

QUANTALE-VALUED GAUGE SPACES

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ABSTRACT. We introduce a quantale-valued generalization of approach spaces in terms of quantale-valued gauges. The resulting category is shown to be topological and to possess an initially dense object. Moreover we show that the category of quantale-valued approach spaces defined recently in terms of quantale-valued closures is a coreflective subcategory of our category and, for certain choices of the quantale, is even isomorphic to our category. Finally, the category of quantale-valued metric spaces is shown to be coreflectively embedded in our category.

1. Introduction

Approach spaces, introduced in [11, 12, 13], form a common supercategory of topological and metric spaces. Recently, a probabilistic generalization was considered [9]. In a recent paper, from the view point of *monoidal topology* [6] the definitions of an approach space and of a probabilistic approach space were generalized to the quantale-valued case by defining them with the help of quantale-valued closure operators [10]. Choosing $L = [0, \infty]$ with the opposite order and extended addition as quantale operation, one recovers Lowen's approach spaces. If one chooses as quantale the set of *distance distribution functions* $L = \Delta^+$ with a triangle function induced by a left-continuous t-norm as quantale operation, then probabilistic approach spaces are recovered. In [10, 9] furthermore these quantale-valued approach spaces were characterized by certain quantale-valued convergence structures, see also [8].

Classically, there are many different but equivalent ways of defining an approach space. One definition in terms of gauges is of particular interest. Such a gauge is an ideal of quasi-metrics that satisfies a so-called local saturation condition. In this paper, after collecting the lattice background and definitions and results about L -approach spaces and L -metric spaces in the next two sections, in section 4 we generalize this definition, by considering L -gauges, i.e. filters of L -metrics that satisfy a suitable generalization of the saturation condition. We show that the resulting category of L -gauge spaces is topological and has an initially dense object. Furthermore in section 5, following the classical lines of proof, we show that the category of L -approach spaces [9] is isomorphic to a coreflective subcategory of the category of L -gauge spaces. We give a condition on the quantale L which guarantees that both categories are isomorphic and show with two examples that

Received: October 2016; Revised: January 2017; Accepted: April 2017

Key words and phrases: L -gauge space, L -approach space, L -metric space, Continuity space.

we cannot omit this condition. In particular, we show that in the probabilistic case, probabilistic approach spaces and probabilistic gauge spaces are not the same. In the final section 6 we show that the category of L -metric spaces can naturally be embedded into our category as a coreflective subcategory.

2. Preliminaries

We consider in this paper completely distributive lattices, i.e. complete lattices L that satisfy the following distributive laws.

$$(CD1) \quad \bigvee_{j \in J} \left(\bigwedge_{i \in I_j} \alpha_{ji} \right) = \bigwedge_{f \in \prod_{j \in J} I_j} \left(\bigvee_{j \in J} \alpha_{jf(j)} \right),$$

$$(CD2) \quad \bigwedge_{j \in J} \left(\bigvee_{i \in I_j} \alpha_{ji} \right) = \bigvee_{f \in \prod_{j \in J} I_j} \left(\bigwedge_{j \in J} \alpha_{jf(j)} \right).$$

We assume that L is non-trivial in the sense that $\top \neq \perp$ for the top element \top and the bottom element \perp . It is well known that, in any complete lattice L , (CD1) and (CD2) are equivalent. In any complete lattice L we can define the *well-below relation* $\alpha \triangleleft \beta$, α is *well-below* β , if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \triangleleft \beta$ and $\alpha \triangleleft \bigvee_{j \in J} \beta_j$ iff $\alpha \triangleleft \beta_i$ for some $i \in J$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee \{ \beta : \beta \triangleleft \alpha \}$ for any $\alpha \in L$, see e.g. Theorem 7.2.3 in [1]. Similarly, we can define the *well-above relation*, β is *well-above* α , $\alpha \prec \beta$ if for all subsets $D \subseteq L$ such that $\bigwedge D \leq \alpha$ there is $\delta \in D$ with $\delta \leq \beta$. Then $\alpha \prec \beta$ implies $\alpha \leq \beta$ and $\bigwedge_{j \in J} \beta_j \prec \alpha$ iff $\beta_j \prec \alpha$ for some $j \in J$. L is completely distributive iff $\alpha = \bigwedge \{ \beta \in L : \alpha \prec \beta \}$ for any $\alpha \in L$. Clearly, in a complete lattice L we have $\alpha \triangleleft \beta$ iff $\beta \prec^{op} \alpha$ in the opposite order. For more results on lattices we refer to [4].

The triple $(L, \leq, *)$, where (L, \leq) is a complete lattice, is called a *quantale* if $(L, *)$ is a semigroup, and $*$ is distributive over arbitrary joins, i.e.

$$\left(\bigvee_{j \in J} \alpha_j \right) * \beta = \bigvee_{j \in J} (\alpha_j * \beta) \quad \text{and} \quad \beta * \left(\bigvee_{j \in J} \alpha_j \right) = \bigvee_{j \in J} (\beta * \alpha_j).$$

A quantale $(L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if the top element of L acts as the unit, i.e. if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$. In any such quantale we can define an implication $\alpha \rightarrow \beta = \bigvee \{ \gamma \in L : \alpha * \gamma \leq \beta \}$. Then $\alpha * \beta \leq \gamma$ iff $\alpha \leq \beta \rightarrow \gamma$. We give a list of properties of the implication.

Lemma 2.1. [7] *Let $(L, \leq, *)$ be an integral and commutative quantale and let $\alpha, \beta, \gamma, \beta_j \in L$ ($j \in J$).*

- (1) *If $\alpha \leq \beta$ then $\alpha \rightarrow \gamma \geq \beta \rightarrow \gamma$ and $\gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$;*
- (2) *$\alpha \leq (\alpha \rightarrow \beta) \rightarrow \beta$;*
- (3) *$\alpha \rightarrow (\bigwedge_{j \in J} \beta_j) = \bigwedge_{j \in J} (\alpha \rightarrow \beta_j)$;*
- (4) *$(\bigvee_{j \in J} \beta_j) \rightarrow \alpha = \bigwedge_{j \in J} (\beta_j \rightarrow \alpha)$.*

Example 2.2. A *triangular norm* or *t-norm* is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $([0, 1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are:

- the minimum t-norm: $\alpha * \beta = \alpha \wedge \beta$,
- the product t-norm: $\alpha * \beta = \alpha \cdot \beta$,
- the Lukasiewicz t-norm: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

Example 2.3. The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale, see e.g. [3]. In this quantale we have $\alpha \rightarrow \beta = (\beta - \alpha) \vee 0$. Furthermore $\bigvee_{j \in J} (\alpha_j \rightarrow \beta) = (\bigwedge_{j \in J} \alpha_j) \rightarrow \beta$ for all $\alpha_j, \beta \in L$.

Example 2.4. A function $\varphi : [0, \infty] \rightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ in the sense that $\varphi(x) = \bigvee \{\varphi(y) : y < x\}$ for all $x \in (0, \infty)$, and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$ is called a *distance distribution function* [17]. The set of all distance distribution functions is denoted by Δ^+ . For example, for each $0 \leq a < \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

are in Δ^+ . The set Δ^+ is ordered pointwise, i.e. for $\varphi, \psi \in \Delta^+$ we define $\varphi \leq \psi$ if for all $x \geq 0$ we have $\varphi(x) \leq \psi(x)$. The bottom element of Δ^+ is ε_∞ and the top element is ε_0 and the set Δ^+ with this order then becomes a complete lattice. We note that $\bigwedge_{i \in I} \varphi_i$ is in general not the pointwise infimum. It is shown in [3] that this lattice is completely distributive.

A binary operation, $*$: $\Delta^+ \times \Delta^+ \rightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and that satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [15, 16, 17]. A triangle function is called *sup-continuous* [17], if $(\bigvee_{i \in I} \varphi_i) * \psi = \bigvee_{i \in I} (\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+$, ($i \in I$), i.e. if $(\Delta^+, \leq, *)$ is a quantale.

We will later use the triangle function τ_* induced by a t-norm $*$, defined by $\tau_*(\varphi, \psi)(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ for all $x \in [0, \infty]$, see [17].

Example 2.5. A frame is a quantale with $* = \wedge$.

Example 2.6. A commutative and integral quantale $(L, \leq, *)$ which satisfies $(\alpha \rightarrow \beta) \rightarrow \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$ is a *complete MV-algebra* [7]. In a complete MV-algebra we have the properties $\bigwedge_{j \in J} (\alpha * \beta_j) = \alpha * \bigwedge_{j \in J} \beta_j$ and $\bigvee_{j \in J} (\alpha_j \rightarrow \beta) = (\bigwedge_{j \in J} \alpha_j) \rightarrow \beta$ for all $\alpha_j, \beta \in L$.

A *value quantale* [3] is a commutative and integral quantale $(L, \leq, *)$ with an underlying completely distributive lattice (L, \leq) such that $\perp \triangleleft \top$ and $\alpha \vee \beta \triangleleft \top$ whenever $\alpha, \beta \triangleleft \top$. Examples for value quantales are $([0, \infty], \geq, +)$ or $(\Delta^+, \leq, *)$ with a sup-continuous triangle function, see [3]. It should be noted that Flagg [3] uses the opposite order. The following result is shown in [3].

Lemma 2.7. [3] *Let $(L, \leq, *)$ be a value quantale. If $\alpha \triangleleft \top$, then there is $\beta \triangleleft \top$ such that $\alpha \triangleleft \beta * \beta$.*

We will later need the following condition.

Definition 2.8. A quantale $(L, \leq, *)$ satisfies the condition (I) if

(I) for all $\perp \prec \beta$ and all $\gamma \triangleleft \top$ we have $\beta \not\leq \gamma * \beta$.

Lemma 2.9. *If the quantale $(L, \leq, *)$ is integral and satisfies the strong cancellation law*

(SCL) for all $\gamma, \alpha \in L, \perp \prec \beta : \gamma * \beta \leq \alpha * \beta$ implies $\gamma \leq \alpha$

and if $\top \not\triangleleft \top$ then the condition (I) is satisfied.

Proof. Let $\perp \prec \beta$ and $\gamma \triangleleft \top$. If we assume $\beta = \top * \beta \leq \gamma * \beta$, then $\gamma = \top$, a contradiction. \square

Example 2.10. (1) The two-point chain $L = \{0, 1\}$ does not satisfy the condition (I) as $1 \triangleleft 1$.

(2) Let $L = [0, \infty]$ with the opposite order and extended addition as quantale operation. Then the strong cancellation law is valid and hence L satisfies the condition (I).

(3) Let $L = [0, 1]$ and multiplication as quantale operation. Then the strong cancellation law is satisfied and hence L satisfies the condition (I).

(4) A frame (L, \leq, \wedge) does in general not satisfy (I). If $\alpha \geq \beta$, then $\beta = \alpha \wedge \beta$.

(5) The 4-element Boolean algebra $\{\perp, \alpha, \beta, \top\}$ with $\alpha \wedge \beta = \perp$ and $\alpha \vee \beta = \top$ satisfies (I), as $\alpha, \beta \not\leq \alpha \wedge \beta$, but does not satisfy the strong cancellation law since $\alpha \wedge \beta \leq \beta \wedge \beta$ but $\alpha \not\leq \beta$.

(6) In an MV-algebra $(L, \leq, *)$ we have $\beta \leq \alpha * \beta$ iff $\beta \wedge (\alpha \rightarrow \perp) = \perp$. Hence an MV-algebra satisfies (I) if and only if $\beta \wedge (\alpha \rightarrow \perp) \neq \perp$ whenever $\alpha \not\triangleleft \top$ and $\perp \not\prec \beta$. In particular, if L has no zero-divisors for \wedge , then $(L, \leq, *)$ satisfies (I).

(7) As a final example we consider the lattice Δ^+ . For $0 < \delta < \infty$ and $0 < \epsilon \leq 1$ we define $f_{\delta\epsilon} \in \Delta^+$ by

$$f_{\delta\epsilon}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \delta \\ \epsilon & \text{if } \delta < x < \infty \\ 1 & \text{if } x = \infty. \end{cases}$$

The following Lemma is then not difficult to show.

Lemma 2.11. (1) $f_{\delta\epsilon} \leq f_{\delta'\epsilon'} \iff \delta' \leq \delta, \epsilon \leq \epsilon'$;

(2) $f_{\delta\epsilon} \triangleleft f_{\delta'\epsilon'} \iff \delta' < \delta, \epsilon < \epsilon'$;

(3) $f_{\delta\epsilon} \triangleleft \varphi \iff \epsilon < \varphi(\delta)$;

(4) $\varphi = \bigvee \{f_{\delta\epsilon} : f_{\delta\epsilon} \triangleleft \varphi\}$ for all $\varphi \in \Delta^+$;

(5) If $\varphi \triangleleft \epsilon_0$ then there is $\epsilon < 1$ such that $\varphi \leq f_{\delta\epsilon}$.

As a consequence, we can show the following result.

Lemma 2.12. *Let $*$ be a t -norm on $[0, 1]$ that satisfies the property (I), i.e. $0 < \beta$ and $\epsilon < 1$ implies $\epsilon * \beta < \beta$. Then (Δ^+, \leq, τ_*) satisfies the condition (I).*

Proof. We first note that in Δ^+ we have $\varepsilon_\infty \not\leq \varepsilon_0$, because $\bigwedge\{\varepsilon_a : a > 0\} = \varepsilon_\infty$ but there is no $a > 0$ such that $\varepsilon_a = \varepsilon_\infty$. Let now $\varepsilon_\infty \prec \psi$, then there is $x \in [0, \infty)$ such that $\psi(x) > 0$. If furthermore $\varphi \triangleleft \varepsilon_0$, then there is $\epsilon < 1$ such that $\varphi \leq f_{\delta\epsilon}$. Hence we conclude

$$\begin{aligned} \tau_*(\varphi, \psi)(x) &= \bigvee_u \varphi(u) * \psi(x-u) \leq \bigvee_u f_{\delta\epsilon}(u) * \psi(x-u) \\ &= \bigvee_{u>\delta} \epsilon * \psi(x-u) = \epsilon * \bigvee_{u>\delta} \psi(x-u) \leq \epsilon * \psi(x). \end{aligned}$$

So if $\psi \leq \tau_*(\varphi, \psi)$, then $\psi(x) \leq \epsilon * \psi(x)$, a contradiction. \square

We will consider in this paper only commutative, integral quantales $(L, \leq, *)$ with completely distributive underlying lattices.

We assume some familiarity with category theory and refer to the textbooks [2] and [14] for more details and notation. A *construct* is a category \mathcal{C} with a faithful functor $U : \mathcal{C} \rightarrow SET$, from \mathcal{C} to the category of sets. We always consider a construct as a category whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : S \rightarrow (S_i, \xi_i))_{i \in I}$ there is a unique structure ξ on S , such that a mapping $g : (T, \eta) \rightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (T, \eta) \rightarrow (S_i, \xi_i)$ is a morphism. We call such a source an *initial source*. An object (S, ξ) in a category \mathcal{C} is called *initially dense in \mathcal{C}* if for any object (T, η) in \mathcal{C} there is an initial source $(f_i : (T, \eta) \rightarrow (S, \xi))_{i \in I}$.

3. L -approach Spaces and L -metric Spaces

In the sequel, let $L = (L, \leq, *)$ be a commutative and integral quantale, where (L, \leq) is completely distributive. For a set X we denote its power set by $P(X)$.

Definition 3.1. [10] An *L -approach space* is a pair (X, c) of a set and a closure operator $c : P(X) \rightarrow L^X$ satisfying, for all $x \in X$, $A, B, A_j \subseteq X$ ($j \in J$), the axioms

$$(LC1) \quad c(\{x\})(x) = \top;$$

$$(LC2) \quad \left(\bigwedge_{y \in B} \bigvee_{j \in J} c(A_j)(y) \right) * c(B)(x) \leq c\left(\bigcup_{j \in J} A_j\right)(x);$$

$$(LC3) \quad c(\emptyset)(x) = \perp;$$

$$(LC4) \quad c(A \cup B) = c(A) \vee c(B).$$

A mapping $f : (X, c) \rightarrow (X', c')$ between two L -approach spaces is called an *L -approach morphism* if $c(A)(x) \leq c'(f(A))(f(x))$ for all $x \in X$ and all $A \subseteq X$. The category with objects the L -approach spaces and morphisms the L -approach morphisms is denoted by $L\text{-AP}$.

Clearly, a closure operator $c : P(X) \rightarrow L^X$ can equivalently be described by an *L -valued point-set distance function* $\delta : X \times P(X) \rightarrow L$, writing $\delta(x, A) = c(A)(x)$. With this in mind, we can give the following characterization, which is more closely related to Lowen's original definition [11].

Lemma 3.2. *A pair (X, δ) with a set X and an L -distance $\delta : X \times P(X) \longrightarrow L$ is an L -approach space if, for all $x \in X$, $A, B \subseteq X$, the following axioms are satisfied.*

$$(LD1) \quad \delta(x, \{x\}) = \top;$$

$$(LD2) \quad \delta(x, \emptyset) = \perp;$$

$$(LD3) \quad \delta(x, A) \vee \delta(x, B) = \delta(x, A \cup B) \text{ for all } A, B \subseteq X;$$

$$(LD4) \quad \delta(x, A) \geq \delta(x, \overline{A}^\alpha) * \alpha \text{ for all } \alpha \in L, \text{ where } \overline{A}^\alpha = \{x \in X : \delta(x, A) \geq \alpha\}.$$

A mapping $f : (X, \delta) \longrightarrow (X', \delta')$ is an L -approach morphism if and only if $\delta(x, A) \leq \delta'(f(x), f(A))$ for all $x \in X, A \subseteq X$.

Proof. We need only show that (LD4) and (LC2) are equivalent. Let first (LD4) be satisfied. We define $\alpha = \bigwedge_{y \in B} \bigvee_{j \in J} \delta(y, A_j)$ and show that $B \subseteq \bigcup_{j \in J} \overline{A_j}^\alpha$. For $y \in B$ we have, as a consequence of (LD3), $\bigvee_{j \in J} \delta(y, A_j) \leq \delta(y, \bigcup_{j \in J} A_j)$ and hence also $\alpha = \bigwedge_{z \in B} \bigvee_{j \in J} \delta(z, A_j) \leq \delta(y, \bigcup_{j \in J} A_j)$. Hence $y \in \overline{\bigcup_{j \in J} A_j}^\alpha$. We conclude $\alpha * \delta(x, B) \leq \alpha * \delta(x, \overline{\bigcup_{j \in J} A_j}^\alpha) \leq \delta(x, \bigcup_{j \in J} A_j)$ by (LD4), which is (LC2).

The converse follows taking $A_j = A$ and $B = \overline{A}^\alpha$. Then $\bigwedge_{y \in B} \delta(y, A) \geq \alpha$ and $\alpha * \delta(x, \overline{A}^\alpha) \leq (\bigwedge_{y \in B} \delta(y, A)) * \delta(x, B) \leq \delta(x, A)$. \square

We give a further characterization of (LD4).

Lemma 3.3. *Let $(X, \delta) \in |L\text{-AP}|$. Then (LD4) is equivalent to*

$$(LD4') \quad \delta(x, B) * \bigwedge_{b \in B} \delta(b, A) \leq \delta(x, A) \text{ for all } A, B \subseteq X \text{ and all } x \in X.$$

Proof. Let first (LD4) be true. We define $\alpha = \bigvee \{\gamma \in L : B \subseteq \overline{A}^\gamma\}$. Then $x \in \overline{A}^\alpha$ iff $\delta(x, A) \geq \gamma$ for all $\gamma \in L$ such that $B \subseteq \overline{A}^\gamma$, i.e. iff $x \in \bigcap_{\gamma: B \subseteq \overline{A}^\gamma} \overline{A}^\gamma \supseteq B$. Moreover, we have $B \subseteq \overline{A}^\gamma$ iff $\bigwedge_{b \in B} \delta(b, A) \geq \gamma$. Hence $\alpha = \bigvee \{\gamma \in L : \gamma \leq \bigwedge_{b \in B} \delta(b, A)\} = \bigwedge_{b \in B} \delta(b, A)$ and we conclude from (LD4) $\delta(x, A) \geq \delta(x, \overline{A}^\alpha) * \alpha \geq \delta(x, B) * \bigwedge_{b \in B} \delta(b, A)$. For the converse, we take $B = \overline{A}^\alpha$. Then $\bigwedge_{b \in B} \delta(b, A) \geq \alpha$ and we conclude $\delta(x, A) \geq \bigwedge_{b \in B} \delta(b, A) * \delta(x, B) \geq \alpha * \delta(x, \overline{A}^\alpha)$, which is (LD4). \square

Definition 3.4. An L -metric space is a pair (X, d) of a set X and an L -metric $d : X \times X \longrightarrow L$ which satisfies the following properties.

$$(LM1) \quad d(x, x) = \top \text{ for all } x \in X \text{ (reflexivity), and}$$

$$(LM2) \quad d(x, y) * d(y, z) \leq d(x, z) \text{ for all } x, y, z \in X \text{ (transitivity).}$$

A mapping between two L -metric spaces, $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called an L -metric morphism if $d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

We denote the category of L -metric spaces with L -metric morphisms by $L\text{-MET}$. We further denote the fibre over X in $L\text{-MET}$ by $L\text{-MET}(X)$. We note that for $d_j \in L\text{-MET}(X)$ ($j \in J$), we have that the pointwise infimum $\bigwedge_{j \in J} d_j \in L\text{-MET}(X)$. As also there is a largest L -metric on X , namely $d(x, y) = \top$ for all $x, y \in X$, the set $L\text{-MET}(X)$ is a complete lattice.

In case $L = \{0, 1\}$, an L -metric space is a preordered set. If $L = [0, \infty]$ with the opposite order and extended addition as quantale operation, an L -metric space is a quasimetric space. If $L = \Delta^+$ and $*$ is a sup-continuous triangle function, an L -metric space is a probabilistic quasimetric space, see [3].

For a value quantale $(L, \leq, *)$, L -metric spaces were introduced under the name *continuity spaces* and L -metric morphisms were called *nonexpansive*, a name which has its justification if one uses the opposite order, in [3]. Often, L -metric spaces are called *L -categories*, e.g. [6, 19], or *L -preordered sets*, see e.g. [18]. Our main examples being quasimetric spaces and probabilistic (quasi-)metric spaces and because we generalize approach spaces, the theory of which has a strong metrical flavour, we prefer to use the term L -metric space.

Example 3.5. An integral quantale $(L, \leq, *)$ becomes an L -metric space if we define, for $\alpha \in L$, $d_\alpha(x, y) = (\alpha \wedge x) \rightarrow (\alpha \wedge y)$, $(x, y \in L)$. In fact, $d_\alpha(x, x) = (\alpha \wedge x) \rightarrow (\alpha \wedge x) = \top$ and $d_\alpha(x, y) * d_\alpha(y, z) = ((\alpha \wedge x) \rightarrow (\alpha \wedge y)) * ((\alpha \wedge y) \rightarrow (\alpha \wedge z)) \leq (\alpha \wedge x) \rightarrow (\alpha \wedge z) = d_\alpha(x, z)$.

Lemma 3.6. Let X be a set and let (X', d') be an L -metric space and let $f : X \rightarrow X'$. Define $d_f(x, y) = d'(f(x), f(y))$ for all $x, y \in X$, i.e. $d_f = d' \circ (f \times f)$. Then (X, d_f) is an L -metric space.

Proof. The proof is straightforward and left for the reader. \square

We note that for $f : X \rightarrow X'$ and $g : X' \rightarrow X''$ and (X'', d'') an L -metric space, we have $d_{g \circ f} = (d_g)_f$.

An L -distance $\delta : X \times P(X) \rightarrow L$ generates in a natural way an L -metric. This L -metric will be useful later.

Lemma 3.7. Let $\delta : X \times P(X) \rightarrow L$ be an L -distance and let $Z \subseteq X$. Then $d_Z(x, y) = \delta(y, Z) \rightarrow \delta(x, Z)$ is an L -metric.

Furthermore, if L satisfies $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$ ($j \in J$), then for any $A \subseteq X$ we have $\delta(x, A) \leq \bigvee_{a \in A} d_Z(x, a)$.

Proof. We have $d_Z(x, x) = \delta(x, Z) \rightarrow \delta(x, Z) = \top$ and $d(x, y) * d(y, z) = (\delta(y, Z) \rightarrow \delta(x, Z)) * (\delta(z, Z) \rightarrow \delta(y, Z)) \leq \delta(z, Z) \rightarrow \delta(x, Z) = d_Z(x, z)$. Hence d is an L -metric on X . Furthermore, from Lemma 3.3 we obtain $\delta(x, A) * \bigwedge_{a \in A} \delta(a, Z) \leq \delta(x, Z)$. Using the condition in the lemma, we obtain $\delta(x, A) \leq (\bigwedge_{a \in A} \delta(a, Z)) \rightarrow \delta(x, Z) = \bigvee_{a \in A} (\delta(a, Z) \rightarrow \delta(x, Z)) = \bigvee_{a \in A} d_Z(x, a)$. \square

We have noted above that e.g. the interval $[0, \infty]$ with the opposite order and extended addition as quantale operation, as well as complete MV-algebras satisfy the condition stated in the lemma.

Finally we are showing that the category L -MET can nicely be embedded into the category L -AP.

Theorem 3.8. L -MET can be embedded into L -AP as a coreflective subcategory.

Proof. Let (X, d) be an L -metric space. We define for $x \in X$ and $A \subseteq X$

$$\delta^d(x, A) = \bigvee_{a \in A} d(x, a).$$

Then (X, δ^d) is an L -approach space. (LD1), (LD2) and (LD3) are easy and left for the reader. We only provide a proof for (LD4). If $y \in \overline{A}^\alpha$, then $\alpha \leq$

$\delta^d(y, A) = \bigvee_{a \in A} d(y, a)$. Hence $\alpha * \delta^d(x, \bar{A}^\alpha) = \alpha * \bigvee_{y \in \bar{A}^\alpha} d(x, y) \leq \bigvee_{a \in A} d(y, a) * \bigvee_{y \in \bar{A}^\alpha} d(x, y) = \bigvee_{y \in \bar{A}^\alpha} \bigvee_{a \in A} d(x, y) * d(y, a) \leq \bigvee_{y \in \bar{A}^\alpha} \bigvee_{a \in A} d(x, a) = \delta^d(x, A)$.

Furthermore, let $(X, d_X), (Y, d_Y) \in |L-MET|$ and let $f : X \rightarrow Y$. Then $f : (X, d_X) \rightarrow (Y, d_Y)$ is an L -metric morphism if and only if $f : (X, \delta^{d_X}) \rightarrow (Y, \delta^{d_Y})$ is an L -approach morphism. If $f : (X, d_X) \rightarrow (Y, d_Y)$ is an L -metric morphism, then for $x \in X$ and $A \subseteq X$ we have $\delta^{d_X}(x, A) = \bigvee_{a \in A} d_X(x, a) \leq \bigvee_{a \in A} d_Y(f(x), f(a)) \leq \bigvee_{b \in f(A)} d_Y(f(x), b) = \delta^{d_Y}(f(x), f(A))$. Hence $f : (X, \delta^{d_X}) \rightarrow (Y, \delta^{d_Y})$ is an L -approach morphism. The converse is obvious using $d(x, y) = \delta^d(x, \{y\})$.

We note that if $(X, d) \neq (X, d')$ for two L -metric spaces, then there are $x, y \in X$ such that $\delta^d(x, \{y\}) = d(x, y) \neq d'(x, y) = \delta^{d'}(x, \{y\})$, i.e. $(X, \delta^d) \neq (X, \delta^{d'})$. Thus the functor

$$G : \begin{cases} L-MET & \rightarrow & L-AP \\ (X, d) & \mapsto & (X, \delta^d) \\ f & \mapsto & f \end{cases}$$

is an embedding functor.

We define now for $(X, \delta) \in |L-AP|$

$$d^\delta(x, y) = \delta(x, \{y\}).$$

Then $(X, d^\delta) \in |L-MET|$. We have $d^\delta(x, x) = \delta(x, \{x\}) = \top$ for all $x \in X$. Furthermore, by (LD1), we have $y \in \overline{\{y\}}^{\delta(y, \{z\})}$ and hence with (LD4) $d^\delta(x, y) * d^\delta(y, z) \leq \delta(x, \overline{\{y\}}^{\delta(y, \{z\})}) * \delta(y, \{z\}) \leq \delta(x, \{y\}) = d^\delta(x, y)$.

It is furthermore not difficult to see that for an L -approach morphism $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$, $f : (X, d^{\delta_X}) \rightarrow (Y, d^{\delta_Y})$ is an L -metric morphism and that we have for $(X, \delta) \in |L-AP|$ that $\delta^{(d^\delta)}(x, A) \leq \delta(x, A)$ and for $(X, d) \in |L-MET|$ we have $d^{(\delta^d)}(x, y) = d(x, y)$. From this the claim follows. \square

4. The Category of L -gauge Spaces

Definition 4.1. Let $\mathcal{H} \subseteq L-MET(X)$ and $d \in L-MET(X)$.

- (1) d is called *locally supported by \mathcal{H}* if for all $x \in X$, $\alpha \triangleleft \top$, $\perp \prec \omega$ there is $e_x^{\alpha, \omega} \in \mathcal{H}$ such that $e_x^{\alpha, \omega}(x, \cdot) * \alpha \leq d(x, \cdot) \vee \omega$;
- (2) \mathcal{H} is called *locally directed* if for all finite subsets $\mathcal{H}_0 \subseteq \mathcal{H}$, $\bigwedge_{d \in \mathcal{H}_0} d$ is locally supported by \mathcal{H} ;
- (3) \mathcal{H} is called *locally saturated* if for $d \in L-MET(X)$ we have $d \in \mathcal{H}$ whenever d is locally supported by \mathcal{H} .
- (4) The set

$$\widehat{\mathcal{H}} = \{d \in L-MET(X) : d \text{ is locally supported by } \mathcal{H}\}$$

is called the *local saturation of \mathcal{H}* .

For $L = [0, \infty]$ and the opposite order, Lowen [11, 12, 13] calls a locally supporting family (*locally dominating*). This expression seems not suitable in our setting why we chose a new term.

We give two characterizations of local support.

Lemma 4.2. *Let $\mathcal{H} \subseteq L\text{-MET}(X)$ and $d \in L\text{-MET}(X)$. Then d is locally supported by \mathcal{H} iff $\bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee_{e \in \mathcal{H}} (e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega)) = \top$.*

Proof. Let first d be locally supported by \mathcal{H} . Then for $x \in X$, $\alpha \triangleleft \top$ and $\perp \prec \omega$ there is $e \in \mathcal{H}$ such that $\alpha \leq e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega)$. Hence, for all $\alpha \triangleleft \top$ we have $\alpha \leq \bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee_{e \in \mathcal{H}} (e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega))$ from which $\top = \bigvee_{\alpha \triangleleft \top} \alpha \leq \bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee_{e \in \mathcal{H}} (e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega))$ follows.

Conversely, let $\bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee_{e \in \mathcal{H}} (e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega)) = \top$. Then for all $x \in X$ and all $\perp \prec \omega$ we have $\bigvee_{e \in \mathcal{H}} (e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega)) = \top$. Hence, for $\alpha \triangleleft \top$, there is $e \in \mathcal{H}$ such that $e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega) \geq \alpha$ and this means that d is locally supported by \mathcal{H} . \square

For the following characterization, we define for a subset $\mathcal{H} \subset L\text{-MET}(X)$ and for $x \in X$, the set $\mathcal{H}(x) = \{f : X \rightarrow L : f(\cdot) \geq d(x, \cdot), d \in \mathcal{H}\}$. The idea of this result goes back to [5].

Lemma 4.3. *Let $\mathcal{H} \subseteq L\text{-MET}(X)$ and $d \in L\text{-MET}(X)$. Then d is locally supported by \mathcal{H} iff $\bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee \{\alpha \in L : \alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$.*

Proof. Let first d be locally supported by \mathcal{H} . Then for all $x \in X$, $\alpha \triangleleft \top$, $\perp \prec \omega$ there is $e \in \mathcal{H}$ such that $e(x, \cdot) \leq \alpha \rightarrow (d(x, \cdot) \vee \omega)$. Therefore $\alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)$ and we have $\bigvee \{\alpha \in L : \alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)\} \geq \bigvee_{\alpha \triangleleft \top} \alpha = \top$. This is true for all $x \in X$ and all $\perp \prec \omega$ and hence $\bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee \{\alpha \in L : \alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$.

Let now the condition of the Lemma be true. Then for all $x \in X$ and all $\perp \prec \omega$ we have $\bigvee \{\alpha \in L : \alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$. Let $\alpha \triangleleft \top$. Then there is $\beta \geq \alpha$ such that $\beta \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)$ and because the set $\mathcal{H}(x)$ is an upper set, we find $\alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)$. Hence there is $e \in \mathcal{H}$ such that $e(x, \cdot) \leq \alpha \rightarrow (d(x, \cdot) \vee \omega)$ and this means that d is locally supported by \mathcal{H} . \square

Corollary 4.4. *Let $\mathcal{H} \subseteq L\text{-MET}(X)$. The following are equivalent.*

- (1) \mathcal{H} is locally saturated.
- (2) $\bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee_{e \in \mathcal{H}} (e(x, \cdot) \rightarrow (d(x, \cdot) \vee \omega)) = \top$ implies $d \in \mathcal{H}$.
- (3) $\bigwedge_{x \in X} \bigwedge_{\perp \prec \omega} \bigvee \{\alpha \in L : \alpha \rightarrow (d(x, \cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$ implies $d \in \mathcal{H}$.

Definition 4.5. Let X be a set. $\mathcal{G} \subseteq L\text{-MET}(X)$ is called an *L-gauge* if \mathcal{G} is a filter in $L\text{-MET}(X)$ and \mathcal{G} is locally saturated. In particular, an *L-gauge* satisfies the axioms

- (LG1) $\mathcal{G} \neq \emptyset$;
- (LG2) $d \in \mathcal{G}$ and $d \leq e$ implies $e \in \mathcal{G}$;
- (LG3) $d, e \in \mathcal{G}$ implies $d \wedge e \in \mathcal{G}$;
- (LG4) \mathcal{G} is locally saturated.

The pair (X, \mathcal{G}) is then called an *L-gauge space*. A mapping between two *L-gauge spaces*, $f : (X, \mathcal{G}) \rightarrow (X', \mathcal{G}')$ is called an *L-gauge morphism* if $d' \circ (f \times f) \in \mathcal{G}$ whenever $d' \in \mathcal{G}'$.

It is not difficult to show that the class of *L-gauge spaces* together with the *L-gauge morphisms* forms a category which shall be denoted *L-GS*.

In case that the quantale L is the interval $[0, \infty]$ with the opposite order and extended addition as quantale operation, then $[0, \infty]$ -gauge spaces are approach spaces defined by means of gauges, [13]. We will study the relation of L -approach spaces and L -gauge spaces in the next section.

Definition 4.6. Let $(X, \mathcal{G}) \in |L\text{-GS}|$ and let $\mathcal{H} \subseteq L\text{-MET}(X)$. If $\widehat{\mathcal{H}} = \mathcal{G}$, then \mathcal{H} is called a *basis for the gauge* \mathcal{G} .

Proposition 4.7. Let L be a value quantale. If $\emptyset \neq \mathcal{H} \subseteq L\text{-MET}(X)$ is locally directed, then $\mathcal{G} = \widehat{\mathcal{H}}$ is a gauge with \mathcal{H} as basis.

Proof. Clearly $\mathcal{H} \subseteq \widehat{\mathcal{H}}$, so that $\mathcal{G} \neq \emptyset$. If $d \in \widehat{\mathcal{H}}$ and $d \leq e$, then for $x \in X$, $\alpha \triangleleft \top$, $\perp \prec \omega$, there is $e_x^{\alpha, \omega} \in \mathcal{H}$ such that $e_x^{\alpha, \omega}(x, \cdot) * \alpha \leq d(x, \cdot) \vee \omega \leq e(x, \cdot) \vee \omega$. Hence e is locally supported by \mathcal{H} and $e \in \widehat{\mathcal{H}}$. Let now $d, e \in \widehat{\mathcal{H}}$. We fix $x \in X$, $\alpha \triangleleft \top$ and $\perp \prec \omega$. Then there is $\beta \triangleleft \perp$ such that $\alpha \triangleleft \beta * \beta$ and hence there are $d_x^{\beta, \omega}, e_x^{\beta, \omega} \in \mathcal{H}$ such that $d_x^{\beta, \omega}(x, \cdot) * \beta \leq d(x, \cdot) \vee \omega$ and $e_x^{\beta, \omega}(x, \cdot) * \beta \leq e(x, \cdot) \vee \omega$. By local directedness then $d_x^{\beta, \omega} \wedge e_x^{\beta, \omega}$ is locally supported by \mathcal{H} and hence there is $f_x^{\beta, \omega} \in \mathcal{H}$ such that $f_x^{\beta, \omega}(x, \cdot) * \beta \leq d_x^{\beta, \omega} \wedge e_x^{\beta, \omega}(x, \cdot) \vee \omega$. We conclude

$$\begin{aligned} f_x^{\beta, \omega}(x, \cdot) * \alpha &\leq f_x^{\beta, \omega}(x, \cdot) * \beta * \beta \leq ((d_x^{\beta, \omega} \wedge e_x^{\beta, \omega}(x, \cdot)) * \beta) \vee (\omega \vee \beta) \\ &\leq ((d_x^{\beta, \omega}(x, \cdot) * \beta) \wedge (e_x^{\beta, \omega}(x, \cdot) * \beta)) \vee \omega \\ &\leq ((d(x, \cdot) \vee \omega) \wedge (e(x, \cdot) \vee \omega)) \vee \omega \leq (d \wedge e)(x, \cdot) \vee \omega. \end{aligned}$$

Hence $d \wedge e$ is locally supported by \mathcal{H} , i.e. $d \wedge e \in \widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}$ is a filter.

We finally show that $\widehat{\mathcal{H}}$ is locally saturated. Let $d \in L\text{-MET}(X)$ be locally supported by $\widehat{\mathcal{H}}$ and let $x \in X$, $\alpha \triangleleft \top$ and $\perp \prec \omega$. There is $\beta \triangleleft \top$ such that $\alpha \triangleleft \beta * \beta$ and hence there is $e_x^{\beta, \omega} \in \widehat{\mathcal{H}}$ such that $e_x^{\beta, \omega}(x, \cdot) * \beta \leq d(x, \cdot) \vee \omega$. As $e_x^{\beta, \omega}$ is locally supported by \mathcal{H} there is $f_x^{\beta, \omega} \in \mathcal{H}$ such that $f_x^{\beta, \omega}(x, \cdot) * \beta \leq e_x^{\beta, \omega}(x, \cdot) \vee \omega$ and we conclude

$$f_x^{\beta, \omega}(x, \cdot) * \alpha \leq f_x^{\beta, \omega}(x, \cdot) * \beta * \beta \leq (e_x^{\beta, \omega}(x, \cdot) \vee \omega) * \beta \leq (e_x^{\beta, \omega}(x, \cdot) * \beta) \vee \omega \leq d(x, \cdot) \vee \omega.$$

Hence d is locally supported by \mathcal{H} , i.e. $d \in \widehat{\mathcal{H}}$. \square

Theorem 4.8. Let L be a value quantale. Then the category $L\text{-GS}$ is topological over SET .

Proof. Let $f_j : X \rightarrow X_j$ ($j \in J$) be a family of mappings and let $(X_j, \mathcal{G}_j) \in |L\text{-GS}|$. We define

$$\mathcal{H} = \left\{ \bigwedge_{j \in K} d_j \circ (f_j \times f_j) : K \subseteq J \text{ finite}, d_j \in \mathcal{G}_j \forall j \in J \right\}.$$

Clearly \mathcal{H} is locally directed, as finite meets of members of \mathcal{H} belong to \mathcal{H} . For $d_j \in \mathcal{G}_j$ we have $d_j \circ (f_j \times f_j) \in \mathcal{H} \subseteq \widehat{\mathcal{H}}$, so that all mappings $f_j : (X, \widehat{\mathcal{H}}) \rightarrow (X_j, \mathcal{G}_j)$ are L -gauge morphisms. Let now $(Y, \mathcal{K}) \in |L\text{-GS}|$ and $g : Y \rightarrow X$ be a mapping such that all $f_j \circ g : (Y, \mathcal{K}) \rightarrow (X_j, \mathcal{G}_j)$ are L -gauge morphisms. Then for $d_j \in \mathcal{G}_j$ we know that $(d_{f_j})_g = d \circ (f_j \times f_j) \circ (g \times g) \in \mathcal{K}$. Let now $d \in \widehat{\mathcal{H}}$. Then for $\alpha \triangleleft \top$,

$\perp \prec \omega$ we have for all $x \in X$ that

$$\left(\bigwedge_{j \in K} d_j \circ (f_j \times f_j)(x, \cdot) \right) * \alpha \leq d(x, \cdot) \vee \omega,$$

with some finite set $K \subseteq J$. We conclude for all $y_1, y_2 \in Y$ that

$$d_g(y_1, y_2) \vee \omega = d(g(y_1), g(y_2)) \vee \omega \geq \left(\bigwedge_{j \in K} d_j \circ (f_j \times f_j) \circ (g \times g)(y_1, y_2) \right) * \alpha.$$

As \mathcal{K} is a filter, we conclude $\bigwedge_{j \in K} d_j \circ (f_j \times f_j) \circ (g \times g) \in \mathcal{K}$. Hence d_g is locally supported by \mathcal{K} and therefore $d_g \in \mathcal{K}$ and $g : (Y, \mathcal{K}) \rightarrow (X, \widehat{\mathcal{H}})$ is an L -gauge morphism. \square

We finally show that L -GS has an initially dense object. To this end, we consider the L -metrics $d_\alpha : L \times L \rightarrow L$ introduced in Example 3.4 and note that $\mathcal{H}_L = \{\bigwedge_{\alpha \in K} d_\alpha : K \subseteq L \text{ finite}\}$ is locally directed. Hence $(L, \widehat{\mathcal{H}}_L)$ is an object in L -GS.

Theorem 4.9. *Let $(L, \leq, *)$ be a value quantale and let $(X, \mathcal{G}) \in |L\text{-GS}|$. Then*

$$\left(d_x(\cdot) = d(x, \cdot) : (X, \mathcal{G}) \rightarrow (L, \widehat{\mathcal{H}}_L) \right)_{x \in X, d \in \mathcal{G}}$$

is an initial source.

Proof. We show that \mathcal{G} is the initial gauge for the source. To this end, we first show that all d_x are L -gauge morphisms. Let $x \in X$ and $d \in \mathcal{G}$. Let further $e \in \widehat{\mathcal{H}}_L$. Then e is locally supported by \mathcal{H}_L , i.e. for all $\eta \in L$, $\alpha \triangleleft \top$ and $\perp \prec \omega$ there is a finite set $K = K_{\eta, \alpha, \omega} \subseteq L$ and $d_\gamma \in \mathcal{H}_L$ ($\gamma \in K$) such that

$$\bigwedge_{\gamma \in K} d_\gamma(\eta, \cdot) * \alpha \leq e(\eta, \cdot) \vee \omega.$$

We show that $e \circ (d_x \times d_x) \in \mathcal{G}$. For any $\kappa \in L$ we have $(\kappa \wedge d(x_1, x_2)) * d(x_1, x_2) \leq \kappa \wedge (d(x, x_1) * d(x_1, x_2)) \leq \kappa \wedge d(x, x_2)$. Hence $d(x_1, x_2) \leq (\kappa \wedge d(x, x_1)) \rightarrow (\kappa \wedge d(x, x_2)) = d_\kappa(d(x, x_1), d(x, x_2))$.

Let now $x_1 \in X$, $\alpha \triangleleft \top$ and $\perp \prec \omega$. Then for all $x_2 \in X$ we have

$$\begin{aligned} e \circ (d_x \times d_x)(x_1, x_2) \vee \omega &= e(d(x, x_1), d(x, x_2)) \vee \omega \\ &\geq \bigwedge_{\gamma \in K_{d(x, x_1), \alpha, \omega}} d_\gamma(\eta, \cdot) * \alpha \geq d(x_1, x_2) * \alpha. \end{aligned}$$

Hence $e \circ (d_x \times d_x)$ is locally supported by \mathcal{G} , and therefore belongs to \mathcal{G} . Consequently, if we denote the initial L -gauge on X for the source $(d_x : X \rightarrow (L, \widehat{\mathcal{H}}_L))_{x \in X, d \in \mathcal{G}}$ by \mathcal{G}_{init} , we have $\mathcal{G}_{init} \subseteq \mathcal{G}$.

Let now $d \in \mathcal{G}$. We show that d is locally supported by \mathcal{G}_{init} . Let $x \in X$, $\alpha \triangleleft \top$ and $\perp \prec \omega$. Then for $x_2 \in X$ we have

$$\begin{aligned} (d_\alpha \circ (d_x \times d_x)(x, x_2)) * \alpha &= ((\alpha \wedge d(x, x)) \rightarrow (\alpha \wedge d(x, x_2))) * \alpha \\ &= \alpha * (\alpha \rightarrow (\alpha \wedge d(x, x_2))) \leq \alpha \wedge d(x, x_2) \leq d(x, x_2) \vee \omega. \end{aligned}$$

Hence we have seen $d_\alpha \circ (d_x \times d_x)(x, \cdot) * \alpha \leq d(x, \cdot) \vee \omega$ and because $d_\alpha \circ (d_x \times d_x) \in \mathcal{G}_{init}$ we conclude that d is locally supported by \mathcal{G}_{init} and therefore $d \in \mathcal{G}_{init}$ and the proof is complete. \square

5. L -approach Spaces as L -gauge Spaces

Proposition 5.1. *Let $(X, \delta) \in |L-AP|$. Define*

$$\mathcal{G}^\delta = \{d \in L-MET(X) : \forall A \subseteq X, x \in X : \delta(x, A) \leq \bigvee_{a \in A} d(x, a)\}.$$

Then $(X, \mathcal{G}^\delta) \in |L-GS|$.

Proof. We first show that \mathcal{G}^δ is a filter in $L-MET(X)$. Clearly $d \equiv \top \in \mathcal{G}^\delta$ and hence $\mathcal{G} \neq \emptyset$. If $d \in \mathcal{G}^\delta$ and $e \geq d$ then $\bigvee_{a \in A} e(x, a) \geq \bigvee_{a \in A} d(x, a) \geq \delta(x, A)$ and hence $e \in \mathcal{G}^\delta$. Finally, let $d_1, d_2 \in \mathcal{G}^\delta$. We denote $\mathcal{G}_0 = \{d_1, d_2\}$. By complete distributivity then

$$\bigvee_{a \in A} \bigwedge_{d \in \mathcal{G}_0} d(x, a) = \bigwedge_{\varphi \in \mathcal{G}_0^A} \bigvee_{a \in A} \varphi(a)(x, a).$$

Now, for $\varphi \in \mathcal{G}_0^A$ we have

$$\begin{aligned} \bigvee_{a \in A} \varphi(a)(x, a) &= \bigvee_{d \in \mathcal{G}_0} \bigvee_{a \in \varphi^\leftarrow(d)} d(x, a) \geq \bigvee_{d \in \mathcal{G}_0} \delta(x, \varphi^\leftarrow(d)) \\ &= \delta(x, \bigcup_{d \in \mathcal{G}_0} \varphi^\leftarrow(d)) = \delta(x, A). \end{aligned}$$

Hence $\bigvee_{a \in A} \bigwedge_{d \in \mathcal{G}_0} d(x, a) = \bigwedge_{\varphi \in \mathcal{G}_0^A} \bigvee_{a \in A} \varphi(a)(x, a) \geq \delta(x, A)$ and therefore $d_1 \wedge d_2 \in \mathcal{G}^\delta$.

Next we show that \mathcal{G}^δ is locally saturated. Let $d \in L-MET(X)$, let $x \in X$, $\alpha \triangleleft \top$ and $\perp \prec \omega$ and let $d_x^{\alpha, \omega} \in \mathcal{G}^\delta$ such that $d_x^{\alpha, \omega}(x, \cdot) * \alpha \leq d(x, \cdot) \vee \omega$. Then

$$\bigvee_{a \in A} d(x, a) \vee \omega \geq \bigvee_{a \in A} d_x^{\alpha, \omega}(x, a) * \alpha \geq \delta(x, A) * \alpha$$

and hence

$$\omega \vee \bigvee_{a \in A} d(x, a) \geq \bigvee_{\alpha \triangleleft \top} \delta(x, A) * \alpha = \delta(x, A) * \bigvee_{\alpha \triangleleft \top} \alpha = \delta(x, A) * \top = \delta(x, A).$$

This is true for any $\perp \prec \omega$ and we conclude

$$\delta(x, A) \leq \bigwedge_{\perp \prec \omega} \left(\omega \vee \bigvee_{a \in A} d(x, a) \right) = \left(\bigvee_{a \in A} d(x, a) \right) \vee \bigwedge_{\perp \prec \omega} \omega = \bigvee_{a \in A} d(x, a).$$

Hence $d \in \mathcal{G}^\delta$ and the proof is complete. \square

Proposition 5.2. *Let $(X, \delta), (X', \delta') \in |L-AP|$ and let $f : (X, \delta) \rightarrow (X', \delta')$ be an L -approach morphism. Then $f : (X, \mathcal{G}^\delta) \rightarrow (X', \mathcal{G}^{\delta'})$ is an L -gauge morphism.*

Proof. Let $d' \in \mathcal{G}^{\delta'}$. Then for all $A' \subseteq X'$ and all $x' \in X'$ we have $\delta'(x', A') \leq \bigvee_{a' \in A'} d'(x', a')$. We want to show that $d_f \in \mathcal{G}^\delta$. Let $x \in X$ and let $A \subseteq X$. Then $\delta(x, A) \leq \delta'(f(x), f(A)) \leq \bigvee_{a \in A} d'(f(x), f(a)) = \bigvee_{a \in A} d_f(x, a)$. Hence $d_f \in \mathcal{G}^\delta$. \square

Hence we can define a functor $E : \begin{cases} L\text{-}AP & \longrightarrow & L\text{-}GS \\ (X, \delta) & \longmapsto & (X, \mathcal{G}^\delta) \\ f & \longmapsto & f \end{cases}$. We will show in the sequel that in the case of a quantale that satisfies $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$, this functor yields an embedding that is coreflective.

Lemma 5.3. *Let L satisfy $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$. Then the functor E is injective on objects.*

Proof. Let $(X, \delta), (X, \delta') \in |L\text{-}AP|$ with $\delta \neq \delta'$. Then there are $x \in X$ and $A \subseteq X$ such that $\delta(x, A) \neq \delta'(x, A)$. Without loss of generality we may assume $\delta(x, A) \not\leq \delta'(x, A)$. From Lemma 3.7 we know that $d_A \in \mathcal{G}^\delta$ where d_A is defined by $d_A(x, y) = \delta(y, A) \rightarrow \delta(x, A)$. Assume that $d_A \in \mathcal{G}^{\delta'}$. Then $\delta'(x, A) \leq \bigvee_{a \in A} d_A(x, a) = \bigvee_{a \in A} (\delta(a, A) \rightarrow \delta(x, A)) = \delta(x, A)$, as for $a \in A$ we have $\delta(a, A) = \top$. This is a contradiction and hence $d_A \notin \mathcal{G}^{\delta'}$ and $(X, \mathcal{G}^\delta) \neq (X, \mathcal{G}^{\delta'})$. \square

Proposition 5.4. *Let $(X, \mathcal{G}) \in |L\text{-}GS|$. If we define $\delta^\mathcal{G} : X \times P(X) \longrightarrow L$ by*

$$\delta^\mathcal{G}(x, A) = \bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(x, a),$$

then $(X, \delta^\mathcal{G}) \in |L\text{-}AP|$.

Proof. (LD1) We have $\delta^\mathcal{G}(x, \{x\}) = \bigwedge_{d \in \mathcal{G}} d(x, x) = \top$.

(LD2) We have $\delta^\mathcal{G}(x, \emptyset) = \bigwedge_{d \in \mathcal{G}} \bigvee \emptyset = \perp$.

(LD3) Clearly $\delta^\mathcal{G}(x, A \cup B) \geq \delta^\mathcal{G}(x, A) \vee \delta^\mathcal{G}(x, B)$. For the converse inequality, let $\delta^\mathcal{G}(x, A) \vee \delta^\mathcal{G}(x, B) \prec \alpha$. Then there are $d_A, d_B \in \mathcal{G}$ such that $\bigvee_{a \in A} d_A(x, a) \prec \alpha$ and $\bigvee_{b \in B} d_B(x, b) \prec \alpha$. As \mathcal{G} is an L -gauge we have $d_A \wedge d_B \in \mathcal{G}$ and by local saturation there is, for $\beta \triangleleft \top$, $\perp \prec \omega$ and $x \in X$ an L -metric $e_x^{\beta, \omega} \in \mathcal{G}$ such that $e_x^{\beta, \omega}(x, \cdot) * \beta \leq d_A \wedge d_B(x, \cdot) \vee \omega$. Hence we conclude

$$\begin{aligned} \delta^\mathcal{G}(x, A \cup B) * \beta &= \left(\bigwedge_{d \in \mathcal{G}} \bigvee_{c \in A \cup B} d(x, c) \right) * \beta \leq \bigwedge_{d \in \mathcal{G}} \left(\bigvee_{c \in A \cup B} d(x, c) * \beta \right) \\ &\leq \bigvee_{c \in A \cup B} e_x^{\beta, \omega}(x, c) * \beta \\ &= \left(\bigvee_{a \in A} e_x^{\beta, \omega}(x, a) \vee \bigvee_{b \in B} e_x^{\beta, \omega}(x, b) \right) * \beta \\ &= \left(\bigvee_{a \in A} e_x^{\beta, \omega}(x, a) * \beta \right) \vee \left(\bigvee_{b \in B} e_x^{\beta, \omega}(x, b) * \beta \right) \\ &\leq \left(\bigvee_{a \in A} d_A(x, a) \vee \omega \right) \vee \left(\bigvee_{b \in B} d_B(x, b) \vee \omega \right) \leq \alpha \vee \omega. \end{aligned}$$

Hence we have seen that for all $\beta \triangleleft \top$ and all $\perp \prec \omega$ we have $\delta(x, A \cup B) * \beta \leq \alpha \vee \omega$. Therefore we conclude

$$\delta^{\mathcal{G}}(x, A \cup B) = \delta^{\mathcal{G}}(x, A \cup B) * \bigvee_{\beta \triangleleft \top} \beta = \bigvee_{\beta \triangleleft \top} \delta^{\mathcal{G}}(x, A \cup B) * \beta \leq \alpha \vee \omega$$

and consequently also $\delta^{\mathcal{G}}(x, A \cup B) \leq \bigwedge_{\perp \prec \omega} (\alpha \vee \omega) = \alpha \vee \bigwedge_{\perp \prec \omega} \omega = \alpha$. From this we obtain $\delta^{\mathcal{G}}(x, A \cup B) \leq \bigwedge \{ \alpha \in L : \delta^{\mathcal{G}}(x, A) \vee \delta^{\mathcal{G}}(x, B) \prec \alpha \} = \delta(x, A) \vee \delta(x, B)$.

(LD4) Let $x \in X$, $A \subseteq X$, $\alpha \in L$ and $\beta \triangleleft \alpha$. For $b \in \bar{A}^\alpha$ we have $\bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(b, a) = \delta^{\mathcal{G}}(b, A) \geq \alpha$. Hence for all $d \in \mathcal{G}$ there is $a_\beta \in A$ such that $d(b, a_\beta) \triangleright \beta$ and we conclude $d(x, a_\beta) \geq d(x, b) * d(b, a_\beta) \geq d(x, b) * \beta$. Therefore $\bigvee_{a \in A} d(x, a) \geq d(x, b) * \beta$. This is true for any $b \in \bar{A}^\beta$ and hence we obtain

$$\bigvee_{a \in A} d(x, a) \geq \bigvee_{b \in \bar{A}^\beta} (d(x, b) * \beta) = (\bigvee_{b \in \bar{A}^\beta} d(x, b)) * \beta.$$

As $\beta \triangleleft \alpha$ was arbitrary, we conclude, using $\bar{A}^\alpha \subseteq \bar{A}^\beta$,

$$\bigvee_{a \in A} d(x, a) \geq \bigvee_{\beta \triangleleft \alpha} \left((\bigvee_{b \in \bar{A}^\alpha} d(x, b)) * \beta \right) = \bigvee_{b \in \bar{A}^\alpha} d(x, b) * \bigvee_{\beta \triangleleft \alpha} \beta = (\bigvee_{b \in \bar{A}^\alpha} d(x, b)) * \alpha.$$

This yields

$$\delta^{\mathcal{G}}(x, A) \geq \bigwedge_{d \in \mathcal{G}} \left((\bigvee_{b \in \bar{A}^\alpha} d(x, b)) * \alpha \right) \geq \left(\bigwedge_{d \in \mathcal{G}} (\bigvee_{b \in \bar{A}^\alpha} d(x, b)) \right) * \alpha = \delta^{\mathcal{G}}(x, \bar{A}^\alpha) * \alpha$$

and (LD4) is true. \square

Proposition 5.5. *Let $(X, \mathcal{G}), (X', \mathcal{G}') \in |L\text{-GS}|$ and let $f : (X, \mathcal{G}) \rightarrow (X', \mathcal{G}')$ be an L -gauge morphism. Then $f : (X, \delta^{\mathcal{G}}) \rightarrow (X', \delta^{\mathcal{G}'})$ is an L -approach morphism.*

Proof. Let $x \in X$ and $A \subseteq X$. We have

$$\delta^{\mathcal{G}'}(f(x), f(A)) = \bigwedge_{d' \in \mathcal{G}'} \bigvee_{a \in A} d'(f(x), f(a)) = \bigwedge_{d' \in \mathcal{G}'} \bigvee_{a \in A} d_f(x, a).$$

As for $d' \in \mathcal{G}'$ we have $d_f \in \mathcal{G}$ we conclude

$$\delta^{\mathcal{G}'}(f(x), f(A)) \geq \bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(x, a) = \delta^{\mathcal{G}}(x, A).$$

\square

Hence we can define a functor $K : \begin{cases} L\text{-GS} & \rightarrow & L\text{-AP} \\ (X, \mathcal{G}) & \mapsto & (X, \delta^{\mathcal{G}}) \\ f & \mapsto & f \end{cases}$.

We will need the following result.

Proposition 5.6. *Let L satisfy $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$. Let $(X, \delta) \in |L\text{-AP}|$ and define \mathcal{G}^δ as in Proposition 5.1. Then for all $A \subseteq X$ and all $x \in X$ we have $\delta(x, A) = \bigwedge_{d \in \mathcal{G}^\delta} \bigvee_{a \in A} d(x, a)$.*

Proof. For $d \in \mathcal{G}^\delta$ we have $\bigvee_{a \in A} d(x, a) \geq \delta(x, A)$ and hence $\bigwedge_{d \in \mathcal{G}^\delta} \bigvee_{a \in A} d(x, a) \geq \delta(x, A)$. For the converse inequality we make use of Lemma 3.7. Then for any $Z \subseteq X$, $d_Z \in \mathcal{G}^\delta$, where $d_Z(x, y) = \delta(y, Z) \rightarrow \delta(x, Z)$. Hence we conclude

$$\begin{aligned} \bigwedge_{d \in \mathcal{G}^\delta} \bigvee_{a \in A} d(x, a) &\leq \bigvee_{Z \subseteq X} \bigvee_{a \in A} d_Z(x, a) = \bigwedge_{Z \subseteq X} \bigvee_{a \in A} (\delta(a, Z) \rightarrow \delta(x, Z)) \\ &\leq \bigvee_{a \in A} (\delta(a, A) \rightarrow \delta(x, A)) = \delta(x, A) \end{aligned}$$

as for $a \in A$ we have by (LD1) that $\delta(a, A) = \top$. \square

Corollary 5.7. *Let L satisfy $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$. Let $(X, \delta) \in |L\text{-AP}|$. Then $\delta(\mathcal{G}^\delta) = \delta$, i.e. we have $K(E((X, \delta))) = (X, \delta)$.*

Proposition 5.8. *Let $(X, \mathcal{G}) \in |L\text{-GS}|$. Then $\mathcal{G} \subseteq \mathcal{G}^{(\delta^\mathcal{G})}$, i.e. we have $E(K((X, \mathcal{G}))) \geq (X, \mathcal{G})$.*

Proof. For $d \in \mathcal{G}$ we have $\delta^\mathcal{G}(x, A) \leq \bigvee_{a \in A} d(x, a)$ and hence $d \in \mathcal{G}^{(\delta^\mathcal{G})}$. \square

As a corollary, we obtain the following theorem.

Theorem 5.9. *Let L satisfy $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$. Then the category $L\text{-AP}$ is isomorphic to a coreflective subcategory of $L\text{-GS}$.*

In general, $\mathcal{G}^{(\delta^\mathcal{G})} \neq \mathcal{G}$, as is shown by the following two examples.

Example 5.10. Let $L = [0, 1] \cup \{\perp = -1, \top = 2\}$ and the order inherited from \mathbb{R} with $\wedge = *$ as the quantale operation. Then $\perp \prec \perp$ and $\top \triangleleft \top$. Let further $X = (0, 1)$ and define, for $x \in X$, the L -metric $e_x : X \times X \rightarrow L$ by

$$e_x(a, b) = \begin{cases} \top & \text{if } a = b \\ x & \text{if } a \neq b \end{cases}.$$

It is easily checked that e_x is an L -metric on X . Furthermore, we have for $A \subseteq X$ and $y \in X$

$$\bigwedge_{x \in X} \bigvee_{a \in A} e_x(y, a) = \bigvee_{a \in A} \bigwedge_{x \in X} e_x(y, a).$$

If $y \in A$, then we have $\bigwedge_{x \in X} \bigvee_{a \in A} e_x(y, a) \geq \bigvee_{a \in A} \bigwedge_{x \in X} e_x(y, a) \geq \bigwedge_{x \in X} e_x(y, y) = \top$. If $y \notin A$, then we have $y \neq a$ for all $a \in A$ and hence $\bigvee_{a \in A} \bigwedge_{x \in X} e_x(y, a) \leq \bigwedge_{x \in X} \bigvee_{a \in A} e_x(y, a) \leq \bigwedge_{x \in X} x = 0$.

We define now

$$\mathcal{H} = \left\{ \bigwedge_{x \in K} e_x : K \subseteq X \text{ finite} \right\}$$

Then \mathcal{H} is locally directed and we denote $\mathcal{G} = \widehat{\mathcal{H}}$. We define $d_0 = \bigwedge_{x \in X} e_x$. For $A \subseteq X$ and $y \in X$ we have

$$\bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(y, a) \leq \bigwedge_{x \in X} \bigvee_{a \in A} e_x(y, a) = \bigvee_{a \in A} \bigwedge_{x \in X} e_x(y, a),$$

and hence $d_0 \in \mathcal{G}^{(\delta^g)}$. However, $d_0 \notin \mathcal{G}$. It is routine to verify that for $y \in X$, $\alpha = \top$ and $\beta = \perp$ there is no finite subset $K \subseteq X$ such that $\bigwedge_{x \in K} e_x(y, \cdot) = \bigwedge_{x \in K} e_x(y, \cdot) \wedge \top \leq d_0(y, \cdot) \vee \perp = d_0(y, \cdot)$. Hence d_0 is not locally supported by \mathcal{H} , i.e. $d_0 \notin \mathcal{G}$. With regard to the following theorem we note that L is a linearly ordered value quantale but does not satisfy the property (I).

Example 5.11. Let $L = \Delta^+$. For $0 \leq \alpha, \beta \leq 1$ we define the distance distribution functions $\varphi_{\alpha\beta} \in \Delta^+$ by

$$\varphi_{\alpha\beta}(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 - \alpha \\ \frac{1}{2\alpha}(x + \alpha - 1) & \text{if } 1 - \alpha < x \leq 1 \\ \frac{1}{2\beta}(x + \beta - 1) & \text{if } 1 < x \leq 1 + \beta \\ 1 & \text{if } 1 + \beta < x \end{cases}.$$

Furthermore, we put $\varphi_\alpha = \varphi_{\alpha\alpha}$ for short. Then $\varphi_\alpha \wedge \varphi_\beta = \varphi_{\alpha \wedge \beta, \alpha \vee \beta}$ and $\bigwedge_{0 < \alpha < 1} \varphi_\alpha = \varphi_{01}$. We consider now, for a set X and $0 < \alpha < 1$, the *equilateral space* [17] (X, d_α) with

$$d_\alpha(p, q) = \begin{cases} \varphi_\alpha & \text{if } p \neq q \\ \varepsilon_0 & \text{if } p = q \end{cases}.$$

It is shown in [17] that for any triangle function τ , an equilateral space is a (Δ^+, τ) -metric space.

For a non-empty $A \subseteq X$ and $p \in X$ we moreover have

$$\bigwedge_{0 < \alpha < 1} \bigvee_{a \in A} d_\alpha(p, a) = \begin{cases} \varphi_{01} & \text{if } p \notin A \\ \varepsilon_0 & \text{if } p \in A \end{cases},$$

and also

$$\bigvee_{a \in A} \bigwedge_{0 < \alpha < 1} d_\alpha(p, a) = \begin{cases} \varphi_{01} & \text{if } p \notin A \\ \varepsilon_0 & \text{if } p \in A \end{cases},$$

and the equality $\bigwedge_{0 < \alpha < 1} \bigvee_{a \in A} d_\alpha(p, a) = \bigvee_{a \in A} \bigwedge_{0 < \alpha < 1} d_\alpha(p, a)$ holds trivially if $A = \emptyset$. We define $\mathcal{H} = \{\bigwedge_{\alpha \in K} d_\alpha : K \subseteq (0, 1) \text{ finite}\}$. Then \mathcal{H} is locally directed and we define $\mathcal{G} = \widehat{\mathcal{H}}$. For $A \subseteq X$ and $p \in X$ we then have

$$\bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(p, a) \leq \bigwedge_{0 < \alpha < 1} \bigvee_{a \in A} d_\alpha(p, a) = \bigvee_{a \in A} \bigwedge_{0 < \alpha < 1} d_\alpha(p, a),$$

and hence $d_0 = \bigwedge_{0 < \alpha < 1} d_\alpha \in \mathcal{G}^{(\delta^g)}$. However, for $\alpha = f_{1/2, 1/2} \triangleleft \varepsilon_0$ (see Lemma 2.11) and $\beta = g_{1/4, 2}$ where $g_{\delta, \gamma} = \begin{cases} \gamma & \text{if } 0 < x \leq \delta \\ 1 & \text{if } \delta < x \end{cases}$, we have $\varepsilon_\infty \prec g_{1/4, 2}$ but there is no finite subset $K \subseteq (0, 1)$ such that

$$\left(\bigwedge_{\alpha \in K} d_\alpha(p, \cdot) \wedge f_{1/2, 1/2} \right)(x) \leq (d_0(p, \cdot) \vee g_{1/4, 2})(x)$$

for all $x \in [0, \infty]$. Indeed, for $p \neq q$ we have with $\delta = \bigwedge_{\alpha \in K} \alpha$ and $\gamma = \bigvee_{\alpha \in K} \alpha$ that $\bigwedge_{\alpha \in K} d_\alpha(p, q) = \varphi_{\delta\gamma}$ and for $1 - \delta/2 < x < 1$ we have $\frac{1}{4} < (\bigwedge_{\alpha \in K} d_\alpha(p, q) \wedge f_{1/2, 1/2})(x) < \frac{1}{2}$ and $(d_0(p, q) \vee g_{1/4, 2})(x) = \frac{1}{4}$. Therefore d_0 is not locally supported by \mathcal{G} and hence $d_0 \notin \mathcal{G}$.

With regard to the following theorem, we note that if we choose the triangle function induced by the product t-norm, $L = \Delta^+$ satisfies the condition (I) but is not linearly ordered.

Under certain assumptions, however, we can guarantee that the categories $L\text{-AP}$ and $L\text{-GS}$ are isomorphic.

Theorem 5.12. *Let $(L, \leq, *)$ be a linearly ordered value quantale that satisfies the condition (I). Let further $\mathcal{G} \subseteq L\text{-MET}(X)$ be an L -gauge. Then $\mathcal{G}^{(\delta^{\mathcal{G}})} = \mathcal{G}$.*

Proof. We have seen above that $\mathcal{G} \subseteq \mathcal{G}^{(\delta^{\mathcal{G}})}$. Now we show that $\mathcal{G}^{(\delta^{\mathcal{G}})} \subseteq \mathcal{G}$. Let $d_0 \in \mathcal{G}^{(\delta^{\mathcal{G}})}$ and assume $d_0 \notin \mathcal{G}$. Then d_0 is not locally supported by \mathcal{G} and hence there is an $x \in X$, $\alpha \triangleleft \top$, $\perp \prec \omega$ such that for all $e \in \mathcal{G}$ we have $e(x, \cdot) * \alpha \not\leq d_0(x, \cdot) \vee \omega$. As L is a value quantale, there is $\beta \triangleleft \top$ such that $\alpha \triangleleft \beta * \beta$ and hence we have for all $e \in \mathcal{G}$

$$e(x, \cdot) * (\beta * \beta) \not\leq d_0(x, \cdot) \vee \omega.$$

Consider a finite subset $\mathcal{D}_0 \subseteq \mathcal{G}$ and define

$$A(\mathcal{D}_0) = \{y \in X : \bigwedge_{d \in \mathcal{D}_0} d(x, y) * \beta \not\leq d_0(x, y) \vee \omega\}.$$

As \mathcal{G} is locally directed, there is $e_0 \in \mathcal{G}$ such that

$$e_0(x, y) * \beta \leq \bigwedge_{d \in \mathcal{D}_0} d(x, y) \vee \omega.$$

As a consequence, if $e_0(x, y) * (\beta * \beta) \not\leq d_0(x, y) \vee \omega$, then $\bigwedge_{d \in \mathcal{D}_0} d(x, y) * \beta \not\leq d_0(x, y) \vee \omega$. For otherwise we had

$$e_0(x, y) * (\beta * \beta) \leq \left(\left(\bigwedge_{d \in \mathcal{D}_0} d(x, y) \right) * \beta \right) \vee \omega \leq d_0(x, y) \vee \omega,$$

a contradiction. It follows that

$$\emptyset \neq \{y \in X : e(x, y) * (\beta * \beta) \not\leq d_0(x, y) \vee \omega\} \subseteq A(\mathcal{D}_0).$$

Moreover we have for finite subsets $\mathcal{D}_0, \mathcal{D}_1 \subseteq \mathcal{G}$ that $A(\mathcal{D}_0 \cup \mathcal{D}_1) \subseteq A(\mathcal{D}_0) \cap A(\mathcal{D}_1)$ and hence the system $\{A(\mathcal{D}_0) : \mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}\}$ is a filter basis on X . We conclude, using $\delta^{(\mathcal{G}^{(\delta^{\mathcal{G}})})} = \delta^{\mathcal{G}}$,

$$\begin{aligned} & \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \delta^{\mathcal{G}}(x, A(\mathcal{D}_0)) \vee \omega \right) * \beta \\ &= \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigwedge_{e \in \mathcal{G}} \bigvee_{a \in A(\mathcal{D}_0)} e(x, a) \right) * \beta \end{aligned}$$

$$\begin{aligned}
&\geq \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigwedge_{\text{finite } e \in \mathcal{G}} \bigvee_{a \in A(\mathcal{D}_0 \cup \{e\})} \left(\bigwedge_{d \in \mathcal{D}_0} (d \wedge e)(x, a) \right) \right) * \beta \\
&= \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigvee_{\text{finite } a \in A(\mathcal{D}_0)} \bigwedge_{d \in \mathcal{D}_0} d(x, a) \right) * \beta \\
&= \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigvee_{\text{finite } a \in A(\mathcal{D}_0)} \left(\bigwedge_{d \in \mathcal{D}_0} d(x, a) * \beta \right).
\end{aligned}$$

As L is linearly ordered, the last expression is

$$\begin{aligned}
&\geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigvee_{\text{finite } a \in A(\mathcal{D}_0)} (d_0(x, a) \vee \omega) \\
&\geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigwedge_{\text{finite } e \in \mathcal{G}^{(\delta^{\mathcal{G}})}} \bigvee_{a \in A(\mathcal{D}_0)} (e(x, a) \vee \omega) \\
&\geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigvee_{\text{finite}} \delta^{\mathcal{G}^{(\delta^{\mathcal{G}})}}(x, A(\mathcal{D}_0)) \vee \omega \geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G}} \bigvee_{\text{finite}} \delta^{\mathcal{G}}(x, A(\mathcal{D}_0)) \vee \omega.
\end{aligned}$$

As L satisfies the property (I), this is a contradiction and hence $d_0 \in \mathcal{G}$. \square

We obtain from Corollary 5.7 and Theorem 5.12 the following result.

Theorem 5.13. *Let $(L, \leq, *)$ be a linearly ordered value quantale that satisfies the condition (I) and $(\bigwedge_{j \in J} \alpha_j) \rightarrow \beta = \bigvee_{j \in J} (\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$. Then the categories L -GS and L -AP are isomorphic.*

In case of $L = [0, \infty]$ and the opposite order and extended addition as quantale operation, we see that in the case of approach spaces [11] the conditions on L are satisfied and hence $([0, \infty]$ -) gauges and $([0, \infty]$ -) approach distances are equivalent concepts. However, as can be seen with Example 5.11, probabilistic approach spaces [9] cannot equivalently be described by Δ^+ -gauges.

6. L -metric Spaces as L -gauge Spaces

Theorem 6.1. *The category L -MET is isomorphic to a coreflective subcategory of L -GS.*

Proof. Let $(X, d) \in |L\text{-MET}|$ and define $\mathcal{G}^d = [d] = \{e \in L\text{-MET}(X) : d \leq e\}$. As $\mathcal{G}^d = [d]$ is a principal filter, it is naturally locally saturated and hence $(X, \mathcal{G}^d) \in |L\text{-GS}|$. Furthermore, let $f : (X, d) \rightarrow (X', d')$ be an L -metric morphism and let $e' \in \mathcal{G}^{d'}$. Then $d' \leq e'$ and hence $e_f(x, y) = e'(f(x), f(y)) \geq d'(f(x), f(y)) \geq d(x, y)$. Hence $e_f \in \mathcal{G}^d$ and $f : (X, \mathcal{G}^d) \rightarrow (X', \mathcal{G}^{d'})$ is an L -gauge morphism.

Hence we can define a functor $F : \begin{cases} L\text{-MET} & \rightarrow & L\text{-GS} \\ (X, d) & \mapsto & (X, \mathcal{G}^d) \\ f & \mapsto & f \end{cases}$. This functor is clearly injective on objects, for if we have two different L -metrics on X , we may

assume $d(x, y) \not\leq d'(x, y)$ for $x, y \in X$. But then $d' \notin \mathcal{G}^d$ whereas $d' \in \mathcal{G}^d$. Let now $(X, \mathcal{G}) \in |L-GS|$ and define $d^{\mathcal{G}} : X \times X \rightarrow L$ by $d^{\mathcal{G}}(x, y) = \bigwedge_{d \in \mathcal{G}} d(x, y)$. Then $(X, d^{\mathcal{G}}) \in |L-MET|$. For $(X, \mathcal{G}), (X', \mathcal{G}') \in |L-GS|$ and an L -gauge morphism $f : (X, \mathcal{G}) \rightarrow (X', \mathcal{G}')$ then $f : (X, d^{\mathcal{G}}) \rightarrow (X', d^{\mathcal{G}'})$ is an L -metric morphism. To see this, let $x, y \in X$. Then, because for $d' \in \mathcal{G}'$ we have $d_f \in \mathcal{G}$, we conclude $d^{\mathcal{G}'}(f(x), f(y)) = \bigwedge_{d' \in \mathcal{G}'} d'(f(x), f(y)) = \bigwedge_{d' \in \mathcal{G}'} d_f(x, y) \geq \bigwedge_{d \in \mathcal{G}} d(x, y) = d^{\mathcal{G}}(x, y)$. Hence we can define a functor $H : \begin{cases} L-GS & \rightarrow & L-MET \\ (X, \mathcal{G}) & \mapsto & (X, d^{\mathcal{G}}) \\ f & \mapsto & f \end{cases}$. For $(X, d) \in |L-MET|$ and $x, y \in X$ we have $d^{(\mathcal{G}^d)}(x, y) = \bigwedge_{e \in \mathcal{G}^d} e(x, y) = \bigwedge_{e \geq d} e(x, y) = d(x, y)$. This shows $d^{(\mathcal{G}^d)} = d$, i.e. $F(H((X, d))) = (X, d)$. For $(X, \mathcal{G}) \in |L-GS|$ and $e \in \mathcal{G}$ we have $d^{\mathcal{G}}(x, y) \leq e(x, y)$ for all $x, y \in X$ and therefore $e \in \mathcal{G}^{(d^{\mathcal{G}})}$. Hence $\mathcal{G} \subseteq \mathcal{G}^{(d^{\mathcal{G}})}$, i.e. $H(F((X, \mathcal{G}))) \geq (X, \mathcal{G})$. \square

Lemma 6.2. *Let $(X, d) \in |L-MET|$. Then $\mathcal{G}^d = \mathcal{G}^{\delta^d}$, i.e. we have $F = E \circ G$.*

Proof. We have $e \in \mathcal{G}^{\delta^d}$ if and only if for all $x \in X$ and all $A \subseteq X$ we have $\delta^d(x, A) \leq \bigvee_{a \in A} e(x, a)$, i.e. if and only if for all $x \in X$ and all $A \subseteq X$ we have $\bigvee_{a \in A} d(x, a) \leq \bigvee_{a \in A} e(x, a)$. Taking for A the one-point sets, we see $d \leq e$, i.e. $e \in \mathcal{G}^d$. Conversely, if $e \in \mathcal{G}^d$, then $d \leq e$ and hence $\delta^d(x, A) \leq \bigvee_{a \in A} e(x, a)$ for all $x \in X$ and all $A \subseteq X$, i.e. $e \in \mathcal{G}^{(\delta^d)}$. \square

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