L-VALUED FUZZY ROUGH SETS

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Abstract. In this paper, we take a GL-quantale as the truth value table to study a new rough set model—L-valued fuzzy rough sets. The three key components of this model are: an L-fuzzy set A as the universal set, an L-valued relation of A and an L-fuzzy set of A (a fuzzy subset of fuzzy sets). Then L-valued fuzzy rough sets are completely characterized via both constructive and axiomatic approaches.

1. Introduction

The concept of rough sets was originally proposed by Pawlak [23, 24] as a formal tool for modelling and processing incomplete information in information systems [2, 15, 17]. Fuzzy set theory[37], on the other hand, is a wide variety of techniques for analyzing imprecise data. It seems therefore natural to combine methods developed within both theories in order to construct hybrid structures capable to deal with both aspects of incompleteness. Such structures, called fuzzy rough sets and rough fuzzy sets, have been proposed in the literature [1, 3, 4, 5, 13, 14, 16, 19, 22, 30, 35, 36, 38].

Fuzzy rough sets were originally proposed by Dubois and Prade in [4, 5]. Combined with rough sets, the three key components of this model are: a universal set X (a crisp set), a fuzzy equivalence relation of X and a fuzzy set of X. Morsi and Yakout [20] were among the first to study fuzzy rough sets based on a left-continuous t-norm and its residual implication. In [26], Radzikowska and Kerre defined a broad family of (I, T)-fuzzy rough sets which are determined by an arbitrary implication I and a t-norm T. Wu et al [32, 33] discussed the axiomatic approach of (S, T)-fuzzy rough sets. However, these fuzzy rough sets are based on the unit interval [0,1]. As Goguen [7] pointed out that sometimes it may be impossible to use the linearly ordered set [0,1] to represent degrees of membership and the concept of L-fuzzy rough sets was then introduced. Many authors have explored and developed L-fuzzy rough sets. In [27], Radzikowska and Kerre proposed fuzzy rough sets based on residuated lattices which were called L-fuzzy rough sets. The three key components of L-fuzzy rough sets are: a universal set X, an L-fuzzy equivalence relation of X and an L-fuzzy set of X, where L is a residuated lattice. As a further discussion on this model, She and Wang [29] studied the axiomatic approach of L-fuzzy rough sets. Topological structure corresponding to L-fuzzy

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rough approximation operators with emphasis to its ditopological was studied in [6]. Ma and Hu [18] investigated topological structures of L-fuzzy rough sets and the relationship between upper (resp. lower) sets and lower (resp. upper) L-fuzzy rough approximation operators. Wu et al. [34]discussed the properties of L-fuzzy rough sets based on complete residuated lattices. Wang and Hu [31] studied fuzzy rough sets based on generalized residuated lattices.

To summarize, the core element of fuzzy rough sets is the fuzzy type of binary relations. In order to study the logical and categorical basis of fuzzy sets, Höhle [9, 10] proposed L-valued set as follows: Let $P : X \times X \to L$, if it satisfies the following conditions: $\forall x, y, z \in X$,

(1) $P(x, y) \leq P(x, x) \wedge P(y, y)$;
(2) $P(x, y) = P(y, x)$;
(3) $P(x, y) * (P(y, y) \to P(y, z)) \leq P(x, z)$.

Then the pair (X, P) is called an L-valued set. Obviously, P is an L-fuzzy equivalence relation if $P(x, x) = 1$ $(\forall x \in X)$. Moreover, $[0, +\infty]^{op}$-valued sets (X, P) is a partial metric space in sense of [21] for $L = [0, +\infty]$ under the inverse order with "+" as the product, which has important applications in theoretical computer science. In 2012, Pu and Zhang [25] defined L-valued relation of L-fuzzy sets based on the theory of quantaloid-enriched category. Moreover, L-fuzzy equivalence relation is a special case of L-valued equivalence relation, and L-valued sets are just L-fuzzy sets endowed with an L-valued equivalence relation. Hence, we can establish the most general model of fuzzy rough sets based on L-valued relation. It needs to be emphasized that mentioned above model of fuzzy rough sets is based on the three key components: a universal set A(an L-set), an L-valued equivalence relation on A and an fuzzy subset of A. It is also valuable to note that Šostak introduced the general concept of an M-approximate system in [28]. As different from the present work, M-approximate systems can be defined without referencing to the universal set and without direct using of an L-relation but just on the basis of an integral commutative cl-monoid. Moreover, Han et al in [8] provided an example to indicate that an M-approximate system cannot be necessary obtained by means of an L-relation. These results provide a new approach to study fuzzy rough set theory.

The aim of the present paper is to propose a new rough set model—L-valued fuzzy rough sets, and to investigate it from both constructive and axiomatic approaches, where L is a GL-quantale. The structure of this paper is organized as follows: In Section 2, we review some basic concepts and results which will be used throughout the paper. In Section 3, the concept of L-valued fuzzy rough approximation operators is introduced and the basic properties are discussed. Furthermore, we discuss connections between some special L-valued relations and the L-valued rough approximation operators. In Section 4, L-valued fuzzy approximation operators are defined in an axiomatic way, and it is established that there is a one-to-one correspondence between the L-valued fuzzy rough approximation operators and L-valued fuzzy relations.

2. Preliminaries

In this section, we recall some basic notions of GL-quantales, L-valued relations and the L-valued power set of an L-set.

Definition 2.1. [9, 12] A *commutative quantale* is a pair $(L, *)$, where L is a complete lattice with the top element 1 and the bottom element 0, and $*$ is a commutative semigroup operation such that

$$\alpha * \Big(\bigvee_{j\in J} \beta_j\Big) = \bigvee_{j\in J} \alpha * \beta_j$$

for all $\alpha \in L$ and $\{\beta_j | \ j \in J\} \subseteq L$. The pair $(L, *)$ is called *unital* if there exists an element e such that $e * \alpha = \alpha$ for all $\alpha \in L$.

Given a commutative quantale $(L, *)$, there is a binary operation $\rightarrow: L \times L \rightarrow L$ defined by

$$\alpha \rightarrow \beta = \bigvee \{x \in L \mid \alpha * x \leq \beta\},$$

which is called the *implication* of $*$. Further, there two operations $*$ and $\rightarrow$ form an adjoint pair in the sense that

$$\alpha * \gamma \leq \beta \Longleftrightarrow \gamma \leq (\alpha \rightarrow \beta)$$

for all $\alpha, \beta, \gamma \in L$.

Some basic properties of $*$ and $\rightarrow$ are listed in the next proposition.

Proposition 2.2. [9, 20] *Let $(L, *)$ be a commutative quantale. Then for all $\alpha, \beta, \gamma, \delta, \alpha_i \in L$ $(i \in K)$, we have*

(1) $\alpha \rightarrow 1 = 1$ *and* $1 \rightarrow \alpha = \alpha$;
(2) $\alpha \leq \beta \Longrightarrow \alpha \rightarrow \beta = 1$;
(3) $\alpha \leq \beta \Longrightarrow \alpha * \gamma \leq \beta * \gamma,\ \alpha \rightarrow \gamma \geq \beta \rightarrow \gamma,\ \gamma \rightarrow \alpha \leq \gamma \rightarrow \beta$;
(4) $(\alpha \rightarrow \beta) * (\gamma \rightarrow \delta) \leq (\alpha * \gamma) \rightarrow (\beta * \delta)$;
(5) $(\alpha * \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma)$;
(6) $(\alpha \rightarrow \beta) * \gamma \leq \alpha \rightarrow (\beta * \gamma)$;
(7) $\alpha \rightarrow (\beta \rightarrow \gamma) = \beta \rightarrow (\alpha \rightarrow \gamma)$;
(8) $(\bigwedge_{i\in K} \alpha_i) \rightarrow \beta \geq \bigvee_{i\in K} (\alpha_i \rightarrow \beta)$;
(9) $(\bigvee_{i\in K} \alpha_i) \rightarrow \beta = \bigwedge_{i\in K} (\alpha_i \rightarrow \beta)$;
(10) $\beta \rightarrow (\bigwedge_{i\in K} \alpha_i) = \bigwedge_{i\in K} (\beta \rightarrow \alpha_i)$;
(11) $\beta \rightarrow (\bigvee_{i\in K} \alpha_i) \geq \bigvee_{i\in K} (\beta \rightarrow \alpha_i)$.

Proposition 2.3. [9, 10, 25] *For a commutative unital quantale $(L, *, e)$ with e being the unit element, the following conditions are equivalent:*

(1) $\forall\ \alpha, \beta \in L,\ \alpha \leq \beta \Longrightarrow \alpha = \beta * (\beta \rightarrow \alpha)$;
(2) $\forall \alpha, \beta, \gamma \in L,\ \alpha, \gamma \leq \beta \Longrightarrow \gamma * (\beta \rightarrow \alpha) = \alpha * (\beta \rightarrow \gamma)$;
(3) $\forall \alpha, \beta \in L,\ \alpha \leq \beta \Longrightarrow \exists \gamma \in L, \alpha = \beta * \gamma$;
(4) $\forall \alpha, \beta \in L,\ \alpha \wedge \beta = \alpha * (\alpha \rightarrow \beta)$ *and* $e = 1$.

A commutative unital quantale $(L,*)$ is called *divisible* if it satisfies one of the conditions (1)–(4) in Proposition 2.3. In this case, $(L,*)$ is also called a *GL-quantale*. Further, if a *GL-quantale* L still satisfies $(a \to 0) \to 0 = a$ for all $a \in L$, then L is an *MV-algebra*[20]. In this paper, we always assume that $(L,*)$ is a GL-quantale.

The concept of *L-set* was introduced by Goguen [7] as a generalization of the notion of Zadeh's fuzzy sets [37]. Let A_0 be a non-empty set. An L-set A of A_0 is defined as a mapping from A_0 to L. Let $B : A_0 \longrightarrow L$ be an L-set and $B(x) \le A(x)$ $(\forall x \in L)$. Then the mapping B is called an *L-subset* of A. The family of all L-subsets of A is denoted by $\mathcal{PA}$. For any $\alpha \in L$, $\widehat{\alpha} \in \mathcal{PA}$ is defined by $\widehat{\alpha}(x) = \alpha$ if $\alpha \le A(x)$ and $\widehat{\alpha}(x) = 0$ otherwise. For any $B_1, B_2 \in \mathcal{PA}$, $B_1 \le B_2$ iff $B_1(x) \le B_2(x)(\forall x \in L)$ (The " $\le$ " is called pointwise order in fuzzy set theory). And the *union*, *intersection*, $*$*-intersection* and $\to$*-implication* of B_1 and B_2 are defined as L-subsets of A by

$$\begin{aligned}
&(B_1 \vee B_2)(x) = B_1(x) \vee B_2(x);\\
&(B_1 \wedge B_2)(x) = B_1(x) \wedge B_2(x);\\
&(B_1 * B_2)(x) = B_1(x) * B_2(x);\\
&(A \wedge (B_1 \to B_2))(x) = A(x) \wedge (B_1(x) \to B_2(x)).
\end{aligned}$$

Let $(L,*)$ be a GL-quantale. For all $B_i \in \mathcal{PA}$, we write $\bigvee_{i\in I} B_i$ and $\bigwedge_{i\in I} B_i$ to denote the L-subsets of A given by

$$\begin{aligned}
(\bigvee_{i\in I} B_i)(x) &= \bigvee_{i\in I} B_i(x);\\
(\bigwedge_{i\in I} B_i)(x) &= \bigwedge_{i\in I} B_i(x).
\end{aligned}$$

Definition 2.4. [25] Let $A : A_0 \longrightarrow L$ and $B : B_0 \longrightarrow L$ be two L-sets. An L-valued relation $P : A \rightharpoonup B$ is a mapping $P : A_0 \times B_0 \longrightarrow L$ such that

$$P(x,y) \le A(x) \wedge B(y)\ (\forall x \in A_0,\ y \in B_0).$$

For an L-valued relation P, a related L-valued relation P^{-1} is defined by $P^{-1}(x,y) = P(y,x)$ for all $x \in A_0$ and $y \in B_0$, called the *inverse* of P.

Definition 2.5. [25] Let A be an L-set and let $P : A \rightharpoonup A$ be an L-valued relation on A.

(1) P is called *reflexive* if $A(x) \le P(x,x)$ for all $x \in A_0$;
(2) P is called *transitive* if $P(x,y)*(A(y) \to P(y,z)) \le P(x,z)$ for all $x,y,z \in A_0$;
(3) P is called *symmetric* if $P(x,y) = P(y,x)$ for all $x,y \in A_0$.

In Definition 2.5, if we take $A = 1_X$, then P is an L-fuzzy relation. Hence, the L-fuzzy relation is a special case of the L-valued relation. Further, Three properties of L-valued relation are consistent with those of L-fuzzy relation.

Definition 2.6. [25] Let A be an L-set and let $P : A \rightharpoonup A$ be an L-valued relation on A.

(1) P is called an *L-valued preorder* on A if it is reflexive and transitive.

(2) P is called an *L-valued equivalence* on A if it is reflexive, transitive and symmetric.

3. L-valued Fuzzy Rough Approximation Operators

In this section, we will define L-valued fuzzy rough sets and study the properties of L-valued lower and L-valued upper fuzzy rough approximation operators with respect to L-valued relations.

Definition 3.1. Let $A : A_0 \longrightarrow L$ be an L-set and let P be an L-valued relation on A. We call the pair (A, P) an *L-valued approximation space.* Define two mappings $\overline{P}_A,\ \underline{P}_A : \mathcal{PA} \longrightarrow \mathcal{PA}$ as follows: for all $B \in \mathcal{PA}$ and all $x \in A_0$,

The operators $\overline{P}_A, \underline{P}_A : \mathcal{PA} \longrightarrow \mathcal{PA}$ are respectively called the *L-valued*

$$\overline{P}_A(B)(x) = \bigvee_{y\in A_0} B(y) * (A(y) \to P(y,x)),$$

$$\underline{P}_A(B)(x) = \bigwedge_{y\in A_0} A(x) * (P(y,x) \to B(y)).$$

upper and the *L-valued lower fuzzy rough approximation operators* of (A, P), and the pair $(\overline{P}_A(B), \underline{P}_A(B))$ is called an *L-valued fuzzy rough set* of B w.r.t. (A, P).

Example 3.2. Let $L = [0, 1]$ and $* = \wedge$. Then residuation implication of $\wedge$ is given by

$$\alpha \to \beta = \begin{cases} 1, & \alpha \le \beta, \\ \beta, & \text{others}. \end{cases}$$

Let A be the identity mapping on L and define $P : A \to A$ by $P(x, y) = \min\{x, y\}$. For $B : [0, 1] \to [0, 1]$

$$B(x) = \begin{cases} A(x), & x \le 1/2, \\ 0, & \text{others}, \end{cases}$$

according to Definition 3.1, $\overline{P}_A(B)(x) = 1/2$ and $\underline{P}_A(B)(x) = 0$.

Proposition 3.3. *For all $a, b, c \in L$, if $a \le b$, then $b \wedge [(b \to a) \to c] = b * (a \to c)$.*

Proof. According to Proposition 2.3(1) and (4), we have

$$\begin{aligned} & b \wedge [(b \to a) \to c] \\ = \ & b * \{b \to [(b \to a) \to c]\} \\ = \ & b * \{[b * (b \to a)] \to c\} \\ = \ & b * (a \to c). \end{aligned}$$

□

According to Proposition 3.3, the L-valued lower fuzzy rough approximation operator $\underline{P}_A$ can be equivalently described as follows:

$$\underline{P}_A(B)(x) = \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to B(y)].$$

Remark 3.4. In Definition 3.1, if $A = 1_X$ and P is an L-fuzzy relation on A, then

$$\overline{P}(B)(x) = \bigvee_{y\in X} B(y) * P(y,x),$$
$$\underline{P}(B)(x) = \bigwedge_{y\in X} P(y,x) \to B(y).$$

Then the pair $(\overline{P}, \underline{P})$ is precisely an L-fuzzy rough set in [26, 29].

The following theorem presents some basic properties of the L-valued upper fuzzy rough approximation operator.

Let $A : A_0 \longrightarrow L$ be an L-set. For every $y \in A_0$, we define a new L-set $A_y : A_0 \longrightarrow L$ by

$$A_y(x) = \begin{cases} A(x), & x = y, \\ 0, & x \neq y. \end{cases}$$

Obviously, A_y is an L-subset of A.

Theorem 3.5. *Let $A : A_0 \longrightarrow L$ be an L-set and let P be an L-valued relation on A. Then the L-valued upper fuzzy rough approximation operator $\overline{P}_A$ has the following properties: $\forall B, B_i \in \mathcal{PA}, i \in I$, where I is the index set,*

(UF1) $\overline{P}_A(\widehat{\alpha} * (\widehat{\beta} \to B)) = \widehat{\alpha} * (\widehat{\beta} \to \overline{P}_A(B))$ *for all* $\alpha, \beta \in L$ *with* $\alpha \leq \beta$ *and* $\bigvee_{x\in X} B(x) \leq \beta$;
(UF2) $\overline{P}_A(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} \overline{P}_A(B_i)$;
(UF3) $B_1 \leq B_2 \Longrightarrow \overline{P}_A(B_1) \leq \overline{P}_A(B_2)$;
(UF4) $\overline{P}_A(\widehat{0}) = \widehat{0}$;
(UF5) $\overline{P}_A(A \wedge \widehat{\alpha}) \leq A \wedge \widehat{\alpha}$ *for all* $\alpha \in L$;
(UF6) $\overline{P}_A(A_y * \widehat{\alpha})(x) = P(y,x) * \widehat{\alpha}$ *for all* $\alpha \in L$ *with* $A_y * \widehat{\alpha} \in \mathcal{PA}$;
(UF7) $\overline{P}_A(A_y)(x) = P(y,x)$.

Proof. (UF1) For any $x \in A_0$, we have

$$\begin{aligned} & \overline{P}_A(\widehat{\alpha} * (\widehat{\beta} \to B))(x) \\ = & \bigvee_{y\in A_0} (\widehat{\alpha} * (\widehat{\beta} \to B))(y) * (A(y) \to P(y,x)) \\ = & (\beta \to \alpha) * \bigvee_{y\in A_0} B(y) * (A(y) \to P(y,x)) \\ = & (\beta \to \alpha) * \overline{P}_A(B)(x) \\ = & (\widehat{\alpha} * (\widehat{\beta} \to \overline{P}_A(B)). \end{aligned}$$

Hence, $\overline{P}_A(\widehat{\alpha} * (\widehat{\beta} \to B)) = \widehat{\alpha} * (\widehat{\beta} \to \overline{P}_A(B))$.

(UF2) For any $x \in A_0$, we have

$$\begin{aligned} & \overline{P}_A(\bigvee_{i\in I} B_i)(x) \\ = & \bigvee_{y\in A_0} (\bigvee_{i\in I} B_i)(y) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} (\bigvee_{i\in I} B_i(y)) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} \bigvee_{i\in I} (B_i(y) * (A(y) \to P(y,x))) \\ = & \bigvee_{i\in I} \bigvee_{y\in A_0} B_i(y) * (A(y) \to P(y,x)) \\ = & \bigvee_{i\in I} \overline{P}_A(B_i)(x). \end{aligned}$$

Hence, $\overline{P}_A(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} \overline{P}_A(B_i)$.

(UF3) It follows from Proposition 2.2(3) directly.

(UF5) For any $x \in A_0$, we have

$$\begin{aligned} & \overline{P}_A(A \wedge \widehat{\alpha})(x) \\ = & \bigvee_{y\in A_0} (A \wedge \widehat{\alpha})(y) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} (\alpha \wedge A(y)) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} (A(y) \to \alpha) * P(y,x) \end{aligned}$$

Since $P(y,x) \le A(x) \wedge A(y)$, we have

$$\bigvee_{y\in A_0} (A(y) \to \alpha) * P(y,x) \le \bigvee_{y\in A_0} (A(y) \to \alpha) * A(x) \le A(x)$$

and

$$\bigvee_{y\in A_0} (A(y) \to \alpha) * P(y,x) \le \bigvee_{y\in A_0} (A(y) \to \alpha) * A(y) \le \alpha.$$

Then $\bigvee_{y\in A_0} (A(y) \to \alpha) * P(y,x) \le A(x) \wedge \alpha$. Hence, $\overline{P}(A \wedge \widehat{\alpha}) \le A \wedge \widehat{\alpha}$.

(UF6) For any $x \in A_0$, we have

$$\begin{aligned} & \overline{P}_A(A_y * \widehat{\alpha})(x) \\ = & \bigvee_{z\in A_0} (A_y * \widehat{\alpha})(z) * (A(z) \to P(z,x)) \\ = & A(y) * \alpha * (A(y) \to P(y,x)) \\ = & \alpha * [A(y) * (A(y) \to P(y,x))] \\ = & \alpha * P(y,x) \end{aligned}$$

(UF7) By taking $\widehat{\alpha} = \widehat{1}$ in (UF6), we have $\overline{P}_A(A_y)(x) = P(y,x)$. □

The following theorem gives some basic properties of the L-valued lower fuzzy rough approximation operator.

Let $A : A_0 \longrightarrow L$ be an L-set. For every $y \in A_0$, $\alpha \in L$ and $\alpha \le A(y)$, we define a new L-set $A - y_\alpha : A_0 \longrightarrow L$ by

$$(A - y_\alpha)(x) = \begin{cases} A(x), & x \ne y, \\ \alpha, & x = y. \end{cases}$$

Obviously, $A - y_\alpha$ is an L-subset of A.

Theorem 3.6. *The L-lower fuzzy rough approximation operator $\underline{P}_A$ has the following properties: $\forall B, B_i, \widehat{\alpha} \in \mathcal{PA}, i \in I$, where I is the index set,*

(LL1) $\underline{P}_A(A \wedge (\widehat{\alpha} \to B)) = A \wedge (\widehat{\alpha} \to \underline{P}_A(B))$;
(LL2) $\underline{P}_A(\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} \underline{P}_A(B_i)$;
(LL3) $B_1 \le B_2 \Rightarrow \underline{P}_A(B_1) \le \underline{P}_A(B_2)$;
(LL4) $\underline{P}_A(A) = A$;
(LL5) $A \wedge \widehat{\alpha} \le \underline{P}_A(A \wedge \widehat{\alpha})$;
(LL6) $\underline{P}_A(A - y_\alpha)(x) = A(x) * (P(y,x) \to \alpha)$.

Proof. (LL1) For any $x \in A_0$, we have

$$\begin{aligned}
& \underline{P}_A(A \wedge (\widehat{\alpha} \to B))(x) \\
= & \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to (A(y) \wedge (\alpha \to B(y)))] \\
= & \bigwedge_{y\in A_0} A(x) \wedge [A(x) \to P(y,x) \to A(y)] \wedge [(A(x) \to P(y,x)) \to (\alpha \to B(y))] \\
= & \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to (\alpha \to B(y))] \\
= & \bigwedge_{y\in A_0} A(x) * [\alpha \to (P(y,x) \to B(y))].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& [A \wedge (\alpha \to \underline{P}_A(B))](x) \\
= & A(x \wedge (\alpha \to \bigwedge_{y\in A_0} [A(x) \wedge (A(x) \to P(y,x) \to B(y))] \\
= & A(x \wedge \bigwedge_{y\in A_0} (\alpha \to A(x)) \wedge (A(x) \to P(y,x) \to B(y))] \\
= & \bigwedge_{y\in A_0} A(x) \wedge (\alpha \to [(A(x) \to P(y,x)) \to B(y)]) \\
= & \bigwedge_{y\in A_0} A(x) \wedge [A(x) \to P(y,x) \to (\alpha \to B(y))] \\
= & \bigwedge_{y\in A_0} A(x) * [\alpha \to (P(y,x) \to B(y))].
\end{aligned}$$

Therefore, $\underline{P}_A(A \wedge (\widehat{\alpha} \to B)) = A \wedge (\widehat{\alpha} \to \underline{P}_A(B))$.

(LL2) For any $x \in A_0$, we have

$$\begin{aligned}
& \underline{P}_A(\bigwedge_{i\in I} B_i)(x) \\
= & \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to \textstyle\bigwedge_{i\in I} B_i(y)] \\
= & \bigwedge_{y\in A_0} A(x) \wedge \bigwedge_{i\in I} [(A(x) \to P(y,x)) \to B_i(y)] \\
= & \bigwedge_{y\in A_0} \bigwedge_{i\in I} A(x) \wedge [(A(x) \to P(y,x)) \to B_i(y)]
\end{aligned}$$

$$\begin{aligned}
= & \bigwedge_{i\in I} \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to B_i(y)] \\
= & \bigwedge_{i\in I} \underline{P}_A(B_i)(x).
\end{aligned}$$

Therefore, $\underline{P}_A(\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} \underline{P}_A(B_i)$.

(LL3) It follows from Proposition 2.2(3) directly.

(LL5) For any $x \in A_0$, we have

$$\begin{aligned}
& \underline{P}_A(A \wedge \widehat{\alpha})(x) \\
= & \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to (A(y) \wedge \alpha)] \\
= & \bigwedge_{y\in A_0} A(x) \wedge \{[(A(x) \to P(y,x)) \to A(y)] \wedge [(A(x) \to P(y,x)) \to \alpha]\} \\
= & \{\bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to A(y)]\} \wedge \{\bigwedge_{y\in A_0} [(A(x) \to P(y,x)) \to \alpha]\} \\
= & \underline{P}_A(A)(x) \wedge \{\bigwedge_{y\in A_0} [(A(x) \to P(y,x)) \to \alpha]\} \\
= & A(x) \wedge \{\bigwedge_{y\in A_0} [(A(x) \to P(y,x)) \to \alpha]\} \\
\geq & A(x) \wedge \bigwedge_{y\in A_0} (1 \to \alpha) \\
= & A(x) \wedge \bigwedge_{y\in A_0} \widehat{\alpha}(x) \\
= & (A \wedge \widehat{\alpha})(x).
\end{aligned}$$

Therefore, $A \wedge \widehat{\alpha} \leq \underline{P}_A(A \wedge \widehat{\alpha})$.

(LL6) For any $x \in A_0$, we have

$$\begin{aligned} & \underline{P}_A(A - y_\alpha)(x) \\ = & \bigwedge_{z \in A_0} A(x) \wedge (P(z,x) \rightarrow (A - y_\alpha)(z)) \\ = & A(x) \wedge (P(y,x) \rightarrow \alpha) \wedge \bigwedge_{z \neq y \in A_0} A(x) \wedge (P(z,x) \rightarrow A(z)) \\ = & A(x) \wedge (P(y,x) \rightarrow \alpha). \end{aligned}$$

Therefore, $\underline{P}_A(A - y_\alpha)(x) = A(x) * (P(y,x) \rightarrow \alpha)$. □

In the following, let L be an MV-algebra. We can define the pseudo complement of B, denoted by

$$\sim B(x) = A(x) * (B(x) \rightarrow 0) \ (\forall x \in A_0).$$

Obviously, for all $x \in A_0$, we have $\sim B(x) \leq A(x)$. Thus $\sim B$ is an L-subset of A.

Further, for all $B_1, B_2 \in \mathcal{PA}$, we have

$$\sim (B_1 \vee B_2) = \sim B_1 \vee \sim B_2; \ \sim (B_1 \wedge B_2) = \sim B_1 \wedge \sim B_2.$$

Then we have the following results.

Theorem 3.7. *Let (A, P) be an L-valued approximation space and let $\overline{P}_A$ (resp. $\underline{P}_A$) be the L-valued upper fuzzy rough approximation operator (resp. L-valued lower fuzzy rough approximation operator). Then for every $B \in \mathcal{PA}$,*

$$\sim \overline{P}_A(\sim B) = \underline{P}_A(B);$$

$$\sim \underline{P}_A(\sim B) = \overline{P}_A(B).$$

Proof. For all $B \in \mathcal{PA}$ and $x \in A_0$, we have

$$\begin{aligned} & \sim \overline{P}_A(\sim B)(x) \\ = & A(x) * (\overline{P}_A(\sim B)(x) \rightarrow 0) \\ = & A(x) * \{[\bigvee_{y \in A_0} A(y) * (B(y) \rightarrow 0) * (A(y) \rightarrow P(y,x))] \rightarrow 0\} \\ = & A(x) * [\bigvee_{y \in A_0} P(y,x) * (B(y) \rightarrow 0) \rightarrow 0] \\ = & A(x) * (\bigwedge_{y \in A_0} (P(y,x) \rightarrow B(y)) \\ = & \bigwedge_{y \in A_0} A(x) * (P(y,x) \rightarrow B(y)) \\ = & \underline{P}_A(B)(x). \end{aligned}$$

Hence, $\sim \overline{P}_A(\sim B) = \underline{P}_A(B)$.

For all $B \in \mathcal{PA}$ and all $x \in A_0$, we have

$$\begin{aligned}
& \sim\underline{P}_A(\sim B)(x) \\
= \;& A(x) * (\underline{P}_A(\sim B)(x) \to 0) \\
= \;& A(x) * [(\bigwedge_{y\in A_0} A(x) * [P(y,x) \to (A(y) * (B(y) \to 0))]) \to 0] \\
= \;& A(x) * [\bigwedge_{y\in A_0} A(x) * (P(y,x) * (A(y) \to B(y)) \to 0) \to 0] \\
= \;& A(x) * \{\bigvee_{y\in A_0} A(x) \to [P(y,x) * (A(y) \to B(y))]\} \\
= \;& \bigvee_{y\in A_0} A(x) \wedge [B(y) * (A(y) \to P(y,x))] \\
= \;& \bigvee_{y\in A_0} B(y) * (A(y) \to P(y,x)) \\
= \;& \overline{P}_A(B)(x).
\end{aligned}$$

Hence, $\sim\underline{P}_A(\sim B) = \overline{P}_A(B)$. □

According to Theorem 3.7, the L-valued upper and lower fuzzy rough approximation operators are dual w.r.t $\sim$ and can be described by each other. Hence, every item in Theorem 3.6 is dual to that in Theorem 3.5 with the same number.

Remark 3.8. In Theorems 3.5, 3.6 and 3.7, if $A = 1_X$ and P is an L-fuzzy relation on A, the corresponding results are known in[29, 31, 33].

The following theorem shows that some properties of L-valued fuzzy rough approximation operators derived from the L-valued approximation space (A, P) can be used to characterize some special properties of the L-valued relation P.

Theorem 3.9. *Let (A, P) be an L-valued approximation space. Then the following statements are equivalent:*
(1) *P is reflexive;*
(2) *$\underline{P}_A(B) \leq B$;*
(3) *$B \leq \overline{P}_A(B)$;*
(4) *$\underline{P^{-1}}_A(B) \leq B$;*
(5) *$B \leq \overline{P^{-1}}_A(B)$.*

Proof. (1) $\Longrightarrow$ (2) For all $B \in \mathcal{PA}$ and $x \in A_0$,

$$\begin{aligned}
& \underline{P}_A(B)(x) \\
= \;& \bigwedge_{y\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to B(y)] \\
= \;& A(x) \wedge [(A(x) \to P(x,x)) \to B(x)] \\
& \quad \wedge\{\bigwedge_{y\neq x\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to B(y)]\} \\
= \;& B(x) \wedge \bigwedge_{y\neq x\in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to B(y)] \\
\leq \;& B(x).
\end{aligned}$$

Therefore, $\underline{P}_A(B) \leq B$.

(2) $\Longrightarrow$ (1) Let $B = A - x_{P(x,x)}$. Then $\underline{P}_A(B)(x) \leq B(x)$, that is

$$\bigwedge_{z\in A_0} A(x) * (P(z,x) \to B(z)) \leq B(x) = P(x,x),$$

and

$$\begin{aligned} & \bigwedge_{z\in A_0} A(x) * (P(z,x) \to B(z)) \\ = & [\bigwedge_{z\neq x\in A_0} A(x) * (P(z,x) \to A(z))] \wedge [A(x) * (P(x,x) \to P(x,x)] \\ = & A(x). \end{aligned}$$

Then $A(x) \leq P(x,x)$. Hence, P is reflexive.

(1) $\Longrightarrow$ (3) For all $B \in \mathcal{PA}$ and $x \in A_0$,

$$\begin{aligned} & \overline{P}_A(B)(x) \\ = & \bigvee_{y\in A_0} B(y) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} B(y) * (P(y,y) \to P(y,x)) \\ = & B(x) \vee \bigvee_{y\neq x\in A_0} B(y) * (P(y,y) \to P(y,x)) \\ = & B(x). \end{aligned}$$

Hence, $B \leq \overline{P}_A(B)$.

(3) $\Longrightarrow$ (1) Let $B = A_x$, by (UF7), we have

$$A(x) = A_x \leq \overline{P}_A(A_x)(x) = P(x,x).$$

Hence, P is reflexive.

(1) $\Longrightarrow$ (5) Let P be reflexive. For all $x \in A_0$, $P^{-1}(x,x) = P(x,x) \geq A(x)$, then P^{-1} is reflexive.

Similarly, the statements of (1) $\Longleftrightarrow$ (4) and (1) $\Longleftrightarrow$ (5) are valid. □

Furthermore, according to (UF7) and (LL6), we have the following conclusion.

Theorem 3.10. *Let (A,P) be an L-valued approximation space. For all $x,y \in A_0$, the following statements are equivalent:*

(1) *P is symmetric;*

(2) *$\overline{P}_A(A_x)(y) = \overline{P}_A(A_y)(x)$;*

(3) *$[A(x) \to \underline{P}_A(A - y_0)(x)] \to 0 = [A(y) \to \underline{P}_A(A - x_0)(y)] \to 0$.*

According to Theorem 3.9, we can use the L-valued upper fuzzy rough approximation operator to characterize the symmetry of P. But the construction of L-valued lower fuzzy rough approximation operator is relatively complex. If $A = 1_X$ and P is an L-fuzzy relation, then this conclusion is consistent with Proposition 3.9 in [27].

Theorem 3.11. *Let (A,P) be an L-valued approximation space. The following statements are equivalent:*

(1) *P is symmetric;*

(2) *$\overline{P}_A(\underline{P}_A(B)) \leq B$;*

(3) $B \leq \underline{P}_A(\overline{P}_A(B))$;
(4) $\overline{P^{-1}}_A(\underline{P^{-1}}_A(B)) \leq B$;
(5) $B \leq \underline{P^{-1}}_A(\overline{P^{-1}}_A(B))$.

Proof. (1) $\Longrightarrow$ (2) If P is symmetric, then for all $B \in \mathcal{PA}$ and all $x \in A_0$, we have

$$\begin{aligned}
& \overline{P}_A(\underline{P}_A(B))(x) \\
= & \bigvee_{y\in A_0} \underline{P}_A(B)(y) * (A(y) \to P(y,x)) \\
= & \bigvee_{y\in A_0} (\bigwedge_{z\in A_0} A(y) * (P(z,y) \to B(z))) * (A(y) \to P(y,x)) \\
\leq & \bigvee_{y\in A_0} (A(y) * (P(x,y) \to B(x))) * (A(y) \to P(y,x)) \\
= & \bigvee_{y\in A_0} A(y) * (A(y) \to P(y,x)) * (P(x,y) \to B(x)) \\
= & \bigvee_{y\in A_0} P(y,x) * (P(x,y) \to B(x)) \\
\leq & \bigvee_{y\in A_0} B(x) \\
= & B(x).
\end{aligned}$$

Hence, $\overline{P}_A(\underline{P}_A(B)) \leq B$.

(2) $\Longrightarrow$ (1) Let $B : A_0 \longrightarrow L$ and $B = Px_0$. For all $y \in A_0$, $B(y) = P(y,x_0) \leq A(y) \wedge A(x_0)$, then $B \in \mathcal{PA}$. Hence, $\overline{P}_A(\underline{P}_A(B)) \leq B$. For all $y \in A_0$, we have

$$\begin{aligned}
& \overline{P}_A(\underline{P}_A(B))(y) \to B(y) \\
= & \bigvee_{x\in A_0} \underline{P}_A(B)(x) * (A(x) \to P(x,y)) \to B(y) \\
= & \bigvee_{x\in A_0} [\bigwedge_{z\in A_0} A(x) * (P(z,x) \to B(z))] * (A(x) \to P(x,y)) \to B(y) \\
\leq & \bigwedge_{z\in A_0} (A(x_0) * (P(z,x_0) \to P(z,x_0))) * (A(x_0) \to P(x_0,y)) \to P(y,x_0) \\
= & \bigwedge_{z\in A_0} A(x_0) * (A(x_0) \to P(x_0,y)) \to P(y,x_0) \\
= & P(x_0,y) \to P(y,x_0).
\end{aligned}$$

Then, for all $x_0 \in A_0$, we have $P(x_0,y) \to P(y,x_0) = 1$. Hence, P is symmetric.

(1) $\Longrightarrow$ (3) If P is symmetric, then for all $B \in \mathcal{PA}$ and all $x \in A_0$, we have

$$\begin{aligned}
& \underline{P}_A(\overline{P}_A(B))(x) \\
= & A(x) \wedge \bigwedge_{y\in A_0} [(A(x) \to P(y,x)) \to \overline{P}_A(B)(y)]
\end{aligned}$$

$$\begin{aligned}
= & A(x) \wedge \bigwedge_{y\in A_0} [(A(x) \to P(y,x)) \to (\bigvee_{z\in A_0} B(z) * (A(z) \to P(z,y)))] \\
\geq & A(x) \wedge \bigwedge_{y\in A_0} (A(x) \to P(y,x)) \to [B(x) * (A(x) \to P(x,y))] \\
\geq & A(x) \bigwedge_{y\in A_0} B(x) \\
= & B(x).
\end{aligned}$$

Therefore, $B \leq \underline{P}_A(\overline{P}_A(B))$.

(3) $\Longrightarrow$ (1) Let $B = A_y$. Then by (UF7), we have

$$\begin{aligned}
& \underline{P}_A(\overline{P}_A(A_y))(y) \\
= \ & A(y) \wedge \bigwedge_{x\in A_0} [(A(y) \to P(x,y)) \to \overline{P}_A(A_y)(x)] \\
\le \ & A(y) \wedge [(A(y) \to P(x,y)) \to P(y,x)] \\
\le \ & (A(y) \to P(x,y)) \to P(y,x).
\end{aligned}$$

Since $A \le \underline{P}_A(\overline{P}_A(A))$, we have $P(x,y) \to P(y,x) = 1$. Hence, P is symmetric.

Let P be symmetric. For all $x, y \in A_0$, $P^{-1}(x,y) = P(y,x) = P(x,y) = P^{-1}(y,x)$, then P^{-1} is symmetric.

Similarly, the statements of (1) $\Longleftrightarrow$ (4) and (1) $\Longleftrightarrow$ (5) are valid. □

Theorem 3.12. *Let (A, P) be an L-valued approximation space. The following statements are equivalent:*

(1) *P is transitive;*
(2) $\underline{P}_A(B) \le \underline{P}_A(\underline{P}_A(B))$;
(3) $\overline{P}_A(\overline{P}_A(B)) \le \overline{P}_A(B)$;
(4) $\underline{P^{-1}}_A(B) \le \underline{P^{-1}}_A(\underline{P^{-1}}_A(B))$;
(5) $\overline{P^{-1}}_A(\overline{P^{-1}}_A(B)) \le \overline{P^{-1}}_A(B)$.

Proof. (1) $\Longrightarrow$ (2) If P is transitive, then for all $B \in \mathcal{PA}$ and all $x \in A_0$, we have

$$\begin{aligned}
& \underline{P}_A(\underline{P}_A(B))(x) \\
= \ & A(x) \wedge \bigwedge_{y\in A_0} [(A(x) \to P(y,x)) \to \underline{P}_A(B)(y)] \\
= \ & A(x) \wedge \bigwedge_{y\in A_0} \{(A(x) \to P(y,x)) \to [A(y) \wedge \bigwedge_{z\in A_0} ((A(y) \to P(z,y)) \to B(z))]\} \\
= \ & A(x) \wedge \bigwedge_{y\in A_0} \{[(A(x) \to P(y,x)) \to A(y)] \\
& \quad \wedge [(A(x) \to P(y,x)) \to \bigwedge_{z\in A_0} ((A(y) \to P(z,y)) \to B(z))]\} \\
= \ & A(x) \wedge \bigwedge_{y\in A_0} \{(A(x) \to P(y,x)) \to \\
& \quad [(A(x) \to P(y,x)) \to \bigwedge_{z\in A_0} ((A(y) \to P(z,y)) \to B(z))]\} \\
= \ & \bigwedge_{y\in A_0} \bigwedge_{z\in A_0} A(x) \wedge \{(A(x) \to P(y,x)) \to \\
& \quad [(A(x) \to P(y,x)) \to ((A(y) \to P(z,y)) \to B(z))]\} \\
= \ & \bigwedge_{y\in A_0} \bigwedge_{z\in A_0} A(x) * (P(y,x) \to \\
& \quad [(A(x) \to P(y,x)) \to ((A(y) \to P(z,y)) \to B(z))])
\end{aligned}$$

$$\begin{aligned}
= \ & \bigwedge_{y\in A_0} \bigwedge_{z\in A_0} A(x) * [P(y,x) * (A(y) \to P(z,y)) \to B(z)] \\
= \ & \bigwedge_{y\in A_0} \bigwedge_{z\in A_0} A(x) * [P(z,y) * (A(y) \to P(y,x)) \to B(z)] \\
\ge \ & \bigwedge_{z\in A_0} A(x) * (P(z,x) \to B(z)) \\
= \ & \underline{P}_A(B)(x).
\end{aligned}$$

Hence, $\underline{P}_A(B) \le \underline{P}_A(\underline{P}_A(B))$.

(2) $\Longrightarrow$ (1) For all $B \in \mathcal{PA}$, we have $\underline{P}_A(B) \leq \underline{P}_A(\underline{P}_A(B))$, then

$$\underline{P}_A(B)(x) \to \underline{P}_A(\underline{P}_A(B))(x) = 1 \ (\forall x \in A_0).$$

For all $x \in A_0$, we have

$$\begin{aligned} & \underline{P}_A(\underline{P}_A(B))(x) \\ = & \bigwedge_{y\in A_0} A(x) * (P(y,x) \to \underline{P}_A(B)(y)) \\ = & \bigwedge_{y\in A_0} A(x) * (P(y,x) \to [A(y) \wedge \bigwedge_{z\in A_0} ((A(y) \to P(z,y)) \to B(z))]) \\ = & \bigwedge_{y\in A_0} A(x) * [(P(y,x) \to A(y)) \wedge \bigwedge_{z\in A_0} (P(y,x) \to ((A(y) \to P(z,y)) \to B(z)))] \\ = & \bigwedge_{y\in A_0} \bigwedge_{z\in A_0} A(x) * [P(z,y) * (A(y) \to P(y,x)) \to B(z)] \end{aligned}$$

and

$\underline{P}_A(B)(x) = \bigwedge_{z\in A_0} A(x) * (P(z,x) \to B(z))$.

By $\underline{P}_A(B)(x) \to \underline{P}_A(\underline{P}_A(B))(x) = 1$ $(\forall x \in A_0)$, we have

$$P(z,y) * (A(y) \to P(y,x)) \to P(z,x) = 1.$$

Hence $P(z,y) * (A(y) \to P(y,x)) \leq P(z,x)$. That is to say, P is transitive.

(1) $\Longrightarrow$ (3) If P is transitive, then for all $B \in \mathcal{PA}$ and all $x \in A_0$, we have

$$\begin{aligned} & \overline{P}_A(\overline{P}_A(B)(x) \\ = & \bigvee_{y\in A_0} \overline{P}_A(B)(y) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} \bigvee_{z\in A_0} B(z) * (A(z) \to P(z,y)) * (A(y) \to P(y,x)) \\ = & \bigvee_{y\in A_0} \bigvee_{z\in A_0} (A(z) \to B(z)) * P(z,y) * (A(y) \to P(y,x)) \\ \leq & \bigvee_{y\in A_0} \bigvee_{z\in A_0} (A(z) \to B(z)) * P(z,x) \\ = & \bigvee_{z\in A_0} B(z) * (A(z) \to P(z,x)) \\ = & \overline{P}_A(B)(x). \end{aligned}$$

Conversely, if (3) holds, then for all $x, z \in A_0$, we have

$$\begin{aligned} P(x,z) & = \overline{P}_A(A_x)(z) \\ & \geq \overline{P}_A(\overline{P}_A(A_x))(z) \\ & = \bigvee_{y\in A_0} P(x,y) * (A(y) \to P(y,z)). \end{aligned}$$

Hence, P is transitive.

If P is transitive, then for all $x, y \in A_0$, we have

$$\begin{aligned} & P^{-1}(x,y) * (A(y) \to P^{-1}(y,z) \\ = & P(y,x) * (A(y) \to P(z,y)) \\ = & P(z,y) * (A(y) \to P(y,x) \\ \leq & P(z,x) \\ = & P^{-1}(x,z). \end{aligned}$$

Then P^{-1} is transitive. Similarly, the statements of (1) $\Longleftrightarrow$ (4) and (1) $\Longleftrightarrow$ (5) are valid. $\square$

When some special classes of L-valued relations are considered, we can obtain the following conclusions.

Corollary 3.13. *Let (A, P) be an L-valued approximation space and P be an L-valued preorder on A. Then the L-valued fuzzy rough approximation operators have the following properties:*

(1) $\underline{P}_A(\underline{P}_A(B)) = \underline{P}_A(B)$;
(2) $\overline{P}_A(\overline{P}_A(B)) = \overline{P}_A(B)$;
(3) $\underline{P^{-1}}_A(B) = \underline{P^{-1}}_A(\underline{P^{-1}}_A(B))$;
(4) $\overline{P^{-1}}_A(\overline{P^{-1}}_A(B)) = \overline{P^{-1}}_A(B)$.

Proof. It follows immediately from Theorems 3.5, 3.9 and 3.12. $\square$

Corollary 3.14. *Let (A, P) be an L-valued approximation space. Then the following statements are equivalent:*

(1) *P is an L-valued equivalence relation;*
(2) $\overline{P}_A(\underline{P}_A(B)) = \underline{P}_A(B)$;
(3) $\underline{P}_A(\overline{P}_A(B)) = \overline{P}_A(B)$;
(4) $\overline{P^{-1}}_A(\underline{P^{-1}}_A(B)) = \underline{P^{-1}}_A(B)$;
(5) $\underline{P^{-1}}_A(\overline{P^{-1}}_A(B)) = \overline{P^{-1}}_A(B)$.

Proof. (1) $\Longrightarrow$ (2) Since P is transitive, for all $B \in \mathcal{PA}$, we have

$$\underline{P}_A(B) \leq \underline{P}_A(\underline{P}_A(B)).$$

Furthermore, by (UF3) and P is symmetric, we have

$$\overline{P}_A(\underline{P}_A(B)) \leq \overline{P}_A(\underline{P}_A(\underline{P}_A(B))) \leq \underline{P}_A(B).$$

Hence, $\overline{P}_A(\underline{P}_A(B)) = \underline{P}_A(B)$.

(2) $\Longrightarrow$ (1) It follows immediately from Theorems 3.9, 3.11 and 3.12. $\square$

4. Axiomatic Characterizations of L-valued Fuzzy Rough Approximation Operators

In this section, we will study axiomatic characterizations of the L-valued fuzzy rough approximation operators. It will be shown that the L-valued fuzzy rough approximation operators can be characterized by a pair of L-fuzzy set operators (H, D), which guarantees the existence of certain types of L-valued relations inducing the same L-valued fuzzy rough approximation operators.

Firstly, we use the residuated implication of $*$ to construct the set-theoretic mappings $H, D : \mathcal{PA} \to \mathcal{PA}$ to characterize the L-valued fuzzy rough approximation operators.

For a mapping $G : \mathcal{PA} \to \mathcal{PA}$, we always consider the following two pairs of conditions: $\forall B, B_i \in \mathcal{PA}$, $i \in I$,

(H1) $G(\widehat{\alpha} * (\widehat{\beta} \to B)) = \widehat{\alpha} * (\widehat{\beta} \to G(B))$ for all $\alpha, \beta \in L$ with $\alpha \leq \beta$ and $\bigvee_{x \in X} B(x) \leq \beta$, (H2) $G(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} G(B_i)$;

(D1) $G(A \wedge (\widehat{\alpha} \to B)) = A \wedge (\widehat{\alpha} \to G(B))$,

(D2) $G(\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} G(B_i)$.

Theorem 4.1. *Let $H : \mathcal{PA} \to \mathcal{PA}$ be a mapping. Then, there exists an L-valued relation P on A such that $H = \overline{P_H}_A$ iff the mapping H satisfies* (H1) *and* (H2).

Proof. The necessity is obvious, we here only prove the sufficiency. Let H satisfy (H1) and (H2). By H we can define a mapping $P_H : A \rightharpoonup A$ as follows:

$$P_H(y, x) = H(A_y)(x) \ (\forall x, y \in A_0).$$

Now we prove P_H is an L-valued relation.

(1) By $H(A_y) \leq A$, we have $P(y, x) \leq A(x)$.

(2) According to (H1) and $A_y = A(y) * (A(y) \to A_y)$, we have

$$H(A(y) * (A(y) \to A_y)) = A(y) * (A(y) \to H(A_y)),$$

and

$$H(A_y) = H(\widehat{A(y)} \wedge A_y) = \widehat{A(y)} \wedge H(A_y) \leq \widehat{A(y)}.$$

Then $P(y, x) = H(A_y)(x) \leq \widehat{A(y)}(x) = A(y)$. Therefore, $P(y, x) \leq A(x) \wedge A(y)$.

For all $B \in \mathcal{PA}$ and $x \in A_0$, we have

$$\begin{aligned} & \overline{P_H}_A(B)(x) \\ = & \bigvee_{y\in A_0} B(y) * (A(y) \to P_H(y, x)) \\ = & \bigvee_{y\in A_0} B(y) * (A(y) \to H(A_y)(x)) \\ = & \bigvee_{y\in A_0} H(B(y) * (A(y) \to A_y))(x) \\ = & H(\bigvee_{y\in A_0} B(y) * (A(y) \to A_y))(x) \\ = & H(B)(x). \end{aligned}$$

Hence, $\overline{P_H}_A = H$. □

Dually, we have

Theorem 4.2. *Let $D : \mathcal{PA} \to \mathcal{PA}$ be a mapping. Then, there exists an L-valued relation P on A such that $D = \underline{P_D}_A$ iff the mapping D satisfies* (D1) *and* (D2).

Proof. The necessity is obvious, we here only proof the sufficiency. Let D satisfy (D1) and (D2). By the mapping D we can define an L-valued relation $P_D : A \rightharpoonup A$ as follows:

$$\begin{aligned} & P_D(y, x) \\ = & [A(x) \to D(A - y_0)(x)] \to 0 \\ = & A(x) \wedge [(A(x) \to D(A - y_0)(x)) \to 0] \ (\forall x, y \in A_0). \end{aligned}$$

Then for all $B \in \mathcal{PA}$ and $x \in A_0$, we have

$$\begin{aligned}
& \underline{P_D}_A(B)(x) \\
= & A(x) \wedge \bigwedge_{z\in A_0} [(A(x) \to P_D(z,x)) \to B(z)] \\
= & A(x) \wedge \bigwedge_{z\in A_0} \{(A(x) \to ([A(x) \to D(A{-}z_0)(x)] \to 0)) \to B(z)\} \\
= & A(x) \wedge \bigwedge_{z\in A_0} [(D(A{-}z_0)(x) \to 0) \to B(z)] \\
= & A(x) \wedge \bigwedge_{z\in A_0} [(B(z) \to 0) \to D(A{-}z_0)(x)] \\
= & \bigwedge_{z\in A_0} D(A \wedge ((B(z) \to 0) \to (A{-}z_0)))(x) \\
= & D(\bigwedge_{z\in A_0} (A \wedge [(B(z) \to 0) \to (A{-}z_0)]))(x) \\
= & D(B)(x).
\end{aligned}$$

Hence, $\underline{P_D}_A = D$. □

Theorem 4.1 (resp., 4.2) shows that a one-to-one correspondence can be constructed between the set-theoretic mappings H (resp., D) and L-valued upper fuzzy rough approximation operators (resp., L-valued lower fuzzy rough approximation operators).

In the following, we assume that L satisfies $(a \to 0) \to 0 = a$ for all $a \in L$, this is to say L is an MV-algebra. We will prove the L-valued upper fuzzy rough approximation operator $\overline{P_H}_A$ and the L-valued lower fuzzy rough approximation operator $\underline{P_D}_A$ are constructed by a same L-valued relation when (H, D) is a pair of dual operators with respect to an order-reversing involution $\sim$, i.e.,

$$\sim H(\sim B) = D(B);\ \sim D(\sim B) = H(B).$$

In this case, we say that H is dual to D.

Theorem 4.3. *Let H be dual to D. Then $P_H = P_D$, where*

$$P_H(y,x) = H(A_y)(x);$$

$$P_D(y,x) = [A(x) \to D(A{-}y_0)(x)] \to 0 \ (\forall x, y \in A_0).$$

Proof. For any $x, y \in A_0$, we have

$$\begin{aligned}
& P_D(y,x) \\
= & A(x) * (\sim H(\sim(A{-}y_0))(x) \to 0) \\
= & A(x) * [A(x) * (H(\sim(A{-}y_0))(x) \to 0) \to 0] \\
= & H(\sim(A{-}y_0))(x)
\end{aligned}$$

and

$$\begin{aligned}
& \sim(A{-}y_0)(z) \\
= & A(x) * ((A{-}y_0)(z) \to 0) \\
= & \begin{cases} A(z) * (A(z) \to 0), & z \neq y, \\ A(y), & z = y \end{cases} \\
= & A_y.
\end{aligned}$$

Then $H(\sim(A{-}y_0))(x) = H(A_y)(x)$. Hence, $P_H = P_D$. □

Theorem 4.3 is very important which indicates that L-valued upper fuzzy rough approximation operator $\overline{P_H}_A$ and L-valued lower fuzzy rough approximation operator $\underline{P_D}_A$ can be constructed by a same L-valued relation.

In what follows, we discuss the relations between special L-valued relations and mappings H, D.

Theorem 4.4. *Let $H : \mathcal{PA} \longrightarrow \mathcal{PA}$ be a mapping satisfying* (H1) *and* (H2). *Then there exists an L-valued relation $P_H : A \rightharpoonup A$ which is reflexive* (*resp., symmetric, transitive*) *such that $H = \overline{P_H}_A$ iff H satisfies* (HR) (*resp.,* (HS), (HT))*:*
(HR) $B \leq H(B)$ $(\forall B \in \mathcal{PA})$;
(HS) $H(A_y)(x) = H(A_x)(y)$ $(\forall x,\ y \in A_0)$;
(HT) $H(H(B)) \leq H(B)$ $(\forall B \in \mathcal{PA})$.

Proof. The necessity and the sufficiency follow immediately from Theorems 3.8, 3.9, 3.11 and 4.1. □

Corollary 4.5. *Let $H : \mathcal{PA} \longrightarrow \mathcal{PA}$ be a mapping satisfying* (H1) *and* (H2). *Then there exists an L-valued equivalence relation $P_H : A \rightharpoonup A$ such that $H = \overline{P_H}_A$ iff H satisfies* (HR),(HS) and (HT).

Theorem 4.6. *Let $D : \mathcal{PA} \longrightarrow \mathcal{PA}$ be a mapping satisfying* (D1) *and* (D2). *Then there exists an L-valued relation $P_D : A \rightharpoonup A$ which is reflexive* (*resp., symmetric, transitive*) *such that $D = \underline{P_D}_A$ iff D satisfies* (DR) (*resp.* (DS), (DT))*:*
(DR) $D(B) \leq B$ $(\forall B \in \mathcal{PA})$;
(DS) $[A(x) \to D(A - y_0)(x)] \to 0 = [A(y) \to D(A - x_0)(y)] \to 0$ $(\forall x, y \in A_0)$;
(DT) $D(B) \leq DD(B)$ $(\forall B \in \mathcal{PA})$.

Proof. The necessity and the sufficiency follow immediately from Theorems 3.8, 3.9, 3.11 and 4.2. □

Corollary 4.7. *Let $D : \mathcal{PA} \longrightarrow \mathcal{PA}$ be a mapping satisfying* (D1) *and* (D2). *Then there exists an L-valued equivalence relation $P_D : A \rightharpoonup A$ such that $D = \underline{P_D}_A$ iff D satisfies* (DR), (DS) and (DT).

Using the composition of H and D, we can get a pair of mapping $HH,\ DD : \mathcal{PA} \to \mathcal{PA}$. This pair is not equivalent to that of (H, D). This means that iterated using of the approximation operators is not reducible. In fact, by the duality of D and H, they can define two pairs of dual operators.

The operators HH and DD are dual to w.r.t. $\sim$, namely, $DD = \sim HH(\sim)$. By the properties of H and D, it follows that HH and DD satisfy (H1), (H2), (D1) and (D2):
(H1) $HH(\widehat{\alpha} * (\widehat{\beta} \to B)) = \widehat{\alpha} * (\widehat{\beta} \to HH(B))$,
(H2) $HH(\bigvee_{i \in I} B_i) = \bigvee_{i \in I} HH(B_i)$;
(D1) $DD(A \wedge (\widehat{\alpha} \to B)) = A \wedge (\widehat{\alpha} \to DD(B))$,
(D2) $DD(\bigwedge_{i \in I} B_i) = \bigwedge_{i \in I} DD(B_i)$.

Therefore, we can use (HH, DD) to construct a pair of L-valued fuzzy rough approximation operators $(\overline{P_{HH}}_A, \underline{P_{DD}}_A)$. By using the same manner, we may obtain many pairs of L-valued fuzzy approximation operators. Once an operator is obtained by iterated using of H, the other operator can be obtained by iterated using of D with the same length, for example the pair set-theoretic mappings $(HHHH, DDDD)$.

5. **Conclusion**

In this paper, we study L-valued fuzzy rough sets based on GL-quantale. By using L-valued relations, we construct L-valued fuzzy rough set in both constructive and axiomatic approaches. The classical L-fuzzy rough set is a special case of L-valued fuzzy rough set. The results presented in this paper can hopefully provide more insight into and a full understanding of fuzzy rough set theory. It may be more useful for potential applications of rough set theory in the fuzzy environment and help us to gain a general mathematical structures of the fuzzy approximation operators. The analysis will facilitate further research in uncertain reasoning under fuzziness.

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