CATEGORY AND SUBCATEGORIES OF (L, M)-FUZZY CONVEX SPACES

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Abstract. In this paper, (L, M)-fuzzy domain finiteness and (L, M)-fuzzy restricted hull spaces are introduced, and several characterizations of the category (L, M)-**CS** of (L, M)-fuzzy convex spaces are obtained. Then, (L, M)-fuzzy stratified (resp. weakly induced, induced) convex spaces are introduced. It is proved that both categories, the category (L, M)-**SCS** of (L, M)-fuzzy stratified convex spaces and the category (L, M)-**WICS** of (L, M)-fuzzy weakly induced convex spaces, are coreflective subcategories of (L, M)-**CS**. It is also proved that three isomorphic categories, namely, the category M-**CS** of M-fuzzifying convex spaces, the category (L, M)-**CGCS** of (L, M)-fuzzy convex spaces induced by M-fuzzifying convex spaces and the category (L, M)-**ICS** of (L, M)-fuzzy induced convex spaces, are coreflective subcategories of both (L, M)-**SCS** and (L, M)-**WICS**.

1. Introduction

Convex Theory, being inspired originally by some simple geometric problems such as shapes of circles and polytopes in Euclidean spaces [4], has been developing along two directions since the last century. One was motivated by concrete problems such as existences of continuous selections and fixed points, or optimization problems [5, 13, 15, 23]; the other was based on an axiomatic point of view, where the notion of axiomatic convex structures was introduced and its theory was established [6, 12, 22, 34]. Later, it is found that some results of this theory, capturing many combinatorial features of convex structures, are quite useful for handling convex combinations [8, 14]. Now, this theory has been abstracted into many mathematical fields, where some new convex structures were discovered, including order convex structures [11], lattice convex structures [33], metric convex structures [34], graph convex structures [7] and median convex structures [17].

In the theory of convex spaces, domain finiteness is an important notion which distinguishes convex structures from closure structures and co-topologies. In addition, it is found that a closure space is a convex space iff it is stable for up-directed unions or its closure operator is domain finite. With the help of this fact, Vel showed that convex spaces and restricted hull operators are one-to-one corresponding, and that many properties of convex spaces can be characterized by finite sets [34].

Convex structures in fuzzy settings have been studied in several ways. The notion of fuzzy convex structures defined by Rosa [24] was further extended by Maruyama

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[16], who introduced the notion of L-convex structures. In fact, both notions were defined in a common way: each notion is actually a family of fuzzy or L-fuzzy sets satisfying certain set of axioms. However, from a totally different way, Shi and Xiu introduced the notion of M-fuzzifying convex structures, where each subset can be regarded as a convex set to some degree [31]. Now, many properties in M-fuzzifying convex spaces have been studied [30, 31, 36, 38, 37, 41, 40, 42]. Later, Xiu and Shi introduced the notion of (L, M)-fuzzy convex structures [32], generalizing of both L-convex structures and M-fuzzifying convex structures. In fact, (L, M)-fuzzy convex structures can be generated by L-fuzzy Alexandroff topologies [9], L-fuzzy groups [3], L-fuzzy sublattices [2], L-fuzzy pre-ordered spaces [10], L-fuzzy vector spaces [16] and L-fuzzy ordered spaces [43], etc.

We will introduce and characterize domain finiteness of (L, M)-fuzzy convex spaces. We also introduce notions of (L, M)-fuzzy restricted hull (resp. stratified, weakly induced, induced, etc) convex spaces, and discuss their relations in a view of category aspect.

This paper is arranged as follows. In Section 2, we recall some notions of (L, M)-fuzzy convex spaces and M-fuzzifying convex spaces. In Sections 3 and 4, we introduce and characterize domain finiteness of both L-convex spaces and (L, M)-fuzzy convex spaces. Also, we characterize (L, M)-**CS** by the category (L, M)-**RHS** of (L, M)-fuzzy restricted hull spaces, the category (L, M)-**GCS** of (L, M)-fuzzy convex spaces generated by (L, M)-fuzzy closure spaces, the category (L, M)-**DFCOS** of (L, M)-fuzzy domain finite closure spaces and the category (L, M)-**CAS** of (L, M)-fuzzy concave spaces. We also show that the category L-**CS** of L-convex spaces is a coreflective subcategory of (L, M)-**CS**. In Sections 5 and 6, we introduce (L, M)-fuzzy stratified (resp. weakly induced) convex spaces. We show that both (L, M)-**SCS** and (L, M)-**WICS** are coreflective subcategories of (L, M)-**CS**. We also show that the category L-**SCS** of L-stratified convex spaces and the category L-**WICS** of L-weakly induced convex spaces are coreflective subcategories of (L, M)-**SCS** and (L, M)-**WICS**, respectively. In Section 7, we introduce (L, M)-fuzzy induced convex spaces and show that three isomorphic categories, the category M-**CS** of M-fuzzifying convex spaces, the category (L, M)-**CGCS** and the category (L, M)-**ICS** of (L, M)-fuzzy induced convex spaces, are coreflective subcategories of both (L, M)-**SCS** and (L, M)-**WICS**. In Section 8, we draw a diagram to show relations among (L, M)-**CS** and its subcategories.

2. Preliminaries

In this paper, X, Y are nonempty sets. 2^X is the power set of X and 2^X_{fin} is the set of all finite subsets of X. Both L and M are completely distributive lattices and M has an inverse involution $'$. The least element and the greatest element of L (resp. M) are denoted by $\bot$ and $\top$. In particular, we write $\mathbf{2}$ for the lattice $\{\bot, \top\}$. If $\varphi \subseteq M$, we denote $\bigvee \varphi = \bigvee_{b\in\varphi} b$ and $\bigwedge \varphi = \bigwedge_{b\in\varphi} b$. Also, we adopt the convention that $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$. An element $a \in M$ is called a prime, if for all $b, c \in M$, $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. We denote $P(M)$ the set of all primes in $M\backslash\{\top\}$, and $J(M) = \{a \in M : a' \in P(M)\}$ [35].

A binary relation $\prec$ on M is defined as: for all $a, b \in M$, $a \prec b$ iff for all $\varphi \subseteq M$, $b \leq \bigvee \varphi$ always implies the existence of $d \in \varphi$ such that $a \leq d$. Further, a binary relation $\prec^{op}$ is defined as: for all $a, b \in M$, $a \prec^{op} b$ iff $b' \prec a'$. Clearly, $\beta(\bigvee_{i\in\Omega} a_i) = \bigcup_{i\in\Omega} \beta(a_i)$ and $\alpha(\bigwedge_{i\in\Omega} a_i) = \bigcup_{i\in\Omega} \alpha(a_i)$ for all $\{a_i\}_{i\in\Omega} \subseteq M$, where $\beta(a) = \{b : b \prec a\}$ and $\alpha(a) = \{b : a \prec^{op} b\}$ for all $a \in M$. We have $\beta(\bot) = \alpha(\top) = \emptyset$ and $a = \bigvee \beta(a) = \bigvee \beta^*(a) = \bigwedge \alpha(a) = \bigwedge \alpha^*(a)$ for all $a \in M$, where $\beta^*(a) = \beta(a) \cap J(M)$ and $\alpha^*(a) = \alpha(a) \cap P(M)$ [26, 35].

L^X is the set of all L-fuzzy sets on X, whose greatest element and least element are denoted by $\underline{\top}$ and $\underline{\bot}$. The characterization function of a subset $U \in 2^X$ is an L-fuzzy set denoted by χ_U. The L-fuzzy set on X with a constant value $\lambda \in L$ is denoted by $\underline{\lambda}$. An L-fuzzy point with a support x and a value $r \in L$ is also an L-fuzzy set which is denoted by x_r. Also, we will denote $J(L^X) = \{x_r \in L^X : r \in J(L)\}$, $\beta(A) = \{x_a \in L^X : a \prec A(x)\}$ and $\beta^*(A) = \beta(A) \cap J(L^X)$ for each $A \in L^X$. Clearly, $A = \bigvee \beta(A) = \bigvee \beta^*(A)$. For convenience, we will write $x_\lambda \not\leq^* A$ for $x_\lambda \in J(L^X)$ and $x_\lambda \not\leq A$. We denote $A_{[a]} = \{x \in X : A(x) \geq a\}$ and $A^{[a]} = \{x \in X : a \notin \alpha(A(x))\}$ for each $A \in L^X$ and each $a \in L$ [26, 27, 28, 29, 31]. For convenience, if we write $\bigvee^{dir}_{i\in\Omega} A_i$, we mean that $\{A_i\}_{i\in\Omega} \subseteq L^X$ is up-directed.

A mapping $f_L^{\rightarrow} : L^X \to L^Y$ is called an L-fuzzy mapping, if there exists a mapping $f : X \to Y$ such that $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $A \in L^X$ and all $y \in Y$. Also, $f_L^{\leftarrow} : L^Y \to L^X$ is defined as: $f_L^{\leftarrow}(B)(x) = B(f(x))$ for all $B \in L^Y$ and all $x \in X$ [27, 28, 29].

Definition 2.1. [29, 30, 32] A pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy closure space, where $\mathcal{C} : L^X \to M$ is a mapping, if

(LMC1) $\mathcal{C}(\underline{\bot}) = \mathcal{C}(\underline{\top}) = \top$;

(LMC2) $\mathcal{C}(\bigwedge_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathcal{C}(A_i)$ for all $\{A_i\}_{i\in\Omega} \subseteq L^X$.

Further, $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex space [32], if

(LMC3) $\mathcal{C}(\bigvee_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathcal{C}(A_i)$ for all totally ordered set $\{A_i\}_{i\in\Omega} \subseteq L^X$.

Let $\mathcal{C}$ and $\mathcal{D}$ be (L, M)-fuzzy convex structures on X. $\mathcal{C}$ is said to be finer than $\mathcal{D}$, denoted by $\mathcal{D} \leq \mathcal{C}$, if $\mathcal{D}(A) \leq \mathcal{C}(A)$ for all $A \in L^X$.

Theorem 2.2. [27, 29] *The closure (resp. hull) operator $co_{\mathcal{C}} : L^X \to M^{J(L^X)}$ (briefly, co) of an (L, M)-fuzzy closure (resp. convex) space $(X, \mathcal{C})$ is defined as:*

$$\forall A \in L^X, \forall x_\lambda \in J(L^X), \quad co(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq^* B \geq A} (\mathcal{C}(B))'.$$

Then, for all $A, B \in L^X$ and $x_\lambda \in J(L^X)$, co satisfies the following conditions:

(LMCO1) $co(\underline{\bot})(x_\lambda) = \bot$;

(LMCO2) $x_\lambda \leq A$ implies that $co(A)(x_\lambda) = \top$;

(LMCO3) $A \leq B$ implies that $co(A) \leq co(B)$;

(LMCO4) $co(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq^* B} co(B)(y_\mu)$.

Conversely, let $co : L^X \to M^{J(L^X)}$ be an operator satisfying (LMCO1)–(LMCO4) and a mapping $\mathcal{C}_{co} : L^X \to M$ be defined as:

$$\forall A \in L^X, \quad \mathcal{C}_{co}(A) = \bigwedge_{x_\lambda \not\leq^* A} (co(A)(x_\lambda))'.$$

Then $\mathcal{C}_{co}$ is an (L,M)-fuzzy closure structure with $co_{\mathcal{C}_{co}} = co$.

Theorem 2.3. [28] *If $co : L^X \to M^{J(L^X)}$ is an operator satisfying (LMCO1)–(LMCO4), then it satisfies (LMCO0).*

(LMCO0) $co(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} co(A)(x_\mu)$ for all $A \in L^X$ and $x_\lambda \in J(L^X)$.

Remark 2.4. (1) An $(L,\mathbf{2})$-fuzzy closure space (resp. $(L,\mathbf{2})$-fuzzy convex space) $(X,\mathcal{C})$ is reduced to an L-closure space (resp. L-convex space [16]). That is, $\mathcal{C}$ satisfies (LC1) and (LC2) (resp. (LC1)–(LC3)).

(LC1) $\perp, \top \in \mathcal{C}$.

(LC2) If $\{A_i\}_{i\in\Omega} \subseteq \mathcal{C}$, then $\bigwedge_{i\in\Omega} A_i \in \mathcal{C}$.

(LC3) If $\{A_i\}_{i\in\Omega} \subseteq \mathcal{C}$ is totally ordered, then $\bigvee_{i\in\Omega} A_i \in \mathcal{C}$.

The L-closure operator $co : L^X \to L^X$ of an L-closure space $(X,\mathcal{C})$ is defined as: $co_{\mathcal{C}}(A) = \bigwedge\{B \in \mathcal{C} : A \leq B\}$ for all $A \in L^X$, which is still denoted by co. Then

(LCO1) $co(\perp) = \perp$;

(LCO2) $A \leq co(A)$ for all $A \in L^X$;

(LCO3) $co(A) \leq co(B)$ for all $A, B \in L^X$ with $A \leq B$;

(LCO4) $co(co(A)) = co(A)$ for all $A \in L^X$.

Conversely, if an operator $co : L^X \to L^X$ satisfies (LCO1)–(LCO4) and $\mathcal{C}_{co} = \{A \in L^X : co(A) = A\}$, then $(X,\mathcal{C}_{co})$ is an L-closure space satisfying $co_{\mathcal{C}_{co}} = co$.

(2) $(X,\mathcal{C})$ is an (L,M)-fuzzy closure space iff $(X,\mathcal{C}_{[a]})$ is an L-closure space for all $a \in M\backslash\{\perp\}$, or $(X,\mathcal{C}^{[a]})$ is an L-closure space for all $a \in \alpha(\perp)$ [32].

(3) A $(\mathbf{2},M)$-fuzzy convex structure is reduced to an M-fuzzifying convex structure [31]. A $(\mathbf{2},\mathbf{2})$-fuzzy convex space is reduced to a convex space [34].

Definition 2.5. [30] An operator $\mathcal{H} : 2^X_{fin} \to M^X$ is called an M-fuzzifying restricted hull operator and the pair $(X,\mathcal{H})$ is called an M-fuzzifying restricted hull space, if

(MRH1) $\mathcal{H}(\emptyset)(x) = \perp$ for all $x \in X$;

(MRH2) $\mathcal{H}(U)(x) = \top$ for all $U \in 2^X_{fin}$ and $x \in U$;

(MRH3) $\mathcal{H}(V)(x) \wedge \bigwedge_{y\in V} \mathcal{H}(U)(y) \leq \mathcal{H}(U)(x)$ for all $U, V \in 2^X_{fin}$ and $x \in X$.

Theorem 2.6. [30] *If $(X,\mathcal{C})$ is an M-fuzzifying convex space, then the restriction $\mathcal{H}_{\mathcal{C}} : 2^X_{fin} \to M^X$ of the hull operator $co_{\mathcal{C}}$ on 2^X_{fin} is an M-fuzzifying restricted hull operator.*

Conversely, if $\mathcal{H} : 2^X_{fin} \to M^X$ is an M-fuzzifying restricted hull operator and $co_{\mathcal{H}} : 2^X \to M^X$ is defined as: $co_{\mathcal{H}}(U)(x) = \bigvee_{V\in 2^U_{fin}} \mathcal{H}(V)(x)$ for all $U \in 2^X$ and $x \in X$, then $co_{\mathcal{H}}$ is the hull operator of an M-fuzzifying convex space $(X,\mathcal{C}_{\mathcal{H}})$ satisfying $\mathcal{H}_{\mathcal{C}_{\mathcal{H}}} = \mathcal{H}$.

Let $(X,\mathcal{H}_X)$ and $(Y,\mathcal{H}_Y)$ be M-fuzzifying restricted hull spaces. $f : X \to Y$ is called an M-fuzzifying RHP mapping, if $\mathcal{H}_X(U)(x) \leq \mathcal{H}_Y(f(U))(f(x))$ for all $U \in 2^X_{fin}$ and $x \in X$.

Definition 2.7. (1) $B \in L^X$ is called a proper subset of $A \in L^X$, denoted by $B \lneqq A$, if $B \leq A$ and $B \neq A$. Clearly, $B \lneqq A$ iff $B \leq A$ and there exists $x_\lambda \prec A$ such that $x_\lambda \not\leq B$.

(2) $F \in L^X$ is called an L-fuzzy finite set relative to $A \in L^X$, if there exists a $\varphi \in 2^{\beta^*(A)}_{fin}$ such that $F = \bigvee \varphi$. We denote $\mathfrak{F}(A) = \{\bigvee \varphi : \varphi \in 2^{\beta^*(A)}_{fin}\}$. In particular, L-fuzzy finite sets relative to $\top$ are simply called L-fuzzy finite sets. We simply denote $\mathfrak{F}(\top)$ by $\mathfrak{F}(X)$.

Proposition 2.8. *Let $A, B \in L^X$ and $F \in \mathfrak{F}(X)$. Then*
(1) $\mathfrak{F}(A)$ is up-directed;
(2) $B \leq A$ iff $\mathfrak{F}(B) \subseteq \mathfrak{F}(A)$;
(3) $\beta^*(A) \subseteq \mathfrak{F}(A)$ and $\bigvee \mathfrak{F}(A) = A$;
(4) $\mathfrak{F}(\bigvee_{i\in\Omega} A_i) = \bigcup_{i\in\Omega} \mathfrak{F}(A_i)$ for all up-directed set $\{A_i\}_{i\in\Omega} \subseteq L^X$.

Proof. (1)–(3) are clear. We only need to prove (4). Clearly, $\bigcup_{i\in\Omega} \mathfrak{F}(A_i) \subseteq \mathfrak{F}(\bigvee_{i\in\Omega} A_i)$. Conversely, if $F \in \mathfrak{F}(\bigvee_{i\in\Omega} A_i)$, then there exists $\varphi \in 2^{\beta^*(\bigvee_{i\in\Omega} A_i)}_{fin} = 2^{\bigcup_{i\in\Omega} \beta^*(A_i)}_{fin}$ such that $F = \bigvee \varphi$. Since the set $\{\beta^*(A_i)\}_{i\in\Omega}$ is also up-directed, there exists $i_0 \in \Omega$ such that $\varphi \in 2^{\beta^*(A_{i_0})}_{fin}$. Thus $F \in \mathfrak{F}(A_{i_0}) \subseteq \bigcup_{i\in\Omega} \mathfrak{F}(A_i)$. Therefore $\bigcup_{i\in\Omega} \mathfrak{F}(A_i) = \mathfrak{F}(\bigvee_{i\in\Omega} A_i)$. □

The following proposition is similar to Lemma 3 in [31].

Proposition 2.9. *Let $p, q \in M$. The following statements are equivalent.*
(1) $p \leq q$.
(2) [31] *$\forall a \in M$, $a \leq p$ implies $a \leq q$.*
(3) [31] *$\forall a \in \alpha(\perp)$, $a \notin \alpha(p)$ implies $a \notin \alpha(q)$.*
(4) [31] *$\forall a \in J(M)$, $a \leq p$ implies $a \leq q$.*
(5) $\forall a \in \beta(\top)$, $a \prec p$ implies $a \leq q$.
(6) $\forall a \in P(M)$, $p \not\leq a$ implies $q \not\leq a$.
(7) $\forall a \in \beta^*(\top)$, $a \prec p$ implies $a \leq q$.

Definition 2.10. [1] Let $\mathbf{A}, \mathbf{B}$ be categories. A mapping $\mathbb{F} : \mathbf{A} \to \mathbf{B}$ is called a functor, if for all $A, B \in O(\mathbf{A})$, $f \in \hom_{\mathbf{A}}(A, B)$ and $g \in \hom_{\mathbf{A}}(B, C)$,
(1) $\mathbb{F}(A) \in O(\mathbf{B})$ and $\mathbb{F}(f) \in \hom_{\mathbf{B}}(\mathbb{F}(A), \mathbb{F}(B))$;
(2) $\mathbb{F}(g \circ f) = \mathbb{F}(g) \circ \mathbb{F}(f)$;
(3) $\mathbb{F}(id_A) = id_{\mathbb{F}(A)}$.

In particular, the functor $\mathbb{I}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$, defined as: $\mathbb{I}_{\mathbf{A}}(A) = A$ and $\mathbb{I}_{\mathbf{A}}(f) = f$ for all $A \in O(\mathbf{A})$ and $f \in \hom_{\mathbf{A}}(A, B)$, is called the identity functor of $\mathbf{A}$.

Definition 2.11. [1] Let $\mathbf{A}$ and $\mathbf{B}$ be categories. A functor $\mathbb{F} : \mathbf{A} \to \mathbf{B}$ is called an isomorphism provided that there is a functor $\mathbb{G} : \mathbf{B} \to \mathbf{A}$ such that $\mathbb{G} \circ \mathbb{F} = \mathbb{I}_{\mathbf{A}}$ and $\mathbb{F} \circ \mathbb{G} = \mathbb{I}_{\mathbf{B}}$. $\mathbf{A}, \mathbf{B}$ are called isomorphic, denoted by $\mathbf{A} \cong \mathbf{B}$, if there exists an isomorphic functor $\mathbb{F} : \mathbf{A} \to \mathbf{B}$.

Definition 2.12. [1] A functor $\mathbb{F} : \mathbf{A} \to \mathbf{B}$ is said to be
(1) an embedding, if $\mathbb{F}$ is injective on both objects and morphisms;
(2) faithful, if $\mathbb{F} : \hom_{\mathbf{A}}(A, B) \to \hom_{\mathbf{B}}(\mathbb{F}(A), \mathbb{F}(B))$ is injective for all $A, B \in O(\mathbf{A})$;
(3) full, if $\mathbb{F} : \hom_{\mathbf{A}}(A, B) \to \hom_{\mathbf{B}}(\mathbb{F}(A), \mathbb{F}(B))$ is surjective for all $A, B \in O(\mathbf{A})$.

Definition 2.13. [1] A concrete category over a category $\mathbf{X}$ is a pair $(\mathbf{A}, \mathbb{U})$, where $\mathbb{U} : \mathbf{A} \to \mathbf{X}$ is a faithful functor. A concrete category over **Set** is called a construct. A concrete functor from $(\mathbf{A}, \mathbb{U})$ to $(\mathbf{B}, \mathbb{V})$ is a functor $\mathbb{F} : \mathbf{A} \to \mathbf{B}$ such that $\mathbb{U} = \mathbb{V} \circ \mathbb{F}$.

Definition 2.14. [1] Let $(\mathbf{A}, \mathbb{U})$ and $(\mathbf{B}, \mathbb{V})$ be constructs and $X \in O(\mathbf{Set})$.

(1) The fibre of X is a preordered class $(\mathfrak{F}_{\mathbf{A}}(X), \ll)$, where $\mathfrak{F}_{\mathbf{A}}(X) = \{A \in O(\mathbf{A}) : \mathbb{U}(A) = X\}$ and $\ll$ is defined as: $A \ll B$ iff $id_X : \mathbb{U}(A) \to \mathbb{U}(B)$ is an $\mathbf{A}$-morphism.

(2) If $\mathbb{F}, \mathbb{G} : \mathbf{A} \to \mathbf{B}$ are concrete functors, then $\mathbb{F}$ is said to be finer than $\mathbb{G}$, denoted by $\mathbb{F} \ll \mathbb{G}$, provided that $\mathbb{F}(A) \ll \mathbb{G}(A)$ for each $A \in O(\mathbf{A})$.

(3) If $\mathbb{F} : \mathbf{A} \to \mathbf{B}$ and $\mathbb{G} : \mathbf{B} \to \mathbf{A}$ are concrete functors, then the pair $(\mathbb{F}, \mathbb{G})$ is called a Galois correspondence provided that $\mathbb{I}_{\mathbf{A}} \ll \mathbb{G} \circ \mathbb{F}$ and $\mathbb{F} \circ \mathbb{G} \ll \mathbb{I}_{\mathbf{B}}$.

(4) A functor $\mathbb{T} : (\mathbf{A}, \mathbb{U}) \to (\mathbf{B}, \mathbb{V})$ is called topological provided that every $\mathbb{T}$-spaced sink $(f_j : B \to \mathbb{T}(A_j))_{j \in J}$ has a unique $\mathbb{T}$-final lift $(\overline{f_j} : A \to A_j)_{j \in J}$.

(5) $(\mathbf{A}, \mathbb{U})$ is called topological provided that $\mathbb{U} : \mathbf{A} \to \mathbf{Set}$ is topological.

Theorem 2.15. [1] *Let $(\mathbf{A}, \mathbb{U})$ be a subcategory of a category $(\mathbf{B}, \mathbb{V})$ with the inclusion functor $\mathbb{E} : (\mathbf{A}, \mathbb{U}) \to (\mathbf{B}, \mathbb{V})$. Then $(\mathbf{A}, \mathbb{U})$ is coreflective in $(\mathbf{B}, \mathbb{V})$ iff there exists a functor $\mathbb{R} : (\mathbf{B}, \mathbb{V}) \to (\mathbf{A}, \mathbb{U})$ such that $(\mathbb{E}, \mathbb{R})$ is a Galois correspondence.*

Remark 2.16. (1) If $co : L^X \to M^{J(L^X)}$ satisfies (LMCO1)–(LMCO4), then (X, co) is called an (L, M)-fuzzy closure operator space. Let (X, co_X) and (Y, co_Y) be (L, M)-fuzzy closure operator spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy closure operator preserving mapping, if $co_X(A)(x_\lambda) \leq co_Y(f_L^{\rightarrow}(A))(f_L^{\rightarrow}(x_\lambda))$ for $A \in L^X$ and $x_\lambda \in J(L^X)$.

(2) Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be (L, M)-fuzzy convex spaces. A mapping $f : X \to Y$ is called an (L, M)-CP mapping, if $f : (X, co_{\mathcal{C}_X}) \to (Y, co_{\mathcal{C}_Y})$ is an (L, M)-fuzzy closure operator preserving mapping [32]. The category whose objects are (L, M)-fuzzy convex spaces and whose morphisms are (L, M)-CP mappings, is denoted by (L, M)-**CS**. The corresponding construct is denoted by $((L, M)$-**CS**,$\mathbb{C})$.

(3) Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-convex spaces. A mapping $f : X \to Y$ is called an L-CP mapping, if $f_L^{\leftarrow}(B) \in \mathcal{C}_X$ for all $B \in \mathcal{C}_Y$ [16]. $(L, \mathbf{2})$-**CS** is denoted by L-**CS** [18].

(4) Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be M-fuzzifying convex spaces. A mapping $f : X \to Y$ is called an M-fuzzifying CP mapping, if $co_{\mathcal{C}_X}(U)(x) \leq co_{\mathcal{C}_Y}(f(U))(f(x))$ for all $U \in 2^X$ and $x \in X$ [31]. $(\mathbf{2}, M)$-**CS** is denoted by M-**CS**. In particular, **2-CS** is denoted by **CS**.

(5) The category, consisting of M-fuzzifying restricted hull spaces as objects and M-fuzzifying RHP mappings as morphisms, is denoted by M-**RHS**. The corresponding construct is denoted by (M-**RHS**,$\mathbb{H}$). In particular, **2-RHS** is denoted by **RHS**.

3. Some properties of L-convex spaces

Theorem 3.1. *Any L-convex space $(X, \mathcal{C})$ is stable for up-directed unions, i.e.,*
(LC3) $\bigvee_{i \in \Omega} A_i \in \mathcal{C}$ for each up-directed family $\{A_i\}_{i \in \Omega} \subseteq \mathcal{C}$.*

Proof. Let $\mathcal{D} = \{D_i\}_{i\in\Omega} \subseteq \mathcal{C}$ be up-directed and $D = \bigvee_{i\in\Omega} D_i$. Define a set

$$\mathcal{M} = \{A \in \mathcal{C} : \forall D_i \in \mathcal{D}, A \vee_{\mathcal{C}} D_i \leq D\},$$

where $A \vee_{\mathcal{C}} D_i = \bigwedge\{B \in \mathcal{C} : A \vee D_i \leq B\}$. Then we have the following results.

(i) $\mathcal{D} \subseteq \mathcal{M}$.

In fact, let $D_{i_0} \in \mathcal{D}$ and $D_i \in \mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{C}$ is up-directed, there exists $D_j \in \mathcal{D}$ such that $D_{i_0}, D_i \leq D_j$. Then $D_{i_0} \vee_{\mathcal{C}} D_i \leq D_j \leq D$, which shows $D_{i_0} \in \mathcal{M}$. Thus $\mathcal{D} \subseteq \mathcal{M}$.

(ii) $A \leq D$ for all $A \in \mathcal{M}$. It directly follows from definition of $\mathcal{M}$.

(iii) $A \vee_{\mathcal{C}} D_i \in \mathcal{M}$ for all $A \in \mathcal{M}$ and $D_i \in \mathcal{D}$.

In fact, by (LC2), $A \vee_{\mathcal{C}} D_i \in \mathcal{C}$. For each $D_j \in \mathcal{D}$, there exists $D_k \in \mathcal{D}$ such that $D_i, D_j \leq D_k$. Thus $D_j, A \vee_{\mathcal{C}} D_i \leq A \vee_{\mathcal{C}} D_k \in \mathcal{C}$. Hence $(A \vee_{\mathcal{C}} D_i) \vee D_j \leq A \vee_{\mathcal{C}} D_k$, which shows $(A \vee_{\mathcal{C}} D_i) \vee_{\mathcal{C}} D_j \leq A \vee_{\mathcal{C}} D_k \leq D$. Therefore $(A \vee_{\mathcal{C}} D_i) \in \mathcal{M}$.

(iv) $\mathcal{M}$ is inductive. i.e., each nonempty chain $\mathcal{K} = \{K_j\}_{j\in\Lambda} \subseteq \mathcal{M}$ is bounded in $\mathcal{M}$.

In fact, $K = \bigvee \mathcal{K} \in \mathcal{C}$ by (LC3). If $D_i \in \mathcal{D}$, the set $\{(K_j \vee_{\mathcal{C}} D_i)\}_{j\in\Lambda}$ is also a chain in $\mathcal{C}$. Thus $\bigvee_{j\in\Lambda}(K_j \vee_{\mathcal{C}} D_i) \in \mathcal{C}$ and $K \vee_{\mathcal{C}} D_i \leq \bigvee_{j\in\Lambda}(K_j \vee_{\mathcal{C}} D_i) \leq D$. Hence $K \in \mathcal{M}$.

By (iv) and Zorn's lemma, $\mathcal{M}$ has a maximal element, namely, $M_{\mathcal{M}}$. Then $D \geq M_{\mathcal{M}}$ by (ii). Conversely, by (iii), we have $M_{\mathcal{M}} \vee_{\mathcal{C}} D_i \in \mathcal{M}$ for all $D_i \in \mathcal{D}$. By maximality of $M_{\mathcal{M}}$ and $M_{\mathcal{M}} \vee_{\mathcal{C}} D_i \geq M_{\mathcal{M}}$, we have $D_i \leq M_{\mathcal{M}} \vee_{\mathcal{C}} D_i = M_{\mathcal{M}}$. Therefore $D = M_{\mathcal{M}} \in \mathcal{C}$. □

Theorem 3.2. *For an L-closure space $(X, \mathcal{C})$, the following results are equivalent.*

(1) $\mathcal{C}$ is an L-convex structure.

(2) $\mathcal{C}$ is stable for up-directed unions.

(3) The closure operator co is domain finite, that is, co satisfies (LDF) as below.

(LDF) $co(A) = \bigvee_{F\in\mathfrak{F}(A)} co(F)$ for all $A \in L^X$.

(4) co is stable for up-directed union, that is, $co(\bigvee_{i\in\Omega} A_i) = \bigvee_{i\in\Omega} co(A_i)$ for all up-directed family $\{A_i\}_{i\in\Omega} \subseteq L^X$.

Proof. (1) ⇒ (2): It follows from Theorem 3.1.

(2) ⇒ (3): Clearly, $\bigvee_{F\in\mathfrak{F}(A)} co(F) \leq co(A)$. Conversely, since $\mathfrak{F}(A)$ is up-directed, the set $\{co(F) : F \in \mathfrak{F}(A)\}$ is also up-directed. Thus $\bigvee_{F\in\mathfrak{F}(A)} co(F) \in \mathcal{C}$. Hence $A = \bigvee_{F\in\mathfrak{F}(A)} F \leq \bigvee_{F\in\mathfrak{F}(A)} co(F)$ and $co(A) \leq \bigvee_{F\in\mathfrak{F}(A)} co(F)$. Therefore $co(A) = \bigvee_{F\in\mathfrak{F}(A)} co(F)$.

(3) ⇒ (4): Let $\{A_i\}_{i\in\Omega} \subseteq L^X$ be an up-directed family and $A = \bigvee_{i\in\Omega} A_i$. By (LDF), $co(A) = \bigvee_{F\in\mathfrak{F}(A)} co(F) = \bigvee_{F\in\bigcup_{i\in\Omega}\mathfrak{F}(A_i)} co(F) = \bigvee_{i\in\Omega}\bigvee_{F\in\mathfrak{F}(A_i)} co(F) = \bigvee_{i\in\Omega} co(A_i)$.

(4) ⇒ (1): If $\{A_i\}_{i\in\Omega} \subseteq \mathcal{C}$ is totally ordered, then it is up-directed. Thus $co(\bigvee_{i\in\Omega} A_i) = \bigvee_{i\in\Omega} co(A_i) = \bigvee_{i\in\Omega} A_i \in \mathcal{C}$. Therefore $\mathcal{C}$ is an (L, M)-fuzzy convex structure. □

Theorem 3.3. *Let $(X, \mathcal{D})$ be an L-closure space and*

$$\mathcal{C}_{\mathcal{D}} = \{A \in L^X : \exists\varphi \subseteq \mathcal{D} \textit{ is up-directed}, A = \bigvee \varphi\}.$$

Then $\mathcal{C}_\mathcal{D}$ is an L-convex structure generated by $\mathcal{D}$.

Proof. (LC1): Clearly, $\perp, \top \in \mathcal{D} \subseteq \mathcal{C}_\mathcal{D}$.

(LC2): Let $\{A_i\}_{i\in\Omega} \subseteq \mathcal{C}_\mathcal{D}$ and $A = \bigwedge_{i\in\Omega} A_i$. By the definition of $\mathcal{C}_\mathcal{D}$, there exists an up-directed subfamily $\mathcal{D}_i \subseteq \mathcal{D}$ such that $A_i = \bigvee \mathcal{D}_i$ for each $i \in \Omega$.

Let $\mathcal{S}$ be the set of all choice mappings $s : \Omega \to \bigcup_{i\in\Omega} \mathcal{D}_i$, where $s(i) \in \mathcal{D}_i$ for each $i \in \Omega$. Let $\mathcal{D}_0 = \{\bigwedge_{i\in\Omega} s(i) : s \in \mathcal{S}\}$. It has the following two properties.

(i) $\mathcal{D}_0$ is up-directed. In fact, let $B_1, B_2 \in \mathcal{D}_0$. Then there exist $s_1, s_2 \in \mathcal{S}$ such that $B_1 = \bigwedge_{i\in\Omega} s_1(i)$ and $B_2 = \bigwedge_{i\in\Omega} s_2(i)$. Thus, for each $i \in \Omega$, there exists $D_i \in \mathcal{D}_i$ such that $s_1(i), s_2(i) \leq D_i$. Define a mapping $s_3 : \Omega \to \bigcup_{i\in\Omega} \mathcal{D}_i$ as: $s_3(i) = D_i$ for all $i \in \Omega$. Then $s_3 \in \mathcal{S}$ and $B_1, B_2 \leq B_3 = \bigwedge_{i\in\Omega} s_3(i) \in \mathcal{D}_0$. This shows $\mathcal{D}_0$ is up-directed.

(ii) $A = \bigvee \mathcal{D}_0$. In fact, if $x_\lambda \prec A = \bigwedge_{i\in\Omega} A_i$, then $x_\lambda \prec A_i$ for each $i \in \Omega$. Thus, for each $i \in \Omega$, there exists $D_i \in \mathcal{D}_i$ such that $x_\lambda \leq D_i$. Let $s : \Omega \to \bigcup_{i\in\Omega} \mathcal{D}_i$ be defined as: $s(i) = D_i$ for each $i \in \Omega$. Then $s \in \mathcal{S}$ and $x_\lambda \leq \bigwedge_{i\in\Omega} D_i = \bigwedge_{i\in\Omega} s(i) \in \mathcal{D}_0$. Hence $x_\lambda \leq \bigvee \mathcal{D}_0$. By Proposition 2.9(5), we have $A \leq \bigvee \mathcal{D}_0$. Conversely, if $y_\mu \prec \bigvee \mathcal{D}_0$, then there exists $s \in \mathcal{S}$ such that $y_\mu \leq \bigwedge_{i\in\Omega} s(i) \leq \bigwedge_{i\in\Omega} A_i = A$. Hence $\bigvee \mathcal{D}_0 \leq A$.

(LC3*): Let $\{A_i\}_{i\in\Omega} \subseteq \mathcal{C}_\mathcal{D}$ be up-directed and $A = \bigvee_{i\in\Omega} A_i$. Since $A_i \in \mathcal{C}_\mathcal{D}$ for each $i \in \Omega$, there exists an up-directed $\mathcal{D}_i \subseteq \mathcal{D}$ such that $A_i = \bigvee \mathcal{D}_i$. Let $\mathcal{D}_* = \bigcup_{i\in\Omega} \mathcal{D}_i$ and $\varphi_F = \{D : F \leq D \in \mathcal{D}_*\}$ for each $F \in \mathfrak{F}(A)$. We firstly check that φ_F is nonempty for all $F \in \mathfrak{F}(A)$. In fact, by Proposition 2.8(4), there exists $i_F \in \Omega$ such that $F \in \mathfrak{F}(A_{i_F}) = \mathfrak{F}(\bigvee \mathcal{D}_{i_F})$. Again, there exists $D_{i_F} \in \mathcal{D}_{i_F} \subseteq \mathcal{D}_*$ such that $F \in \mathfrak{F}(D_{i_F})$. Hence $D_{i_F} \in \varphi_F$.

Since $F \leq D_{i_F} \leq A_{i_F} \leq A$ for all $F \in \mathfrak{F}(A)$, we have $F \leq \bigwedge \varphi_F \leq A$. Thus $A = \bigvee_{F\in\mathfrak{F}(A)} F \leq \bigvee_{F\in\mathfrak{F}(A)} \bigwedge \varphi_F \leq A$ which implies that $A = \bigvee_{F\in\mathfrak{F}(A)} \bigwedge \varphi_F$. So we have $A \in \mathcal{C}_\mathcal{D}$ since $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\} \subseteq \mathcal{D}$ is up-directed. Hence $\mathcal{C}_\mathcal{D}$ is an L-convex space. □

Definition 3.4. Let $(X, \mathcal{C})$ be an L-convex space and & $\subseteq \mathcal{D}$. & is called a

(1) subbase of $\mathcal{C}$, if $\mathcal{C}$ is the coarsest L-convex structure on X containing &;

(2) base of $\mathcal{C}$, if elements of $\mathcal{C}$ are supermums of up-directed subfamilies of &.

By Theorem 3.3, an L-closure space is a base of its generated L-convex space. Also, by Definition 3.4 and Theorem 3.1, a base of an L-convex space is a subbase.

Theorem 3.5. *Let $(X, \mathcal{C})$ be an L-convex space and & $\subseteq \mathcal{C}$.*

(1) If $\{co(F) : F \in \mathfrak{F}(X)\backslash\{\perp\}\} \subseteq$ &, then & is a base of $\mathcal{C}$.

(2) If $co(F) = \bigwedge \&_1$, where $\&_1 \subseteq$ & for all $F \in \mathfrak{F}(X)\backslash\{\perp\}$, then & is a subbase of $\mathcal{C}$.

Proof. We show M_I satisfies (1) and (2).

(1): Let $C \in \mathcal{C}$. By Proposition 2.8(3) and Theorem 3.2(4), we have $C = co(C) = co(\bigvee_{F\in\mathfrak{F}(C)} F) = \bigvee_{F\in\mathfrak{F}(C)} co(F) = \bigvee_{F\in\mathfrak{F}(C)\backslash\{\perp\}} co(F)$. Since the set $\{co(F) : F \in \mathfrak{F}(C)\backslash\{\perp\}\} \subseteq$ & is up-directed, we know that & is a base of $\mathcal{C}$.

(2): Let $\mathcal{D} = \{\bigwedge \varphi \in L^X : \varphi \subseteq \&\}$. Since $\{co(F) : F \in \mathfrak{F}(X)\backslash\{\perp\}\} \subseteq \mathcal{D} \subseteq \mathcal{C}$, we know that $\mathcal{D}$ is a base of $\mathcal{C}$ by (1). Thus each element of $\mathcal{C}$ is the supermum of an up-directed subset of $\mathcal{D}$. To show that & is a subbase of $\mathcal{C}$, let $(X, \mathcal{F})$ be an L-convex space with & $\subseteq \mathcal{F}$. Then $\mathcal{D} \subseteq \mathcal{F}$ which shows $\mathcal{C} \subseteq \mathcal{F}$. Therefore & is a subbase of $\mathcal{C}$. □

The inverse results of Theorem 3.5 are not true. We have the following example.

Example 3.6. Let $X = \{x\}$ and $L = [0, 1]$. Then & $= \{x_t : t \in [0, \frac{1}{3}) \cup [\frac{1}{2}, 1)\}$ is a base of the L-convex structure $\mathcal{C} = [0, \frac{1}{3}]^X \cup [\frac{1}{2}, 1]^X$. Thus & is also a subbase of $\mathcal{C}$. However, we have $x_{\frac{1}{3}} \in \mathfrak{F}(X)$, $co(x_{\frac{1}{3}}) = x_{\frac{1}{3}} \notin$ & and $x_{\frac{1}{3}} \neq \bigwedge \varphi$ for any $\varphi \subseteq$ &.

4. Characterizations of (L, M)-fuzzy convex spaces

Theorem 4.1. *The follows are equivalent for an (L, M)-fuzzy closure space $(X, \mathcal{C})$.*
(1) $\mathcal{C}$ is an (L, M)-fuzzy convex structure.
(2) co is domain finite, that is, co fulfills (LMDF) below.
(LMDF) $\forall A \in L^X, \forall x_\lambda \in J(L^X)$, $co(A)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu)$.
(3) $\mathcal{C}$ is stable for up-directed unions, that is, $\mathcal{C}$ satisfies (LMC3) below.*
(LMC3) $\mathcal{C}(\bigvee_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathcal{C}(A_i)$ for each up-directed subset $\{A_i\}_{i\in\Omega} \subseteq L^X$.*
(4) co is stable for up-directed unions, that is,

$$co(\bigvee_{i\in\Omega} A_i)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{i\in\Omega} co(A_i)(x_\mu)$$

for each up-directed subset $\{A_i\}_{i\in\Omega} \subseteq L^X$ and all $x_\lambda \in J(L^X)$.

Proof. (1) $\Rightarrow$ (2): Let $A \in L^X$, $x_\lambda \in J(L^X)$ and $a \in \beta(\top)$ with $a \prec co(A)(x_\lambda)$. Then

$$\begin{aligned} co(A)(x_\lambda) &\Rightarrow \forall x_\lambda \not\leq B \geq A, \ \ B \notin \mathcal{C}^{[a']} \\ &\Rightarrow x_\lambda \leq co_{\mathcal{C}^{[a']}}(A) = \bigvee_{F\in\mathfrak{F}(A)} co_{\mathcal{C}^{[a']}}(F) \\ &\Rightarrow \forall \mu \in \beta^*(\lambda), \ \ \exists F_\mu \in \mathfrak{F}(A), \ s.t. \ x_\mu \leq co_{\mathcal{C}^{[a']}}(F_\mu) \\ &\Rightarrow \forall \mu \in \beta^*(\lambda), \ \ \exists F_\mu \in \mathfrak{F}(A), \ s.t. \ \forall x_\mu \not\leq D \geq F_\mu, \ \ D \notin \mathcal{C}^{[a']} \\ &\Rightarrow \forall \mu \in \beta^*(\lambda), \ \ \exists F_\mu \in \mathfrak{F}(A), \ s.t. \ co(F_\mu)(x_\mu) \geq a \\ &\Rightarrow \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{F\in\mathfrak{F}(A)} co(F)(x_\mu) \geq a. \end{aligned}$$

By Proposition 2.9(5), we have $co(A)(x_\lambda) \leq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{F\in\mathfrak{F}(A)} co(F)(x_\mu)$. Conversely, we have $co(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} co(A)(x_\mu) \geq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{F\in\mathfrak{F}(A)} co(F)(x_\mu)$.

(2) $\Rightarrow$ (3): Let $\{A_i\}_{i\in\Omega} \subseteq L^X$ be up-directed and $A = \bigvee_{i\in\Omega} A_i$. If $a \in M$ with $\mathcal{C}(A) \not\geq a$, then there exists $b \prec a$ such that $\mathcal{C}(A) \not\geq b$. By Proposition 2.8(4), we

have

$$\begin{aligned}\mathcal{C}(A) = &\bigwedge_{x_\lambda \not\leq^* A} (co(A)(x_\lambda))' \not\geq b \\ &\Rightarrow \exists x_\lambda \not\leq^* A, \quad \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{F\in\mathfrak{F}(A)} co(F)(x_\mu) \not\leq b' \\ &\Rightarrow \exists x_\lambda \not\leq^* A, \quad s.t. \ \ \forall \mu \in \beta^*(\lambda), \ \exists F_\mu \in \mathfrak{F}(A), co(F_\mu)(x_\mu) \not\leq b' \\ &\Rightarrow \exists x_\lambda \not\leq^* A, \quad s.t. \ \ \forall \mu \in \beta^*(\lambda), \ \exists F_\mu \in \mathfrak{F}(A_{i_\mu})(i_\mu \in \Omega), \ (co(F_\mu)(x_\mu))' \not\geq b.\end{aligned}$$

Thus there exists $x_\lambda \not\leq^* A$ and $\bigvee_{\mu\prec\lambda} \bigwedge_{F\in\mathfrak{F}(A_{i_\mu})} (co(F)(x_\mu))' \not\geq a$. Hence we have

$$\bigwedge_{i\in\Omega} \mathcal{C}(A_i) = \bigwedge_{i\in\Omega} \bigwedge_{x_\lambda \not\leq A_i} (co_{\mathcal{C}}(A_i)(x_\lambda))' \leq \bigwedge_{x_\lambda \not\leq A} \bigvee_{\mu\in\beta^*(\lambda)} \bigwedge_{F\in\mathfrak{F}(A_{i_\mu})} (co_{\mathcal{C}}(F)(x_\mu))' \not\geq a.$$

This shows $\bigwedge_{i\in\Omega} \mathcal{C}(A_i) \not\geq a$. By Proposition 2.9(2), we have $\bigwedge_{i\in\Omega} \mathcal{C}(A_i) \leq \mathcal{C}(A)$.

(2) $\Rightarrow$ (4): Let $\{A_i\}_{i\in\Omega} \subseteq L^X$ be up-directed, $A = \bigvee_{i\in\Omega} A_i$ and $x_\lambda \in J(L^X)$. Thus

$$\begin{aligned} co(A)(x_\lambda) &= \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{i\in\Omega} \bigvee_{F\in\mathfrak{F}(A_i)} co(F)(x_\mu) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigwedge_{\eta\in\beta^*(\mu)} \bigvee_{i\in\Omega} \bigvee_{F\in\mathfrak{F}(A_i)} co(F)(x_\eta) \\ &\geq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{i\in\Omega} \bigwedge_{\eta\in\beta^*(\mu)} \bigvee_{F\in\mathfrak{F}(A_i)} co(F)(x_\eta) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{i\in\Omega} co(A_i)(x_\mu).\end{aligned}$$

Conversely, by Proposition 2.8(4), (LMCO3) and (LMDF), we have

$$co(A)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{i\in\Omega} \bigvee_{F\in\mathfrak{F}(A_i)} co(F)(x_\mu) \leq \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{i\in\Omega} co(A_i)(x_\mu).$$

(4) $\Rightarrow$ (2): Let $A \in L^X$ and $x_\lambda \in J(L^X)$. By Proposition 2.8(1) and (3), we have

$$co(A)(x_\lambda) = co(\bigvee_{F\in\mathfrak{F}(A)} F)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{F\in\mathfrak{F}(A)} co(F)(x_\mu).$$

(3) $\Rightarrow$ (1): Clear. □

Definition 4.2. An operator $\mathcal{H} : \mathfrak{F}(X) \to M^{J(L^X)}$ is called an (L, M)-fuzzy restricted hull operator and the pair $(X, \mathcal{H})$ is called an (L, M)-fuzzy restricted hull space, if for all $x_\lambda \in J(L^X)$ and $F, G \in \mathfrak{F}(X)$, $\mathcal{H}$ satisfies the following conditions.

(LMRH0) $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \mathcal{H}(F)(x_\mu)$.

(LMRH1) $\mathcal{H}(\perp)(x_\lambda) = \perp$.

(LMRH2) If $x_\lambda \leq F$, then $\mathcal{H}(F)(x_\lambda) = \top$.

(LMRH3) $\mathcal{H}(G)(x_\lambda) \wedge \bigwedge_{y_\mu\in\beta^*(G)} \mathcal{H}(F)(y_\mu) \leq \mathcal{H}(F)(x_\lambda)$.

(LMRH4) $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu\in\beta^*(\lambda)} \bigvee_{G\in\mathfrak{F}(F)} \mathcal{H}(G)(x_\mu)$.

Remark 4.3. (1) A $(\mathbf{2}, M)$-fuzzy restricted hull operator is an M-fuzzifying restricted hull operator (Definition 2.5). An $(L, \mathbf{2})$-fuzzy restricted hull operator is reduced to an L-restricted hull operator $\mathcal{H} : \mathfrak{F}(X) \to L^X$, which satisfies (LRH1)-(LRH4).

(LRH1) $\mathcal{H}(\perp) = \perp$.

(LRH2) $F \leq \mathcal{H}(F)$ for all $F \in \mathfrak{F}(X)$.

(LRH3) $\mathcal{H}(G) \leq \mathcal{H}(F)$ for all $F, G \in \mathfrak{F}(X)$ with $G \leq \mathfrak{F}(\mathcal{H}(F))$.

(LRH4) $\mathcal{H}(F) = \bigvee_{G\in\mathfrak{F}(F)} \mathcal{H}(G)$ for all $F \in \mathfrak{F}(X)$.

(2) Let $(X, \mathcal{H}_X)$ and $(Y, \mathcal{H}_Y)$ be (L,M)-fuzzy restricted hull spaces. A mapping $f : X \to Y$ is called an (L,M)-RHP mapping, if $\mathcal{H}_X(F)(x_\lambda) \leq \mathcal{H}_Y(f_L^{\to}(F))(f_L^{\to}(x_\lambda))$ for all $F \in \mathfrak{F}(X)$ and all $x_\lambda \in J(L^X)$. In particular, if $M = \mathbf{2}$, then $f : X \to Y$ is called an L-RHP mapping. That is, $f_L^{\to}(\mathcal{H}_X(F)) \leq \mathcal{H}_Y(f_L^{\to}(F))$ for all $F \in \mathfrak{F}(X)$.

Theorem 4.4. *Let $(X, \mathcal{C})$ be an (L,M)-fuzzy convex space. Then the restriction of co on $\mathfrak{F}(X)$ is an (L,M)-fuzzy restricted hull operator, which is denoted by $\mathcal{H}_\mathcal{C}$.*

Proof. Since $\mathcal{H}_\mathcal{C}(F) = co(F)$ for all $F \in \mathfrak{F}(X)$, we only need to check (LMRH3).

(LMRH3): Let $F, G \in \mathfrak{F}(X)$ and $x_\lambda \in J(L^X)$. If $x_\lambda \leq F$, then (LMRH3) is trivial. If $x_\lambda \leq G$, then $\mathcal{H}_\mathcal{C}(G)(x_\lambda) = \top$. This implies $\bigwedge_{y_\mu\in\beta^*(G)} \mathcal{H}_\mathcal{C}(F)(y_\mu) \leq \bigwedge_{\mu\in\beta^*(\lambda)} \mathcal{H}_\mathcal{C}(F)(x_\mu) = \bigwedge_{\mu\in\beta^*(\lambda)} co(F)(x_\mu) = \mathcal{H}_\mathcal{C}(F)(x_\lambda)$. Thus (LMRH3) holds.

Assume that $x_\lambda \not\leq F \vee G$. Then $\mathcal{H}_\mathcal{C}(F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq A \geq F} \bigvee_{y_\mu \not\leq^* A} co(A)(y_\mu)$ by (LMCO4). Let $x_\lambda \not\leq A \geq F$. If $G \not\leq A$, then the set $\varphi = \{y_\mu \in \beta^*(G) : y_\mu \not\leq^* A\}$ is not empty and

$$\bigvee_{y_\mu \not\leq^* A} co(A)(y_\mu) \geq \bigvee_{y_\mu\in\varphi} co(A)(y_\mu) \geq \bigvee_{y_\mu\in\varphi} co(F)(y_\mu) \geq \bigwedge_{y_\mu\in\beta^*(G)} \mathcal{H}_\mathcal{C}(F)(y_\mu).$$

If $G \leq A$, then $\bigvee_{y_\mu \not\leq^* A} co(A)(y_\mu) \geq co(A)(x_\lambda) \geq co(G)(x_\lambda) = \mathcal{H}_\mathcal{C}(G)(x_\lambda)$.

Combining the above two inequalities, we conclude that (LMRH3) holds. □

Theorem 4.5. *Let $(X, \mathcal{H})$ be an (L,M)-fuzzy restricted hull space and $\mathcal{C}_\mathcal{H} : L^X \to M$ be defined as:*

$$\forall A \in L^X, \quad \mathcal{C}_\mathcal{H}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F\in\mathfrak{F}(A)} (\mathcal{H}(F)(x_\lambda))'.$$

Then $\mathcal{C}_\mathcal{H}$ is an (L,M)-fuzzy convex structure with $\mathcal{H}_{\mathcal{C}_\mathcal{H}} = \mathcal{H}$.

Proof. (LMC1): $\mathcal{C}_\mathcal{H}(\bot) = \bigwedge_{x_\lambda\in J(L^X)} (\mathcal{H}(\bot)(x_\lambda))' = \top$ and $\mathcal{C}_\mathcal{H}(\top) = \bigwedge \emptyset = \top$.

(LMC2): Let $\{A_i\}_{i\in\Omega} \subseteq L^X$ and $A = \bigwedge_{i\in\Omega} A_i$. Then $\mathfrak{F}(A) \subseteq \bigcap_{i\in\Omega} \mathfrak{F}(A_i)$. If $x_\lambda \not\leq^* A$, then there exists $i_0 \in \Omega$ such that $x_\lambda \not\leq^* A_{i_0}$. Thus, if $F \in \mathfrak{F}(A)$, then

$$(\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{G\in\mathfrak{F}(A_{i_0})} (\mathcal{H}(G)(x_\lambda))' \geq \mathcal{C}_\mathcal{H}(A_{i_0}) \geq \bigwedge_{i\in\Omega} \mathcal{C}_\mathcal{H}(A_i).$$

Hence $\mathcal{C}_\mathcal{H}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F\in\mathfrak{F}(A)} (\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{i\in\Omega} \mathcal{C}_\mathcal{H}(A_i)$.

(LMC3*): Let $\{A_i\}_{i\in\Omega} \subseteq L^X$ be up-directed and $A = \bigvee_{i\in\Omega} A_i$. If $x_\lambda \not\leq^* A$, then $x_\lambda \not\leq^* A_i$ for each $i \in \Omega$. If $F \in \mathfrak{F}(A)$, then there exists $i_F \in \Omega$ such that $F \in \mathfrak{F}(A_{i_F})$. So

$$(\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{G\in\mathfrak{F}(A_{i_F})} (\mathcal{H}(G)(x_\lambda))' \geq \mathcal{C}_\mathcal{H}(A_{i_F}) \geq \bigwedge_{i\in\Omega} \mathcal{C}_\mathcal{H}(A_i).$$

Hence $\mathcal{C}_\mathcal{H}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F\in\mathfrak{F}(A)} (\mathcal{H}(F)(x_\lambda))' \geq \bigwedge_{i\in\Omega} \mathcal{C}_\mathcal{H}(A_i)$.

In order to prove that $\mathcal{H}_{\mathcal{C}_\mathcal{H}} = \mathcal{H}$, let $F \in \mathfrak{F}(X)$ and $x_\lambda \in J(L^X)$ with $x_\lambda \not\leq F$. Then

$$\mathcal{H}_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda) = co_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq F} (\mathcal{C}_\mathcal{H}(B))' = \bigwedge_{x_\lambda \not\leq B \geq F} \bigvee_{y_\mu \not\leq^* B} \bigvee_{G \in \mathfrak{F}(B)} \mathcal{H}(G)(y_\mu).$$

If $B \in L^X$ with $x_\lambda \not\leq B \geq F$, then there exists $\mu_B \in \beta^*(\lambda)$ such that $x_{\mu_B} \not\leq^* B$. Thus

$$\begin{aligned}\mathcal{H}_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda) = co_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda) &= \bigwedge_{x_\lambda \not\leq B \geq F} (\mathcal{C}_\mathcal{H}(B))' = \bigwedge_{x_\lambda \not\leq B \geq F} \bigvee_{y_\mu \not\leq^* B} \bigvee_{G \in \mathfrak{F}(B)} \mathcal{H}(G)(y_\mu) \\ &\geq \bigwedge_{x_\lambda \not\leq B \geq F} \bigvee_{G \in \mathfrak{F}(B)} \mathcal{H}(G)(x_{\mu_B}) \geq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{G \in \mathfrak{F}(F)} \mathcal{H}(G)(x_\mu) = \mathcal{H}(F)(x_\lambda).\end{aligned}$$

Conversely, let $a \in \beta(\top)$ with $a \prec \mathcal{H}_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda)$. We say $a \leq \mathcal{H}(F)(x_\lambda)$.

Assume that $x_\lambda \notin \mathcal{H}(F)_{[a]}$. If $x_\lambda \leq \bigvee \mathcal{H}(F)_{[a]} = B_0$, then $x_\mu \in \beta^*(B_0)$ for each $\mu \in \beta^*(\lambda)$. Thus $\mathcal{H}(F)(x_\mu) \geq a$ and $\mathcal{H}(F)(x_\lambda) = \bigwedge_{\mu \in \beta^*(\lambda)} \mathcal{H}(F)(x_\mu) \geq a$ by (LMRH0). Hence $x_\lambda \in \mathcal{H}(F)_{[a]}$ which is a contradiction. Thus $x_\lambda \not\leq B_0 \geq F$. By $a \prec \mathcal{H}_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda)$, there exists $y_\mu \not\leq^* B_0$ such that $a \leq \bigvee_{G \in \mathfrak{F}(B_0)} \mathcal{H}(G)(y_\mu)$. Thus there exists $G_b \in \mathfrak{F}(B_0)$ such that $b \leq \mathcal{H}(G_b)(y_\mu)$ for each $b \prec a$. If $z_\zeta \prec G_b$, then there exists $z_\rho \in \beta^*(B_0)$ such that $z_\zeta \prec z_\rho$. Hence there further exists $z_\eta \in \mathcal{H}(F)_{[a]}$ such that $z_\rho \in \beta^*(z_\eta)$. So $\mathcal{H}(z_\zeta) \geq \mathcal{H}(z_\rho) \geq \mathcal{H}(z_\eta) \geq a \succ b$ by (LMRH0). Therefore $b \leq \mathcal{H}(G_b)(y_\mu) \wedge \bigwedge_{z_\zeta \in \beta^*(G_b)} \mathcal{H}(F)(z_\zeta) \leq \mathcal{H}(F)(y_\mu)$ by (LMRH3). However, by Proposition 2.9(5), $a \leq \mathcal{H}(F)(y_\mu)$ which contradicts $y_\mu \not\leq B_0$. Hence $a \leq \mathcal{H}(F)(x_\lambda)$. By Proposition 2.9(5) again, $\mathcal{H}_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda) \leq \mathcal{H}(F)(x_\lambda)$. Therefore $\mathcal{H}_{\mathcal{C}_\mathcal{H}}(F)(x_\lambda) = \mathcal{H}(F)(x_\lambda)$ and so $\mathcal{H}_{\mathcal{C}_\mathcal{H}} = \mathcal{H}$. □

Theorem 4.6. *If $(X, \mathcal{C})$ is an (L, M)-fuzzy convex space, then $\mathcal{C}_{\mathcal{H}_\mathcal{C}} = \mathcal{C}$.*

Proof. Let $A \in L^X$ and $x_\lambda \in J(L^X)$. In order to prove that $\mathcal{C}_{\mathcal{H}_\mathcal{C}}(A) \leq \mathcal{C}(A)$, we firstly prove that $co_{\mathcal{C}_{\mathcal{H}_\mathcal{C}}}(A)(x_\lambda) \geq co_\mathcal{C}(A)(x_\lambda)$. Let $x_\lambda \not\leq^* A$. If $B \in L^X$ with $x_\lambda \not\leq B \geq A$, then there exists $\mu_B \in \beta^*(\lambda)$ such that $x_{\mu_B} \not\leq^* B$. By Theorem 4.5 and (LMDF), we have

$$\begin{aligned}co_{\mathcal{C}_{\mathcal{H}_\mathcal{C}}}(A)(x_\lambda) &= \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\eta \not\leq^* B} \bigvee_{F \in \mathfrak{F}(B)} \mathcal{H}_\mathcal{C}(F)(y_\eta) \geq \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{F \in \mathfrak{F}(A)} co_\mathcal{C}(F)(x_{\mu_B}) \\ &\geq \bigwedge_{\mu \in \beta^*(\lambda)} \bigvee_{F \in \mathfrak{F}(A)} co(F)(x_\mu) = co_\mathcal{C}(A)(x_\lambda).\end{aligned}$$

Hence $\mathcal{C}_{\mathcal{H}_\mathcal{C}}(A) \leq \mathcal{C}(A)$. Conversely, we have

$$\mathcal{C}_{\mathcal{H}_\mathcal{C}}(A) = \bigwedge_{x_\lambda \not\leq^* A} \bigwedge_{F \in \mathfrak{F}(A)} (co_\mathcal{C}(F)(x_\lambda))' \geq \bigwedge_{x_\lambda \not\leq^* A} (co_\mathcal{C}(A))' = \mathcal{C}(A).$$

Thus $\mathcal{C}_{\mathcal{H}_\mathcal{C}}(A) = \mathcal{C}(A)$. Therefore $\mathcal{C}_{\mathcal{H}_\mathcal{C}} = \mathcal{C}$. □

Theorem 4.7. *Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be (L, M)-fuzzy convex spaces. If a mapping $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is an (L, M)-CP mapping, then $f : (X, \mathcal{H}_{\mathcal{C}_X}) \to (Y, \mathcal{H}_{\mathcal{C}_Y})$ is an (L, M)-RHP mapping.*

Proof. Let $F\in\mathfrak{F}(X)$ and $x_\lambda\in J(L^X)$. Then $f_L^{\rightarrow}(F)\in\mathfrak{F}(Y)$, $f_L^{\rightarrow}(x_\lambda)\in J(Y)$ and

$$\mathcal{H}_{\mathcal{C}_X}(F)(x_\lambda)=co_{\mathcal{C}_X}(F)(x_\lambda)\le co_{\mathcal{C}_Y}(f_L^{\rightarrow}(F))(f_L^{\rightarrow}(x_\lambda))=\mathcal{H}_{\mathcal{C}_Y}(f_L^{\rightarrow}(F))(f_L^{\rightarrow}(x_\lambda)).$$

Thus f is an (L,M)-RHP mapping. □

Theorem 4.8. *Let $(X,\mathcal{H}_X)$ and $(Y,\mathcal{H}_Y)$ be (L,M)-fuzzy restricted hull spaces. If $f:(X,\mathcal{H}_X)\to(Y,\mathcal{H}_Y)$ be an (L,M)-RHP mapping, then $f:(X,\mathcal{C}_{\mathcal{H}_X})\to(Y,\mathcal{C}_{\mathcal{H}_Y})$ is an (L,M)-CP mapping.*

Proof. Let $A\in L^X$, $F\in\mathfrak{F}(A)$ and $x_\lambda\in J(L^X)$. Then $f_L^{\rightarrow}(F)\in\mathfrak{F}(f_L^{\rightarrow}(A))$ and $f_L^{\rightarrow}(x_\lambda)\in\mathfrak{F}(Y)$. Thus, by Theorem 4.5, we have

$$\begin{aligned}co_{\mathcal{C}_{\mathcal{H}_X}}(A)(x_\lambda)&=\bigwedge_{\mu\in\beta^*(\lambda)}\bigvee_{F\in\mathfrak{F}(A)}co_{\mathcal{C}_{\mathcal{H}_X}}(F)(x_\mu)=\bigwedge_{\mu\in\beta^*(\lambda)}\bigvee_{F\in\mathfrak{F}(A)}\mathcal{H}_{\mathcal{C}_{\mathcal{H}_X}}(F)(x_\mu)\\&=\bigwedge_{\mu\in\beta^*(\lambda)}\bigvee_{F\in\mathfrak{F}(A)}\mathcal{H}_X(F)(x_\mu)\le\bigwedge_{\mu\in\beta^*(\lambda)}\bigvee_{F\in\mathfrak{F}(A)}\mathcal{H}_Y(f_L^{\rightarrow}(F))(f_L^{\rightarrow}(x_\mu))\\&\le\bigwedge_{\mu\in\beta^*(\lambda)}\bigvee_{G\in\mathfrak{F}(f_L^{\rightarrow}(A))}\mathcal{H}_Y(G)(f_L^{\rightarrow}(x_\mu))=co_{\mathcal{C}_{\mathcal{H}_Y}}(f_L^{\rightarrow}(A))(f_L^{\rightarrow}(x_\lambda)).\end{aligned}$$

Hence $f:(X,\mathcal{C}_{\mathcal{H}_X})\to(Y,\mathcal{C}_{\mathcal{H}_Y})$ is an (L,M)-CP mapping. □

Theorem 4.9. *Let $(X,\mathcal{D})$ be an (L,M)-fuzzy closure space, and $\mathcal{C}_\mathcal{D}:L^X\to M$ be defined as:*

$$\forall A\in L^X,\quad \mathcal{C}_\mathcal{D}(A)=\bigvee_{\bigvee^{dir}_{j\in J}B_j=A}\ \bigwedge_{j\in J}\mathcal{D}(B_j).$$

Then $\mathcal{C}_\mathcal{D}$ is an (L,M)-fuzzy convex structure.

Proof. (LMC1) directly follows from the fact that $\mathcal{C}_\mathcal{D}\ge\mathcal{D}$.

(LMC2): Let $\{A_i\}_{i\in\Omega}\subseteq L^X$ and $a\in\beta(\top)$ with $a\prec\bigwedge_{i\in\Omega}\mathcal{C}_\mathcal{D}(A_i)$. Then $a\prec\mathcal{C}_\mathcal{D}(A_i)$ for each $i\in\Omega$. Thus, for each $i\in\Omega$, there exists an up-directed set $\{B_{i,j}:j\in J_i\}\subseteq L^X$ such that $A_i=\bigvee^{dir}_{j\in J_i}B_{i,j}$ and $\mathcal{D}(B_{i,j})\ge a$ for all $j\in J_i$. Hence

$$\bigwedge_{i\in\Omega}A_i=\bigwedge_{i\in\Omega}\bigvee_{j\in J_i}B_{i,j}=\bigvee_{f\in\Pi_{i\in\Omega}J_i}\bigwedge_{i\in\Omega}B_{i,f(i)}.$$

Clearly, $\{B_{i,f(i)}:f\in\Pi_{i\in\Omega}J_i\}$ is up-directed and $\mathcal{D}(B_{i,f(i)})\ge a$. Hence

$$\mathcal{C}_\mathcal{D}(\bigwedge_{i\in\Omega}A_i)=\mathcal{C}_\mathcal{D}(\bigvee_{f\in\Pi_{i\in\Omega}J_i}\bigwedge_{i\in\Omega}B_{i,f(i)})\ge\bigwedge_{f\in\Pi_{i\in\Omega}J_i}\bigwedge_{i\in\Omega}\mathcal{D}(B_{i,f(i)})\ge a.$$

Therefore, $\mathcal{C}_\mathcal{D}(\bigwedge_{i\in\Omega}A_i)\ge\bigwedge_{i\in\Omega}\mathcal{C}_\mathcal{D}(A_i)$ by Proposition 2.9(5).

(LMC3*): Let $\{A_i\}_{i\in\Omega}\subseteq L^X$ be up-directed, $A=\bigvee_{i\in\Omega}A_i$ and $a\in\beta(\top)$ with $a\prec\bigwedge_{i\in\Omega}\mathcal{C}_\mathcal{D}(A_i)$. Then $a\prec\mathcal{C}_\mathcal{D}(A_i)$ for each $i\in\Omega$. Thus there exists an up-directed set $\{B_{i,j}:j\in J_i\}\subseteq L^X$ such that $\bigvee^{dir}_{j\in J_i}B_{i,j}=A_i$ and $\mathcal{D}(B_{i,j})\ge a$ for each $j\in J_i$.

Let $\mathcal{B}_*=\{B_{i,j}:i\in\Omega,j\in J_i\}$ and $\varphi_F=\{B\in\mathcal{B}_*:F\le B\}$ for all $F\in\mathfrak{F}(A)$. Next, we check that φ_F is nonempty for all $F\in\mathfrak{F}(A)$. In fact, by Proposition 2.8(4), there exists $i_F\in\Omega$ such that $F\in\mathfrak{F}(A_{i_F})=\mathfrak{F}(\bigvee^{dir}_{j\in J_{i_F}}B_{i_F,j})$. Thus there exists

$B_{i_F,j_0} \in \mathcal{B}_*$ such that $F \in \mathfrak{F}(B_{i_F,j_0})$. So $B_{i_F,j_0} \in \varphi_F$. Now, since $F \leq B_{i_F,j_0} \leq A_{i_F} \leq A$ for all $F \in \mathfrak{F}(A)$, we have $F \leq \bigwedge \varphi_F \leq A$. Thus $A = \bigvee_{F \in \mathfrak{F}(A)} F \leq \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F \leq A$ and $A = \bigvee_{F \in \mathfrak{F}(A)} \bigwedge \varphi_F$.

If $F, G \in \mathfrak{F}(A)$, then $F \vee G \in \mathfrak{F}(A)$. Thus $\varphi_{F\vee G} \subseteq \varphi_F \cap \varphi_G$ and $\bigwedge \varphi_F, \bigwedge \varphi_G \leq \bigwedge \varphi_{F\vee G}$. Hence the set $\{\bigwedge \varphi_F : F \in \mathfrak{F}(A)\} \subseteq L^X$ is an up-directed set. It follows that

$$\mathcal{C}_{\mathcal{D}}(A) \geq \bigwedge_{F \in \mathfrak{F}(A)} \mathcal{D}(\bigwedge \varphi_F) \geq \bigwedge_{F \in \mathfrak{F}(A)} \bigwedge_{B \in \varphi_F} \mathcal{D}(B) \geq a.$$

Thus $\mathcal{C}_{\mathcal{D}}(A) \geq \bigwedge_{i\in\Omega} \mathcal{C}_{\mathcal{D}}(A_i)$ by Proposition 2.9(5). □

Corollary 4.10. *An (L, M)-fuzzy closure space $(X, \mathcal{D})$ is an (L, M)-fuzzy convex space iff $\mathcal{C}_{\mathcal{D}} = \mathcal{D}$.*

Remark 4.11. (1) The category, consisting of (L, M)-fuzzy restricted hull spaces as objects and (L, M)-RHP mappings as morphisms, is denoted by (L, M)-**RHS**. Its construct is denoted by $((L, M)$-**RHS**,$\mathbb{H})$. In particular, when $M = \mathbf{2}$, we obtain a category L-**RHS**.

(2) The category, consisting of (L, M)-fuzzy domain finite closure operator spaces (refer to Remark 2.16 and Theorem 4.1(2)) as objects and (L, M)-closure operator preserving mappings as morphisms, is denoted by (L, M)-**DFCOS**. When $M = \mathbf{2}$, we obtain a category L-**DFCOS** (refer to Theorem 3.2(3) and Remark 2.14(1)). When $L = \mathbf{2}$, we obtain a category M-**DFCOS**.

Theorem 4.12. (L, M)-**CS** $\cong$ (L, M)-**RHS** $\cong$ (L, M)-**DFCOS**.

Proof. Let $\mathbb{F}_h$:(L, M)-**RHS**$\to$$(L, M)$-**CS** be defined as: $\mathbb{F}_h((X, \mathcal{H})) = (X, \mathcal{C}_{\mathcal{H}})$ and $\mathbb{F}_h(f) = f$ for all $(X, \mathcal{H}) \in O((L, M)$-**RHS**$)$ and $f \in \hom_{LM_{\mathbf{RHS}}}((X, \mathcal{H}_X), (Y, \mathcal{H}_Y))$.

Conversely, let $\mathbb{G}_h$:(L, M)-**CS**$\to$$(L, M)$-**RHS** be defined as: $\mathbb{G}_h((X, \mathcal{C})) = (X, \mathcal{H}_{\mathcal{C}})$ and $\mathbb{G}_h(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$-**CS**$)$ and $f \in \hom_{LM_{\mathbf{CS}}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

By Theorems 4.4–4.8, $\mathbb{G}_h \circ \mathbb{F}_h = \mathbb{I}_{LM_{\mathbf{RHS}}}$ and $\mathbb{F}_h \circ \mathbb{G}_h = \mathbb{I}_{LM_{\mathbf{CS}}}$. Thus (L, M)-**RHS**$\cong$ (L, M)-**CS**. Also, (L, M)-**DFCOS**$\cong$$(L, M)$-**CS** by Theorems 2.2 and 4.1(2). □

Corollary 4.13. *(1)* L-**CS** $\cong$ L-**RHS** $\cong$ L-**DFCOS**.
(2) M-**CS** $\cong$ M-**RHS** $\cong$ M-**DFCOS**. *In particular,* **RHS** $\cong$ **CS**.

Next, we discuss relations between L-**CS** and (L, M)-**CS**.

Let $(X, \mathcal{C}) \in \mathfrak{F}_{LM_{\mathbf{CS}}}(X)$ and $\&_{\mathcal{C}} = \{\bigwedge \varphi : \varphi \subseteq \bigcup_{a\in J(L)} \mathcal{C}_{[a]}\}$.

Clearly, we have $\underline{\perp}, \underline{\top} \in \&_{\mathcal{C}}$. If $\{A_i\}_{i\in\Omega} \subseteq \&_{\mathcal{C}}$, then there exists $\varphi_i \subseteq \bigcup_{a\in J(L)} \mathcal{C}_{[a]}$ such that $A_i = \bigwedge \varphi_i$ for each $i \in \Omega$. Thus $\varphi = \bigcup_{i\in\Omega} \varphi_i \subseteq \bigcup_{a\in J(L)} \mathcal{C}_{[a]}$ and $\bigwedge_{i\in\Omega} A_i = \bigwedge_{i\in\Omega} \bigwedge \varphi_i = \bigwedge \varphi \in \&_{\mathcal{C}}$. Hence $(X, \&_{\mathcal{C}})$ is an L-closure structure. By Theorem 3.3, $(X, \&_{\mathcal{C}})$ is a base of an L-convex space which is denoted by $(X, \&^*_{\mathcal{C}})$.

An (L, M)-fuzzy convex structure $\mathcal{C}$ on X is called an (L, M)-fuzzy crisp convex structure and $(X, \mathcal{C})$ is called an (L, M)-fuzzy crisp convex space, if there exists an L-convex structure $\mathcal{D}$ such that $\mathcal{C} = \chi_{\mathcal{D}}$, where $\chi_{\mathcal{D}}$ is the characterization function of $\mathcal{D}$.

Theorem 4.14. $\hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}),(Y,\mathcal{D})) \subseteq \hom_{L_{\mathbf{CS}}}((X,\&^*_{\mathcal{C}}),(Y,\&^*_{\mathcal{D}}))$.

Proof. Let $f \in \hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}),(Y,\mathcal{D}))$. If $B \in \&^*_{\mathcal{D}}$, then there exists an up-directed family $\{C_i\}^{dir}_{i\in\Omega} \subseteq \&_{\mathcal{D}}$ such that $B = \bigvee^{dir}_{i\in\Omega} C_i$. Further, there exists $\{D_{ij}\}_{j\in J_i} \subseteq \bigcup_{a\in J(L)} \mathcal{D}_{[a]}$ such that $C_i = \bigwedge_{j\in J_i} D_{ij}$ for each $i \in \Omega$. Thus $B = \bigvee^{dir}_{i\in\Omega} \bigwedge_{j\in J_i} D_{ij}$. Hence $f^{\leftarrow}_L(B) = f^{\leftarrow}_L(\bigvee^{dir}_{i\in\Omega} \bigwedge_{j\in J} D_{ij}) = \bigvee^{dir}_{i\in\Omega} \bigwedge_{j\in J_i} f^{\leftarrow}_L(D_{ij})$.

Since $D_{ij} \in \bigcup_{a\in J(L)} \mathcal{D}_{[a]}$ for each $i \in \Omega$ and each $j \in J_i$, there exists $a \in J(L)$ such that $\mathcal{D}(D_{ij}) \geq a$. Thus $\mathcal{C}(f^{\leftarrow}_L(D_{ij})) \geq \mathcal{D}(D_{ij}) \geq a$. Hence $f^{\leftarrow}_L(D_{ij}) \in \mathcal{C}_{[a]}$ and $f^{\leftarrow}_L(B) = \bigvee^{dir}_{i\in\Omega} \bigwedge_{j\in J_i} f^{\leftarrow}_L(D_{ij}) \in \&^*_{\mathcal{C}}$. Therefore $f : (X,\&^*_{\mathcal{C}}) \to (Y,\&^*_{\mathcal{D}})$ is an L-CP mapping. □

Theorem 4.15. $\hom_{L_{\mathbf{CS}}}((X,\mathcal{C}),(Y,\mathcal{D})) = \hom_{LM_{\mathbf{CS}}}((X,\chi_{\mathcal{C}}),(Y,\chi_{\mathcal{D}}))$ *for all* $(X,\mathcal{C})$, $(X,\mathcal{D}) \in \mathfrak{F}_{L_{\mathbf{CS}}}(X)$.

Define $\mathbb{E}_c$:L-**CS**→(L,M)-**CS** as: $\mathbb{E}_c((X,\mathcal{C})) = (X,\chi_{\mathcal{C}})$ and $\mathbb{E}(f) = f$ for all $(X,\mathcal{C}) \in O(L\text{-}\mathbf{CS})$ and all $f \in \hom_{L\mathbf{CS}}((X,\mathcal{C}),(Y,\mathcal{D}))$.

Conversely, define $\mathbb{T}$:(L,M)-**CS**→L-**CS** as: $\mathbb{T}((X,\mathcal{C})) = (X,\&^*_{\mathcal{C}})$ and $\mathbb{T}(f) = f$ for all $(X,\mathcal{C}) \in O((L,M)\text{-}\mathbf{CS})$ and all $f \in \hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}),(Y,\mathcal{D}))$.

Theorem 4.16. $(\mathbb{E}_c,\mathbb{T})$ *is a Galois's correspondence and* $\mathbb{T}$ *is a left inverse of* $\mathbb{E}_c$.

Proof. By Theorems 4.14 and 4.15, $\mathbb{E}_c \circ \mathbb{T} \ll \mathbb{I}_{LM_{\mathbf{CS}}}$ and $\mathbb{T} \circ \mathbb{E}_c = \mathbb{I}_{L_{\mathbf{CS}}}$. Therefore $(\mathbb{E}_c,\mathbb{T})$ is a Galois's correspondence and $\mathbb{T}$ is a left inverse of $\mathbb{E}_c$. □

Corollary 4.17. L-**CS** *is a coreflective subcategory of* (L,M)-**CS**. *In particular,* **CS** *is a coreflective subcategory of* M-**CS**.

Now, we introduce and characterize (L,M)-fuzzy concave spaces.

Definition 4.18. A mapping $\mathcal{T} : L^X \to M$ is an (L,M)-fuzzy concave structure and the pair $(X,\mathcal{T})$ is called an (L,M)-fuzzy concave space, if

(LMCA1) $\mathcal{T}(\underline{\perp}) = \mathcal{T}(\underline{\top}) = \top$;

(LMCA2) $\mathcal{T}(\bigvee_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathcal{T}(A_i)$ for each subset $\{A_i\}_{i\in\Omega} \subseteq L^X$;

(LMCA3) $\mathcal{T}(\bigwedge_{i\in\Omega} A_i) \geq \bigwedge_{i\in\Omega} \mathcal{T}(A_i)$ for each totally ordered subset $\{A_i\}_{i\in\Omega} \subseteq L^X$.

Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be (L,M)-fuzzy concave structures. A mapping $f : X \to Y$ is called an (L,M)-fuzzy concave preserving mapping, if $\mathcal{T}_X(f^{\rightarrow}_L(A)) \geq \mathcal{T}_Y(A)$ for all $A \in L^Y$. The category, consisting of (L,M)-fuzzy concave spaces as objects and (L,M)-fuzzy concave preserving mappings as morphisms, is denoted by (L,M)-**CAS**. In particular, when $M = \mathbf{2}$, then (L,M)-**CAS** is reduced to the category L-**CAS**, consisting of L-concave spaces as objects and L-concave preserving mappings as morphisms [19].

Theorem 4.19. (L,M)-**CS** $\cong$ (L,M)-**CAS**.

5. The category of (L,M)-fuzzy stratified convex spaces

Definition 5.1. An (L,M)-fuzzy convex structure $\mathcal{C}$ on X is called an (L,M)-fuzzy stratified convex structure and $(X,\mathcal{C})$ is called an (L,M)-fuzzy stratified convex space, if

(LMSC) $\mathcal{C}(\underline{\lambda}) = \top$ for all $\lambda \in L$.

The finest (L,M)-convex structures is an (L,M)-fuzzy stratified convex structure. Also, a mapping $\mathcal{C} : L^X \to M$, defined as: $\mathcal{C}(\underline{\lambda}) = \top$ for all $\lambda \in L$ and $\mathcal{C}(A) = \bot$ otherwise, is an (L,M)-fuzzy stratified convexity. The full subcategory of (L,M)-**CS**, consisting of (L,M)-fuzzy stratified convex spaces as objects and (L,M)-CP mappings as morphisms, is denoted by (L,M)-**SCS**. Its construct is denoted by $((L,M)\text{-}\mathbf{SCS}, \mathbb{C}_s)$.

Theorem 5.2. *Both $\mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$ and $\mathfrak{F}_{LM_{\mathbf{CS}}}(X)$ are complete lattices.*

Proof. Let $\{(X, \mathcal{C}_\lambda)\}_{\lambda \in \Lambda} \subseteq \mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$. Define $\mathcal{C}_{\wedge_s}, \mathcal{C}_{\vee_s} : L^X \to M$ as:

$$\begin{aligned} \forall A \in L^X,\ \ \mathcal{C}_{\wedge_s}(A) &= \bigwedge\{\mathcal{C}(A) : (X,\mathcal{C}) \in \mathfrak{F}_{LM_{\mathbf{SCS}}}(X), \forall \lambda \in \Lambda, \mathcal{C} \ll \mathcal{C}_\lambda\}; \\ \mathcal{C}_{\vee_s}(A) &= \bigwedge_{\lambda \in \Lambda} \mathcal{C}_\lambda(A). \end{aligned}$$

We have $(X, \mathcal{C}_{\wedge_s}), (X, \mathcal{C}_{\vee_s}) \in \mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$ with $\mathcal{C}_{\wedge_s} \ll \mathcal{C}_\lambda \ll \mathcal{C}_{\vee_s}$ for all $\lambda \in \Lambda$. If $(X, \mathcal{D}_1)$, $(X, \mathcal{D}_2) \in \mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$ satisfies $\mathcal{D}_1 \ll \mathcal{C}_\lambda \ll \mathcal{D}_2$ for all $\lambda \in \Lambda$, then $\mathcal{D}_1 \ll \mathcal{C}_{\wedge_s}$ and $\mathcal{C}_{\vee_s} \ll \mathcal{D}_2$. Thus $\mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$ is a complete lattice. Similarly, $\mathfrak{F}_{LM_{\mathbf{CS}}}(X)$ is a complete lattice. □

Theorem 5.3. *Both (L,M)-**SCS** and (L,M)-**CS** are topological categories.*

Proof. Let $(X, (f_k, (X_k, \mathcal{C}_{X_k}))_{k \in K})$ be a $\mathbb{C}_s$-structured sink (i.e., $\{(X_k, \mathcal{C}_{X_k})\}_{k \in K}$ is a family of (L,M)-fuzzy stratified convex spaces and $f_k : X_k \to X$ is a mapping for each $k \in K$). We need to prove that there exists an (L,M)-fuzzy stratified convex space $(X, \mathcal{C}_s)$ such that for each (L,M)-fuzzy stratified convex space $(Y, \mathcal{C}_Y)$ and each mapping $g : X \to Y$, $g \in \hom_{LM_{\mathbf{SCS}}}((X, \mathcal{C}_s), (Y, \mathcal{C}_Y))$ iff $g \circ f_k \in \hom_{LM_{\mathbf{SCS}}}((X_k, \mathcal{C}_{X_k}), (Y, \mathcal{C}_Y))$ for all $k \in K$. To this end, define $\mathcal{C}_s : L^X \to M$ as:

$$\forall A \in L^X,\ \mathcal{C}_s(A) = \bigwedge_{k \in K} \mathcal{C}_k(f_k^{\leftarrow}(A)).$$

We firstly check that $\mathcal{C}_s$ is an (L,M)-fuzzy stratified convex space.

(LMC1). $\mathcal{C}_s(\underline{\bot}_X) = \bigwedge_{k \in K} \mathcal{C}_k(\underline{\bot}_{X_k}) = \top$ and $\mathcal{C}_s(\underline{\top}_X) = \bigwedge_{k \in K} \mathcal{C}_k(\underline{\top}_{X_k}) = \top$.

(LMC2). Let $\{A_i\}_{i \in \Omega} \subseteq L^X$ and $A = \bigwedge_{i \in \Omega} A_i$. Then we have

$$\mathcal{C}_s(A) = \bigwedge_{k \in K} \mathcal{C}_k(\bigwedge_{i \in \Omega} (f_k)_L^{\leftarrow}(A_i)) \geq \bigwedge_{k \in K} \bigwedge_{i \in \Omega} \mathcal{C}_k((f_k)_L^{\leftarrow}(A_i)) = \bigwedge_{i \in \Omega} \mathcal{C}_s(A_i).$$

(LMC3). If $\{A_i\}_{i \in \Omega} \subseteq L^X$ is totally ordered and $A = \bigvee_{i \in \Omega} A_i$, then

$$\mathcal{C}_s(A) = \bigwedge_{k \in K} \mathcal{C}_k(\bigvee_{i \in \Omega} (f_k)_L^{\leftarrow}(A_i)) \geq \bigwedge_{k \in K} \bigwedge_{i \in \Omega} \mathcal{C}_k((f_k)_L^{\leftarrow}(A_i)) = \bigwedge_{i \in \Omega} \mathcal{C}_s(A_i).$$

(LMSC). If $\lambda \in L$, then $(f_k)_L^{\leftarrow}(\underline{\lambda}_X) = \underline{\lambda}_{X_k}$ for all $k \in K$. So $\mathcal{C}_s(\underline{\lambda}_X) = \bigwedge_{k \in K} \mathcal{C}_k((f_k)_L^{\leftarrow}(\underline{\lambda}_X)) = \bigwedge_{k \in K} \mathcal{C}_k(\underline{\lambda}_{X_k}) = \top$. Thus $\mathcal{C}_s$ is stratified.

If $g \in \hom_{LM_{\mathbf{SCS}}}((X, \mathcal{C}_s), (Y, \mathcal{C}_Y))$ and $B \in L^Y$, then

$$\mathcal{C}_Y(B) \leq \mathcal{C}_s(g_L^{\leftarrow}(B)) = \bigwedge_{k \in K} \mathcal{C}_k(((f_k)_L^{\leftarrow} \circ g_L^{\leftarrow})(B))) \leq \mathcal{C}_k(((f_k)_L^{\leftarrow} \circ g_L^{\leftarrow})(B))).$$

Conversely, if $g\circ f_k\in \hom_{LM_{\mathbf{SCS}}}((X_k,\mathcal{C}_{X_k}),(Y,\mathcal{C}_Y))$ for each $k\in K$, then for each $B\in L^Y$,

$$\mathcal{C}_Y(B)\le \bigwedge_{k\in K}\mathcal{C}_k(((f_k)_L^{\leftarrow}\circ g_L^{\leftarrow})(B))=\bigwedge_{k\in K}\mathcal{C}_k((f_k)_L^{\leftarrow}(g_L^{\leftarrow}(B)))=\mathcal{C}_s(g_L^{\leftarrow}(B)).$$

Therefore (L,M)-**SCS** is topological. Similarly, (L,M)-**CS** is topological. □

Corollary 5.4. *Let $(X,\mathcal{C}_X)\in\mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$ and $f:X\to Y$ be a mapping. Define $\mathcal{C}_X^*:L^Y\to M$ as: $\mathcal{C}_X^*(B)=\mathcal{C}_X(f_L^{\leftarrow}(B))$ for all $B\in L^Y$. Then $(X,\mathcal{C}_X^*)\in\mathfrak{F}_{LM_{\mathbf{SCS}}}(Y)$.*

Theorem 5.5. *Let $(X,\mathcal{C})\in\mathfrak{F}_{LM_{\mathbf{CS}}}(X)$ and $\overline{\mathcal{C}}:L^X\to M$ be defined as:*

$$\forall A\in L^X,\ \overline{\mathcal{C}}(A)=\bigwedge\{\mathcal{D}(A):(X,\mathcal{D})\in\mathfrak{F}_{LM_{\mathbf{SCS}}}(X),\mathcal{D}\ll\mathcal{C}\}.$$

Then $(X,\overline{\mathcal{C}})\in\mathfrak{F}_{LM_{\mathbf{SCS}}}(X)$ satisfying $\overline{\mathcal{C}}\ll\mathcal{C}$.

Corollary 5.6. *An (L,M)-fuzzy convex space $(X,\mathcal{C})$ is stratified iff $\overline{\mathcal{C}}=\mathcal{C}$.*

Theorem 5.7. $\hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))\subseteq\hom_{LM_{\mathbf{SCS}}}((X,\overline{\mathcal{C}_X}),(Y,\overline{\mathcal{C}_Y}))$.

Proof. Let $f\in\hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))$. By Corollary 5.4, we have $\overline{\mathcal{C}_X}^*(B)=\overline{\mathcal{C}_X}(f_L^{\leftarrow}(B))\ge\mathcal{C}_X(f_L^{\leftarrow}(B))\ge\mathcal{C}_Y(B)$ for all $B\in L^Y$. Thus $\mathcal{C}_Y\le\overline{\mathcal{C}_X}^*$ and $(Y,\overline{\mathcal{C}_X}^*)\in\mathfrak{F}_{LM_{\mathbf{SCS}}}(Y)$. Hence $\overline{\mathcal{C}_Y}(B)\le\overline{\mathcal{C}_X}^*(B)=\overline{\mathcal{C}_X}(f_L^{\leftarrow}(B))$. Therefore $f\in\hom_{LM_{\mathbf{SCS}}}((X,\overline{\mathcal{C}_X}),(Y,\overline{\mathcal{C}_Y}))$. □

Remark 5.8. (1) Let $\mathbb{E}_s$:(L,M)-**SCS**→(L,M)-**CS** be defined as: $\mathbb{E}_s((X,\mathcal{C}))=(X,\mathcal{C})$ and $\mathbb{E}_s(f)=f$ for all $(X,\mathcal{C})\in O((L,M)\text{-}\mathbf{SCS})$ and all $f\in\hom_{LM_{\mathbf{SCS}}}(\mathcal{C}_X,\mathcal{C}_Y)$. Since$(L,M)$-**SCS** is a full subcategory of (L,M)-**CS**, $\mathbb{E}_s$ is a full embedding satisfying $\mathbb{C}_s=\mathbb{C}\circ\mathbb{E}_s$:(L,M)-**SCS**→**Set**. So (L,M)-**SCS** is a concrete subcategory of (L,M)-**CS**.

(2) Let $\mathbb{S}_1$:(L,M)-**CS**→(L,M)-**SCS** be defined as: $\mathbb{S}_1((X,\mathcal{C}))=(X,\overline{\mathcal{C}})$ and $\mathbb{S}_1(f)=f$ for all $(X,\mathcal{C})\in O((L,M)\text{-}\mathbf{CS})$ and all $f\in\hom_{LM_{\mathbf{CS}}}(\mathcal{C}_X,\mathcal{C}_Y)$. By Theorems 5.5 and 5.7, $\mathbb{S}_1:((L,M)\text{-}\mathbf{SCS},\mathbb{C}_s)\to((L,M)\text{-}\mathbf{CS},\mathbb{C})$ is a concrete functor with $\mathbb{C}_s=\mathbb{C}\circ\mathbb{S}_1$.

Theorem 5.9. *$(\mathbb{E}_s,\mathbb{S}_1)$ is a Galois correspondence and $\mathbb{S}_1$ is a left inverse of $\mathbb{E}_s$.*

Proof. $\mathbb{E}_s\circ\mathbb{S}_1\ll\mathbb{I}_{LM_{\mathbf{CS}}}$ by Theorem 5.5, and $\mathbb{S}_1\circ\mathbb{E}_s=\mathbb{I}_{LM_{\mathbf{SCS}}}$ by Corollary 5.6. □

Corollary 5.10. *(L,M)-**SCS** is a coreflective subcategory of (L,M)-**CS**.*

6. The category of (L,M)-fuzzy weakly induced convex spaces

Definition 6.1. An (L,M)-fuzzy convex structure $\mathcal{C}$ on X is called an (L,M)-fuzzy weakly induced convex structure and the pair $(X,\mathcal{C})$ is called an (L,M)-fuzzy weakly induced convex space, if

(LMWIC) $\mathcal{C}(A)\le\bigwedge_{a\in L}\mathcal{C}(\chi_{A_{[a]}})$ for all $A\in L^X$.

The finest (L,M)-fuzzy convex structure is an (L,M)-fuzzy weakly induced convex structure. Also, if $(X,\mathcal{C})\in\mathfrak{F}_{M_{\mathbf{CS}}}$, then a mapping $\mathcal{C}:L^X\to M$, defined as: $\mathcal{C}(A)=\mathcal{C}(U)$ if $A=\chi_U$ for some $U\in 2^X$ and $\mathcal{C}(A)=\bot$ otherwise, is an (L,M)-fuzzy weakly induced convex structure.

The full subcategory of (L,M)-**CS**, consisting of (L,M)-fuzzy weakly induced convex spaces as objects and (L,M)-CP mappings as morphisms, is denoted by (L,M)-**WICS**. Its construct is denoted by $((L,M)$-**WICS**,$\mathbb{C}_w)$.

Theorem 6.2. $\mathfrak{F}_{LM_{\mathbf{WICS}}}(X)$ *is a complete lattice.*

Proof. Let $\{(X,\mathcal{C}_\lambda)\}_{\lambda\in\Lambda}\subseteq\mathfrak{F}_{LM_{\mathbf{WICS}}}(X)$. Define $\mathcal{C}_{\wedge_w},\mathcal{C}_{\vee_w}:L^X\to M$ as:

$$\forall A\in L^X,\ \ \mathcal{C}_{\wedge_w}(A) = \bigwedge\{\mathcal{C}(A):(X,\mathcal{C})\in\mathfrak{F}_{LM_{\mathbf{WICS}}}(X),\forall\lambda\in\Lambda,\mathcal{C}\ll\mathcal{C}_\lambda\}$$

$$\mathcal{C}_{\vee_w}(A) = \bigwedge_{\lambda\in\Lambda}\mathcal{C}_\lambda(A).$$

Then $(X,\mathcal{C}_{\wedge_w}),(X,\mathcal{C}_{\vee_w})\in\mathfrak{F}_{LM_{\mathbf{WICS}}}(X)$ with $\mathcal{C}_{\wedge_w}\ll\mathcal{C}_\lambda\ll\mathcal{C}_{\vee_w}$ for each $\lambda\in\Lambda$. Further, let $(X,\mathcal{D}_1),(X,\mathcal{D}_2)\in\mathfrak{F}_{LM_{\mathbf{WICS}}}(X)$ with $\mathcal{D}_1\ll\mathcal{C}_\lambda\ll\mathcal{D}_2$ for each $\lambda\in\Lambda$. Then $\mathcal{D}_1\ll\mathcal{C}_{\wedge_s}$ and $\mathcal{C}_{\vee_s}\ll\mathcal{D}_2$. Therefore $\mathfrak{F}_{LM_{\mathbf{WICS}}}(X)$ is a complete lattice. $\square$

Theorem 6.3. (L,M)-**WICS** *is a topological category.*

Proof. Let $(X,(f_k,(X_k,\mathcal{C}_k))_{k\in K})$ be a $\mathbb{C}_w$-structured sink, (that is, $\{(X_k,\mathcal{C}_k)\}_{k\in K}$ is a family of (L,M)-fuzzy weakly induced convex spaces and $f_k:X_k\to X$ is a mapping). We shall prove that there exists an (L,M)-fuzzy weakly induced convex space $(X,\mathcal{C}_w)$ such that for each (L,M)-fuzzy weakly induced convex space $(Y,\mathcal{C}_Y)$ and each mapping $g:X\to Y$, $g\in\hom_{LM_{\mathbf{WICS}}}((X,\mathcal{C}_w),(Y,\mathcal{C}_Y))$ iff $g\circ f_k\in\hom_{LM_{\mathbf{WICS}}}((X_k,\mathcal{C}_k),(Y,\mathcal{C}_Y))$ for all $k\in K$. To this end, define $\mathcal{C}_w:L^X\to M$ as:

$$\forall A\in L^X,\ \mathcal{C}_w(A)=\bigwedge_{k\in K}\mathcal{C}_k((f_k)_L^{\leftarrow}(A)).$$

It is easy to check that $\mathcal{C}_w$ is an (L,M)-fuzzy convex structure. In addition, for each $A\in L^X$, $\bigwedge_{a\in L}\mathcal{C}_w(\chi_{A_{[a]}})=\bigwedge_{a\in L}\bigwedge_{k\in K}\mathcal{C}_k(\chi_{(f_k)_L^{\leftarrow}(A)_{[a]}})\geq\bigwedge_{k\in K}\mathcal{C}_k((f_k)_L^{\leftarrow}(A))=\mathcal{C}_w(A)$. Thus $\mathcal{C}_w$ is weakly induced. In addition, similar to Theorem 5.3, $g\in\hom_{LM_{\mathbf{WICS}}}((X,\mathcal{C}_w),(Y,\mathcal{C}_Y))$ iff $g\circ f_k\in\hom_{LM_{\mathbf{WICS}}}((X_k,\mathcal{C}_k),(Y,\mathcal{C}_Y))$ for all $k\in K$. $\square$

Corollary 6.4. *If* $(X,\mathcal{C}_X)\in\mathfrak{F}_{LM_{\mathbf{WICS}}}(X)$ *and* $f:X\to Y$ *is a mapping, then* $(Y,\mathcal{C}_X^*)\in\mathfrak{F}_{LM_{\mathbf{WICS}}}(Y)$*, where* $\mathcal{C}_X^*:L^Y\to M$*:* $\mathcal{C}_X^*(B)=\mathcal{C}_X(f_L^{\leftarrow}(B))$ *for all* $B\in L^Y$.

Theorem 6.5. *Let* $(X,\mathcal{C})\in\mathfrak{F}_{LM_{\mathbf{CS}}}(X)$*. Define* $\underline{\mathcal{C}}:L^X\to M$ *as:*

$$\forall A\in L^X,\ \ \underline{\mathcal{C}}(A)=\bigwedge\{\mathcal{D}(A):(X,\mathcal{D})\in\mathfrak{F}_{LM_{\mathbf{WIC}}}(X),\mathcal{D}\ll\mathcal{C}\}.$$

Then $\underline{\mathcal{C}}$ *is an* (L,M)*-fuzzy weakly induced convex structure satisfying* $\underline{\mathcal{C}}\ll\mathcal{C}$.

Corollary 6.6. *An* (L,M)*-fuzzy convex space* $(X,\mathcal{C})$ *is weakly induced iff* $\underline{\mathcal{C}}=\mathcal{C}$.

Theorem 6.7. $\hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))\subseteq\hom_{LM_{\mathbf{WICS}}}((X,\underline{\mathcal{C}_X}),(Y,\underline{\mathcal{C}_Y}))$.

Proof. We have $\mathcal{C}_X \leq \underline{\mathcal{C}_X}$ by Theorem 6.5. If $f \in \hom_{LM_{\mathbf{CS}}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$, then $\underline{\mathcal{C}_X}(f_L^{\leftarrow}(B)) \geq \mathcal{C}_X(f_L^{\leftarrow}(B)) \geq \mathcal{C}_Y(B)$ for all $B \in L^Y$. Thus $\underline{\mathcal{C}_Y}(B) \leq \mathcal{C}_Y^*(B) = \mathcal{C}_X(f_L^{\leftarrow}(B)) \leq \underline{\mathcal{C}_X}(f_L^{\leftarrow}(B))$ by Corollary 6.4. So $f \in \hom_{LM_{\mathbf{WICS}}}((X, \underline{\mathcal{C}_X}), (Y, \underline{\mathcal{C}_Y}))$. □

Remark 6.8. (1) Let $\mathbb{E}_w$:(L, M)-**WICS**→(L, M)-**CS** be defined as: $\mathbb{E}_w((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{E}_w(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$-**WICS**$)$ and all $f \in \hom_{LM_{\mathbf{WICS}}}$ $((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. Since (L, M)-**WICS** is a full subcategory of (L, M)-**CS**, $\mathbb{E}_w$ is a full embedding and $\mathbb{C}_w = \mathbb{C} \circ \mathbb{E}_w$ is the forgetful functor of (L, M)-**WICS**. In this sense, (L, M)-**WICS** can be naturally regarded as a concrete subcategory of (L, M)-**CS**.

(2) Let $\mathbb{W}_1$:(L, M)-**CS**→(L, M)-**WICS** be defined as: $\mathbb{W}_1((X, \mathcal{C})) = (X, \underline{\mathcal{C}})$ and $\mathbb{W}_1(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)$-**CS**$)$ and $f \in \hom_{\mathbf{CS}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. By Theorems 6.5 and 6.7, $\mathbb{W}_1 : ((L, M)$-**WICS**$, \mathbb{C}_w) \to ((L, M)$-**CS**$, \mathbb{C})$ is a concrete functor with $\mathbb{C}_w = \mathbb{C} \circ \mathbb{W}_1$.

Theorem 6.9. $(\mathbb{E}_w, \mathbb{W}_1)$ *is a Galois correspondence, where* $\mathbb{W}_1$ *is a left inverse of* $\mathbb{E}_w$.

Proof. We have $\mathbb{E}_w \circ \mathbb{W}_1 \ll \mathbb{I}_{LM_{\mathbf{CS}}}$ by Theorem 6.5 and $\mathbb{W}_1 \circ \mathbb{E}_w = \mathbb{I}_{LM_{\mathbf{WICS}}}$ by Corollary 6.6. Thus $(\mathbb{E}_w, \mathbb{W}_1)$ is a Galois correspondence and $\mathbb{W}_1$ is a left inverse of $\mathbb{E}_w$. □

Corollary 6.10. (L, M)-**WICS** *is concretely coreflective in* (L, M)-**CS**.

Theorem 6.11. *(1)* L-**SCS** *is coreflective in* (L, M)-**SCS**.
(2) L-**WICS** *is coreflective in* (L, M)-**WICS**.

7. The category of (L, M)-fuzzy induced convex spaces

In this section, L is a β-lattice, that is, $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for all $a, b \in L$.

Definition 7.1. An (L, M)-fuzzy convex structure $\mathcal{C}$ on X is called an (L, M)-fuzzy induced convex structure and the pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy induced convex space, if $\mathcal{C}(A) = \bigwedge_{a \in L} \mathcal{C}(\chi_{A_{[a]}})$ for all $A \in L^X$.

Clearly, an (L, M)-fuzzy induced convex space is both weakly induced and stratified.

Let $(X, \mathcal{C}) \in \mathfrak{F}_{LM_{\mathbf{CS}}}(X)$. Define $\varphi_{\mathcal{C}} : 2^X \to M$ as: $\varphi_{\mathcal{C}}(U) = \bigvee_{a \in L} \bigvee_{A_{[a]} = U} \mathcal{C}(A)$ for all $U \subseteq 2^X$. The M-fuzzifying convex structure with $\varphi_{\mathcal{C}}$ as its subbase is denoted by $\iota(\mathcal{C})$.

Theorem 7.2. *Let* $(X, \mathcal{C}) \in \mathfrak{F}_{LM_{\mathbf{CS}}}(X)$*. Define* $[\mathcal{C}] : 2^X \to M$ *as:* $[\mathcal{C}](U) = \mathcal{C}(\chi_U)$ *for all* $U \in 2^X$*. Then* $[\mathcal{C}]$ *is an* M*-fuzzifying convex structure on* X.

Theorem 7.3. [32] *Let* $(X, \mathcal{C}) \in \mathfrak{F}_{M_{\mathbf{CS}}}(X)$*, and* $\mathcal{C}_{\mathcal{C}} : L^X \to M$ *be defined as:*

$$\forall A \in L^X, \quad \mathcal{C}_{\mathcal{C}}(A) = \bigwedge_{a \in L} \mathcal{C}(A_{[a]}).$$

Then $(X, \mathcal{C}_{\mathcal{C}})$ *is an* (L, M)*-fuzzy convex space induced by* $\mathcal{C}$.

Theorem 7.4. *Let* $(X,\mathcal{C})\in\mathfrak{F}_{LM_{\mathbf{CS}}}(X)$ *and* $(X,\mathcal{C})\in\mathfrak{F}_{M_{\mathbf{CS}}}(X)$. *Then*

(1) $[\mathcal{C}]\leq\varphi_{\mathcal{C}}\leq\iota(\mathcal{C})$;

(2) $\mathcal{C}$ *is weakly induced iff* $[\mathcal{C}]=\iota(\mathcal{C})$;

(3) $\iota(\mathcal{C}_{\mathcal{C}})=[\mathcal{C}_{\mathcal{C}}]=\mathcal{C}$ *and* $\mathcal{C}\leq\mathcal{C}_{\iota(\mathcal{C})}$;

(4) $\mathcal{C}$ *is induced by an* M*-fuzzifying convex structure iff* $\mathcal{C}_{\iota(\mathcal{C})}=\mathcal{C}$ *iff* $\mathcal{C}_{[\mathcal{C}]}=\mathcal{C}$ *iff* $\mathcal{C}$ *is an* (L,M)*-fuzzy induced convex structure.*

Proof. (1) is direct. Also, (4) directly follows from (3).

(2): If $\mathcal{C}$ is weakly induced, then $\varphi_{\mathcal{C}}(U)\leq\bigvee_{a\in L}\bigvee_{A_{[a]}=U}\bigwedge_{b\in L}\mathcal{C}(\chi_{A_{[b]}})\leq[\mathcal{C}](U)$ for all $U\in 2^X$. So $\varphi_{\mathcal{C}}(U)=[\mathcal{C}](U)$ by (1), and $\iota(\mathcal{C})=[\mathcal{C}]$. Conversely, if $\iota(\mathcal{C})=[\mathcal{C}]$, then $\bigwedge_{a\in L}\mathcal{C}(\chi_{A_{[a]}})=\bigwedge_{a\in L}[\mathcal{C}](A_{[a]})=\bigwedge_{a\in L}\iota(\mathcal{C})(A_{[a]})\geq\mathcal{C}(A)$. Thus $\mathcal{C}$ is weakly induced.

(3): $\varphi_{\mathcal{C}_{\mathcal{C}}}(U)=\bigvee_{a\in L}\bigvee_{A_{[a]}=U}\bigwedge_{b\in L}\mathcal{C}(A_{[b]})\leq\bigvee_{a\in L}\bigvee_{A_{[a]}=U}\mathcal{C}(U)=\mathcal{C}(U)$. Conversely, $\varphi_{\mathcal{C}_{\mathcal{C}}}(U)=\bigvee_{a\in L}\bigvee_{A_{[a]}=U}\mathcal{C}_{\mathcal{C}}(A)\geq\mathcal{C}_{\mathcal{C}}(\chi_U)=\mathcal{C}(U)$. Thus $\varphi_{\mathcal{C}_{\mathcal{C}}}(U)=\mathcal{C}(U)$, which shows $\varphi_{\mathcal{C}_{\mathcal{C}}}=\mathcal{C}$. Hence $\iota(\mathcal{C}_{\mathcal{C}})=\mathcal{C}$. Also, $[\mathcal{C}_{\mathcal{C}}](U)=\bigwedge_{a\in L}\mathcal{C}((\chi_U)_{[a]})=\mathcal{C}(U)$. Thus $[\mathcal{C}_{\mathcal{C}}]=\mathcal{C}$. Finally, $\mathcal{C}_{\iota(\mathcal{C})}(A)=\bigwedge_{a\in L}\iota(\mathcal{C})(A_{[a]})\geq\bigwedge_{a\in L}\varphi_{\mathcal{C}}(A_{[a]})\geq\mathcal{C}(A)$. □

Theorem 7.5. *(1)* $\hom_{M_{\mathbf{CS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))\subseteq\hom_{LM_{\mathbf{ICS}}}((X,\mathcal{C}_{\mathcal{C}_X}),(Y,\mathcal{C}_{\mathcal{C}_Y}))$.

(2) $\hom_{LM_{\mathbf{CS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))\subseteq\hom_{M_{\mathbf{CS}}}((X,\iota(\mathcal{C}_X)),(Y,\iota(\mathcal{C}_Y)))$.

Remark 7.6. (1) The full subcategory of (L,M)-**CS**, consisting of (L,M)-fuzzy induced convex spaces as objects and (L,M)-CP mappings as morphisms, is denoted by (L,M)-**ICS**. Its construct is denoted by $((L,M)$-**ICS**, $\mathbb{C}_i)$; the full subcategory of (L,M)-**CS**, consisting of (L,M)-fuzzy convex spaces induced by M-fuzzifying convex spaces as objects and (L,M)-CP mappings as morphisms, is denoted by (L,M)-**CGCS**.

(2) Define $\mathbb{E}_i^1$:(L,M)-**ICS**→(L,M)-**SCS** as: $\mathbb{E}_i^1((X,\mathcal{C}))=(X,\mathcal{C})$ and $\mathbb{E}_i^1(f)=f$ for all $(X,\mathcal{C})\in O((L,M)$-**ICS**$)$ and all $f\in\hom_{LM_{\mathbf{ICS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))$. Since (L,M)-**ICS** is a full subcategory of (L,M)-**SCS**, $\mathbb{E}_i^1$ is a full embedding with $\mathbb{C}_i=\mathbb{C}_s\circ\mathbb{E}_i^1$. Thus (L,M)-**ICS** is a concrete subcategory of (L,M)-**SCS**.

(3) Define $\mathbb{E}_i^2$:(L,M)-**ICS**→(L,M)-**WICS** as: $\mathbb{E}_i^2((X,\mathcal{C}))=(X,\mathcal{C})$ and $\mathbb{E}_i^2(f)=f$ for $(X,\mathcal{C})\in O((L,M)$-**ICS**$)$ and all $f\in\hom_{LM_{\mathbf{ICS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))$. Since $\mathbb{E}_i^2$ is a full embedding with $\mathbb{C}_i=\mathbb{C}_w\circ\mathbb{E}_i^2$, (L,M)-**ICS** is a concrete subcategory of (L,M)-**WICS**.

(4) Define $\mathbb{S}_2$:(L,M)-**SCS**→(L,M)-**ICS** as: $\mathbb{S}_2((X,\mathcal{C}))=(X,\mathcal{C}_{\iota(\mathcal{C})})$ and $\mathbb{S}_2(f)=f$ for all $(X,\mathcal{C})\in O((L,M)$-**SCS**$)$ and all $f\in\hom_{LM_{\mathbf{SCS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))$.

(5) Define $\mathbb{W}_2$:(L,M)-**WICS**→(L,M)-**ICS** as: $\mathbb{W}_2((X,\mathcal{C}))=(X,\mathcal{C}_{[\mathcal{C}]})$ and $\mathbb{W}_2(f)=f$ for $(X,\mathcal{C})\in O((L,M)$-**WICS**$)$ and $f\in\hom_{LM_{\mathbf{WICS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))$.

Theorem 7.7. M*-***CS** $\cong$ (L,M)*-***ICS** $\cong$ (L,M)*-***CGCS**.

Proof. Define $\mathbb{F}_3$:M-**CS**→(L,M)-**ICS** as: $\mathbb{F}_3((X,\mathcal{C}))=(X,\mathcal{C}_{\mathcal{C}})$ and $\mathbb{F}_3(f)=f$ for all $(X,\mathcal{C})\in O(M$-**CS**$)$ and all $f\in\hom_{M_{\mathbf{CS}}}((X,\mathcal{C}_X),(Y,\mathcal{C}_Y))$.

Conversely, define $\mathbb{G}_3$:(L,M)-**ICS**→M-**CS** as: $\mathbb{G}_3((X,\mathcal{C}))=(X,\iota(\mathcal{C}))$ and $\mathbb{G}_3(f)=f$ for all $(X,\mathcal{C})\in O((L,M)$-**ICS**$)$ and all $f\in\hom_{LM_{\mathbf{ICS}}}((X,\mathcal{C}_X),(Y,\mathcal{D}_Y))$. By Theorems 7.3-7.5, $\mathbb{F}_3,\mathbb{G}_3$ are concrete functors satisfying $\mathbb{F}_3\circ\mathbb{G}_3=\mathbb{I}_{LM_{\mathbf{ICS}}}$ and $\mathbb{G}_3\circ\mathbb{F}_3=\mathbb{I}_{M_{\mathbf{CS}}}$. Thus (L,M)-**ICS** and M-**CS** are isomorphic.

Define $\mathbb{F}_4$:(L, M)-**CGCS**$\rightarrow(L, M)$-**ICS** as: $\mathbb{F}_4((X, \mathcal{C})) = (X, \mathcal{C})$ and $\mathbb{F}_4(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-}\mathbf{CGCS})$ and all $f \in \hom_{LM_{\mathbf{CGCS}}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$.

Conversely, define $\mathbb{G}_4$:(L, M)-**ICS**$\rightarrow(L, M)$-**CGCS** as: $\mathbb{G}_4((X, \mathcal{C})) = (X, \mathcal{C}_{[\mathcal{C}]})$ and $\mathbb{G}_4(f) = f$ for all $(X, \mathcal{C}) \in O((L, M)\text{-}\mathbf{ICS})$ and $f \in \hom_{LM_{\mathbf{ICS}}}((X, \mathcal{C}_X), (Y, \mathcal{C}_Y))$. By Theorem 7.4(4), $\mathbb{F}_4 \circ \mathbb{G}_4 = \mathbb{I}_{LM_{\mathbf{ICS}}}$ and $\mathbb{G}_4 \circ \mathbb{F}_4 = \mathbb{I}_{LM_{\mathbf{CGCS}}}$. Hence $\mathbb{F}_4$ is isomorphic. □

Theorem 7.8. *(1) $(\mathbb{E}_i^1, \mathbb{S}_2)$ is a Galois correspondence and $\mathbb{S}_2$ is a left inverse of $\mathbb{E}_i^1$.*

(2) $(\mathbb{E}_i^2, \mathbb{W}_2)$ is a Galois correspondence and $\mathbb{W}_2$ is a left inverse of $\mathbb{E}_i^2$.

Corollary 7.9. *(1) $(\mathbb{E}_i^1 \circ \mathbb{F}_3, \mathbb{G}_3 \circ \mathbb{S}_2)$ is a Galois correspondence and $\mathbb{G}_3 \circ \mathbb{S}_2$ is a left inverse of $\mathbb{E}_i^1 \circ \mathbb{F}_3$. So M-**CS** can be embedded in (L, M)-**SCS** as a coreflective subcategory.*

*(2) $(\mathbb{E}_i^2 \circ \mathbb{F}_3, \mathbb{G}_3 \circ \mathbb{W}_2)$ is a Galois correspondence and $\mathbb{G}_3 \circ \mathbb{W}_2$ is a left inverse of $\mathbb{E}_i^2 \circ \mathbb{F}_3$. So M-**CS** can be embedded in (L, M)-**WICS** as a coreflective subcategory.*

8. Conclusion

Fuzzy convex spaces has been studied in various aspects [16, 20, 21, 24, 25, 30, 31, 32, 36, 37, 38, 39, 41, 42, 40]. Apparently, studies on M-fuzzifying convex spaces are in a deeper and broader level than that on (L, M)-fuzzy convex spaces. This because that domain finiteness of convex spaces has been introduced successfully in M-fuzzifying convex spaces [31]. Yet, it hasn't been studied in either L-convex convex spaces or (L, M)-fuzzy convex spaces. We define and characterize domain finiteness of both L-convex spaces and (L, M)-fuzzy convex spaces. We find that domain finiteness of (L, M)-fuzzy convex spaces perfectly matches that of L-convex spaces (resp. M-fuzzifying convex spaces) when $M = \mathbf{2}$ (resp. $L = \mathbf{2}$).

Research on fuzzy convex spaces in a view point of category is also meaningful. Pang and Shi have studied relations among the category of L-convex spaces and its subcategories [18]. Since the notion of (L, M)-fuzzy convex spaces has been introduced [32], many new categories related to (L, M)-fuzzy convex spaces emerged. Thus relations among the category and subcategories of (L, M)-fuzzy convex spaces need to be discussed. In this paper, we introduce the category and several subcategories of (L, M)-fuzzy convex spaces. In a view of category aspect, we study their relations which can be showed by the diagram and tables in the following page where $L(\beta)$ stands for L if L is a β-lattice. Clearly, it contains the diagram described in [18]. Also, we conclude from Theorems 5.2, 5.3, 6.2 and 6.3 that all categories in the diagram are topological constructs whose fibres are complete lattices.

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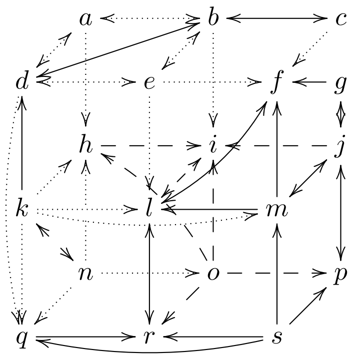

FIGURE 1. Relations

Symbol	Meaning
$\longrightarrow$	$\xrightarrow{coreflective}$
$\cdots\cdots>$	$\xrightarrow[L(\beta)]{coreflective}$
$\longleftrightarrow$	$\xleftrightarrow{\cong}$
— —	——

TABLE 1. Symbols

No.	Category	No.	Category
a	(L, M)-**CGCS**	j	L-**DFCOS**
b	M-**RHS**	k	**CS**
c	M-**DFCOS**	l	(L, M)-**CS**
d	M-**CS**	m	L-**CS**
e	(L, M)-**ICS**	n	**RHS**
f	(L, M)-**CAS**	o	L-**SCS**
g	L-**CAS**	p	L-**RHS**
h	(L, M)-**SCS**	q	(L, M)-**WICS**
i	(L, M)-**RHS**	r	(L, M)-**DFCOS**
s	L-**WICS**		

TABLE 2. Categories

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