# SINGLE MACHINE DUE DATE ASSIGNMENT SCHEDULING PROBLEM WITH PRECEDENCE CONSTRAINTS AND CONTROLLABLE PROCESSING TIMES IN FUZZY ENVIRONMENT

J. LI, D. XU AND H. LI

Abstract. In this paper, a due date assignment scheduling problem with precedence constraints and controllable processing times in uncertain environment is investigated, in which the basic processing time of each job is assumed to be the symmetric trapezoidal fuzzy number, and the linear resource consumption function is used. The objective is to minimize the crisp possibilistic mean (or expected) value of a cost function that includes the costs of earliness, tardiness, makespan and resource consumption jointly by scheduling the jobs under precedence constraints and determining the due date and the resource allocation amount satisfying resource constraints for each job. First, the problem is shown to be NP-hard. Furthermore, an optimal algorithm with polynomial time for the special case of this problem is put forward. Moreover, an efficient 2-approximation algorithm is presented based on solving the relaxation of the problem. Finally, the numerical experiment is given, whose results show that our method is promising.

## 1. Introduction

Just-In-Time(JIT) management system is usually employed by modern enterprises in supply chain management, which demands that the suppliers delivery goods as close to the required dates as possible in order to reduce the inventory costs [3]. The suppliers should meet the deliver dates or face large penalties [29]. Hence, meeting due dates is important goal in product scheduling and supply chain management [33]. A tardy job completion may result in tardiness penalties such as contractual penalties and reduced sales. Similarly, an early job completion may result in earliness penalties such as storage costs and insurance costs [27]. One important goal of the JIT scheduling model is to discourage early and tardy jobs so as to minimize the earliness-tardiness penalties [36]. Due date assignment schedule strategies are usually adopted, especially when the goal is to schedule the jobs and determine the due dates of jobs such that the total cost is to be minimized [2, 41, 39]. Pioneering researches in the area of due date assignment scheduling problems were done by

Received: May 2017; Revised: October 2017; Accepted: December 2017

*Key words and phrases:* Fuzzy scheduling, Fuzzy number, Possibilistic mean value and variance, Due date assignment scheduling, Precedence constraints, Controllable processing times.

This work was supported by National Natural Science Foundation of China (11401030), Outstanding Young Teacher Training Program of Colleges and Universities in Guangdong Province (2014) and Teacher Research Capacity Promotion Plan of Beijing Normal University Zhuhai.

Seidmann et al. [28] and Panwalkar et al. [23] in 1890s. Seidmann et al. investigated a distinct due date assignment scheduling problem that assigning a due date for each job and finding an optimal schedule of all jobs such that the total penalties are minimized. Panwalkar et al. examined a common due date assignment scheduling problem that assigning a common due date to all jobs and scheduling the jobs such that the total penalties reach minimal value. Since then, numerous extensions in deterministic environment have been studied. The readers may refer to the survey papers written by Gordon et al. [10] and Janiak et al. [15, 16], which give us an current research situation in related fields.

Owing to the uncertainty inherent in practical applications, the data for the processing times of the jobs are usually recorded or collected under the influence of some unexpected noises. Hence, the processing times are not known exactly in advance. As a result, due date assignment models under uncertain environment are proposed by some researchers, in which the processing times are usually assumed to be random variables or fuzzy numbers [6, 35, 24, 4, 14, 18, 19]. For example, Cheng [6] considers a single machine due date assignment scheduling problem in which the processing times are stochastically independent random variables, and the objective is to determine the optimal due dates and the optimal schedule of jobs that jointly minimize the expected total cost. Soroush [35] investigates a due date assignment scheduling problem with stochastic processing times on a single machine, whose objective is to find the optimal schedule and determine the optimal due dates for jobs such that the expected total earliness and tardiness costs are minimized. Some efficient heuristics are given to find candidates for the optimal sequence, and one of these heuristics is shown to be asymptotically optimal by Portougal and Trietsch [24]. Furthermore, a branch and bound algorithm to find the optimal solution is put forward for this problem in [4]. Iranpoor [14] examines a schedule problem including assigning due dates and scheduling the jobs and maintenance activity on a single machine with stochastic processing times. For the cases that the historical data are not abundant for the processing time of a job to get the probability distribution, and intuition and judgment of the experienced workers play an important role in determining the processing time of a job, the due date assignment scheduling models in fuzzy environment are proposed [18, 19], in which the processing times are presented by fuzzy variables and the objective is to schedule the jobs and determine the due dates to jointly minimize the mean value of the earliness and tardiness penalties.

In practice, the customer may demand that the job is completed before a deadline, which would make the enterprise to allocate extra resources such as labor and/or capital to the job in order to speed up its processing time [21, 38, 17, 37, 32]. If the processing time is speed up by allocating extra resources, it is usually called the controllable processing time in resource allocation scheduling problems. The actual processing time of a job usually has two forms in resource allocation scheduling problems. One is linear resource consumption function, that is, the job $J_j$'s actual processing time is defined as $p_j = \bar{p}_j - s_j u_j$, where $\bar{p}_j$ is the basic processing time, $s_j$ is the compression rate, and $u_j$ is the quantity of resources allocated for job $J_j$ [40]. The other form is convex resource consumption function, that is, the job $J_j$'s actual processing time is given by $p_j = (w_j/u_j)^k$, where $w_j$ is the workload along with the quantity $u_j$ of resources allocated to job $J_j$ and the given constant $k$ [20]. A survey of scheduling problems with controllable processing times is given by Shabtay and Steiner [31]. In deterministic environment, the due date assignment scheduling problems with controllable processing times are investigated by many scholars [7, 30, 34, 26]. For example, Cheng et al. [7] investigate a scheduling problem with a linear resource consumption under both CON and SLK due date assignment methods. Shabtay et al. [30] discuss due date assignment and resource allocation in a group technology scheduling environment. Also, Shabtay et al. [34] examine a single due date assignment scheduling problem with controllable processing times, whose objective is to minimize the total weighted number of tardy jobs plus due date assignment costs subject to an upper bound on the value of total resource consumption. Morteza [26] considers an integrated due date assignment, production and batch delivery scheduling problem with controllable processing times for multiple customers in a supply chain. A pseudo-polynomial dynamic programming algorithm is put forward to solve the problem.

Note that, in resource allocation scheduling problems, the basic processing time of a job is not known exactly in advance due to some unexpected noises or/and insufficient historical data for the processing time. Always, the basic processing time is estimated by the experienced worker. For example, the basic processing time of the job is usually about 5 hours (denoted as $\tilde{5}$ hours) by one worker. If the customer demands that the job must be completed in 3 hours, the firm has to speed up its processing time in order to satisfy the customer's demand. The firm may allocate the extra resources such as other workers to the job. For example, the actual processing time may be about 2.5 hours (denoted as $\tilde{2.5}$ hours) by allocating one more worker to the job. If the uncertain processing time of the job is speed up by allocating extra resources, it is called the controllable processing time in uncertain environment. Besides, there are usually precedence constraints among jobs in real-life applications [42, 22]. In order to improve the satisfaction of the customers and reduce the inventor costs for the enterprises, it is important to schedule the jobs under the precedence constraints and determine the due date with controllable processing time for each job in uncertain environment. Taking all factors above into account, we study the due date assignment scheduling problem with precedence constraints and controllable processing times in uncertain environment, in which the basic processing time of each job is assumed to be fuzzy number, and the linear resource consumption function is used. The objective is to minimize the crisp possibilistic mean (or expected) value of a cost function that includes the costs of earliness, tardiness, makespan and resource consumption jointly by scheduling the jobs under precedence constraints and determining the due date and the resource allocation amount satisfying resource constraints for each job. First, we show that the problem of this paper is NP-hard. Furthermore, an optimal algorithm with polynomial time for special case of this problem is put forward. Also, an efficient 2-approximation algorithm is presented for the problem with general constraints based on solving a relaxation of the problem. Finally, the numerical experiment is given, whose results show that our method is promising.

The rest of the paper is organized as follows. In next section, some notions of the fuzzy sets theory used in this paper are introduced, and the formal description of the considered problem is given. In section 3, the computational complexity of the problem is discussed. In section 4, an optimal algorithm with polynomial time is proposed for special case of the considered problem. In section 5, a 2-approximation algorithm is presented for the problem with general constraints based on solving a relaxation of this kind of scheduling problems. In section 6 and section 7, the numerical experiment and some conclusions are given, respectively.

## 2. **Preliminary and Problem Formulation**

2.1. **Preliminary.** A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $R$ with a normal, fuzzy convex and continuous membership function of bounded support, and the family of fuzzy numbers of the real line $R$ is denoted by $\mathcal{F}(R)$ [5]. Let $\tilde{A} \in \mathcal{F}(R)$ and $\gamma \in [0,1]$, the crisp set $\tilde{A}_\gamma$ defined by $\{t \in R | \tilde{A}(t) \geq \gamma\}$ is called the $\gamma$-cut of $\tilde{A}$ [9]. It follows from the definition of fuzzy number that $\tilde{A}_\gamma$ is an interval, that is $\tilde{A}_\gamma = [a_1(\gamma), a_2(\gamma)]$. In [5], a trapezoidal fuzzy number $\tilde{A}$, denoted by $(a, b, \alpha, \beta)$ with peak $[a, b]$, left-width $\alpha > 0$ and right-width $\beta > 0$, is defined with the membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a+\alpha}{\alpha}, & x \in [a-\alpha, a); \\ 1, & x \in [a, b); \\ \dfrac{x-b-\beta}{-\beta}, & x \in [b, b+\beta]; \\ 0, & \text{ortherwise}. \end{cases} \tag{1}$$

When $\alpha = \beta$, the trapezoidal fuzzy number is called the symmetric trapezoidal fuzzy number denoted by $(a, b, \alpha)$. For the symmetric trapezoidal fuzzy number $(a, b, \alpha)$, $(b - a)$ is called peak width, and $\lambda = (b - a)/\alpha$ is called the ratio of peak-width and left-width. In this paper, the symmetric trapezoidal fuzzy number $(a, b, \alpha)$ with the ratio $\lambda$ of peak-width and left-width is always simplistically called the symmetric trapezoidal fuzzy number $(a, b, \alpha)$ with ratio $\lambda$.

The following arithmetic operations of fuzzy numbers are given in [13]. For $\tilde{A}, \tilde{B} \in \mathcal{F}(R)$, let $\tilde{A}_\gamma = [a_1(\gamma), a_2(\gamma)]$ and $\tilde{B}_\gamma = [b_1(\gamma), b_2(\gamma)]$, then

$$\tilde{A} + \tilde{B} = \bigcup_{\gamma \in [0,1]} \gamma[a_1(\gamma) + b_1(\gamma), a_2(\gamma) + b_2(\gamma)]; \tag{2}$$

$$\tilde{A} - \tilde{B} = \bigcup_{\gamma \in [0,1]} \gamma[a_1(\gamma) - b_2(\gamma), a_2(\gamma) - b_1(\gamma)]; \tag{3}$$

$$c\tilde{A} = \bigcup_{\gamma \in [0,1]} \gamma[ca_1(\gamma), ca_2(\gamma)] \text{ for } c \geq 0; \tag{4}$$

$$\max\{\tilde{A}, \tilde{B}\} = \bigcup_{\gamma \in [0,1]} \gamma[a_1(\gamma) \vee b_1(\gamma), a_2(\gamma) \vee b_2(\gamma)]. \tag{5}$$

According to the operations above, for the trapezoidal fuzzy numbers $(a_1, b_1, \alpha_1)$ and $(a_2, b_2, \alpha_2)$, it is easy to get

$$(a_1, b_1, \alpha_1) + (a_2, b_2, \alpha_2) = (a_1 + a_2, b_1 + b_2, \alpha_1 + \alpha_2); \tag{6}$$

$$c(a, b, \alpha) = (ca, cb, c\alpha) \text{ for } c \geq 0. \tag{7}$$

**Definition 2.1.** [5] The crisp possibilistic mean value of the fuzzy number $\tilde{A}$, denoted by $\bar{M}(\tilde{A})$, is defined as

$$\bar{M}(\tilde{A}) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma. \tag{8}$$

According to the arithmetic operations of fuzzy numbers and Definition 2.1, it is easy to verify that

$$\bar{M}(\tilde{A} + \tilde{B}) = \bar{M}(\tilde{A}) + \bar{M}(\tilde{B}) \text{ and } \bar{M}(c\tilde{A}) = c\bar{M}(\tilde{A}) \text{ for } c \geq 0. \tag{9}$$

**Definition 2.2.** [5] The possibilistic variance of fuzzy number $\tilde{A}$, denoted by $\text{Var}(\tilde{A})$, is defined by

$$\text{Var}(\tilde{A}) = \frac{1}{2}\int_0^1 \gamma(a_1(\gamma) - a_2(\gamma))^2 d\gamma. \tag{10}$$

The standard deviation of $\tilde{A}$ is defined by $\sigma_{\tilde{A}} = \sqrt{\text{Var}(\tilde{A})}$.

In [5], it is pointed out that, for the trapezoidal fuzzy number $\tilde{A} = (a, b, \alpha, \beta)$, $\bar{M}(\tilde{A}) = \frac{a+b}{2} + \frac{\beta-\alpha}{6}$ and $\text{Var}(\tilde{A}) = [\frac{b-a}{2} + \frac{\beta+\alpha}{6}]^2 + \frac{(\alpha+\beta)^2}{72}$, respectively. Also, for the symmetric trapezoidal fuzzy number $\tilde{A} = (a, b, \alpha)$ with ratio $\lambda = \frac{b-a}{\alpha}$, it is easy to get $\bar{M}(\tilde{A}) = \frac{a+b}{2}$, $\text{Var}(\tilde{A}) = [\frac{b-a}{2} + \frac{\alpha}{3}]^2 + \frac{\alpha^2}{18} = [\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}]\alpha^2$ and $\sigma_{\tilde{A}} = \alpha\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}$.

2.2. **Problem Formulation.** In this paper, a single machine due date assignment scheduling problem with precedence constraints and controllable processing times in uncertain environment is investigated, which can be formulated as follows. There is a set of jobs $\mathcal{J} = \{J_1, J_2, \cdots, J_n\}$ to be processed on a single machine, in which the preemption of the jobs and idle machine time are not allowed, and the machine can process at most one job at a time; there are precedence constraints between jobs. The precedence constraints are specified in the form of a directed acyclic graph $G$ same as ones used in [1], in which each vertex represents a job. We say that $J_i$ precedes $J_j$ or $J_j$ succeeds $J_i$, written $J_i \to J_j$, whenever there is a directed path from vertex $J_i$ to vertex $J_j$ in $G$; the additional resources with total amount $U_0$ will be allotted to jobs; $u_i \geq 0$ denotes the amount of additional resources assigned to job $J_i$; if there are no additional resources to jobs $J_i$, its processing time belongs to $[a_i - \alpha_i, b_i + \alpha_i](b_i \geq a_i \geq \alpha_i > 0)$, and is assume to be a symmetric trapezoidal fuzzy number $(a_i, b_i, \alpha_i)$, which is called the basic processing time of job $J_i$, denoted as $\tilde{\bar{p}}_i = (a_i, b_i, \alpha_i)(i = 1, \cdots, n)$; if there are additional resources with amount $u_i$

assigned to job $J_i$, its actual processing time $\tilde{p}_i$ is assumed to be linear function of the amount of the allocated resources, i.e., the resource consumption function is of the form

$$\tilde{p}_i(u_i) = \bar{\tilde{p}}_i - s_i u_i, \tag{11}$$

where $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ is the basic processing time, $s_i$ is the positive compression rate of job $J_i$, $0 \le u_i \le \dfrac{a_i - \alpha_i}{s_i}$ and $\sum_{i=1}^{n} u_i \le U_0$; a schedule is said to be feasible if all jobs can be successfully processed with respect to precedence constraints and resource constraints $\sum_{i=1}^{n} u_i \le U_0$ and $0 \le u_i \le \dfrac{a_i - \alpha_i}{s_i} (i = 1, \cdots, n)$; for a feasible schedule $\pi$, the deterministic due date of job $J_i$ is denoted by $d_i$; fuzzy numbers $\tilde{C}_i$, $\tilde{E}_i = \max\{d_i - \tilde{C}_i, 0\}$ and $\tilde{T}_i = \max\{\tilde{C}_i - d_i, 0\}$ represent the completion time, the earliness and the tardiness of job $J_j$, respectivly; the per unit earliness penalties and the per unit tardiness penalties are $e_i$ and $t_i$, respectively. The objective is to determine the due dates $\vec{d} = (d_1, d_2, \cdots, d_n)$ and the resource allocation amounts $\vec{u} = (u_1, u_2, \cdots, u_n)$ satisfying resource constraints and schedule the jobs under precedence constraints such that the crisp possibilistic mean (or expected) value of a cost function that includes the costs of earliness, tardiness, makespan and resource consumption is minimized. The objective function is denoted as follows:

$$f(\pi, \vec{d}, \vec{u}) = \bar{M}\Big(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i\Big), \tag{12}$$

where $\tilde{\mathrm{C}}_{\max} = \max\{\tilde{C}_1, \cdots, \tilde{C}_n\}$ is the maximal completion time (makespan), $\delta \ge 0$ is the cost of one unit of operation time, and $v_i \ge 0$ is the cost of one unit of resources allocated to job $J_i$. The problem discussed in this paper can be described by the following optimization problem:

$$\begin{aligned} \min \quad & f(\pi, \vec{d}, \vec{u}) = \bar{M}\Big(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i\Big) \\ \text{s.t.} \quad & 0 \le u_i \le \frac{a_i - \alpha_i}{s_i}; \\ & \sum_{i=1}^{n} u_i \le U_0; \\ & \text{all jobs in } \pi \text{ satisfy the constriants } G. \end{aligned} \tag{13}$$

Extending the standard scheduling notations [11], this problem is denoted as

$$1|prec, \tilde{p}, resource, \vec{d}|\bar{M}\Big(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i\Big). \tag{14}$$

**Definition 2.3.** Let the completion time and the due date of a job be $\tilde{C}$ and $d$. If the crisp possibilistic mean value of the job's earliness-tardiness penalties

$$\bar{M}(e\max\{d - \tilde{C}, 0\} + t\max\{\tilde{C} - d, 0\}) \tag{15}$$

reaches the minimal value when $d = d^*$, then $d^*$ is called the optimal due date with respect to the completion time $\tilde{C}$.

**Definition 2.4.** For a feasible schedule $\pi$ of jobs, assign the due date $d_i$ and the amount $u_i$ of additional resources to $J_i(i = 1, \cdots, n)$ and let $\vec{d} = (d_1, \cdots, d_n)$ and $\vec{u} = (u_1, \cdots, u_n)$. $< \pi^*, \vec{d^*}, \vec{u^*} >$ is call the optimal schedule of jobs, if $f(\pi, \vec{d}, \vec{u}) = \bar{M}(\sum_{i=1}^n e_i\tilde{E}_i + \sum_{i=1}^n t_i\tilde{T}_i + \delta\tilde{\mathrm{C}}\mathrm{max} + \sum_{i=1}^n v_iu_i)$ reaches the minimal value when $< \pi, \vec{d}, \vec{u} >=< \pi^*, \vec{d^*}, \vec{u^*} >$. For convenience, $\pi$ is also called an optimal schedule with $\vec{d}$ and $\vec{u}$, or $\pi$ is optimal.

## 3. Computational Complexity Results

In this section, we examine the computational complexity of the problem discussed in this paper, and show that it is NP-hard.

**Lemma 3.1.** *Let the per unit earliness penalties, the per unit tardiness penalties and the completion time of a job be* $e, t,$ *and* $\tilde{C} = (a, b, \alpha)$ *with ratio* $\lambda = (b - a)/\alpha$*, then the optimal due date* $d^*$ *of the job with respect to the completion time* $\tilde{C}$ *is as follows:*

$$d^* = \bar{M}(\tilde{C}) + \frac{k^* - \frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}}\sigma_{\tilde{C}}, \tag{16}$$

*where*

$$k^* = \begin{cases} \sqrt{\frac{2t}{t+e}} - 1, & t \le e; \\ (\lambda + 1 - \sqrt{\frac{2e}{t+e}}), & t > e. \end{cases} \tag{17}$$

*Proof.* Do linear transform that $d = \bar{M}(\tilde{C}) + \frac{k - \frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}}\sigma_{\tilde{C}}$. According to $\bar{M}(\tilde{C}) = \frac{a+b}{2}$, $\sigma_{\tilde{C}} = \sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}\alpha$ and $\lambda = \frac{b-a}{\alpha}$, the linear transformation can be repressed by

$$d = \bar{M}(\tilde{C}) + \frac{k - \frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}}\sigma_{\tilde{C}} = \frac{a+b}{2} + (k - \frac{\lambda}{2})\alpha = a + k\alpha. \tag{18}$$

By equation (18) above, the crisp possibilistic mean value of the job's earliness and tardiness penalties can be expressed as follows:

$$\begin{aligned} &\bar{M}(e\max\{d - \tilde{C}, 0\} + t\max\{\tilde{C} - d, 0\}) \\ &= \bar{M}(e\max\{a + k\alpha - \tilde{C}, 0\} + t\max\{\tilde{C} - (a + k\alpha), 0\}) \triangleq g(k). \end{aligned} \tag{19}$$

Then, we have

$$
\begin{aligned}
g(k) &= \bar{M}(e\max\{(a+k\alpha)-\tilde{C},0\}+t\max\{\tilde{C}-(a+k\alpha),0\}) \\
&= \begin{cases}
t\left(\frac{\lambda}{2}-k\right)\alpha, & k<-1; \\
t[\int_0^{1+k}\gamma\,(\lambda+1-k-\gamma)\,\alpha\mathrm{d}\gamma+\int_{1+k}^{1}\gamma\,(\lambda-2k)\,\alpha\mathrm{d}\gamma] & \\
+e[\int_0^{1+k}\gamma\,(0+1+k-\gamma)\,\alpha\mathrm{d}\gamma], & -1\le k<0; \\
e[\int_0^{1}\gamma\,(1+k-\gamma)\,\alpha\mathrm{d}\gamma]+t[\int_0^{1}\gamma\,(\lambda+1-k-\gamma)\,\alpha\mathrm{d}\gamma], & 0\le k<\lambda; \\
e[\int_0^{1+\lambda-k}\gamma\,(1+k-\gamma)\,\alpha\mathrm{d}\gamma+\int_{1+\lambda-k}^{1}\gamma\,(2k-\lambda)\,\alpha\mathrm{d}\gamma] & \\
+t[\int_0^{1+\lambda-k}\gamma\,(1+\lambda-k-\gamma)\,\alpha\mathrm{d}\gamma], & \lambda\le k<1+\lambda; \\
e\alpha\left(k-\frac{\lambda}{2}\right), & k\ge\lambda+1,
\end{cases} \\
&= \alpha\begin{cases}
t\left(\dfrac{\lambda}{2}-k\right), & k<-1; \\
t\left[\dfrac{(1+k)^3}{6}-k+\dfrac{\lambda}{2}\right]+e\dfrac{(1+k)^3}{6}, & -1\le k<0; \\
t\left(\dfrac{\lambda}{2}-\dfrac{k}{2}+\dfrac{1}{6}\right)+e\dfrac{1+3k}{6}, & 0\le k<\lambda; \\
t\dfrac{(1+\lambda-k)^3}{6}+e\left[\dfrac{(1+\lambda-k)^3}{6}+k-\dfrac{\lambda}{2}\right], & \lambda\le k<\lambda+1; \\
e\left(k-\dfrac{\lambda}{2}\right), & k\ge\lambda+1,
\end{cases} \\
&= \alpha h(e,t,\lambda,k),
\end{aligned}
\tag{20}
$$

where

$$
h(e,t,\lambda,k)=\begin{cases}
t\left(\dfrac{\lambda}{2}-k\right), & k<-1; \\
t\left[\dfrac{(1+k)^3}{6}-k+\dfrac{\lambda}{2}\right]+e\dfrac{(1+k)^3}{6}, & -1\le k<0; \\
t\left(\dfrac{\lambda}{2}-\dfrac{k}{2}+\dfrac{1}{6}\right)+e\dfrac{1+3k}{6}, & 0\le k<\lambda; \\
t\dfrac{(1+\lambda-k)^3}{6}+e\left[\dfrac{(1+\lambda-k)^3}{6}+k-\dfrac{\lambda}{2}\right], & \lambda\le k<\lambda+1; \\
e\left(k-\dfrac{\lambda}{2}\right), & k\ge\lambda+1.
\end{cases}
\tag{21}
$$

It is easy to get that

$$(22)\qquad g'(k)=\alpha\begin{cases}-t, & k<-1;\\ t\dfrac{(1+k)^2}{2}+e\dfrac{(1+k)^2}{2}-t, & -1\le k<0;\\ \dfrac{e}{2}-\dfrac{t}{2}, & 0\le k<\lambda;\\ e-t\dfrac{(1+\lambda-k)^2}{2}-e\dfrac{(1+\lambda-k)^2}{2}, & \lambda\le k<\lambda+1;\\ e, & k\ge\lambda+1,\end{cases}$$

and $g''(k)\ge 0$. Hence, $g(k)$ reaches the minimum when

$$(23)\qquad k=k^*=\begin{cases}\sqrt{\dfrac{2t}{t+e}}-1, & t\le e;\\ (\lambda+1-\sqrt{\dfrac{2e}{t+e}}), & t>e.\end{cases}$$

It follows that the optimal due date is

$$(24)\qquad d^*=a+k^*\alpha=\bar{M}(\tilde{C})+\frac{k^*-\frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}}.$$

□

Note that the linear transformation $d=\bar{M}(\tilde{C})+\frac{k-\frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}}=a+k\alpha$ plays an important role in obtaining the optimal due date with respect to $\tilde{C}=(a,b,\alpha)$ with the ratio $\lambda=\frac{b-a}{\alpha}$. In fact, $k=\frac{d-a}{\alpha}=\frac{\lambda}{2}+\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}\frac{[d-\bar{M}(\tilde{C})]}{\sigma_{\tilde{C}}}$ is an important index which can express the level of the due date minimizing the earliness and tardiness penalties with respect to the completion time in some sense. In practice, $k$ is sometimes called the customer service level of the due date with respect to the completion time in uncertain environment [14, 18].

Also, by the linear transformation $d=\bar{M}(\tilde{C})+\frac{k-\frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}}$, the earliness $\tilde{E}$ and tardiness $\tilde{T}$ can also be represented as $\tilde{E}=\max\{d-\tilde{C},0\}=\max\{\bar{M}(\tilde{C})+\frac{k-\frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}}-\tilde{C},0\}$, and $\tilde{T}=\max\{\tilde{C}-d,0\}=\max\{\tilde{C}-(\bar{M}(\tilde{C})+\frac{k-\frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}}),0\}$, respectively. Hence, the crisp possibilistic mean value of the job's earliness-tardiness penalties $\bar{M}(e\tilde{E}+t\tilde{T})=\bar{M}(e\max\{\bar{M}(\tilde{C})+\frac{(k-\frac{\lambda}{2})\sigma_{\tilde{C}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}-\tilde{C},0\}+t\max\{\tilde{C}-(\bar{M}(\tilde{C})+\frac{(k-\frac{\lambda}{2})\sigma_{\tilde{C}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}),0\})$.

**Definition 3.2.** For a job with the due date $d$ and the completion time $\tilde{C}=(a,b,\alpha)$ with ratio $\lambda=\frac{b-a}{\alpha}$,

$$(25)\qquad k=\frac{\lambda}{2}+\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}\frac{[d-\bar{M}(\tilde{C})]}{\sigma_{\tilde{C}}}$$

is called the customer service level of the due date $d$ respect to the completion time $\tilde{C}$; $k^*$ is called the optimal customer service level with respect to the completion time $\tilde{C}$, if the crisp possibilistic mean value of the job's earliness-tardiness penalties

$$\bar{M}(e\max\{\bar{M}(\tilde{C})+\frac{(k-\frac{\lambda}{2})\sigma_{\tilde{C}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}-\tilde{C},0\}+t\max\{\tilde{C}-(\bar{M}(\tilde{C})+\frac{(k-\frac{\lambda}{2})\sigma_{\tilde{C}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}),0\}) \tag{26}$$

reaches the minimal value when $k = k^*$.

It is easy to verify that when $d^*$ is the optimal due date with respect to the completion time $\tilde{C} = (a, b, \alpha)$ with $\lambda = (b - a)/\alpha$, $k^* \triangleq (d^* - a)/\alpha = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}\frac{d^*-\bar{M}(\tilde{C})}{\sigma_{\tilde{C}}}$ is the optimal customer service level with respect to $\tilde{C}$. On the other hand, if $k^*$ is the optimal customer service level with respect to $\tilde{C} = (a, b, \alpha)$ with $\lambda = (b-a)/\alpha$, then $d^* = a + k^*\alpha = \bar{M}(\tilde{C}) + \frac{k^*-\frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}}$ is the optimal due date with respect to $\tilde{C}$.

Let $\vec{k} = (k_1, k_2, \cdots, k_n)$ where $k_i$ is the customer service level of the due date $d_i$ with the completion $\tilde{C}_i$ with ratio $\lambda_i$. According to Definition 3.2, the objective function $f(\pi, \vec{d}, \vec{u})$ can be represented by $\vec{k}$ as follows:

$$\begin{aligned}
f(\pi,\vec{d},\vec{u}) &= \bar{M}(\sum_{i=1}^{n} e_i\tilde{E}_i + \sum_{i=1}^{n} t_i\tilde{T}_i + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_iu_i) \\
&= \bar{M}(\sum_{i=1}^{n} e_i\max\{d_i-\tilde{C}_i,0\}+t_i\max\{\tilde{C}_i-d_i,0\})+\delta\bar{M}(\tilde{\mathrm{C}}_{\max})+\sum_{i=1}^{n} v_iu_i \\
&= \sum_{i=1}^{n}\bar{M}(e_i\max\{\bar{M}(\tilde{C}_i)+\frac{(k_i-\frac{\lambda_i}{2})\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda_i^2}{4}+\frac{\lambda_i}{3}+\frac{1}{6}}}-\tilde{C}_i,0\} \\
&\qquad +t_i\max\{\tilde{C}_i-(\bar{M}(\tilde{C}_i)+\frac{(k_i-\frac{\lambda_i}{2})\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda_i^2}{4}+\frac{\lambda_i}{3}+\frac{1}{6}}}),0\}) \\
&\qquad +\delta\bar{M}(\max\{\tilde{C}_1,\cdots,\tilde{C}_n\})+\sum_{i=1}^{n} v_iu_i \\
&\triangleq g(\pi,\vec{k},\vec{u}).
\end{aligned} \tag{27}$$

Our objective is also to sequence the jobs under precedence constraints and determine the customer service levels $\vec{k} = (k_1, k_2, \cdots, k_n)$ and the amounts $\vec{u} = (u_1, u_2, \cdots, u_n)$ of resources assigned to jobs to minimize the function $g(\pi, \vec{k}, \vec{u})$.

Furthermore, without loss of generality, let $\pi = (J_1, J_2, \cdots, J_n)$ with the basic processing time $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ with $\lambda_i = (b_i - a_i)/\alpha_i$ and $u_i$ be the amount of the additional resources allocated to job $J_i$. Then the completion time of job $J_i$ is $\tilde{C}_i = \sum_{j=1}^{i}\tilde{p}_i = (\sum_{j=1}^{i}(a_j-s_ju_j), \sum_{j=1}^{i}(b_j-s_ju_j), \sum_{j=1}^{i}\alpha_j)$ $(i = 1, \cdots, n)$. Given customer service level $k_i$ for $J_i$, the due date of $J_i$ under $k_i$ is

$$d_i = \bar{M}(\tilde{C}_i) + \frac{k_i - \frac{\lambda_{\tilde{C}_i}}{2}}{\sqrt{\frac{\lambda^2_{\tilde{C}_i}}{4} + \frac{\lambda_{\tilde{C}_i}}{3} + \frac{1}{6}}}\sigma_{\tilde{C}_i} = \sum_{j=1}^{i}(a_j - s_j u_j) + k_i \sum_{j=1}^{i} \alpha_j, \tag{28}$$

where $\lambda_{\tilde{C}_i} = \dfrac{\Sigma_{j=1}^{i}(b_j - a_j)}{\Sigma_{j=1}^{i}\alpha_j}$. It follows by Lemma 3.1 that

$$\begin{aligned} g(\pi, \vec{k}, \vec{u}) &= \bar{M}(\sum_{i=1}^{n} e_i \max\{\sum_{j=1}^{i}(a_j - s_j u_j) + k_i \sum_{j=1}^{i} \alpha_j - \tilde{C}_i, 0\} \\ &\qquad + t_i \max\{\tilde{C}_i - (\sum_{j=1}^{i}(a_j - s_j u_j) + k_i \sum_{j=1}^{i} \alpha_j), 0\}) \\ &\quad + \delta \bar{M}(\tilde{\mathrm{C}}\mathrm{max}) + \sum_{i=1}^{n} v_i u_i \\ &= \sum_{i=1}^{n} \bar{M}(e_i \max\{\sum_{j=1}^{i}(a_j - s_j u_j) + k_i \sum_{j=1}^{i} \alpha_j - \tilde{C}_i, 0\} \\ &\qquad + t_i \max\{\tilde{C}_i - (\sum_{j=1}^{i}(a_j - s_j u_j) + k_i \sum_{j=1}^{i} \alpha_j), 0\}) \\ &\quad + \delta \bar{M}(\tilde{C}_n) + \sum_{i=1}^{n} v_i u_i \\ &= \sum_{i=1}^{n}\{h(e_i, t_i, \lambda_{\tilde{C}_i}, k_i) \sum_{j=1}^{i} \alpha_j\} + \delta[\sum_{i=1}^{n} \frac{a_i + b_i}{2} - \sum_{i=1}^{n} s_i u_i] \\ &\quad + \sum_{i=1}^{n} v_i u_i \\ &= \sum_{i=1}^{n}\{h(e_i, t_i, \lambda_{\tilde{C}_i}, k_i) \frac{\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda^2_{\tilde{C}_i}}{4} + \frac{\lambda_{\tilde{C}_i}}{3} + \frac{1}{6}}}\} + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} \\ &\quad + \sum_{i=1}^{n}(v_i - \delta s_i) u_i. \end{aligned} \tag{29}$$

**Lemma 3.3.** *Let $\pi$ be an optimal schedule of jobs set $\mathcal{J} = \{J_1, J_2, \cdots, J_n\}$ with $\vec{k} = (k_1, k_2, \cdots, k_n)$ and $\vec{u} = (u_1, u_2, \cdots, u_n)$, in which $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ with $\lambda_i = (b_i - a_i)/\alpha_i (i = 1, \cdots, n)$. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, then*

$$k_i = k_i^* \triangleq \begin{cases} \sqrt{\frac{2t_i}{t_i + e_i}} - 1, & t_i \le e_i; \\ (\lambda_i + 1 - \sqrt{\frac{2e_i}{t_i + e_i}}), & t_i > e_i, \end{cases} \tag{30}$$

*and $\vec{u} = (u_1, u_2, \cdots, u_n)$ is a solution of the linear programming as follows:*

$$\begin{aligned} & min && \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i \\ & s.t. && 0 \le u_i \le \frac{a_i - \alpha_i}{s_i}; \\ & && \sum_{i=1}^{n} u_i \le U_0. \end{aligned} \tag{31}$$

*Proof.* According to $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ for the jobs with $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)(i = 1, \cdots, n)$, it is easy to verify that $\lambda_{\tilde{C}_i} = \dfrac{b_{c_i} - a_{c_i}}{\alpha_{c_i}} = \lambda$ for the completion time $\tilde{C}_i = (a_{c_i}, b_{c_i}, \alpha_{c_i})$ for job $J_i$ in any feasible schedule of jobs. According to Lemma 3.1 and equation (20), we have $h(e_{i_j}, t_{i_j}, \lambda, k_{i_j}) \ge h(e_{i_j}, t_{i_j}, \lambda, k^*_{i_j})$. Suppose there exists $j_0$ such that $k_{j_0} \ne k^*_{j_0}$. Without loss of generality, let $j_0 = 1$. It follows that

$$\begin{aligned} g(\pi, \vec{k}, \vec{u}) =& \sum_{l=1}^{n} \{h(e_{i_l}, t_{i_l}, \lambda_{i_l}, k_{i_l}) \sum_{j=1}^{l} \alpha_{i_j}\} + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i \\ &> h(e_{i_1}, t_{i_1}, \lambda_{i_1}, k^*_{i_1}) \alpha_{i_1} + \sum_{l=2}^{n} \{h(e_{i_l}, t_{i_l}, \lambda_{i_l}, k^*_{il}) \sum_{j=1}^{l} \alpha_{i_j}\} \\ &\quad + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i \\ &= g(\pi, \vec{k}^*, \vec{u}), \end{aligned} \tag{32}$$

where $\vec{k}^* = (k^*_1, k^*_2, \cdots, k^*_n)$. A contradiction appears since $\pi$ is optimal.

Also, suppose $\vec{u}$ is not a solution of the linear programming (31). Since $\pi$ is feasible, it is easy to get $0 \le u_i \le (a_i - \alpha_i)/s_i$ and $\sum_{i=1}^{n} u_i \le U_0$, which implies $\vec{u}$ is in feasible domain of linear programming (31). Let $\vec{u}^*$ be a solution of the linear programming (31). Then, we have

$$\begin{aligned} g(\pi, \vec{k}, \vec{u}) &= \sum_{i=1}^{n} \{h(e_i, t_i, \lambda_{\tilde{C}_i}, k_i) \sum_{j=1}^{i} \alpha_j\} + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i \\ &> \sum_{i=1}^{n} \{h(e_i, t_i, \lambda_{\tilde{C}_i}, k_i) \sum_{j=1}^{i} \alpha_j\} + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u^*_i \\ &= g(\pi, \vec{k}, \vec{u}^*). \end{aligned} \tag{33}$$

A contradiction also appears since $\pi$ is optimal. □

In the rest of the paper, we use $\vec{k}^* = (k^*_1, k^*_2, \cdots, k^*_n)$ to denote the optimal service level vector for an optimal schedule $\pi$ of the jobs, where $k^*_i$ is obtained by equation (30).

By the lemma above, a reduction based on the problem $1|prep|\sum_{i=1}^{n} w_i C_i$ can be obtained.

**Theorem 3.4.** *The problem* $1|prec, \tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{C}max + \sum_{i=1}^{n} v_i u_i)$ *is NP-hard, even if* $\lambda_{\bar{\tilde{p}}_i} = (b_i - a_i)/\alpha_i = \lambda$ *for the basic processing time* $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ *of* $J_i (i = 1, \cdots, n)$.

*Proof.* We will show that the NP-hard problem $1|prep|\sum_{i=1}^{n} w_i C_i$ can be polynomially reducible to the problem $1|prec, \tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}\mathrm{max} + \sum_{i=1}^{n} v_i u_i)$. Let us recall that an instance $I$ of $1|prep|\sum_{i=1}^{n} w_i C_i$, in which jobs set $\mathcal{J} = \{J_1, \cdots, J_n\}$, the precedence constraints between jobs is $G$, job $J_i$'s processing time $p_i > 0$, and the weight of job $J_i$ is $w_i (i = 1, \cdots, n)$. The corresponding instance $I'$ of the problem $1|prec, \tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}\mathrm{max} + \sum_{i=1}^{n} v_i u_i)$ is constructed as follows: let jobs set $\mathcal{J}' = \{J'_1, J'_2, \cdots, J'_n\}$ where $J'_i$ corresponds $J_i$ of instance $I$; for $J'_i$, let $\bar{\tilde{p}}_i = (p_i + 1, 2p_i + 1, p_i)$, $t_i = e_i = \frac{6}{5} w_i$, $\delta = 1, s_i = 1, v_i = 2, 0 \le u_i \le 1 (i = 1, \cdots, n)$, and $\sum_{i=1}^{n} u_i \le n$; let the precedence constraints between jobs in $\mathcal{J}'$ denoted as $G'$ be same as $G$.

Without of generality, let a sequence of the instance $I$ be $\pi = (J_1, J_2, \cdots, J_n)$. Then the corresponding sequence of instance $I'$ is $\pi' = (J'_1, J'_2, \cdots, J'_n)$. Since two instances have the same precedence constraints, $\pi$ is feasible schedule of $I$ if and only if $\pi'$ is feasible one of $I'$. Also, for job $J_i$, it is easy to get the ratio for $\bar{\tilde{p}}_i = (p_i + 1, 2p_i + 1, p_i)$ is 1; it follows that the ratio of $\tilde{C}_i$ is also 1; by equation (30), we have the optimal customer service level $k_i^* = 0$; it follows that $h(e_i, t_i, 1, 0) = \frac{5e_i}{6} = w_i$. Let $\vec{u}^*$ be a solution of the following linear programming

$$
\begin{aligned}
\text{min} \quad & \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i = \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} \frac{3p_i + 2}{2} \\
\text{s.t.} \quad & 0 \le u_i \le \frac{p_i + 1 - p_i}{s_i} = 1; \\
& \sum_{i=1}^{n} u_i \le U_0 = n.
\end{aligned} \tag{34}
$$

It is easy to verify that the above linear programming has only one solution $\vec{u} = (0, \cdots, 0)$. Hence, $\vec{u}^* = (0, \cdots, 0)$. Then we have

$$
\begin{aligned}
f(\pi', \vec{d}^*, \vec{u}^*) &= g(\pi', \vec{k}^*, \vec{u}^*) = \sum_{i=1}^{n} \{h(e_i, t_i, 1, 0) \sum_{j=1}^{i} p_j\} + \sum_{i=1}^{n} \frac{3p_i + 2}{2} \\
&= \sum_{i=1}^{n} w_i (\sum_{j=1}^{i} p_j) + \sum_{i=1}^{n} \frac{3p_i + 2}{2} = \sum_{i=1}^{n} w_i C_i + \sum_{i=1}^{n} \frac{3p_i + 2}{2}.
\end{aligned} \tag{35}
$$

Equation (35) implies that $\pi$ is an optimal schedule of $I$ if and only if $\pi'$ is an optimal one of $I'$. It is easy to verify that instance $I'$ can be obtained from instance $I$ in polynomial time. This means that having a polynomial algorithm for $1|prec, resource, \tilde{p}, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}\mathrm{max} + \sum_{i=1}^{n} v_i u_i)$, we would be able to solve the NP-hard problem $1|prep|\sum_{i=1}^{n} w_i C_i$ in polynomial time. Therefore, the problem $1|prec, \tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}\mathrm{max} + \sum_{i=1}^{n} v_i u_i)$ is NP-hard. □

## 4. The Polynomially Solvable Case

In above section, we show that the problem $1|prec, \tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i)$ is NP-hard, even if $\lambda_{\bar{\tilde{p}}_i} = \lambda(i = 1, \cdots, n)$. In this section, we focus on the problem with no precedence constraints between jobs, which is denoted as $1|\tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i)$.

In the rest of this section, it is assume that $\lambda_{\bar{\tilde{p}}_i} = (b_i - a_i)/\alpha_i = \lambda$ for the basic processing time $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ of $J_i(i = 1, \cdots, n)$. Indeed, we can design an optimal algorithm with polynomial time for the case that $\lambda_{\bar{\tilde{p}}_i} = \lambda(i = 1, \cdots, n)$ as follows:

**Algorithm 1.**

Step 0. Input the cost of one unit of operation time $\delta$, the total amount of resources $U_0$, the basic processing time $(a_i, b_i, \alpha_i)$, the per unit earliness penalties $e_i$, the per unit tardiness penalties $t_i$, the cost of one unit of resources $v_i$, and the compression rate $s_i$ for job $J_i(i = 1, \cdots, n)$.

Step 1. Construct the linear programming induce by the problem $1|\tilde{p}, resource, \vec{d}|$ $\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i)$ according to (31) as follows:

$$\begin{aligned} &\min \quad \delta\sum_{i=1}^{n}\frac{a_i + b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i \\ &\text{s.t.} \quad 0 \le u_i \le \frac{a_i - \alpha_i}{s_i}(i = 1, \cdots, n); \\ &\qquad \sum_{i=1}^{n} u_i \le U_0. \end{aligned} \tag{36}$$

Solve the linear programming above and obtain the solution $\vec{u} = (u_1, \cdots, u_n)$. Then compute the actual processing time $\tilde{p}_i = (a_i, b_i, \alpha_i) - s_i u_i = (a_i - s_i u_i, b_i - s_i u_i, \alpha_i)$ for $J_i(i = 1, \cdots, n)$.

Step 2. Compute the optimal customer service level $k_i$ according to equation (30), then let $w_i = h(e_i, t_i, \lambda, k_i)$ where the function $h(e, t, \lambda, k)$ is given by (21) and $\lambda = (b_i - a_i)/\alpha_i(i = 1, \cdots, n)$.

Step 3. Schedule the jobs according to the nonincreasing order of $w_i/\alpha_i$. This schedule is denoted as $\pi$.

Step 4. Compute the completion time $\tilde{C}_i$ of $J_i$ in the schedule $\pi$ obtained in step 3, then assign the due date $d_i$ to $J_i$ by $d_i = \bar{M}(\tilde{C}_i) + \frac{k_i - \frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}}\sigma_{\tilde{C}_i}(i = 1, \cdots, n)$.

Step 5. Return $\pi$, $\vec{d} = (d_1, d_2, \cdots, d_n)$, $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{k} = (k_1, k_2, \cdots, k_n)$.

**Theorem 4.1.** ***Algorithm 1*** *is an optimal algorithm with polynomial time for the problem* $1|\tilde{p}, resource, \vec{d}|$ $\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{C}max + \sum_{i=1}^{n} v_i u_i)$ *with* $\lambda_{\bar{\tilde{p}}_i} = (b_i - a_i)/\alpha_i = \lambda$ *for the basic processing time* $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ *of* $J_i(i = 1, \cdots, n)$.

*Proof.* It is easy to know that the time spent on the Step 0, Step 2, Step 4 and Step 5 of Algorithm 1 is O($n$), and the time spent on Step 3 is O($nlog(n)$). Also, according to [8], the linear programming (36) can be solve in polynomial time. Hence, Algorithm 1 is a polynomial time algorithm.

Assume that there exists an optimal schedule $\pi$ with $\vec{k}$ and $\vec{u}$ not obtained by Algorithm 1. Known from Lemma 3.3, $k_i$ is the same as one obtained by Algorithm 1 for $J_i$. Also, $\vec{u}$ is a solution of the linear programming of the step 4. According to the logic of Algorithm 1, it follows that there are at least two adjacent jobs $J_j$ and $J_i$ ($J_j$ before $J_i$ in $\pi$) such that $(w_j/\alpha_j) < (w_i/\alpha_i)$. Next, let schedule $\pi'$ be obtained by interchanging $J_j$ and $J_i$ without changing other jobs. According to equation (29), it is easy to get

$$g(\pi,\vec{k},\vec{u}) - g(\pi',\vec{k},\vec{u}) = \alpha_j\alpha_i(\frac{w_i}{\alpha_i} - \frac{w_j}{\alpha_j}) > 0. \tag{37}$$

Thus, a contradiction appears. □

## 5. Approximation Algorithm

Note that the problem $1|prec,\tilde{p},resource,\vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\text{C}}\text{max} + \sum_{i=1}^{n} v_iu_i)$ is NP-hard even if $\lambda_{\bar{\tilde{p}}_i} = \frac{b_i-a_i}{\alpha_i} = \lambda$ for the basic processing time $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$. Motivated by some polyhedral methods for single machine scheduling problems [12, 25], we can design an approximation algorithm for the problem $1|prec,\tilde{p},resource,\vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\text{C}}\text{max} + \sum_{i=1}^{n} v_iu_i)$ when $\lambda_{\bar{\tilde{p}}_i} = \lambda$ based on solving a linear relaxation of the problem, whose worst case ratio is shown to be 2.

In a feasible $\pi$, the actual processing time and completion time of $J_i (i = 1,\cdots,n)$ are $\tilde{p}_i$ and $\tilde{C}_i$. And, the standard deviation of the actual processing time and the completion time of job $J_i$ are denoted as $\sigma_{\tilde{p}_i}$ and $\sigma_{\tilde{C}_i}$, respectively. By Definition 2.2, for $\tilde{p}_i$ and $\tilde{p}_j$, we have $\sigma_{\tilde{p}_i} = \sigma_{\bar{\tilde{p}}_i}$, and $\sigma_{\tilde{p}_i+\tilde{p}_j} = \sigma_{\tilde{p}_i} + \sigma_{\tilde{p}_j}$; furthermore, for $\tilde{C}_i$ and $\tilde{C}_j$, we also have $\sigma_{\tilde{C}_i+\tilde{C}_j} = \sigma_{\tilde{C}_i} + \sigma_{\tilde{C}_j}$. It follows that $\sigma_{\tilde{C}_i} \le \sigma_{\tilde{C}_j}$ if $J_i$ is processed before $J_j$ in a feasible schedule $\pi$, which means that the schedule $\pi$ is in accord with the sequence of $\sigma_{\tilde{C}_1},\cdots,\sigma_{\tilde{C}_n}$ which is sorted from small to large. Also, the precedence constraints $G$ can also be presented as ones of $\sigma_{\tilde{C}_1},\cdots,\sigma_{\tilde{C}_n}$ as follows: $\sigma_{\tilde{C}_i} \ge \sigma_{\tilde{p}_i}$ for $i = 1,\cdots,n$; $\sigma_{\tilde{C}_k} \ge \sigma_{\tilde{C}_i} + \sigma_{\tilde{p}_k}$ for each arc $J_iJ_k$ in $G$, and $\sigma_{\tilde{C}_k} \ge \sigma_{\tilde{C}_i} + \sigma_{\tilde{p}_k}$ or $\sigma_{\tilde{C}_i} \ge \sigma_{\tilde{C}_k} + \sigma_{\tilde{p}_i}$ for any two jobs $J_k$ and $J_i$. According to Lemma 3.1 and equation (29), for the schedule $\pi$ with the customer service levels $\vec{k}$ and assigned additional resource amounts $\vec{u}$, the objective function is $g(\pi,\vec{k},\vec{u}) = \sum_{i=1}^{n}\{h(e_i,t_i,\lambda_{\tilde{C}_i},k_i)\frac{\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda_{\tilde{C}_i}^2}{4}+\frac{\lambda_{\tilde{C}_i}}{3}+\frac{1}{6}}}\} + \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i$. Hence, we can formulate $1|prec,\tilde{p},resource,\vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\text{C}}\text{max} + \sum_{i=1}^{n} v_iu_i)$ in the following way, where $\sigma_{\tilde{C}_i}$ are used as the variables and the constraints ensure that the variables $\sigma_{\tilde{C}_1},\cdots,\sigma_{\tilde{C}_n}$ specify a feasible set of standard deviations of the completion times:

$$\min\sum_{i=1}^{n}\{h(e_i,t_i,\lambda_{\tilde{C}_i},k_i)\frac{\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda_{\tilde{C}_i}^2}{4}+\frac{\lambda_{\tilde{C}_i}}{3}+\frac{1}{6}}}\} + \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i \tag{38}$$

$$\text{(39)} \quad \text{s.t.} \quad 0 \le u_i \le \frac{a_i - \alpha_i}{s_i};$$

$$\text{(40)} \quad \sum_{i=1}^{n} u_i \le U_0;$$

$$\text{(41)} \quad \sigma_{\tilde{C}_i} \ge \sigma_{\tilde{p}_i} \text{ for } i = 1, \cdots, n;$$

$$\text{(42)} \quad \sigma_{\tilde{C}_k} \ge \sigma_{\tilde{C}_i} + \sigma_{\tilde{p}_k} \text{ for each arc } J_iJ_k \text{ in } G;$$

$$\text{(43)} \quad \sigma_{\tilde{C}_k} \ge \sigma_{\tilde{C}_i} + \sigma_{\tilde{p}_k} \text{ or } \sigma_{\tilde{C}_i} \ge \sigma_{\tilde{C}_k} + \sigma_{\tilde{p}_i} \text{ for any two jobs } J_k, J_i,$$

where $u_1, u_2, \cdots, u_n$ are also variables and $h(e, t, \lambda, k)$ is given in (21).

Note that equation (43) is not linear inequality. However, for a feasible scheduling $\pi = \{J_{\pi(1)}, J_{\pi(2)}, \cdots, J_{\pi(n)}\}$ of jobs set $\mathcal{J}$, let $S \subset \mathcal{J}$, then we schedule the jobs of $S$ according to the order of the jobs in $\pi$, which is denoted as $J_{\pi(s_1)} - J_{\pi(s_2)} - \cdots - J_{\pi(s_l)}$. It follows that $\sigma_{\tilde{C}_{\pi(k)}} = \sum_{j=1}^{k} \sigma_{\tilde{p}_{\pi(j)}}$ and $\sigma_{\tilde{C}_{\pi(s_j)}} = \sum_{i=1}^{s_j} \sigma_{\tilde{p}_{\pi(i)}}$. For convenience, the following notations are used: for any set $S \subset \mathcal{J} = \{J_1, \cdots, J_n\}$,

$$\text{(44)} \quad \sigma_{\tilde{p}}(S) \triangleq \sum_{J_i \in S} \sigma_{\tilde{p}_i}, \quad \sigma^2_{\tilde{p}}(S) \triangleq \sum_{J_i \in S} \sigma^2_{\tilde{p}_i} \text{ and } \sigma_{\tilde{p}}(S)^2 \triangleq [\sum_{J_i \in S} \sigma_{\tilde{p}_i}]^2.$$

Then, by $\sigma_{\tilde{p}_i} = \sigma_{\bar{\tilde{p}}_i}$, we have

$$\text{(45)} \quad \begin{aligned} \sum_{J_i \in S} \sigma_{\tilde{p}_i} \sigma_{\tilde{C}_i} &= \sum_{j=1}^{l} \sigma_{\tilde{p}_{\pi(s_j)}} \sigma_{\tilde{C}_{\pi(s_j)}} = \sum_{j=1}^{l} \sigma_{\tilde{p}_{\pi(s_j)}} (\sum_{i=1}^{s_j} \sigma_{\tilde{p}_{\pi(i)}}) \\ &= \sigma_{\tilde{p}_{\pi(s_1)}} [\sigma_{\tilde{p}_{\pi(1)}} + \sigma_{\tilde{p}_{\pi(2)}} + \cdots + \sigma_{\tilde{p}_{\pi(s_1)}}] \\ &\quad + \sigma_{\tilde{p}_{\pi(s_2)}} [\sigma_{\tilde{p}_{\pi(1)}} + \sigma_{\tilde{p}_{\pi(2)}} + \cdots \sigma_{\tilde{p}_{\pi(s_1)}} + \cdots + \sigma_{\tilde{p}_{\pi(s_2)}}] + \cdots \\ &\quad + \sigma_{\tilde{p}_{\pi(s_l)}} [\sigma_{\tilde{p}_{\pi(1)}} + \sigma_{\tilde{p}_{\pi(2)}} + \cdots + \sigma_{\tilde{p}_{\pi(s_{l-1})}} + \cdots + \sigma_{\tilde{p}_{\pi(s_l)}}] \\ &\ge \sigma_{\tilde{p}_{\pi(s_1)}} [\sigma_{\tilde{p}_{\pi(s_1)}}] + \sigma_{\tilde{p}_{\pi(s_2)}} [\sigma_{\tilde{p}_{\pi(s_1)}} + \sigma_{\tilde{p}_{\pi(s_2)}}] + \cdots \\ &\quad + \sigma_{\tilde{p}_{\pi(s_l)}} [\sigma_{\tilde{p}_{\pi(s_1)}} + \sigma_{\tilde{p}_{\pi(s_2)}} + \cdots + \sigma_{\tilde{p}_{\pi(s_l)}}] \\ &= \frac{1}{2} [\sum_{j=1}^{l} \sigma^2_{\tilde{p}_{\pi(s_j)}} + (\sum_{j=1}^{l} \sigma_{\tilde{p}_{\pi(s_j)}})^2] = \frac{1}{2} [\sigma^2_{\tilde{p}}(S) + \sigma_{\tilde{p}}(S)^2]. \end{aligned}$$

Based on the above linear inequalities, using the standard deviations of the completion times as the variables, a linear programming relaxation of the problem$1|prec, \tilde{p}, resource, \vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\mathrm{C}}_{\max} + \sum_{i=1}^{n} v_i u_i)$ can be represented as follows:

$$\begin{aligned}
&\min\ \sum_{i=1}^{n} h(e_i,t_i,\lambda_{\tilde{C}_i},k_i)\frac{\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda^2_{\tilde{C}_i}}{4}+\frac{\lambda_{\tilde{C}_i}}{3}+\frac{1}{6}}}+\delta\sum_{i=1}^{n}\frac{a_i+b_i}{2}+\sum_{i=1}^{n}(v_i-\delta s_i)u_i\\
&\text{s.t.}\quad 0\le u_i\le \frac{a_i-\alpha_i}{s_i};\\
&\sum_{i=1}^{n}u_i\le U_0;\\
&\sigma_{\tilde{C}_i}\ge\sigma_{\tilde{p}_i}\ \text{for } i=1,\cdots,n;\\
&\sigma_{\tilde{C}_k}\ge\sigma_{\tilde{C}_i}+\sigma_{\tilde{p}_k}\ \text{for each arc } J_iJ_k \text{ in } G;\\
&\sum_{J_j\in S}\sigma_{\tilde{p}_j}\sigma_{\tilde{C}_j}\ge\frac{1}{2}(\sigma^2_{\tilde{p}}(S)+\sigma_{\tilde{p}}(S)^2)\ \text{for each } S\subset\mathcal{J}.
\end{aligned}\tag{46}$$

It is easy to verify that $(\sigma_{\tilde{C}_1},\sigma_{\tilde{C}_2},\cdots,\sigma_{\tilde{C}_n},u_1,u_2,\cdots,u_n)$ is a solution of (46) if and only if $(\sigma_{\tilde{C}_1},\sigma_{\tilde{C}_2},\cdots,\sigma_{\tilde{C}_n})$ and $(u_1,u_2,\cdots,u_n)$ are the solutions of the following linear programs (47) and (48), respectively:

$$\begin{aligned}
&\min\quad \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2}+\sum_{i=1}^{n}(v_i-\delta s_i)u_i\\
&\text{s.t.}\quad 0\le u_i\le\frac{a_i-\alpha_i}{s_i};\\
&\sum_{i=1}^{n}u_i\le U_0,
\end{aligned}\tag{47}$$

and

$$\begin{aligned}
&\min\quad \sum_{i=1}^{n}\{h(e_i,t_i,\lambda_{\tilde{C}_i},k_i)\frac{\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda^2_{\tilde{C}_i}}{4}+\frac{\lambda_{\tilde{C}_i}}{3}+\frac{1}{6}}}\}\\
&\text{s.t.}\quad \sigma_{\tilde{C}_i}\ge\sigma_{\tilde{p}_i}\ \text{for } i=1,\cdots,n;\\
&\sigma_{\tilde{C}_k}\ge\sigma_{\tilde{C}_i}+\sigma_{\tilde{p}_k}\ \text{for each arc } J_iJ_k \text{ in } G;\\
&\sum_{J_j\in S}\sigma_{\tilde{p}_j}\sigma_{\tilde{C}_j}\ge\frac{1}{2}(\sigma^2_{\tilde{p}}(S)+\sigma_{\tilde{p}}(S)^2)\ \text{for each } S\subset\mathcal{J}.
\end{aligned}\tag{48}$$

Based on the linear programmes above, we can design an approximation algorithm for $1|prec,\tilde{p},\ resource,\vec{d}|\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i+t_i\tilde{T}_i)+\delta\tilde{\mathrm{C}}\mathrm{max}+\sum_{i=1}^{n}v_iu_i)$ with $\lambda_{\bar{\tilde{p}}_i}=\lambda(1,\cdots,n)$.

**Algorithm 2.**

Step 0. Input the cost of one unit of operation time $\delta$, the total amount of additional resources $U_0$, the precedence constraints $G$ among jobs, the basic processing time $(a_i,b_i,\alpha_i)$, the per unit earliness penalties $e_i$, the per unit tardiness penalties $t_i$, the cost of one unit of resources $v_i$, and the compression rate $s_i$ for job $J_i(i=1,\cdots,n)$.

Step 1. Construct the linear programming induce by $1|prec, \tilde{p}, resource, \vec{d}|$ $\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\text{C}}\text{max} + \sum_{i=1}^{n} v_i u_i)$ according to (47) as follows:

$$\begin{aligned}
&\min \quad && \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2}+\sum_{i=1}^{n}(v_i-\delta s_i)u_i \\
(49) \qquad &\text{s.t.} \quad && 0 \le u_i \le \frac{a_i-\alpha_i}{s_i}(i=1,\cdots,n); \\
& && \sum_{i=1}^{n} u_i \le U_0.
\end{aligned}$$

Solve the linear programming (49) and obtain the solution $\vec{u} = (u_1, \cdots, u_n)$. Then compute the actual processing time $\tilde{p} = (a_i, b_i, \alpha_i) - s_i u_i = (a_i - s_i u_i, b_i - s_i u_i, \alpha_i)$ for $J_i(i = 1, \cdots, n)$.

Step 2. Compute the optimal customer service level $k_i$ according to equation (30), then let $w_i = h(e_i, t_i, \lambda, k_i)$ where $\lambda = (b_i - a_i)/\alpha_i(i = 1, \cdots, n)$.

Step 3. Construct the linear programming induced by $1|prec, \tilde{p}, resource, \vec{d}|$ $\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{\text{C}}\text{max} + \sum_{i=1}^{n} v_i u_i)$ according to (48) as follows:

$$\begin{aligned}
&\min \quad && \sum_{i=1}^{n}\{h(e_i,t_i,\lambda,k_i)\frac{\sigma_{\tilde{C}_i}}{\sqrt{\frac{\lambda_{\tilde{C}_i}^2}{4}+\frac{\lambda_{\tilde{C}_i}}{3}+\frac{1}{6}}}\} \\
(50) \qquad &\text{s.t.} \quad && \sigma_{\tilde{C}_i} \ge \sigma_{\tilde{p}_i} \text{ for } i=1,\cdots,n; \\
& && \sigma_{\tilde{C}_k} \ge \sigma_{\tilde{C}_i}+\sigma_{\tilde{p}_k} \text{ for each arc } J_iJ_k \text{ in } G; \\
& && \sum_{J_j\in S}\sigma_{\tilde{p}_j}\sigma_{\tilde{C}_j} \ge \frac{1}{2}(\sigma_{\tilde{p}}^2(S)+\sigma_{\tilde{p}}(S)^2) \text{ for each } S\subset\mathcal{J}.
\end{aligned}$$

Solve the linear programming (50) and obtain the solution $(\bar{\sigma}_{\tilde{C}_1}, \bar{\sigma}_{\tilde{C}_2}, \cdots, \bar{\sigma}_{\tilde{C}_n})$.

Step 4. Schedule the jobs according to the nondecreasing order of $\bar{\sigma}_{\tilde{C}_j}$, where ties are broken by choosing an order that is consistent with the precedence constraints $G$. This schedule is denoted as $\pi$.

Step 5. Compute the completion time $\tilde{C}_i$ of $J_i$ in the schedule $\pi$ obtained in step 4, then assign the due date $d_i$ to $J_i$ by $d_i = \bar{M}(\tilde{C}_i) + \frac{k_i - \frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\sigma_{\tilde{C}_i}(i = 1, \cdots, n)$.

Step 6. Return $\pi$, $\vec{d}=(d_1, d_2,\cdots,d_n)$, $\vec{u} = (u_1, u_2,\cdots,u_n)$ and $\vec{k} = (k_1, k_2,\cdots,k_n)$.

**Theorem 5.1.** *The worst case ratio of **Algorithm 2** is 2 for* $1|prec, \tilde{p}, resource, \vec{d}|$ $\bar{M}(\sum_{i=1}^{n}(e_i\tilde{E}_i + t_i\tilde{T}_i) + \delta\tilde{C}max + \sum_{i=1}^{n} v_i u_i)$ *with* $\lambda_{\bar{\tilde{p}}_i} = (b_i - a_i)/\alpha_i = \lambda$ *for the basic processing time* $\bar{\tilde{p}}_i = (a_i, b_i, \alpha_i)$ *of* $J_i(i = 1, \cdots, n)$.

*Proof.* Let the schedule $\pi = \{J_{\pi(1)}, J_{\pi(2)}, \cdots, J_{\pi(n)}\}$ with $\vec{d}$, $\vec{u}$ and $\vec{k}$ be obtained by Algorithm 2, and $(\bar{\sigma}_{\tilde{C}_1}, \bar{\sigma}_{\tilde{C}_2}, \cdots, \bar{\sigma}_{\tilde{C}_n})$ be a solution of the linear programming in step 3 of Algorithm 2. It follows that $\sum_{J_j\in S}\sigma_{\tilde{p}_j}\bar{\sigma}_{\tilde{C}_j} \ge \frac{1}{2}(\sigma_{\tilde{p}}^2(S)+\sigma_{\tilde{p}}(S)^2)$ for each $S \subset \mathcal{J}$. Additionally, by step 4, we have $\bar{\sigma}_{\tilde{C}_{\pi(1)}} \le \bar{\sigma}_{\tilde{C}_{\pi(2)}} \le \cdots \le \bar{\sigma}_{\tilde{C}_{\pi(n)}}$. Hence, for the set $S = \{J_{\pi(1)}, J_{\pi(2)}, \cdots, J_{\pi(i)}\} \subset J(i = 1, 2, \cdots, n)$, we have

$$\sum_{j=1}^{i} \sigma_{\tilde{p}_{\pi(j)}} \bar{\sigma}_{\tilde{C}_{\pi(j)}} \geq \frac{1}{2}(\sigma_{\tilde{p}}^2(S) + \sigma_{\tilde{p}}(S)^2) \geq \frac{1}{2}\sigma_{\tilde{p}}(S)^2 = \frac{1}{2}(\sum_{j=1}^{i} \sigma_{\tilde{p}_{\pi(j)}})^2. \tag{51}$$

It follows that $\bar{\sigma}_{\tilde{C}_{\pi(i)}} \sum_{j=1}^{i} \sigma_{\tilde{p}_{\pi(j)}} \geq \sum_{j=1}^{i} \sigma_{\tilde{p}_{\pi(j)}} \bar{\sigma}_{\tilde{C}_{\pi(j)}} \geq \frac{1}{2}(\sum_{j=1}^{i} \sigma_{\tilde{p}_{\pi(j)}})^2$, which implies that, for any $i \in \{1, 2, \cdots, n\}$,

$$\sigma_{\tilde{C}_{\pi(i)}} = \sum_{j=1}^{i} \sigma_{\tilde{p}_{\pi(j)}} \leq 2\bar{\sigma}_{\tilde{C}_{\pi(i)}}. \tag{52}$$

Then, we have

$$\begin{aligned} f(\pi, \vec{d}, \vec{u}) =& \sum_{i=1}^{n} \frac{h(e_{\pi(i)}, t_{\pi(i)}, \lambda, k_{\pi(i)})\sigma_{\tilde{C}_{\pi(i)}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}} + \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i \\ \leq& \sum_{i=1}^{n} \frac{2h(e_{\pi(i)}, t_{\pi(i)}, \lambda, k_{\pi(i)})\bar{\sigma}_{\tilde{C}_{\pi(i)}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}} + \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i \\ \leq& 2[\sum_{i=1}^{n} \frac{h(e_{\pi(i)}, t_{\pi(i)}, \lambda, k_{\pi(i)})\bar{\sigma}_{\tilde{C}_{\pi(i)}}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}} + \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i] \\ =& 2[\sum_{i=1}^{n} \frac{h(e_i, t_i, \lambda, k_i)\bar{\sigma}_{\tilde{C}_i}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}} + \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i]. \end{aligned} \tag{53}$$

On the other hand, for the optimal schedule $\pi^* = \{J_{\pi^*(1)}, J_{\pi^*(2)}, \cdots, J_{\pi^*(n)}\}$ with $\vec{d}^*$, $u^*$ and $\vec{k}^*$, Lemma 3.1 implies that $\vec{k}^* = \vec{k}$ and $u^*$ is a solution of the linear programming (49), which means that

$$\delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i^* = \delta\sum_{i=1}^{n}\frac{a_i+b_i}{2} + \sum_{i=1}^{n}(v_i - \delta s_i)u_i. \tag{54}$$

Also, $(\sigma_{\tilde{C}_1^*}, \cdots, \sigma_{\tilde{C}_n^*})$ satisfies $\sum_{J_j \in S} \sigma_{\tilde{p}_j}\sigma_{\tilde{C}_j^*} \geq \frac{1}{2}(\sigma_{\tilde{p}}^2(S) + \sigma_{\tilde{p}}(S)^2)$ for each $S \subset \mathcal{J}$, which implies that $(\sigma_{\tilde{C}_1^*}, \cdots, \sigma_{\tilde{C}_n^*})$ is in the feasible domain of the linear programming of (50). It follows that

$$\sum_{i=1}^{n}\{h(e_i, t_i, \lambda, k_i)\frac{\bar{\sigma}_{\tilde{C}_i}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\} \leq \sum_{i=1}^{n}\{h(e_i, t_i, \lambda, k_i^*)\frac{\sigma_{\tilde{C}_i^*}}{\sqrt{\frac{\lambda^2}{4}+\frac{\lambda}{3}+\frac{1}{6}}}\}. \tag{55}$$

According to (53),(54) and (55), we have

$$
\begin{aligned}
f(\pi, \vec{d}, \vec{u}) &\leq 2[\sum_{i=1}^{n} \frac{h(e_i, t_i, \lambda, k_i)\bar{\sigma}_{\tilde{C}_i}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}} + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i]. \\
&\leq 2[\sum_{i=1}^{n} \frac{h(e_i, t_i, \lambda, k_i^*)\sigma_{\tilde{C}_i^*}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}} + \delta \sum_{i=1}^{n} \frac{a_i + b_i}{2} + \sum_{i=1}^{n} (v_i - \delta s_i) u_i^*] \\
&= 2f(\pi^*, \vec{d}^*, \vec{u}^*).
\end{aligned} \tag{56}
$$

□

Next, an example is given to show how Algorithm 2 works.

**Example 1.** Let $\mathcal{J} = \{J_1, \cdots, J_{12}\}$ with the basic processing times $\bar{\tilde{p}}_1 = (3,4,2), \bar{\tilde{p}}_2 = (7,10,6), \bar{\tilde{p}}_3 = (5,7,4), \bar{\tilde{p}}_4 = (9,13,8), \bar{\tilde{p}}_5 = (4,5,2)$, $\bar{\tilde{p}}_6 = (8,11,6)$, $\bar{\tilde{p}}_7 = (6,8,4), \bar{\tilde{p}}_8 = (10,14,8)$, $\bar{\tilde{p}}_9 = (7,9,4), \bar{\tilde{p}}_{10} = (9,12,6), \bar{\tilde{p}}_{11} = (11,15,8)$ and $\bar{\tilde{p}}_{12} = (5,6,2)$; the per unit earliness penalty vector for jobs is $(e_1, \cdots, e_{12}) = (6,7,3,8,4,11,9,6,3,5,3,8)$; the per unit tardiness penalty vector $(t_1, \cdots, t_{12})$ for jobs is $(5,9,5,7,8,14,4,5,8,6,9,3)$; the total amount of resources is $U_0 = 50$; the cost of one unit of operation time is $\delta = 2$; the compression rate vector of jobs is $(s_1, \cdots, s_{12}) = (0.1,0.3,0.2,0.4,0.1,0.3,0.2,0.4,0.2,0.3,0.4,0.1)$; the cost vector of one unit of resources is $(v_1, \cdots, v_{12})$=$(0.5,0.7,0.3,0.2,0.1,0.8,0.9,0.5,0.3,0.5,0.9,1)$; and the precedence constraints between jobs are described as Figure 1.

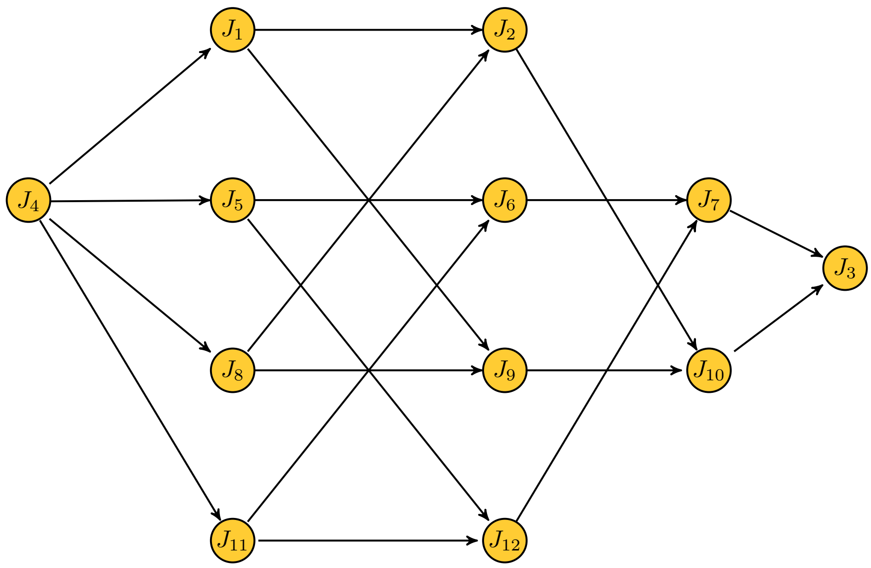


FIGURE 1. The precedence constraints for Example 1.

First, according to the step 1 of Algorithm 2, find a solution of the linear programming (49) induced by the problem about the amounts of the resources allocated to jobs as follows:

$$(u_1, \cdots, u_{12}) = (0, 0, 3.463, 2.500, 17.318, 0, 0, 5.000, 12.503, 9.216, 0, 0). \quad (57)$$

Then compute the actual processing times for jobs by the resource consumption function $\tilde{p}_i(u_i) = \bar{\tilde{p}}_i - s_i u_i (i = 1, \cdots, 12)$, and we get $\tilde{p}_1 = (3, 4, 2), \tilde{p}_2 = (7, 10, 6)$, $\tilde{p}_3 = (4.307, 6.307, 4), \tilde{p}_4 = (8, 12, 8), \tilde{p}_5 = (2.268, 3.268, 2), \tilde{p}_6 = (8, 11, 6), \tilde{p}_7 = (6, 8, 4), \tilde{p}_8 = (8, 12, 8), \tilde{p}_9 = (4.499, 6.499, 4), \tilde{p}_{10} = (6.235, 9.235, 6), \tilde{p}_{11} = (11, 15, 8)$ and $\tilde{p}_{12} = (5, 6, 2)$.

Next, compute the optimal customer service level for each job and find a solution of the linear programming of step 3 in Algorithm 2; the solution $(\bar{\sigma}_{\tilde{C}_1}, \cdots, \bar{\sigma}_{\tilde{C}_{12}})$ is (6.501,29.778,37.749,5.243,6.501,16.987,19.504,26.005,28.522,35.233,13.212, 14.471). Next, scheduling the jobs according to nondecreasing order of $\bar{\sigma}_{\tilde{C}_j}$, and we get the schedule

$$\pi = (J_4, J_5, J_1, J_{11}, J_{12}, J_6, J_7, J_8, J_9, J_2, J_{10}, J_3). \quad (58)$$

Finally, compute the completion times for jobs in the schedule $\pi$ and assign the due date to each job by $d_i = \bar{M}(\tilde{C}_i) + \frac{k_i - \frac{\lambda}{2}}{\sqrt{\frac{\lambda^2}{4} + \frac{\lambda}{3} + \frac{1}{6}}} \sigma_{\tilde{C}_i} (i = 1, \cdots, 12)$. Then, the due date vector $(d_1, d_2, \cdots, d_{12})$ is $(12.710, 90.997, 111.349, 7.729, 17.103, 53.002, 36.371, 49.407, 89.272, 99.609, 40.126, 23.516)$.

**Remark 5.2.** Note that the constraints of the linear programming (50) induced by the problem is the exponentially large class of constraints. However, as it is pointed in [25], an effective ellipsoid algorithm based on a polynomial-time separation algorithm is given for solving the linear programming with the exponentially large class of constraints. Hence, the solution for the example above can be quickly obtained even the size of the constraints matrix is $4113 \times 12$.

## 6. **Experiment**

In [24, 18], two methods are put forward to predict the due date for each job with the uncertain processing time. In [24], a due date assignment model in stochastic environment is proposed in which the job's processing time is assumed as a random variable with normal distribution; the due date for each job is given by $d = E(C) + k\sigma_C$ where $E(C)$ is the mean value of the completion time $C$, $\sigma_C$ is the standard variance of the completion time $C$, and $k$ is the value such that $\Phi(k) = \frac{t}{e+t}$ in which $\Phi(x)$ is the standard normal cumulative distribution function, $e$ is the per unit earliness penalties, and $t$ is the per unit tardiness penalties. Besides, in [18], the due date assignment model in fuzzy environment is given in which the symmetric triangular fuzzy numbers are used to model the uncertain processing times; the due date is assigned to each job by $d = a + k\alpha$ with respect to the completion time $(a, \alpha)$ where $k$ is determined by $e$ and $t$. Without special statement, the method in [24], the method in [18] and the method of our paper are briefly written as method 1, method 2, and method 3 in the following.

Next, the three methods above will be tested on the same data come from Example 1 [24] to examine their effectivity by comparing the total costs for the same data, which can be denoted by the the flow diagram in Figure 2.

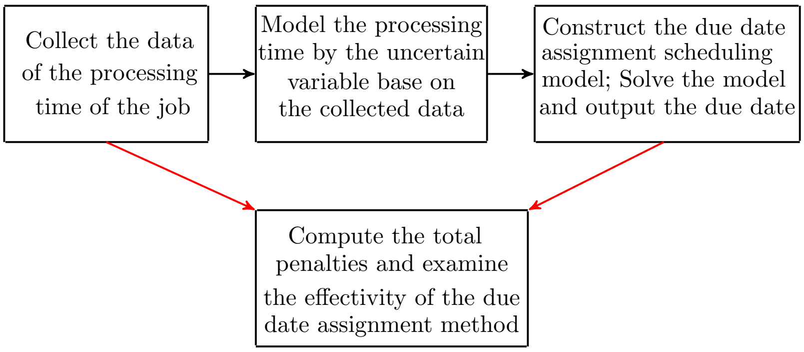


FIGURE 2. The flow diagram for examining the effectivity of the due date assignment method.

In Example 1 [24], the processing time of the job is the random variable with normal distribution $N(10, 3^2)$; the per unit earliness penalties and tardiness penalties are $e = 3\sqrt{2\pi}$ and $t = 27\sqrt{2\pi}$. Next, we produce 6 samples from the normal distribution $N(10, 3^2)$ same as ones in [18], whose sample sizes are 46, 81, 180, 344, 666 and 1332, respectively. The sampling method for each sample is as follows. First, according to the $3\sigma$ principe, determine the interval which the processing time of the job is almost in. It is easy to get that the interval is [1,19]. Second, divide the interval [1,19] into 6 subintervals as $[1, 4], [4, 7], [7, 10], [10, 13], [13, 16]$ and $[16, 19]$. At last, choose processing times from 6 subintervals accord with the probability of the random variable $N(10, 3^2)$ lying in each interval.

Next, model the uncertain processing time of the job by the uncertain variable based on the collected data and determine the due dates of the job by use of the three methods. For method 1 [24], the uncertain processing time of the job is modeled by the random variable with normal distribution $N(10, 3^2)$ for each sample, and the due date for the job determined by method 1 is 13.8460; for method 2 [18], the uncertain processing time of this job is modeled by the symmetrical triangle number $(10, 8.5)$ for each sample, and the due date determined according to method 2 is 14.6987 [18]; for method 3, we use the symmetric trapezoidal fuzzy number $(10, 10.8095, 8.0952)$ to model the uncertain processing time for each sample, and the due date for this job is 14.4749 according to the method of our paper.

Then, compute the earliness-tardiness penalties of all jobs in each sample, which are are shown in Figure 3. Compute the total penalties of all jobs in each sample for these three methods, and compute the ratios of the total penalties of jobs for method 3 and the ones for other methods which is given in Table 1 with $r_{i3} = \dfrac{\text{the total penalties of jobs for method 3}}{\text{the total penalties of jobs for method } i}$ for each sample $(i = 1, 2)$; Known from ratios of Table 1, for the same data above, the method proposed in this paper is better than the other two methods.

Further, according to Algorithm 2, our model can deal with the due date assignment problems with the precedence constraints and controllable processing times

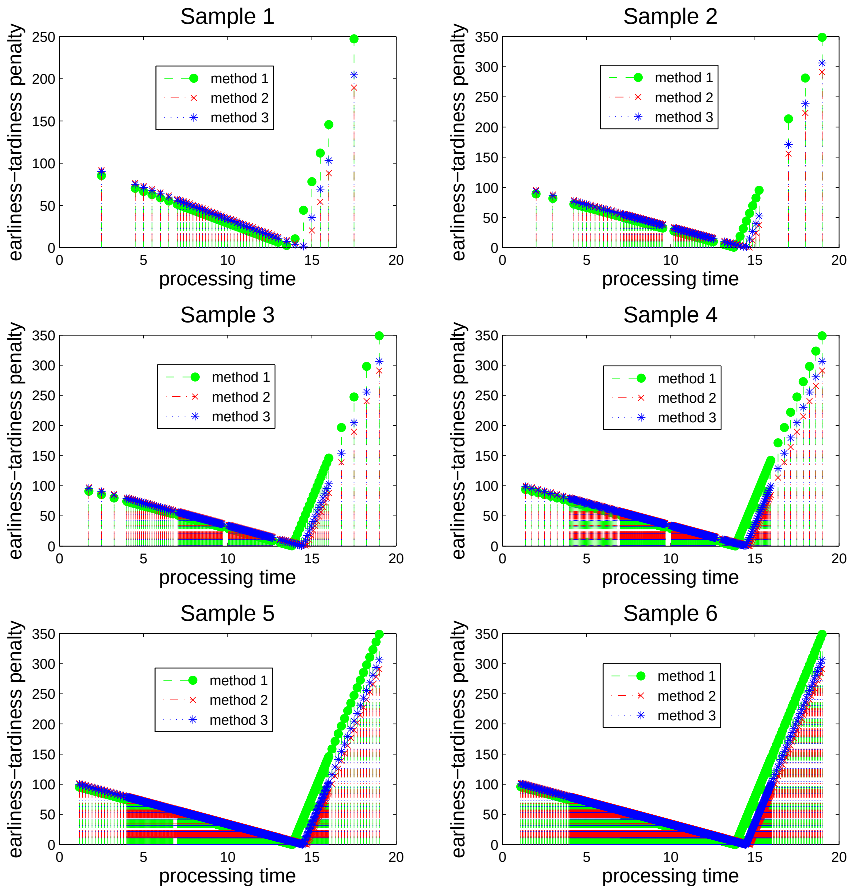


FIGURE 3. The earliness-tardiness penalties of jobs in samples1-6.

TABLE 1. The ratios of the total penalties of the jobs in each sample for three methods

| | Sample 1<br>46 jobs | Sample 2<br>81 jobs | Sample 3<br>180 jobs | Sample 4<br>344 jobs | Sample 5<br>666 jobs | Sample 6<br>1332 jobs |
|---|---|---|---|---|---|---|
| $r_{13}$ | 0.9847 | 0.9844 | 0.9788 | 0.9724 | 0.9888 | 0.9921 |
| $r_{23}$ | 0.9888 | 0.9891 | 0.9794 | 0.9725 | 0.9933 | 0.9967 |

in uncertain environment, whereas there are no results for this case by the other two ones [24, 18] as far as we know. Moreover, we use the Example 1 in section 5 to test Algorithm 2. All algorithms are implemented using MATLAB language (no explicit use of parallel toolbox), tested in MATLAB 2014a, 8GB memory and Intel core i7-4790 CPU. Then, the schedule with the objective function value 1407.14 is obtained by Algorithm 2 in 1 second. On the other side, the optimal schedule $\pi^* = (J_4, J_1, J_5, J_{11}, J_6, J_{12}, J_7, J_8, J_2, J_9, J_{10}, J_3)$ with the objective function value 1402.64 can be obtained by enumeration, but the time is about 1271 seconds due to examining 3628800 permutations. Moreover, the ratio of two objective function values is $\frac{1407.14}{1402.64} = 1.003$, which shows that Algorithm 2 is effective.

## 7. Conclusions

In this paper, we study a due date assignment scheduling problem with precedence constraints and controllable processing times in fuzzy environment, in which the basic processing time of each job is assumed to be the symmetric trapezoidal fuzzy number, and the linear resource consumption function is used. First, the condition for the optimal due date with respect to the completion is given. Then, the properties of the optimal schedule are obtained. Based on these properties, we show that the problem is NP-hard. Furthermore, we identified a polynomially solvable case of the problem. For the general cases, we develop an efficient approximation algorithm based on solving the relaxation of the problem, whose worst case ratio is show to be 2.

However, Algorithm 2 is required to solve a kind of linear programming with exponentially large class of constraints. Our work may accordingly focus on designing more effective algorithm for the due date assignment scheduling with general precedence constraints and controllable processing times in uncertain environment.

**Acknowledgements.** The authors would like to thank the editors and the anonymous referees for their valuable comments and suggestions.

## References

[1] L.D. Adolphson, *Sinle machine job sequencing with precedence constraints*, SIAM J. Comput., 6 (1977), 40-54.

[2] R. Alvarez-Valdes, J.M. Tamarit, F. Villa, *Minimizing weighted earliness-tardiness on parallel machines using hybrid metaheuristics*, Comput. Oper. Res., 54(2015), 1-11.

[3] P. Assarzadegan, M. Rasti-Barzoki, *Minimizing sum of the due date assignment costs, maximumtardiness and distribution costs in a supply chain scheduling problem*, Appl. Soft Comput., 47(2016), 343-356.

[4] K.R. Baker, *Minimizing earliness and tardiness costs in stochastic scheduling*, Eur. J. Oper. Res., 236(2014), 445-452.

[5] C. Carlsson, R. Fullér, *On possibilistic mean value and variance of fuzzy numbers*, Fuzzy Sets Syst., 122 (2001), 315-326.

[6] T.C.E. Cheng, *Optimal assignment of slack due-dates and sequencing of jobs with random processing times on a single machine*, Eur. J. Oper. Res., 51 (1991), 348-353.

[7] T.C.E. Cheng, C. Oğuz, X. Qi, *Due-date assignment and single machine scheduling with compressible processing times*, Int. J. Prod. Econ., 43(1996), 107-113.

[8] D. Du, Ker-I Ko, X. Hu, *Design and Analysis of Approximation Algorithms*, Springer Press, New York, 2012.

[9] R. Fullér, P. Majlender, *On weighted possibilistic mean and variance of fuzzy numbers*, Fuzzy Sets Syst., 136 (2003), 363-374.

[10] V. Gordon, J.-M. Proth, Chengbin Chu, *A survey of the state-of-the-art of common due date assignment and scheduling research*, Eur. J. Oper. Res., 139(2002), 1-25.

[11] R.L. Graham, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, *Optimization and approximation in deterministic sequencing and scheduling: a survey*, Ann. Discrete Math., 5 (1979), 287-326.

[12] L.A. Hall, A.S. Schulz, D.B. Shmoys, J. Wein, *Scheduling to Minimize Average Completion Time: Off-Line and On-Line Approximation Algorithms*, Math. Oper. Res., 22( 1997 ), 513-544.

[13] M. Hanss, *Applied Fuzzy Arithmetic: An Introduction with Engineering Applications*, Springer Press, Berlin, 2005.

[14] M. Iranpoor, S.M.T. FatemiGhomi, M. Zandieh, *Due-date assignment and machine scheduling in a low machine-rate situation with stochastic processing times*, Comput. Oper. Res., 40(2013), 1100-1108.

[15] A. Janiak, W.A. Janiak, T. Krysiaka, T. Kwiatkowski, *A survey on scheduling problems with due windows*, Eur. J. Oper. Res., 242(2015), 347-57.

[16] A. Janiak, T. Kwiatkowski, M. Lichtenstein, *Scheduling problems with a common due window assignment: a survey*, Int. J. Appl. Math. Comput. Sci., 23(2013), 231-241.

[17] M. Ji, J.J. Ge, K. Chen, T.C.E. Cheng, *Single-machine due-window assignment and scheduling with resource allocation, aging effect, and a deteriorating rate-modifyingactivity*, Comput. Ind. Eng., 66(2013), 952-961.

[18] J. Li, K. Sun, D. Xu, H. Li, *Single machine due date assignment scheduling problem with customer service level in fuzzy environment*, Appl. Soft Comput., 10(2010), 849-858.

[19] J, Li, X, Yuan, E.S. Lee, D. Xu, *Setting due dates to minimize the total weighted possibilistic mean value of the weighted earliness-tardiness costs on a single machine*, Comput. Math. Appl., 62(2011), 4126-4139.

[20] L. Liu, J. Wang, F. Liu, M. Liu, *Single machine due window assignment and resource allocation scheduling problems with learning and general positional effects*, J. Manuf. Syst., 43 (2017), 1-14.

[21] L. Liu, J. Wang, X. Wang, *Single machine due-window assignment scheduling with resource-dependent processing times to minimize total resource consumption cost*, Int. J. Prod. Res., 54(2016), 1-10.

[22] I.N. Lushchakova, *Two machine preemptive scheduling problem with release dates, equal processing times and precedence constraints*, Eur. J. Oper. Res., 171 (2006), 107-122.

[23] S.S. Panwalker, M.L. Smith, A. Seidmann, *Common due date assignment to minimize total penalty for the one machine scheduling problem*, Oper. Res., 30(1982), 391-399.

[24] V. Portougal, D. Trietsch, *Setting due dates in a stochastic single machine environment*, Comput. Oper. Res., 33 (2006), 1681-1694.

[25] M. Queyranne, *Structure of a simple scheduling polyhedron*, Math. Program., 58(1993), 263-285.

[26] M. Rasti-Barzoki, S. Hejazi, *Pseudo-polynomial dynamic programming for an integrated due date assignment, resource allocation, production, and distribution scheduling model in supply chain scheduling*, Appl. Math. Model., 39(2015), 3280-3289.

[27] B. F. Rosa, M. J. F. Souza, S. R. de Souza, et al., *Algorithms for job scheduling problems with distinct time windows and general earliness/tardiness penalties*, Comput. Oper Res., 81(2017), 203-215.

[28] A. Seidmann, S.S. Panwalker, M.L. Smith, *Optimal assignment of due-dates for a single processor scheduling problem*, Int. J. Prod. Res., 19(1981), 393-399.

[29] D. Shabtay, *Due date assignments and scheduling a single machine with a general earliness/tardiness cost function*, Comput. Oper. Res., 35(2008), 1539-1545.

[30] D. Shabtay, Y. Itskovich, L. Yedidsion, D. Oron, *Optimal due date assignment and resource allocation in a group technology scheduling environment*, Comput. Oper. Res., 37(2010), 2218-2228.

[31] D. Shabtay, G. Steiner, *A survey of scheduling with controllable processing times*, Discrete Appl. Math., 155(2007), 1643-1666.

[32] D. Shabtay, G. Steiner, *The single-machine earliness-tardiness scheduling problem with due date assignment and resource-dependent processing times*, Ann. Oper. Res., 159 (2008), 25-40.

[33] D. Shabtay, G. Steiner, *Two due date assignment problems in scheduling a single machine*, Oper. Res. Lett., 34 (2006), 683-691.

[34] D. Shabtay, G. Steiner, R. Zhang, *Optimal coordination of resource allocation, due date assignment and scheduling decisions*, Omega, 65(2016), 41-54.

[35] H. Soroush, *Sequencing and due-date determination in the stochastic single machine problem with earliness and tardiness costs*, Eur. J. Oper. Res., 113(1999), 450-468.

[36] N.H. Tuong, Ameur Soukha, *Due dates assignment and JIT scheduling with equal-size jobs*, Eur. J. Oper. Res., 205 (2010), 280-289.

[37] D.L. Yang, C.J. Lai, S.J. Yang, *Scheduling problems with multiple due windows assignment and controllable processing times on a single machine*, Int. J. Prod. Econ., 150 (2014), 96-103.

[38] Y. Yin, T.C.E. Cheng, C.C. Wu, S.R. Cheng, *Single-machine due window assignment and scheduling with a common flow allowance and controllable job processing time*, J. Oper. Res. Soc., 65 (2014), 1-13.

[39] Y. Yin, T.C.E. Cheng, D. Xu, C. Wu, *Common due date assignment and scheduling with a rate-modifying activity to minimize the due date, earliness, tardiness, hoding and batch delivery cost*, Comput. Ind. Eng., 63(2012), 223-234.

[40] Q. Yue, G. Wan, *Order schedule with controllable processing times, common due date and processing deadline*, J. Syst. Sci. Syst. Eng., 26(2017), 199-218.

[41] C. Zhao, C. Hsu, S. Cheng, Y. Yin, C. Wu, *Due date assignment and single machine scheduling with deteriorating jobs to minimize the weighted number of tardy jobs*, Appl. Math. Comput., 248(2014), 503-510.

[42] C. Zhao, H. Tang, *Two-machine flow shop scheduling with deteriorating jobs and chain precedence constraints*, Int. J. Prod. Econ., 136(2012), 131-136.

Jinquan Li, School of Applied Mathematics, Zhuhai Municipal Key Laboratory of Intelligent Control, Beijing Normal University Zhuhai, Zhuhai 519087, P.R. China
*E-mail address*: ljq_1023@mail.bnu.edu.cn

Dehua Xu, School of International Economics and Business, Nanjing University of Finance & Economics, Nanjing 210023, P.R. China
*E-mail address*: 9120171025@nufe.edu.cn, ** Corresponding Author

Hongxing Li, School of Electronic and Information Engineering, Dalian University of Technology, Dalian 116024, P.R. China
*E-mail address*: lihx@dlut.edu.cn