \alpha_x \,\not\hat{q}\, a \end{cases}$$

An example of an s-stratified LM-filter is the following [7]. We define, for $a \in L^X$, $[x]_s(a) = s(a(x))$. Then $[x]_s \in \mathcal{F}^s_{LM}(X)$ is called the *s-stratified point LM-filter of x*. Note that in the case $X = \{x\}$, $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$ implies $[x]_s(a) = s(a(x)) \leq \mathcal{F}(a(x)) = \mathcal{F}(a)$ for $a \in L^X$, i.e. $[x]_s \leq \mathcal{F}$.

For a mapping $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$ we define $f(\mathcal{F}) \in \mathcal{F}^s_{LM}(Y)$ by $f(\mathcal{F})(b) = \mathcal{F}(f^{\leftarrow}(b))$. If furthermore $\mathcal{G} \in \mathcal{F}^s_{LM}(Y)$, then we define, for $a \in L^Y$, $f^{\leftarrow}(\mathcal{G})(a) = \bigvee\{\mathcal{F}(b) \;:\; f^{\leftarrow}(b) \leq a\}$. Then $f^{\leftarrow}(\mathcal{F}) \in \mathcal{F}^s_{LM}(X)$ if and only if $\mathcal{G}(b) = \bot^M$ whenever $f^{\leftarrow}(b) = \bot^L_X$. In particular this condition is satisfied if $f : X \longrightarrow Y$ is a surjection and in this case we also have $f(f^{\leftarrow}(\mathcal{G})) = \mathcal{G}$. A non-surjective example is the embedding mapping for $Y \subseteq X$, $i_Y : Y \longmapsto X$, $i(y) = y$ for all $y \in Y$. For $\mathcal{G} \in \mathcal{F}^s_{LM}(X)$ we denote, in case of existence, $\mathcal{G}_Y = i^{\leftarrow}_Y(\mathcal{G})$ and call it the *trace of $\mathcal{G}$ on Y*.

For two s-stratified LM-filters $\mathcal{F} \in \mathcal{F}^s_{LM}(X), \mathcal{G} \in \mathcal{F}^s_{LM}(Y)$ we define their product, $\mathcal{F} \times \mathcal{G} : L^{X\times Y} \longrightarrow M$, by

$$\mathcal{F} \times \mathcal{G}(a) = \bigvee\{\mathcal{F}(f) \wedge \mathcal{G}(g) \;:\; f \times g \leq a\}, \qquad (a \in L^{X\times Y})$$

with $f \times g(x, y) = f(x) \wedge g(y)$ for $f \in L^X$ and $g \in L^Y$.

Lemma 2.2. [8] *Let $s : L \longrightarrow M$ and $t : M \longrightarrow L$ be stratification mappings such that $s \circ t \geq id_M$ and $t \circ s \geq id_L$. If $\mathcal{F} \in \mathcal{F}^s_{LM}(X), \mathcal{G} \in \mathcal{F}^s_{LM}(Y)$ then $\mathcal{F} \times \mathcal{G} \in \mathcal{F}^s_{LM}(X \times Y)$.*

We give below some examples of frames L, M with pairs of stratification mappings $s : L \longrightarrow M$ and $t : M \longrightarrow L$ such that $s \circ t \geq id_M$ and $t \circ s \geq id_L$.

Example 2.3. (1) Let L, M be frames with prime bottom elements and define $s : L \longrightarrow M$ and $t : M \longrightarrow L$ as the pointwisely largest stratification mappings, i.e. $s(\alpha) = \top^M$ iff $\alpha \neq \bot^L$ and $t(\beta) = \top^L$ iff $\beta \neq \bot^M$. Then $s \circ t \geq id_M$ and $t \circ s \geq id_L$.

(2) Let $L = M$ be linearly ordered frames and define, for $\alpha \neq \top$, the stratification mapping $s_\alpha : L \longrightarrow L$ by

$$s_\alpha(\beta) = \begin{cases} \top & \text{if} \quad \alpha < \beta \\ \beta & \text{if} \quad \beta \leq \alpha \end{cases}$$

It is routine to verify that s_α is a stratification mapping and that $s_\alpha \circ s_\alpha \geq id_L$.

(3) Let $M = \Delta^+$ be the set of distance distribution functions [12] , i.e. $\varphi : [0, \infty] \longrightarrow [0, 1]$ is in Δ^+ if and only if $\varphi(0) = 0$, $\varphi(\infty) = 1$ and φ is left-continuous,

i.e. for all $x \in (0,\infty)$ we have $\varphi(x) = \sup\{\varphi(y) \;:\; y < x\}$. With the pointwise minimum and supremum, then Δ^+ is a frame with smallest element ε_∞ defined by $\varepsilon_\infty(x) = 0$ for $0 \le x < \infty$ and $\varepsilon_\infty(\infty) = 1$ and largest element ε_0 defined by $\varepsilon_0(0) = 0$ and $\varepsilon_0(x) = 1$ for $0 < x \le \infty$. We consider further the interval $L = [0,\infty]$ with the opposite order and define the mapping $s : [0,\infty] \longrightarrow \Delta^+$ by $s(\alpha) = \varepsilon_\alpha$ with the distance distribution function $\varepsilon_\alpha(x) = 0$ if $0 \le x \le \alpha$ and $\varepsilon_\alpha(x) = 1$ for $\alpha < x \le \infty$. Then $s(\bot^L) = s(\infty) = \epsilon_\infty = \bot^M$ and $s(\top^L) = s(0) = \epsilon_0 = \top^M$ and $s(\alpha \wedge \beta) = s(\max\{\alpha,\beta\}) = \varepsilon_{\max\{\alpha,\beta\}} = \varepsilon_\alpha \wedge \varepsilon_\beta$ and hence s is non-decreasing. The mapping $t : \Delta^+ \longrightarrow [0,\infty]$, $t(\varphi) = \sup\{\alpha \in [0,\infty] \;:\; \varphi(\alpha) = 0\}$ satisfies $s \circ t(\varphi) \ge \varphi$ for all $\varphi \in \Delta^+$ and $t \circ s(\alpha) = t(\varepsilon_\alpha) = \alpha$. Moreover, we have for $\varphi, \psi \in \Delta^+$ that $\varphi \wedge \psi(x) = 0$ if and only if $\varphi(x) = 0$ or $\psi(x) = 0$ and hence $t(\varphi \wedge \psi) = \max\{t(\varphi), t(\psi)\} = t(\varphi) \wedge t(\psi)$. We thus have a pair of stratification mappings that satisfy $s \circ t \ge id_M$ and $t \circ s \ge id_L$.

(4) A *Galois connection* between the frames L, M is a pair (t,s) of order-preserving mappings $s : L \longrightarrow M$, $t : M \longrightarrow L$ such that for all $\alpha \in L$ and all $\beta \in M$, $\beta \le s(\alpha)$ is equivalent to $t(\beta) \le \alpha$, [4]. In this case s is called the *upper adjoint* and t is called the *lower adjoint*. Moreover, either of the mappings uniquely determines the other and s prerves arbitrary meets and t preserves arbitrary joins. Hence in this case s is a stratification mapping. A pair of order-preserving mappings (t,s) is a Galois connection between L, M if and only if $t \circ s \le id_L$ and $id_M \le s \circ t$. If s is injective, then $t \circ s = id_L$. However, note that t does not necessarily preserve finite meets, i.e. is not a stratification mapping in general.

Another condition that ensures that $\mathcal{F} \times \mathcal{G} \in \mathcal{F}^s_{LM}(X \times Y)$ is that the bottom element $\bot^L \in L$ is prime, see [10]. This condition applies in particular for a complete chain L, but it excludes complete Boolean algebras with more than two points.

3. Three Different Definitions of Stratified Lattice-valued Convergence Spaces Based on LM-filters

In [8] we gave the following definition.

Definition 3.1. Let L, M be frames, let N be a complete lattice and let $s : L \longrightarrow M$ be a stratification mapping. A pair $(X, \overline{q})$, of a set X and $\overline{q} = (q_\alpha : \mathcal{F}^s_{LM}(X) \longrightarrow P(X))_{\alpha \in N}$, is an *$s$-stratified LMN-convergence tower space* if

(CT1) $x \in q_\alpha([x]_s)$ for all $x \in X$, $\alpha \in N$;
(CT2) $q_\alpha(\mathcal{F}) \subseteq q_\alpha(\mathcal{G})$ whenever $\mathcal{F} \le \mathcal{G}$;
(CT3) $q_\beta(\mathcal{F}) \subseteq q_\alpha(\mathcal{F})$ whenever $\alpha \le \beta$;
(CT4) $q_{\bot^N}(\mathcal{F}) = X$ for all $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$.

A mapping $f : (X, \overline{q}) \longrightarrow (X', \overline{q'})$ between two s-stratified LMN-convergence tower spaces is called *continuous* if $f(q_\alpha(\mathcal{F})) \subseteq q'_\alpha(f(\mathcal{F}))$ for all $\alpha \in N$, $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$. We denote the category with objects the s-stratified LMN-convergence tower spaces and morphisms the continuous mappings by *$sLMN$-CTS*.

$(X, \overline{q}) \in |sLMN\text{-}CTS|$ is called *left-continuous* if

(CTL) $\displaystyle\bigcap_{\beta \in A} q_\beta(\mathcal{F}) \subseteq q_{\bigvee A}(\mathcal{F})$ whenever $A \subseteq N$.

In [10, 11] the following definitions were given.

Definition 3.2. [10] Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. A mapping $c : \mathcal{F}_{LM}(X) \longrightarrow P(J(L^X))$ is called an *LM-fuzzy convergence structure* on X if it satisfies:
(LFC1) $\alpha_x \in c(\hat{q}(\alpha_x))$ for all $\alpha \in J(L)$ and $x \in X$;
(LFC2) $\mathcal{F} \leq \mathcal{G}$ implies $c(\mathcal{F}) \subseteq c(\mathcal{G})$.
If moreover the axiom
(ELFC) $\mathcal{F}(\beta_X) = \top^M$ if $\alpha_x \in c(\mathcal{F})$ and $\alpha \not\leq \beta'$
is satisfied, then c is called an *enriched LM-fuzzy convergence structure* on X and the pair (X, c) is called an *enriched LM-fuzzy convergence space.* A mapping $f : (X, c_X) \longrightarrow (Y, c_Y)$ between two enriched LM-fuzzy convergence spaces (X, c_X), (Y, c_Y) is called *continuous* if for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $\alpha_x \in J(L^X)$, $\alpha_x \in c_X(\mathcal{F})$ implies $\alpha_{f(x)} \in c_Y(f(\mathcal{F}))$. The category with objects the enriched LM-fuzzy convergence spaces and morphisms the continuous mappings is denoted by *eLM-FCS*.

Definition 3.3. [11] Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. An *LM-fuzzy Q-convergence structure on X* is a mapping $qc : \mathcal{F}_{LM}(X) \longrightarrow L^X$ which satisfies:
(LFQC1) $\alpha \leq qc(\hat{q}(\alpha_x))(x)$ for $\alpha \in J(L)$ and $x \in X$;
(LFQC2) $\mathcal{F} \leq \mathcal{G}$ implies $qc(\mathcal{F}) \leq qc(\mathcal{G})$;
An LM-fuzzy Q-convergence structure qc on X is called *stratified* provided that for all $\alpha \in J(L)$
(SLFQC) $\mathcal{F}(\beta_X) = \top^M$ if $\alpha \leq qc(\mathcal{F})(x)$ and $\alpha \not\leq \beta'$.
The pair (X, qc) is then called a *stratified LM-fuzzy Q-convergence space.* A mapping $f : (X, qc_X) \longrightarrow (Y, qc_Y)$ between stratified LM-fuzzy Q-convergence spaces is called *continuous* provided that $\alpha \leq qc_X(\mathcal{F})(x)$ implies $\alpha \leq qc_Y(f(\mathcal{F}))(f(x))$ for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$, $x \in X$ and $\alpha \in J(L)$. The category of stratified LM-fuzzy Q-convergence spaces and their continuous mappings is denoted by *sLM-FQC*.

Both Definitions 3.2 and 3.3 generalize the concept of an *LM-fuzzy quasi-coincident neighborhood space.*

Definition 3.4 ([2] for $M = L$, [10])**.** An *LM-fuzzy quasi-coincident neighborhood system on X* is defined to be a set $\overline{Q} = \{Q_{\alpha_x} : \alpha_x \in J(L^X)\}$ of mappings $Q_{\alpha_x} : L^X \longrightarrow M$ satisfying the conditions (LFQ1) $Q_{\alpha_x} \in \mathcal{F}_{LM}(X)$ and (LFQ2) $Q_{\alpha_x}(a) \neq \bot^M$ implies $\alpha_x \, \hat{q} \, a$.
An LM-fuzzy quasi-coincident neighbourhood system on X is called *enriched* if for all $\alpha \in J(L)$
(ELFQ) $Q_{\alpha_x}(\beta_X) = \top^M$ for all $\alpha \not\leq \beta'$.

In all three definitions, the enrichement resp. the stratification conditions can be formulated more lucidly if we use the following concept.

Definition 3.5. [11] Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. For $\alpha \in J(L)$, an LM-filter $\mathcal{F} \in \mathcal{F}_{LM}(X)$ is called *α-enriched* if $\mathcal{F}(\beta_X) = \top^M$ whenever $\alpha \not\leq \beta'$.

We then can formulate the axioms as follows.
(ELFC) $\mathcal{F}$ is α-enriched if $\alpha_x \in c(\mathcal{F})$.
(SLFQC) $\mathcal{F}$ is α-enriched if $\alpha \le qc(\mathcal{F})(x)$.
(ELFQ) Q_{α_x} is α-enriched.

We will now show that the concept of an α-enriched LM-filter is a special case of an s-stratified LM-filter. We therefore do not need to develop a new theory for α-enriched LM-filters. Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. For $\alpha \in J(L)$ we denote $L^1_\alpha = \{\beta \in L \,:\, \alpha \not\le \beta'\}$ and $L^2_\alpha = L \setminus L^1_\alpha$. We define $s_\alpha : L \longrightarrow M$ by $s_\alpha(\beta) = \top^M$ if $\beta \in L^1_\alpha$ and $s_\alpha(\beta) = \bot^M$ if $\beta \in L^2_\alpha$.

Lemma 3.6. *Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. If $\alpha \in J(L)$, then s_α is a stratification mapping.*

Proof. (S1) We have $\alpha \le \top^L = (\bot^L)'$ and hence $\bot^L \in L^2_\alpha$ and $s_\alpha(\bot^L) = \bot^M$. (S2) As $\alpha \in J(L)$, we have $(\top^L)' = \bot^L < \alpha$, i.e. $\top^L \in L^1_\alpha$ and hence $s_\alpha(\top^L) = \top^M$. (S3) If $s_\alpha(\beta \wedge \gamma) = \top^M$ then $\beta \wedge \gamma \in L^1_\alpha$. But then also $\beta, \gamma \in L^1_\alpha$ and therefore $s_\alpha(\beta) \wedge s_\alpha(\gamma) = \top^M$. If $s_\alpha(\beta \wedge \gamma) = \bot^M$, then $\beta \wedge \gamma \in L^2_\alpha$, i.e. $\alpha \le (\beta \wedge \gamma)' = \beta' \vee \gamma'$. α being coprime then implies $\alpha \le \beta'$ or $\alpha \le \gamma'$, i.e. $s_\alpha(\beta) = \bot^M$ or $s_\alpha(\gamma) = \bot^M$. Hence also $s_\alpha(\beta) \wedge s_\alpha(\gamma) = \bot^M$. □

Lemma 3.7. *Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. Then $\mathcal{F} \in \mathcal{F}_{LM}(X)$ is α-enriched if and only if $\mathcal{F}$ is s_α-stratified.*

Proof. We have that $\mathcal{F}$ is s_α-stratified iff for all $\beta \in L$, $s_\alpha(\beta) \le \mathcal{F}(\beta_X)$, i.e. iff for all $\beta \in L$, $s_\alpha(\beta) = \top^M$ implies $\mathcal{F}(\beta_X) = \top^M$. This is equivalent to $\mathcal{F}(\beta_X) = \top^M$ whenever $\alpha \not\le \beta'$, i.e. to $\mathcal{F}$ being α-enriched. □

Lemma 3.8. *Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution and let $x \in X$ and $\alpha \in J(L)$. Then $[x]_{s_\alpha} = \hat{q}(\alpha_x)$.*
Proof. We have, for $a \in L^X$,

$$[x]_{s_\alpha}(a) = s_\alpha(a(x)) = \left\{ \begin{array}{ll} \top^M & \text{if } a(x) \in L^1_\alpha \\ \bot^M & \text{if } a(x) \in L^2_\alpha \end{array} \right\} = \left\{ \begin{array}{ll} \top^M & \text{if } \alpha \not\le (a(x))' \\ \bot^M & \text{if } \alpha \le (a(x))' \end{array} \right\}$$
$$= \left\{ \begin{array}{ll} \top^M & \text{if } \alpha_x \,\hat{q}\, a \\ \bot^M & \text{if } \alpha_x \,\not\!\hat{q}\, a \end{array} \right\} = \hat{q}(\alpha_x)(a).$$

□

We would like to point out that in the situation of this section, for the stratification mapping $s_\alpha : L \longrightarrow M$ there is no stratification mapping $t_\alpha : M \longrightarrow L$ such that $t_\alpha \circ s_\alpha \ge id_L$. Hence in order to ensure the existence of products of s_α-stratified LM-filters, we have to demand that the bottom element of L is prime, as is done in [10, 11].

4. Level Structures for Enriched LM-fuzzy Convergence Spaces and for Stratified LM-fuzzy Q-convergence Spaces

In this section, let L, M be completely distributive lattices and let $(L,')$ be equipped with an order-reversing involution. We further consider the stratification mappings s_α from the last section. First we look at enriched LM-fuzzy convergence spaces. The proofs of the following lemmas are straightforward and not shown.

Lemma 4.1. *Let $(X,c) \in |eLM\text{-}FCS|$. We define for $\alpha \in J(L)$ the mapping $q^c_\alpha : \mathcal{F}^{s_\alpha}_{LM}(X) \longrightarrow P(X)$ by $x \in q^c_\alpha(\mathcal{F}) \iff \alpha_x \in c(\mathcal{F})(x)$. Then*
(LLFC1) *$x \in q^c_\alpha([x]_{s_\alpha})$ for all $x \in X$ and all $\alpha \in J(L)$;*
(LLFC2) *$q^c_\alpha(\mathcal{F}) \subseteq q^c_\alpha(\mathcal{G})$, whenever $\mathcal{F} \le \mathcal{G}$, $\mathcal{F},\mathcal{G} \in \mathcal{F}^{s_\alpha}_{LM}(X)$.*

Lemma 4.2. *Let $(X,c_X),(Y,c_Y) \in |eLM\text{-}FCS|$ and let $f : (X,c_X) \longrightarrow (Y,c_Y)$ be continuous. Then, for all $\alpha \in J(L)$, $f(x) \in q^{c_Y}_\alpha(f(\mathcal{F}))$ whenever $x \in q^{c_X}_\alpha(\mathcal{F})$.*

The category $\overline{s}LM\text{-}LFCS$ has as objects the spaces $(X,\overline{q})$, where $\overline{q} = (q_\alpha : \mathcal{F}^{s_\alpha}_{LM}(X) \longrightarrow P(X))_{\alpha\in J(L)}$ satisfies the axioms
(LLFC1) $x \in q_\alpha([x]_{s_\alpha})$ for all $x \in X$ and all $\alpha \in J(L)$;
(LLFC2) $q_\alpha(\mathcal{F}) \subseteq q_\alpha(\mathcal{G})$, whenever $\mathcal{F} \le \mathcal{G}$, $\mathcal{F},\mathcal{G} \in \mathcal{F}^{s_\alpha}_{LM}(X)$,
and as morphisms the mappings $f : (X,\overline{q^X}) \longrightarrow (Y,\overline{q^Y})$ that satisfy, for all $\alpha \in J(L)$, $f(x) \in q^Y_\alpha(f(\mathcal{F}))$ whenever $x \in q^X_\alpha(\mathcal{F})$. The two lemmas above then show that we can define a functor

$$A : eLM\text{-}FCS \longrightarrow \overline{s}LM\text{-}LFCS,\ A((X,c)) = (X,\overline{q^c}),\ A(f) = f.$$

Theorem 4.3. *The categories $eLM\text{-}FCS$ and $\overline{s}LM\text{-}LFCS$ are isomorphic.*

Proof. Let $(X,\overline{q}) \in |\overline{s}LM\text{-}LFCS|$. We define $\alpha_x \in c^{\overline{q}}(\mathcal{F}) \iff x \in q_\alpha(\mathcal{F})$. It is not difficult to show that $(X,c^{\overline{q}}) \in |eLM\text{-}FCS|$. Furthermore, for a continuous mapping $f : (X,\overline{q^X}) \longrightarrow (Y,\overline{q^Y})$ then also $f : (X,c^{\overline{q^X}}) \longrightarrow (Y,c^{\overline{q^Y}})$ is continuous. Hence we can define a functor

$$B : \overline{s}LM\text{-}LFCS \longrightarrow eLM\text{-}FCS,\ B((X,\overline{q})) = (X,c^{\overline{q}}),\ B(f) = f.$$

It is a simple matter to show that $A\circ B = id_{\overline{s}LM\text{-}LFCS}$ and $B\circ A = id_{eLM\text{-}FCS}$. □

We now turn our attention to stratified LM-fuzzy Q-convergence spaces. We first note the following properties of the stratification mappings s_α.

Lemma 4.4. (1) *We have $s_\alpha \le s_\beta$ whenever $\alpha \le \beta$.*
(2) *If $A \subseteq J(L)$ and $\bigvee A \in J(L)$, then $s_{\bigvee A} \le \bigvee_{\alpha\in A} s_\alpha$.*

Proof. (1) If $s_\alpha(\delta) = \top^M$, then $\alpha \not\le \delta'$ and hence also $\beta \not\le \delta'$, i.e. $s_\beta(\delta) = \top^M$.
(2) If $s_{\bigvee A}(\delta) = \top^M$, then $\bigvee A \not\le \delta'$. If for all $\alpha \in A$ we had $\alpha \le \delta'$, then also $\bigvee A \le \delta'$, a contradiction. Hence there is $\gamma \in A$ such that $\gamma \not\le \delta'$ and then $\bigvee_{\alpha\in A} s_\alpha(\delta) \ge s_\gamma(\delta) = \top^M$. □

Lemma 4.5. *Let $(X,qc) \in |sLM\text{-}FQC|$. We define, for $\alpha \in J(L)$, the mapping $q^{qc}_\alpha : \mathcal{F}^{s_\alpha}_{LM}(X) \longrightarrow P(X)$ by $x \in q^{qc}_\alpha(\mathcal{F}) \iff \alpha \le qc(\mathcal{F})(x)$. Then*
(LLFQC1) *$x \in q^{qc}_\alpha([x]_{s_\alpha})$ for all $\alpha \in J(L)$ and all $x \in X$;*
(LLFQC2) *$q^{qc}_\alpha(\mathcal{F}) \subseteq q^{qc}_\alpha(\mathcal{G})$ whenever $\mathcal{F} \le \mathcal{G}$;*
(LLFQC3) *$q^{qc}_\beta(\mathcal{F}) \subseteq q^{qc}_\alpha(\mathcal{F})$ whenever $\alpha \le \beta$;*
(LLFQC4) *If $A \subseteq J(L)$ and $\bigvee A \in J(L)$, then $x \in q^{qc}_{\bigvee A}(\mathcal{F})$ whenever $x \in q^{qc}_\alpha(\mathcal{F})$ for all $\alpha \in A$.*

Proof. Also this proof is straightforward and not presented. □

Lemma 4.6. *Let $(X, qc_X), (Y, qc_Y) \in |sLM\text{-}FQC|$ and let $f : (X, qc_X) \longrightarrow (Y, qc_Y)$ be continuous. Then, for all $\alpha \in J(L)$, $f(x) \in q_\alpha^{qc_Y}(f(\mathcal{F}))$ whenever $x \in q_\alpha^{qc_X}(\mathcal{F})$.*

The category $\overline{s}LM\text{-}LFQCS$ has as objects the spaces $(X, \overline{q})$, where $\overline{q} = (q_\alpha : \mathcal{F}_{LM}^{s_\alpha}(X) \longrightarrow P(X))_{\alpha \in J(L)}$ satisfies the axioms
(LLFQC1) $x \in q_\alpha([x]_{s_\alpha})$ for all $\alpha \in J(L)$ and all $x \in X$;
(LLFQC2) $q_\alpha(\mathcal{F}) \subseteq q_\alpha(\mathcal{G})$ whenever $\mathcal{F} \leq \mathcal{G}$;
(LLFQC3) $q_\beta(\mathcal{F}) \subseteq q_\alpha(\mathcal{F})$ whenever $\alpha \leq \beta$;
(LLFQC4) If $A \subseteq J(L)$ and $\bigvee A \in J(L)$, then $x \in q_{\bigvee A}(\mathcal{F})$ whenever $x \in q_\alpha(\mathcal{F})$ for all $\alpha \in A$;

and as morphisms the mappings $f : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$ that satisfy, for all $\alpha \in J(L)$, $f(x) \in q_\alpha^Y(f(\mathcal{F}))$ whenever $x \in q_\alpha^X(\mathcal{F})$. The two lemmas above then show that we can define a functor

$$C : sLM\text{-}FQC \longrightarrow \overline{s}LM\text{-}LFQCS,\ A((X, qc)) = (X, \overline{q^{qc}}),\ A(f) = f.$$

Lemma 4.7. *Let $(X, \overline{q}) \in |\overline{s}LM\text{-}LFCS|$ and define $qc^{\overline{q}}(\mathcal{F})(x) = \bigvee\{\alpha \in J(L) : x \in q_\alpha(\mathcal{F})\}$. Then $(X, qc^{\overline{q}}) \in |sLM\text{-}FQC|$.*

Proof. We only prove the property (LFQC3) and leave the others to the reader. Let, for $\alpha \in J(L)$, $\alpha \leq qc^{\overline{q}}(\mathcal{F})(x) = \bigvee\{\beta \in J(L) : x \in q_\beta(\mathcal{F})\}$. Let $\gamma \in J(L)$, $\gamma \lhd \alpha$. Then there is $\beta \in J(L)$, $\gamma \leq \beta$ with $x \in q_\beta(\mathcal{F})$. By (LLFQC3) then $x \in q_\gamma(\mathcal{F})$ and hence, using (LLFQC4), we obtain $x \in q_{\bigvee\{\gamma \in J(L) : \gamma \lhd \alpha\}}(\mathcal{F}) = q_\alpha(\mathcal{F})$. Therefore $\mathcal{F}$ is s_α-stratified. □

Lemma 4.8. *Let $f : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$ be continuous. Then $f : (X, qc^{\overline{q^X}}) \longrightarrow (Y, qc^{\overline{q^Y}})$ is continuous.*

Proof. Also this proof is easy and not presented. □

Hence we can define a functor

$$D : \overline{s}LM\text{-}LFQCS \longrightarrow sLM\text{-}FQC,\ D((X, \overline{q})) = (X, qc^{\overline{q}}),\ D(f) = f.$$

Theorem 4.9. *The categories $sLM\text{-}FQC$ and $\overline{s}LM\text{-}LFQCS$ are isomorphic.*

Proof. Let $(X, \overline{q}) \in |\overline{s}LM\text{-}LFQCS|$, let $\alpha \in J(L), x \in X$ and $\mathcal{F} \in \mathcal{F}_{LM}^{s_\alpha}(X)$. If $x \in q_\alpha^{qc^{\overline{q}}}(\mathcal{F})$, then $\alpha \leq \bigvee\{\beta \in J(L) : x \in q_\beta(\mathcal{F})\}$. Let $\gamma \lhd \alpha$, $\gamma \in J(L)$. Then there is $\beta \in J(L), \gamma \leq \beta$ such that $x \in q_\beta(\mathcal{F})$. With (LLFQC3) then $x \in q_\gamma(\mathcal{F})$. From (LLFQC4) we conclude, as $\alpha = \bigvee\{\gamma \in J(L) : \gamma \lhd \alpha\}$, that $x \in q_\alpha(\mathcal{F})$. Conversely, if $x \in q_\alpha(\mathcal{F})$, then $\alpha \leq qc^{\overline{q}}(\mathcal{F})(x)$ and hence $x \in q_\alpha^{qc^{\overline{q}}}(\mathcal{F})$. This shows that $C \circ D = id_{\overline{s}LM\text{-}LFQCS}$.

Let now $(X, qc) \in |sLM\text{-}FQC|$. Let $\alpha \leq qc^{\overline{q^{qc}}}(\mathcal{F})(x)$. Then $\alpha \leq \bigvee\{\beta \in J(L) : \beta \leq qc(\mathcal{F})\}$. Let $\gamma \lhd \alpha$. Then there is $\beta \in J(L)$ such that $\gamma \leq \beta \leq qc(\mathcal{F})$. Hence $\alpha = \bigvee\{\gamma \in L : \gamma \lhd \alpha\} \leq qc(\mathcal{F})$. Conversely, let $\alpha \leq qc(\mathcal{F})(x)$. Let $\gamma \in J(L), \gamma \lhd \alpha$. Then $\gamma \leq qc(\mathcal{F})(x)$ and we have $x \in q_\gamma^{qc}(\mathcal{F})$. Hence $\gamma \leq qc^{\overline{q^{qc}}}(\mathcal{F})(x)$ and we obtain $\alpha = \bigvee\{\gamma \in J(L) : \gamma \lhd \alpha\} \leq qc^{\overline{q^{qc}}}(\mathcal{F})(x)$. This proves $D \circ C = id_{sLM\text{-}FQC}$. □

5. A General Framework for s-stratified LMN-convergence Tower Spaces, Enriched LM-fuzzy Convergence Spaces and Stratified LM-fuzzy Q-convergence Spaces

In this section we slightly alter the definition of s-stratified LMN-convergence tower space given in [8], in order to capture the examples discussed in the previous section. We consider the following situation. Let L, M be frames and let N be a poset. We call a family of stratification mappings $\overline{s} = (s_\alpha : L \longrightarrow M)_{\alpha\in N}$ *increasing* if $s_\alpha \le s_\beta$ whenever $\alpha \le \beta$.

Definition 5.1. Let an increasing family of stratification mappings $\overline{s} = (s_\alpha : L \longrightarrow M)_{\alpha\in N}$ be given. A family of mappings $\overline{q} = (q_\alpha : \mathcal{F}^{s_\alpha}_{LM}(X) \longrightarrow P(X))_{\alpha\in N}$ is called an *$\overline{s}$-stratified LMN-convergence tower on X* if

(CT1) $x \in q_\alpha([x]_{s_\alpha})$ for all $\alpha \in N$, $x \in X$;
(CT2) $q_\alpha(\mathcal{F}) \subseteq q_\alpha(\mathcal{G})$ whenever $\mathcal{F} \le \mathcal{G}$;
(CT3) $q_\beta(\mathcal{F}) \subseteq q_\alpha(\mathcal{F})$ whenever $\alpha \le \beta$.

The pair $(X, \overline{q})$ is then called an *$\overline{s}$-stratified LMN-convergence tower space.* If additionally

(CTL) $s_{\bigvee A} \le \bigvee_{\alpha\in A} s_\alpha$ and $x \in q_{\bigvee A}(\mathcal{F})$ whenever $x \in q_\alpha(\mathcal{F})$ for all $\alpha \in A \subseteq N$ and $\bigvee A \in N$

then $(X, \overline{q})$ is called *left-continuous.*

A mapping $f : (X, \overline{q^X}) \longrightarrow (Y, \overline{q^Y})$ between two $\overline{s}$-stratified LMN-convergence tower spaces is called continuous if $f(x) \in q^Y_\alpha(f(\mathcal{F}))$ whenever $x \in q^X_\alpha(\mathcal{F})$. The category with objects the $\overline{s}$-stratified LMN-convergence tower spaces and morphisms the continuous mappings is denoted by $\overline{s}LMN\text{-}CTS$ and the subcategory of all left-continuous $\overline{s}$-stratified LMN-convergence tower spaces is denoted by $\overline{s}LMN\text{-}LCTS$.

Example 5.2. Let L, M be frames and let N be a complete lattice and let $s_\alpha = s$ for all $\alpha \in N$, then a (left-continuous) $\overline{s}$-stratified LMN-convergence tower space is a (left-continuous) s-stratified LMN-convergence tower space [8]. In this way, all examples listed in [8] are contained in our new theory.

Example 5.3. Let L, M be completely distributive lattices and let $(L, ')$ be equipped with an order-reversing involution. Consider for $\alpha \in J(L)$ the stratification mapping $s_\alpha(\beta) = \top^M$ if $\alpha \not\le \beta'$ and $= \bot^M$ if $\alpha \le \beta'$. Let further $N = J(L)$ with the discrete order $\alpha \le \beta \iff \alpha = \beta$. Then an $\overline{s}$-stratified LMN-convergence tower space is an enriched LM-fuzzy convergence space [10].

Example 5.4. Let L, M be completely distributive lattices and let $(L, ')$ be equipped with an order-reversing involution. Consider $N = J(L)$ with the inherited order from L and for $\alpha \in J(L)$ let $s_\alpha(\beta) = \top^M$ if $\alpha \not\le \beta'$ and $= \bot^M$ if $\alpha \le \beta'$. Then a left-continous $\overline{s}$-stratified LMN-convergence tower space is a stratified LM-fuzzy Q-convergence space [11].

We finally state the important categorical properties. However, we do not give detailed proofs as they are routine and can simply be adapted from previous works [6, 3].

Theorem 5.5. *The category $\overline{s}LMN$-CTS is a well-fibred topological category.*

Proof. For the terminal separator property, let $X = \{x\}$ be a one-point set. Then $\mathcal{F} \in \mathcal{F}^{s_\alpha}_{LM}(X)$ satisfies $\mathcal{F} \geq [x]_{s_\alpha}$ and hence, by (CT1) and (CT2), there is only one $\overline{s}$-stratified LMN-convergence tower on X, namely $x \in q_\alpha(\mathcal{F}) \iff \mathcal{F} \geq [x]_{s_\alpha}$. Furthermore, for a source $(f_j : X \longrightarrow (X_j, \overline{q^j}))_{j \in J}$ the initial structure $\overline{q}$ on X is given by $x \in q_\alpha(\mathcal{F}) \iff f_j(x) \in q^j_\alpha(f_j(\mathcal{F}))$ for all $j \in J$. □

In many instances $\overline{s}LMN$-CTS is also Cartesian closed, i.e. possesses natural function spaces. We only have to make sure that products of s_α-stratified LM-filters are again s_α-stratified LM-filters. As mentioned above, this can be ensured e.g. if the bottom element in L is prime or if there are, for each $\alpha \in N$, stratification mappings $t_\alpha : M \longrightarrow L$ such that $s_\alpha \circ t_\alpha \geq id_M$ and $t_\alpha \circ s_\alpha \geq id_L$.

Theorem 5.6. *Let L, M be frames and let N be a poset and let $\overline{s} = (s_\alpha : L \longrightarrow M)_{\alpha \in N}$ be an increasing family of stratification mappings. If for all $\alpha \in N$ products of s_α-stratified LM-filters are s_α-stratified, then the category $\overline{s}LMN$-CTS is Cartesian closed.*

Proof. We only mention the function space structures. Let, for $(X, \overline{q^X}), (Y, \overline{q^Y}) \in |\overline{s}LMN\text{-}CTS|$ denote $C(X,Y) = \{f : X \longrightarrow Y \ : \ f \text{ is continuous}\}$ and denote $ev : C(X,Y) \times X \longrightarrow Y$ the evaluation mapping defined by $ev(f,x) = f(x)$. For $\alpha \in N$, $\Phi \in \mathcal{F}^{s_\alpha}_{LM}(C(X,Y))$ and $f \in C(X,Y)$ we define

$$f \in c_\alpha(\Phi) \iff f(x) \in q^Y_\beta(ev(\Phi \times \mathcal{F})) \text{ whenever } \beta \leq \alpha, \beta \in N, x \in q^X_\alpha(\mathcal{F}).$$

It is not difficult to show that $(C(X,Y), \overline{c}) \in |\overline{s}LMN\text{-}CTS|$ and that the evaluation mapping $ev : C(X,Y) \times X \longrightarrow Y, (f,x) \longmapsto f(x)$ is continuous. Furthermore, for $f : X \times Y \longrightarrow Z$ continuous, the mapping $\hat{f} : X \longrightarrow C(Y,Z)$, $\hat{f}(x) = f_x$ with $f_x(y) = f(x,y)$ is continuous. As a well-fibred construct (note that here we need the $\overline{s}$-stratification) $\overline{s}LMN$-CTS is Cartesian closed iff it has function spaces [1]. □

Theorem 5.7. *The category $\overline{s}LMN$-CTS is extensional.*

Proof. Also this proof can be adapted from a standard one and we only sketch the construction. Let $(Z, \overline{p})$ be a subspace of $(X, \overline{q})$ in $|\overline{s}LMN\text{-}CTS|$ and let $f : (Z, \overline{p}) \longrightarrow (Y, \overline{r})$ be continuous. We consider the one-point extension $Y^* = Y \cup \{\infty\}$ and $f_* : X \longrightarrow Y^*$ by $f_*(x) = \begin{cases} f(x) & \text{if } x \in Z \\ \infty & \text{if } x \in X \setminus Z \end{cases}$. We define the following $\overline{s}LMN$-CTS structure on Y^*: for $\mathcal{G} \in \mathcal{F}^{s_\alpha}_{LM}(Y^*)$ we define

- $y \in r^*_\alpha(\mathcal{G}) \iff \mathcal{G}_Y$ exists and $y \in r_\alpha(\mathcal{G}_Y)$ $(y \in Y)$
- $y \in r^*_\alpha(\mathcal{G}) \iff \mathcal{G}_Y$ does not exist $(y \in Y)$
- $\infty \in r^*_\alpha(\mathcal{G})$

It is then not difficult to show that $(Y^*, \overline{r^*}) \in |\overline{s}LMN\text{-}CTS|$ and that f_* is continuous, cf. [3]. □

6. On the Relations of $\overline{s}$-stratified LMN-convergence Spaces and LM(-fuzzy) Topological Spaces

Let X be a set and $\Delta \subseteq L^X$. The pair (X, Δ) is called a *stratified L-topological space* [5] if $g \wedge h \in \Delta$ for all $g, h \in \Delta$; $\bigvee_{i\in J} g_i \in \Delta$ whenever $g_i \in \Delta$ for all $i \in J$ and if $\alpha_X \in \Delta$ whenever $\alpha \in L$. A mapping $f : (X, \Delta) \longrightarrow (X', \Delta')$ between the stratified L-topological spaces $(X, \Delta), (X', \Delta')$ is called *continuous* whenever $b \circ f \in \Delta$ for all $b \in \Delta'$. The category of stratified L-topological spaces with continuous mappings is denoted by $sL\text{-}TOP$.

An *enriched LM-fuzzy topological space* [5] is a pair (X, τ) where $\tau : L^X \longrightarrow M$ satisfies the axioms $\tau(\alpha_X) = \top^L$ for all $\alpha \in L$, $\tau(g \wedge h) \geq \tau(g) \wedge \tau(h)$ for all $g, h \in L^X$ and $\tau(\bigvee_{i\in J} g_i) \geq \bigwedge_{i\in J} \tau(g_i)$ for all $g_i \in L^X$, $(i \in J)$. A mapping $f : (X, \tau) \longrightarrow (X', \tau')$ between the enriched LM-fuzzy topological spaces $(X, \tau), (X', \tau')$ is called *continuous* if $\tau(b \circ f) \geq \tau'(b)$ for all $b \in L^{X'}$. The category of enriched LM-fuzzy topological spaces with continuous mappings is denoted by $eLM\text{-}FTOP$.

It is a natural question, what relations exist between enriched LM-fuzzy topological spaces or stratified L-topological spaces and our $\overline{s}$-stratified LMN-convergence tower spaces. To understand this problem, we first note that the categories $sL\text{-}TOP$ of stratified L-topological spaces resp. $eLM\text{-}FTOP$ of enriched LM-fuzzy topological spaces have one, resp. two variables, namely the frames L and M. Our category of $\overline{s}$-stratified LMN-convergence tower spaces has four variables, namely additionally to the frames L and M the poset N and the family $\overline{s}$ of stratification mappings, which, in a sense, ties these four variables together. Based on this observation, in order to study the relations between $sL\text{-}TOP$, $eLM\text{-}FTOP$ and $\overline{s}LMN\text{-}CTS$, we have a certain freedom how to choose the "free variables" N and $\overline{s}$. We first state some results known from the literature where different choices lead to different embeddings and then develop, for a special choice of $\overline{s}$, a general approach to embed $eLM\text{-}FTOP$ into $\overline{s}LMN\text{-}CTS$. This approach is based on the ideas in [11] but is more general.

The following results are known from the literature.

Theorem 6.1. [6] *Let $L = M = N$ be frames and consider the stratification mappings $s_\alpha = id_L$ for all $\alpha \in L$. The category $sL\text{-}TOP$ of stratified L-topological spaces can be embedded into the category of left-continuous $\overline{s}$-stratified LLL-convergence tower spaces as a reflective subcategory.*

Theorem 6.2. [8] *Let $s : L \longrightarrow M$ and $t : M \longrightarrow L$ be stratification mappings such that $t \circ s = id_L$ and s preserves arbitrary joins. Define for $\alpha \in M$, $s_\alpha = s$ for a stratification mapping $s : L \longrightarrow M$. Then $eLM\text{-}FTOP$ is isomorphic to a subcategory of $\overline{s}LMM^{op}\text{-}CTS$.*

Theorem 6.3. [11] *Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. For $\alpha \in J(L)$ consider the stratification mapping s_α from Example 5.3. Consider further the poset $J(L)$ with the order inherited from L. Then the category $eLM\text{-}FTOP$ can be embedded into the category $\overline{s}LMJ(L)\text{-}LCTS$ of left-continuous $\overline{s}$-stratified $LMJ(L)$-convergence tower spaces as a reflective subcategory.*

Theorem 6.4. [10] *Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. For $\alpha \in J(L)$ consider the stratification mapping s_α from Example 5.3. Consider further $N = J(L)$ with the discrete order. Then the category eLM-FTOP can be embedded into the category $\overline{s}LMJ(L)$-CTS as a reflective subcategory.*

We are now going to develop our general approach.

Let $(X,\tau) \in |eLM\text{-}FTOP|$ and consider a family of stratification mappings $\overline{s} = (s_\alpha)_{\alpha \in N}$ where for each $\alpha \in N$, $s_\alpha(L) = \{\bot^M, \top^M\}$. We then call $\overline{s}$ a *family of two-valued stratification mappings.* We first give some examples of such stratification mappings.

Example 6.5. (1) Let $L = [0,1]$ and fix an element $\alpha > 0$. We define $s_\alpha(\beta) = \top^M$ if $\beta \geq \alpha$ and $s_\alpha(\beta) = \bot^M$ if $\beta < \alpha$. Then s_α is a stratification mapping. For another poset N, consider a non-decreasing mapping $\varphi : N \longrightarrow (0,1]$. Then $\overline{s} = (s_{\varphi(\nu)})_{\nu \in N}$ is an increasing family of two-valued stratification mappings. We note that for $L = [0,1]$, for any stratification mapping $s : L \longrightarrow M$ with $s(L) = \{\bot^M, \top^M\}$ we have that $s^{-1}(\top^M)$ is an interval. Hence the only other possible such stratification mappings are of the form $s^{-1}(\top^M) = (\alpha, 1]$ for some $\alpha > 0$.

(2) Let $L = \{\bot^L, \alpha, \beta, \top^L\}$ be the complete Boolean algebra with $\bot^L \leq \alpha, \beta \leq \top^L$ and $\alpha \wedge \beta = \bot^L$, $\alpha \vee \beta = \top^L$, α, β uncomparable. Then there are three different stratification mappings $s : L \longrightarrow M$ with $s(L) = \{\bot^M, \top^M\}$ given in the following table.

	$\bot^L$	α	β	$\top^L$
s_1	$\bot^M$	$\bot^M$	$\bot^M$	$\top^M$
s_2	$\bot^M$	$\bot^M$	$\top^M$	$\top^M$
s_3	$\bot^M$	$\top^M$	$\bot^M$	$\top^M$

.

(3) Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. For $\alpha \in J(L)$ we define $s_\alpha : L \longrightarrow M$ by $s_\alpha(\beta) = \top^M$ if $\alpha \not\leq \beta'$ and $s_\alpha(\beta) = \bot^M$ if $\alpha \leq \beta'$. Then $\overline{s} = (s_\alpha)_{\alpha \in J(L)}$ is a family of two-valued stratification mappings.

With regard to the existence of s-stratified product LM-filters, we note that for stratification mappings $s : L \longrightarrow M$ with $s(L) = \{\bot^M, \top^M\}$ there is in general no stratification mapping $t : M \longrightarrow L$ such that $s \circ t \geq id_L$. Hence in this situation, we must demand that the bottom element of L is prime for the Cartesian closedness of the category $\overline{s}LMN$-CTS.

We define, for $x \in X$ and $\alpha \in N$ the mapping $\mathcal{U}_\alpha^{\tau,x} : L^X \longrightarrow M$ by

$$\mathcal{U}_\alpha^{\tau,x}(a) = \bigvee \{\tau(b) \;:\; b \leq a, s_\alpha(b(x)) = \top^M\}.$$

Proposition 6.6. *Let $(X,\tau) \in |eLM\text{-}FTOP|$ and consider a family of two-valued stratification mappings $\overline{s} = (s_\alpha)_{\alpha \in N}$. Then for each $\alpha \in N$ and $x \in X$, $\mathcal{U}_\alpha^{\tau,x} \in \mathcal{F}_{LM}^{s_\alpha}(X)$.*

Proof. We have $\mathcal{U}_\alpha^{\tau,x}(\top_X^L) \geq \tau(\top_X^L) = \top^M$ and $\mathcal{U}_\alpha^{\tau,x}(\top_X^L) = \bigvee \emptyset = \bot^M$. Furthermore, it follows straight from the definition that $a \leq b$ entails $\mathcal{U}_\alpha^{\tau,x}(a) \leq \mathcal{U}_\alpha^{\tau,x}(b)$.

Let now $a, b \in L^X$. Then, using the frame law, we get

$$\begin{aligned}
\mathcal{U}_\alpha^{\tau,x}(a) \wedge \mathcal{U}_\alpha^{\tau,x}(b) &= \bigvee_{c \le a, s_\alpha(c(x)) = \top^M} \bigvee_{d \le b, s_\alpha(d(x)) = \top^M} \tau(c) \wedge \tau(d) \\
&\le \bigvee_{c \wedge d \le a \wedge b, s_\alpha(c \wedge d(x)) = \top^M} \tau(c \wedge d) \\
&\le \mathcal{U}_\alpha^{\tau,x}(a \wedge b).
\end{aligned}$$

Finally, if $s_\alpha(\beta) = \top^M$, then $\mathcal{U}_\alpha^{\tau,x}(\beta_X) \ge \tau(\beta_X) = \top^M$, i.e. $\mathcal{U}_\alpha^{\tau,x}(\beta_X) \ge s_\alpha(\beta)$. □

We note that if $\mathcal{U}_\alpha^{\tau,x}(a) > \bot^M$, then we have in particular $s_\alpha(a(x)) = \top^M$. Hence $\mathcal{U}_\alpha^{\tau,x} \le [x]_{s_\alpha}$. Furthermore, if $\alpha \le \beta$ implies $s_\alpha \le s_\beta$, then $\mathcal{U}_\alpha^{\tau,x} \le \mathcal{U}_\beta^{\tau,x}$.

We define now, for $(X, \tau) \in |eLM\text{-}FTOP|$ and $x \in X$ and $\mathcal{F} \in \mathcal{F}_{LM}^{s_\alpha}(X)$,

$$x \in q_\alpha^\tau(\mathcal{F}) \iff \mathcal{F} \ge \mathcal{U}_\alpha^{\tau,x}.$$

Theorem 6.7. *Let $\overline{s} = (s_\alpha)_{\alpha \in N}$ be an increasing family of two-valued stratification mappings and let $(X, \tau), (X', \tau') \in |eLM\text{-}FTOP|$. Then $(X, \overline{q^\tau}) \in |\overline{s}LMN\text{-}CTS|$.*

Furthermore, if $f : (X, \tau) \longrightarrow (X', \tau')$ is continuous, then also $f : (X, \overline{q^\tau}) \longrightarrow (X', \overline{q^{\tau'}})$ is continuous.

Proof. The first part is obvious. In order to show the second part, we show that for a continuous mapping $f : (X, \tau) \longrightarrow (X', \tau')$ we have $f(\mathcal{U}_\alpha^{\tau,x}) \ge \mathcal{U}_\alpha^{\tau',f(x)}$ for all $x \in X$ and $\alpha \in N$. Let $b \in L^{X'}$. Then

$$\begin{aligned}
\mathcal{U}_\alpha^{\tau',f(x)}(b) &= \bigvee_{c \le b, s_\alpha(c(x)) = \top^M} \tau'(c) \\
&\le \bigvee_{c \le b, s_\alpha(c \circ f(x)) = \top^M} \tau(c \circ f) \\
&\le \bigvee_{c \circ f \le b \circ f, s_\alpha(c \circ f(x)) = \top^M} \tau(c \circ f) \\
&\le \bigvee_{d \le b \circ f, s_\alpha(d(x)) = \top^M} \tau(d) \\
&= \mathcal{U}_\alpha^{\tau,x}(b \circ f) \;=\; f(\mathcal{U}_\alpha^{\tau,x})(b).
\end{aligned}$$

Let now $x \in q_\alpha^\tau(\mathcal{F})$. Then $\mathcal{F} \ge \mathcal{U}_\alpha^{\tau,x}$ and hence $f(\mathcal{F}) \ge f(\mathcal{U}_\alpha^{\tau,x}) \ge \mathcal{U}_\alpha^{\tau',f(x)}$, i.e. $f(x) \in q_\alpha^{\tau'}(f(\mathcal{F}))$. □

As a consequence, we can define a functor $E : eLM\text{-}FTOP \longrightarrow \overline{s}LMN\text{-}CTS$ by $E((X, \tau)) = (X, \overline{q^\tau})$ and $E(f) = f$.

In order to ensure that this functor is an embedding, we need to impose the following condition on the family $\overline{s}$. We say that the family of stratification mappings $\overline{s} = (s_\alpha)_{\alpha \in N}$ is *lifting* if for all $\beta \ne \bot^L$ there is $\alpha \in N$ such that $s_\alpha(\beta) = \top^M$.

Example 6.8. Let L, M be completely distributive and let $(L,')$ be equipped with an order-reversing involution. For $\alpha \in J(L)$ we consider $\overline{s}$ with $s_\alpha : L \longrightarrow M$ defined by $s_\alpha(\beta) = \top^M$ if $\alpha \not\le \beta'$ and $s_\alpha(\beta) = \bot^M$ if $\alpha \le \beta'$. Then $\overline{s}$ is lifting.

Let $\beta \neq \bot^L$ and assume that for all $\alpha \in J(L)$ we have $\alpha \leq \beta'$. Then $\top^L = \bigvee\{\alpha \in J(L) \,:\, \alpha \leq \top^L\} \leq \beta'$ and hence $\beta \leq (\top^L)' = \bot^L$, a contradiction.

Theorem 6.9. *Let the increasing family of two-valued stratification mappings be lifting. Then the functor E is injective on objects.*

Proof. Let $(X, \tau_1), (X, \tau_2) \in |eLM\text{-}FTOP|$ and let $\tau_1 \neq \tau_2$. Then there is $a \in L^X$ such that $\tau_1(a) \neq \tau_2(a)$. In particular, $a \neq \bot^L_X$. We fix $x \in X$ such that $a(x) > \bot^L$ and $\alpha \in N$ such that $s_\alpha(a(x)) = \top^M$. We may assume $\tau_1(a) \not\leq \tau_2(a)$. We then have $\mathcal{U}^{\tau_2,x}_\alpha(a) \geq \tau_2(a)$ and $\mathcal{U}^{\tau_1,x}_\alpha(a) \geq \tau_1(a)$. Hence $\mathcal{U}^{\tau_1,x}_\alpha(a) \not\leq \mathcal{U}^{\tau_2,x}_\alpha(a)$ and we conclude that $x \in q^{\tau_2}_\alpha(\mathcal{U}^{\tau_2,x}_\alpha)$ but $x \notin q^{\tau_1}_\alpha(\mathcal{U}^{\tau_2,x}_\alpha)$. □

Corollary 6.10. *Let $\overline{s} = (s_\alpha)_{\alpha \in N}$ be an increasing family of two-valued stratification mappings that is lifting. Then $eLM\text{-}FTOP$ can be embedded into $\overline{s}LMN\text{-}CTS$ as a subcategory.*

We are now going to construct a functor $F : \overline{s}LMN\text{-}CTS \longrightarrow eLM\text{-}FTOP$. In order to achieve this goal, we have to impose a further condition on the stratification mappings. We call a stratification mapping $s : L \longrightarrow M$ *join-preserving* if for all $\beta_i \in L$, $i \in J$, we have $s(\bigvee_{i\in J} \beta_i) = \bigvee_{i \in J} s(\beta_i)$. It is not difficult to see that in the situation of Example 6.8, for each $\alpha \in J(L)$, the stratification mapping s_α is join-preserving.

For $(X, \overline{q}) \in |\overline{s}LMN\text{-}CTS|$ we define $\tau^{\overline{q}} : L^X \longrightarrow M$ by

$$\tau^{\overline{q}}(a) = \bigwedge_{x,\alpha : s_\alpha(a(x)) = \top^M} \; \bigwedge_{\mathcal{F} : x \in q_\alpha(\mathcal{F})} \mathcal{F}(a).$$

Theorem 6.11. *Let $\overline{s} = (s_\alpha)_{\alpha \in N}$ be an increasing family of two-valued join-preserving stratification mappings and let $(X, \overline{q}), (X', \overline{q'}) \in |\overline{s}LMN\text{-}CTS,|$. Then $(X, \tau^{\overline{q}}) \in |eLM\text{-}FTOP|$. Moreover, if $f : (X, \overline{q}) \longrightarrow (X', \overline{q'})$ is continuous, then also $f : (X, \tau^{\overline{q}}) \longrightarrow (X', \tau^{\overline{q'}})$ is continuous.*

Proof. We first show that $\tau^{\overline{q}}$ is an enriched LM-fuzzy topology. Let $\beta \in L$. Then

$$\tau^{\overline{q}}(\beta_X) = \bigwedge_{s_\alpha(\beta) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(\beta_X) \geq \bigwedge_{s_\alpha(\beta) = \top^M} s_\alpha(\beta) = \top^M.$$

For $a, b \in L^X$ we further have

$$\begin{aligned}
\tau^{\overline{q}}(a \wedge b) &= \bigwedge_{s_\alpha(a \wedge b(x)) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(a \wedge b) \\
&= \bigwedge_{s_\alpha(a(x)) \wedge s_\alpha(b(x)) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(a) \wedge \mathcal{F}(b) \\
&\geq \bigwedge_{s_\alpha(a(x)) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(a) \wedge \bigwedge_{s_\alpha(b(x)) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(b) \; = \; \tau^{\overline{q}}(a) \wedge \tau^{\overline{q}}(b).
\end{aligned}$$

Finally, using that the family $\overline{s}$ is join-preserving, we obtain for $a_i \in L^X$, $i \in J$,

$$\begin{aligned}
\tau^{\overline{q}}(\bigvee_{i \in J} a_i) &= \bigwedge_{s_\alpha(\bigvee_{i \in J} a_i(x)) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(\bigvee_{i \in J} a_i) \\
&\geq \bigwedge_{i \in J} \; \bigwedge_{s_\alpha(a_i(x)) = \top^M} \; \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(a_i) \; = \; \bigwedge_{i \in J} \tau^{\overline{q}}(a_i).
\end{aligned}$$

To show the preservation of continuity, let $b \in L^{X'}$. Then

$$\begin{aligned}
\tau^{\overline{q}}(b \circ f) &= \bigwedge_{s_\alpha(b \circ f(x)) = \top^M} \bigwedge_{x \in q_\alpha(\mathcal{F})} \mathcal{F}(b \circ f) \\
&= \bigwedge_{s_\alpha(b(f(x))) = \top^M} \bigwedge_{x \in q_\alpha(\mathcal{F})} f(\mathcal{F})(b) \\
&\geq \bigwedge_{s_\alpha(b(f(x))) = \top^M} \bigwedge_{f(x) \in q'_\alpha(f(\mathcal{F}))} f(\mathcal{F})(b) \\
&\geq \bigwedge_{s_\alpha(b(y)) = \top^M} \bigwedge_{y \in q'_\alpha(\mathcal{G})} \mathcal{G}(b) \ = \ \tau^{\overline{q'}}(b).
\end{aligned}$$

□

As a consequence, we can define a functor $F : \overline{s}LMN\text{-}CTS \longrightarrow eLM\text{-}FTOP$ by $F((X, \overline{q})) = (X, \tau^{\overline{q}})$ and $F(f) = f$.

Theorem 6.12. *Let $\overline{s} = (s_\alpha)_{\alpha \in N}$ be an increasing family of two-valued join-preserving stratification mappings and consider the functors*

$$E : eLM\text{-}FTOP \longrightarrow \overline{s}LMN\text{-}CTS \quad \text{and} \quad F : \overline{s}LMN\text{-}CTS \longrightarrow eLM\text{-}FTOP$$

defined above. Then the pair (E, F) is a Galois correspondance, i.e. we have $E \circ F \leq id_{\overline{s}LMN\text{-}CTS}$ and $F \circ E \geq id_{eLM\text{-}FTOP}$.

Proof. We first show that the identity mapping $id_X : (X, \overline{q}) \longrightarrow (X, \overline{q^{\tau^{\overline{q}}}})$ is continuous. Let $x \in q_\alpha(\mathcal{F})$. Then

$$\begin{aligned}
\mathcal{U}_\alpha^{\tau^{\overline{q}}, x}(a) &= \bigvee_{b \leq a, s_\alpha(b(x)) = \top^M} \tau^{\overline{q}}(b) \\
&= \bigvee_{b \leq a, s_\alpha(b(x)) = \top^M} \bigwedge_{s_\beta(b(y)) = \top^M} \bigwedge_{y \in q_\alpha(\mathcal{G})} \mathcal{G}(b) \\
&\leq \bigvee_{b \leq a, s_\alpha(b(x)) = \top^M} \bigwedge_{x \in q_\alpha(\mathcal{G})} \mathcal{G}(b) \\
&\leq \bigwedge_{x \in q_\alpha(\mathcal{G})} \mathcal{G}(a) \ \leq \ \mathcal{F}(a).
\end{aligned}$$

Hence $x \in q_\alpha^{\tau^{\overline{q}}}(\mathcal{F})$.

Next we show that $id_X : (X, \tau^{\overline{q^\tau}}) \longrightarrow (X, \tau)$ is continuous. Let $a \in L^X$. Then we have

$$\begin{aligned}
\tau^{\overline{q^\tau}}(a) &= \bigwedge_{s_\alpha(a(x)) = \top^M} \bigwedge_{x \in q_\alpha^\tau(\mathcal{F})} \mathcal{F}(a) \\
&= \bigwedge_{s_\alpha(a(x)) = \top^M} \bigwedge_{\mathcal{F} \geq \mathcal{U}_\alpha^{\tau, x}} \mathcal{F}(a) \\
&= \bigwedge_{s_\alpha(a(x)) = \top^M} \mathcal{U}_\alpha^{\tau, x}(a) \\
&= \bigwedge_{s_\alpha(a(x)) = \top^M} \bigvee_{b \leq a, s_\alpha(b(x)) = \top^M} \tau(b) \ \geq \ \tau(a).
\end{aligned}$$

□

7. Conclusions

In this paper, slightly generalizing a previous definition [8], we provided a general framework for studying various stratified lattice-valued convergence spaces. It appears that unless very special reasons and properties require to consider one particular of the examples contained in this general framework, none of these should be studied separately and suitable notions like separation axioms or compactness or subcategories like pretopological or topological convergence spaces or supercategories like semi-uniform convergence spaces should be studied within this general framework rather than re-doing the same work over and over again for each special case.

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